

# Quantum Markov fields on graphs

Luigi Accardi,  
Centro Vito Volterra,  
Università di Roma “Tor Vergata”,  
Roma I-00133, Italy  
E-mail: [accardi@volterra.uniroma2.it](mailto:accardi@volterra.uniroma2.it)

Hiromichi Ohno,  
Faculty of Engineering, Shinshu University  
4-17-1 Wakasato, Nagano 380-8553, Japan  
E-mail: [h\\_ohno@shinshu-u.ac.jp](mailto:h_ohno@shinshu-u.ac.jp)

Farrukh Mukhamedov,  
Department of Computational & Theoretical Sciences,  
Faculty of Science, International Islamic University Malaysia,  
P.O. Box, 141, 25710, Kuantan, Pahang, Malaysia  
E-mail: [far75m@yandex.ru](mailto:far75m@yandex.ru), [farrukh.m@iiu.edu.my](mailto:farrukh.m@iiu.edu.my)

## Abstract

We introduce generalized quantum Markov states and generalized  $d$ -Markov chains which extend the notion quantum Markov chains on spin systems to that on  $C^*$ -algebras defined by general graphs. As examples of generalized  $d$ -Markov chains, we construct the entangled Markov fields on tree graphs. The concrete examples of generalized  $d$ -Markov chains on Cayley trees are also investigated.

*Mathematics Subject Classification:* 46L53, 60J99, 46L60, 60G50, 82B10, 81Q10, 94A17.

*Key words:* generalized Markov state; graph; entangled markov fields;  $d$ -Markov chains; Cayley tree.

## 1 Introduction

Markov fields play an important role in classical probability, in physics, in biological and neurological models and in an increasing number of technological problems such as image recognition.

It is quite natural to forecast that the quantum analogue of these models will also play a relevant role.

The papers [13], [3], [4],[7] are a first attempts to construct a quantum analogue of classical Markov fields. These papers extend to fields the notion of *quantum Markov state* introduced in [6] as a sub-class of the *quantum Markov chains* introduced in [1]. As remarked in [13], the peculiarity of the former class of states with respect to the latter consists in the fact that they admit a Umegaki conditional expectation *into* rather than *onto* their range.

This small difference allows, when applied to states on infinite tensor products of  $C^*$ -algebras, to obtain nontrivial (i.e. non product) states while maintaining most of the simple algebraic properties related to classical Markovianity.

The prize one has to pay for this simplification is that the resulting class of states, although non trivial, has very poor entanglement properties so that they cannot exhibit some of the most interesting properties which distinguish the quantum from the classical world.

On the contrary the quantum Markov chains or, more generally, the *generalized quantum Markov states* in the sense of [15] may exhibit very strong entanglement properties. In particular the paper [14] shows that this is indeed the case for the *entangled Markov chains* constructed in [2]. A degree of entanglement of entangled Markov chains is considered in [8].

The above considerations naturally suggest the study of following two problems:

- (i) the extension to fields of the notion of generalized Markov state (or Markov chain)
- (ii) the extension to fields of the construction of entangled Markov chains produced in [2]

The present paper is a first step towards the solution of these problems. We introduce a hierarchy of notions of Markovianity for states on discrete infinite tensor products of  $C^*$ -algebras (Section 4) and for each of these notions we construct some explicit examples. We show that the construction of [2] can be generalized to trees (Section 5). It is interesting to notice that, in a different context and for quite different purposes, the special role of trees was already emphasized in [13]. Note that in [11] finitely correlated states are constructed as ground states of VBS-model on Cayley tree. As well as, such shift invariant  $d$ -Markov chains can be also considered as an extension of  $C^*$ -finitely correlated states defined in [12] to the Cayley trees. In the classical case, Markov fields on trees are also considered in [16]-[20].

A comment on the notion of generalized quantum Markov state introduced in Definition 4.1 may help understanding the logic leading to this definition and in particular condition (4.8) which otherwise might, at first sight, seem artificial.

The point is that, as we know from Dobrushin's seminal work [10], the natural localization for fields on a discrete set  $L$  is given by the finite subsets of  $L$  and their complements. This localization, when restricted to the 1-dimensional case, does not lead to the usual probabilistic localization but, in a certain sense to its *dual* (or *time reversal*), corresponding to the conditioning of the past on the future rather than conversely. This leads to different structures of the Markov chains in the two cases, a fact already noted in [1] where these two types were called *Markov chains* and *inverse Markov chains* respectively.

In particular the role played by the time zero algebra in the usual Markov processes is played by the algebra at infinity in the multi-dimensional case.

But, while the time zero algebra has a meaning independent of the state, the algebra at infinity can be (meaningfully) defined only in the GNS representation of the given state. Therefore, if one wants to give a constructive and local definition of a state one cannot make use of a global notion such as the algebra at infinity.

In the ergodic cases, corresponding physically to the *pure phases* in Dobrushin's theory, one expects that the algebra at infinity is trivial and that the sequence of conditional expectations appearing in (4.8) converges weakly to a single state (asymptotic independence of the boundary) so that the resulting state is in fact independent of the sequence of states  $(\hat{\varphi}_{\Lambda_n^c})$  which plays the role of the single "state"  $\hat{\varphi}_{L^c} = \hat{\varphi}_\infty$ , not available at a  $C^*$ -level.

Let us briefly mention about the organization of the paper. In Sections 2 and 3, we introduce definition of graphs and bundles of graphs, and in Section 4 *generalized quantum Markov states* and *d-Markov chains* on graphs are defined. In the further Sections 5 we provide examples of generalized quantum Markov chains which extend the entangled Markov chains, defined in [2], to tree graphs and general graphs. In Section 6, we consider a particular case of tree, so called Cayley tree. Over such a tree we give a construction of  $d$ -Markov chains, in next sections 7 and 8 we provide some more concrete examples of such chains, which are shift invariant and have the clustering property.

## 2 Graphs

Let  $\mathcal{G} = (L, E)$  be a (non-oriented simple) graph, that is,  $L$  is a non-empty at most countable set and

$$E \subset \{\{x, y\} : x, y \in L, x \neq y\}.$$

Elements of  $L$  and of  $E$  are called *vertices* and *edges*, respectively. Two vertices  $x, y \in L$  are called *adjacent*, or *nearest neighbors*, if  $\{x, y\} \in E$ , and in that case we also write  $x \sim y$ .

For each  $x \in L$ , the set of nearest neighbors of  $x$  will be denoted by

$$N(x) := \{y \in L : y \sim x\}.$$

The *degree* of  $x \in L$ , denoted by  $\kappa(x)$ , is the number of vertices adjacent to  $x$ , namely,

$$\kappa(x) := |N(x)| = |\{y \in L : y \sim x\}|,$$

where  $|\cdot|$  is the cardinality.

A graph can be equivalently assigned by giving the pair

$$(L, \sim)$$

of its vertices and the binary symmetric relation  $\sim$ .

A *path* or a *trajectory* or a *walk* connecting two points  $x, y \in L$  is a finite sequence of vertices such that  $x = x_1 \sim x_2 \sim \cdots \sim x_n = y$ . In this case  $n - 1$  is called the *length* of the walk. For two distinct vertices  $x, y \in L$ , the distance  $\text{dist}(x, y)$  is defined to be the shortest length of a walk connecting  $x$  and  $y$ . By definition  $\text{dist}(x, x) = 0$ .

Throughout the paper we always assume that a graph is locally finite, i.e.,  $\kappa(x) < \infty$  for all  $x \in L$ , and is connected, i.e., for any pair of vertices, there exists a walk connecting them. We will write

$$\Lambda \subseteq_{\text{fin}} L, \quad \Lambda \subseteq_{\text{fin,c}} L$$

to mean that  $\Lambda$  is a finite subset and a finite connected subset of  $L$ , respectively. Given  $\Lambda \subseteq_{\text{fin}} L$  we define the external boundary of  $\Lambda$  by

$$\vec{\partial}\Lambda := \{x \in \Lambda^c : y \sim x, \exists y \in \Lambda\}$$

and the closure of  $\Lambda$  by

$$\bar{\Lambda} := \Lambda \cup \vec{\partial}\Lambda.$$

We will write

$$\Lambda \subset\subset \Lambda_1$$

to mean that  $\bar{\Lambda} \subset \Lambda_1$ . Notice that, by definition

$$\Lambda \cap \vec{\partial}\Lambda = \emptyset,$$

$$\vec{\partial}\{x\} =: \vec{\partial}x = N(x).$$

### 3 Bundles on graphs

To each  $x \in L$  it is associated an Hilbert space  $\mathcal{H}_x$  of dimension  $d_{\mathcal{H}}(x) \in \mathbb{N}$ . In the present paper we will assume that

$$d := d_{\mathcal{H}}(x) = d_{\mathcal{H}} < +\infty \quad (\text{independent of } x).$$

Given  $\Lambda \subseteq_{\text{fin}} L$  we define

$$\mathcal{H}_{\Lambda} := \bigotimes_{x \in \Lambda} \mathcal{H}_x.$$

For each  $x$  in  $L$ , we fix an orthonormal basis of  $\mathcal{H}_x$ :

$$\{e_j(x)\} \quad ; \quad j \in S(x) := \{1, \dots, d\}.$$

When we consider  $S$  as a total space,  $\pi_S : S \rightarrow L$  is the bundle whose fibers are the finite sets  $\pi_S^{-1}(x) := S(x)$  and the sections of this bundle are the maps:

$$\mathcal{F}(\Lambda, S) := \{\omega_{\Lambda} : x \in \Lambda \mapsto \omega_{\Lambda}(x) \in S(x)\}.$$

A section  $\omega_{\Lambda}$  is also called a *configuration* in the volume  $\Lambda$ . For  $\omega_{\Lambda} \in \mathcal{F}(\Lambda, S)$ , the vector  $e_{\omega_{\Lambda}}$  is defined by

$$e_{\omega_{\Lambda}} := \bigotimes_{x \in \Lambda} e_{\omega_{\Lambda}(x)}(x) \in \mathcal{H}_{\Lambda} \quad (3.1)$$

and we will use the symbol  $P_{\omega_{\Lambda}}$  for the corresponding rank one projection:

$$P_{\omega_{\Lambda}} := |e_{\omega_{\Lambda}}\rangle\langle e_{\omega_{\Lambda}}| = e_{\omega_{\Lambda}} e_{\omega_{\Lambda}}^*. \quad (3.2)$$

Then the set

$$\{e_{\omega_{\Lambda}} : \omega_{\Lambda} \in \mathcal{F}(\Lambda, S)\} \quad (3.3)$$

is an orthonormal basis of  $\mathcal{H}_{\Lambda}$ . Thus the generic vector of  $\mathcal{H}_{\Lambda}$  has the form

$$\sum_{\omega_{\Lambda} \in \mathcal{F}(\Lambda, S)} \lambda_{\omega_{\Lambda}} e_{\omega_{\Lambda}}.$$

We will use the notation

$$\mathcal{B}_{\Lambda} := \mathcal{B}(\mathcal{H}_{\Lambda})$$

for each  $\Lambda \subseteq_{\text{fin}} L$  and  $\mathcal{B}_L$  is the inductive  $C^*$ -algebra, that is,

$$\mathcal{B}_L := \varinjlim \mathcal{B}_{\Lambda}$$

for  $\Lambda \uparrow L$ . As a  $C^*$ -algebra  $\mathcal{B}_L$  is isomorphic to the (unique) infinite  $C^*$ -tensor product  $\bigotimes_{x \in L} \mathcal{B}_x$ , the natural embedding of  $\mathcal{B}_x$  into  $\mathcal{B}_L$  will be denoted by

$$j_x : b \in \mathcal{B}_x \mapsto j_x(b) = b \otimes I_{\{x\}^c} \in \mathcal{B}_L. \quad (3.4)$$

Similarly, for  $\Lambda \subseteq_{\text{fin}} L$ , we define

$$j_{\Lambda} := \bigotimes_{x \in \Lambda} j_x.$$

To simplify the notations, in the following we will often identify each  $\mathcal{B}_\Lambda$  to the subalgebra  $j_\Lambda(\mathcal{B}_\Lambda)$  of  $\mathcal{B}_L$ , through the identification

$$\mathcal{B}_\Lambda \equiv \mathcal{B}_\Lambda \otimes I_{\Lambda^c} = j_\Lambda(\mathcal{B}_\Lambda).$$

With these notations the elements of the  $*$ -subalgebra of  $\mathcal{B}_L$  defined by

$$\mathcal{B}_{L,loc} := \bigcup_{\Lambda \subseteq_{\text{fin}} L} \mathcal{B}_\Lambda$$

will be called a local algebra or local operators (observables if self-adjoint).

In what follows, by  $\mathcal{S}(\mathcal{B}_\Lambda)$  we will denote the set of all states defined on the algebra  $\mathcal{B}_\Lambda$ .

## 4 Definition of generalized quantum Markov state

Consider a triplet  $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$  of unital  $C^*$ -algebras. Recall that a *quasi-conditional expectation* with respect to the given triplet is a completely positive identity preserving linear (CP1) map  $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{B}$  such that

$$\mathcal{E}(ca) = c\mathcal{E}(a), \quad a \in \mathcal{A}, c \in \mathcal{C}. \quad (4.5)$$

Notice that, as the quasi-conditional expectation  $\mathcal{E}$  is a real map, one has

$$\mathcal{E}(ac) = \mathcal{E}(a)c, \quad a \in \mathcal{A}, c \in \mathcal{C}.$$

as well.

**Definition 4.1.** *A state  $\varphi$  on  $\mathcal{B}_L$  is called a generalized quantum Markov state on  $\mathcal{B}_L$  if there exist an increasing sequence of finite sets  $\Lambda_n \uparrow L$  with  $\Lambda_n \subset \Lambda_{n+1}$  and, for each  $\Lambda_n$ , a quasi-conditional expectation  $\mathcal{E}_{\Lambda_n^c}$  with respect to the triplet*

$$\mathcal{B}_{\Lambda_n^c} \subseteq \mathcal{B}_{\Lambda_n} \subseteq \mathcal{B}_{\Lambda_{n-1}^c} \quad (4.6)$$

and a state

$$\hat{\varphi}_{\Lambda_n^c} \in \mathcal{S}(\mathcal{B}_{\Lambda_n^c})$$

such that for any  $n \in \mathbb{N}$  one has

$$\hat{\varphi}_{\Lambda_n^c}|_{\mathcal{B}_{\Lambda_{n+1} \setminus \Lambda_n}} = \hat{\varphi}_{\Lambda_{n+1}^c} \circ \mathcal{E}_{\Lambda_{n+1}^c}|_{\mathcal{B}_{\Lambda_{n+1} \setminus \Lambda_n}} \quad (4.7)$$

and

$$\varphi = \lim_{n \rightarrow \infty} \hat{\varphi}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_{n-1}^c} \circ \cdots \circ \mathcal{E}_{\Lambda_1^c} \quad (4.8)$$

in the weak- $*$  topology.

In this definition, a generalized quantum Markov state  $\varphi$  generated by  $\mathcal{E}_{\Lambda_n^c}$  and  $\varphi_{\Lambda_n^c}$  is well-defined. Indeed, we have

$$\hat{\varphi}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_n^c}|_{\mathcal{B}_{\Lambda_n}} = \hat{\varphi}_{\Lambda_{n+1}^c} \circ \mathcal{E}_{\Lambda_{n+1}^c} \circ \mathcal{E}_{\Lambda_n^c}|_{\mathcal{B}_{\Lambda_n}}$$

by (4.7) and a following remark so that, for  $\Lambda \subset \Lambda_k$  and  $a \in \mathcal{B}_\Lambda$ ,

$$\lim_{n \rightarrow \infty} \hat{\varphi}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_{n-1}^c} \circ \cdots \circ \mathcal{E}_{\Lambda_1^c}(a) = \hat{\varphi}_{\Lambda_k^c} \circ \mathcal{E}_{\Lambda_k^c} \circ \mathcal{E}_{\Lambda_{k-1}^c} \circ \cdots \circ \mathcal{E}_{\Lambda_1^c}(a).$$

**Remark.** Markov states on multi-dimensional lattice  $\mathbb{Z}^{\nu}$  introduced in [3] are generalized quantum Markov states. Indeed, define an increasing sequence of finite sets  $\Lambda_n \uparrow L$ . Then for any  $\Lambda_n$ , there is a conditional expectation  $\mathcal{E}_{\Lambda_n^c}$  from  $\mathcal{B}_L$  to  $\mathcal{B}_{\Lambda_n^c}$  with  $\mathcal{E}_{\Lambda_n^c}(\mathcal{B}_L) \subset \mathcal{B}_{\Lambda_n^c}$  and

$$\varphi \circ \mathcal{E}_{\Lambda_n^c} = \varphi.$$

Let  $\hat{\varphi}_{\Lambda_n^c} = \varphi|_{\mathcal{B}_{\Lambda_n^c}}$ . Then the Markov state  $\varphi$  is a generalized quantum Markov state generated by  $\mathcal{E}_{\Lambda_n^c}$  and  $\hat{\varphi}_{\Lambda_n^c}$ .

**Remark.** In the case of infinite tensor products (the only one considered here) one has, for any subset,  $I \subseteq L$ :

$$\mathcal{B}_{I^c} = \mathcal{B}'_I \quad \text{the commutant of } \mathcal{B}_I. \quad (4.9)$$

From (4.5) for the quasi-conditional expectation  $\mathcal{E}_{\Lambda_n^c} : \mathcal{B}_L \rightarrow \mathcal{B}_{\Lambda_n^c}$  with respect to the triplet (4.6) one has

$$\mathcal{E}_{\Lambda_n^c}(a_{\bar{\Lambda}_n^c} a_{\Lambda_n}) = a_{\bar{\Lambda}_n^c} \mathcal{E}_{\Lambda_n^c}(a_{\Lambda_n}). \quad (4.10)$$

Because of (4.9) the last equality implies that  $\mathcal{E}_{\Lambda_n^c}(\mathcal{B}_{\Lambda_n}) \subseteq (\mathcal{B}_{\bar{\Lambda}_n^c})' = \mathcal{B}_{(\bar{\Lambda}_n^c)^c} = \mathcal{B}_{\bar{\Lambda}_n}$ .

Consequently,

$$\mathcal{E}_{\Lambda_n^c}(\mathcal{B}_{\Lambda_n}) \subseteq \mathcal{B}_{\Lambda_n^c} \cap \mathcal{B}_{\bar{\Lambda}_n} = \mathcal{B}_{\bar{\partial}\Lambda_n}$$

which is the natural quantum generalization of the multidimensional (discrete) Markov property as originally formulated by Dobrushin [10].

The above argument shows that, whenever (4.9) holds (e.g. in the case of infinite tensor products) the Markov property

$$\mathcal{E}_{\Lambda_n^c}(\mathcal{B}_{\Lambda_n}) \subseteq \mathcal{B}_{\bar{\partial}\Lambda}$$

follows from the basic property (4.10) of the quasi-conditional expectations. This is not true in general when (4.9) does not hold (e.g. in the abelian case or in the case of CAR algebras, see [5]). In all these cases the Markov property should be included in the definition of the various notions of Markov states as an additional requirement [5].

Next, we introduce the definition of  $d$ -Markov chains extending the definition in [1] to the graph case. Assume  $\{\Lambda_n\}_{n=1}^{\infty}$  is an increasing sequence of finite sets of  $L$  such that  $\bar{\Lambda}_n = \Lambda_{n+1}$  then  $\Lambda_n \uparrow L$ .

**Definition 4.2.** A state  $\varphi$  on  $\mathcal{B}_L$  is called a  $d$ -Markov chain associated to  $\{\Lambda_n\}$  if there exist a quasi-conditional expectation  $\mathcal{E}_n$  with respect to the triple  $\mathcal{B}_{\Lambda_{n-1}} \subset \mathcal{B}_{\Lambda_n} \subset \mathcal{B}_{\Lambda_{n+1}}$  for each  $n \in \mathbb{N}$  and an initial state  $\rho$  on  $\mathcal{B}_{\Lambda_1}$  such that

$$\varphi = \lim_{n \rightarrow \infty} \rho \circ \mathcal{E}_1 \circ \mathcal{E}_2 \circ \cdots \circ \mathcal{E}_n$$

in the weak-\* topology.

In this definition, the state  $\varphi$  is well-defined. Indeed, since  $\mathcal{E}_k(a) = a$  for any  $a \in \mathcal{B}_{\Lambda_n}$  and  $k \geq n+1$ , we have

$$\lim_{k \rightarrow \infty} \rho \circ \mathcal{E}_1 \circ \mathcal{E}_2 \circ \cdots \circ \mathcal{E}_k(a) = \rho \circ \mathcal{E}_1 \circ \mathcal{E}_2 \circ \cdots \circ \mathcal{E}_n(a).$$

## 5 Entangled Markov fields on trees

In this section we prove that, for a very special class of graphs, i.e. the trees, the construction of entangled Markov chains proposed in [2] can be generalized. The simplification coming from considering trees rather than general graphs manifests itself in the fact that the analogue of the basic isometries, used in the construction of [2], in this case commute.

Recall that a tree is a connected graph without loops. This definition implies that any finite connected subset  $\Lambda \subseteq_{\text{fin},c} L$  enjoys the following fundamental property:

### Tree Property

For any  $\Lambda \subseteq_{\text{fin},c} L$  and for arbitrary  $x \in \vec{\partial}\Lambda$ , there exists a unique point  $y \in \Lambda$  such that  $x \sim y$ .

$\mathbb{N}$ ,  $\mathbb{Z}$  and Cayley trees are examples of tree graphs and general tree graphs have a form as in Fig 1.

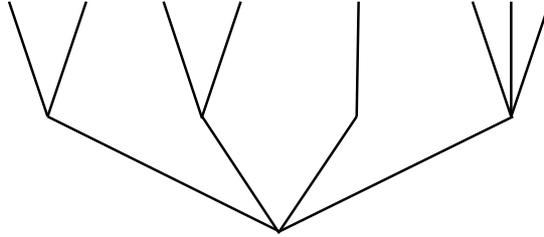


Fig 1: example of tree graphs

The fact that Tree Property is the main ingredient used in the proofs of the results below justifies the expectation that our results could be generalized to any graph such that there exists a sequence of  $\Lambda_n \subseteq_{\text{fin},c} L$  such that  $\Lambda_n \uparrow L$  and each  $\Lambda_n$  enjoys Tree Property (maybe with the exception of a *small* set of points).

The trouble with Tree Property is that, if  $\Lambda$  has Tree Property and  $x \in \vec{\partial}\Lambda$ , unfortunately it is not true that also  $\Lambda \cup x$  has Tree Property. However trees have a very special property given by the following Lemma.

**Lemma 5.1.** *In a tree every finite connected subset  $\Lambda \subseteq_{\text{fin},c} L$  enjoys Tree Property.*

*Proof.* Let  $\Lambda \subseteq_{\text{fin},c} L$  be a finite connected subset and let  $x \in \vec{\partial}\Lambda$ . If there exist  $y, z \in \Lambda$  such that  $y \sim x$ ,  $z \sim x$ , then since a tree is connected, there is a path between  $y$  and  $z$  and this would give a loop. Against the definition of tree.  $\square$

We keep the notations and assumptions of Section 2. Let  $(L, E)$  be a graph and let, for each  $\{x, y\} \in E$ , be given a complex  $d \times d$  matrix  $(\psi_{xy}(i, j))$  such that the matrix  $(|\psi_{xy}(i, j)|^2)$  is bi-stochastic, i.e.

$$\sum_{i=1}^d |\psi_{xy}(i, j)|^2 = \sum_{j=1}^d |\psi_{xy}(i, j)|^2 = 1.$$

$(\psi_{xy}(i, j))$  will be called an *amplitude matrix*: notice that unitarity of the matrices  $(\psi_{xy}(i, j))_{i,j}$  is not required. Define the vector

$$\psi_{xy} = \sum_{i,j=1}^d \psi_{xy}(i, j) \cdot e_i(x) \otimes e_j(y) \in \mathcal{H}_x \otimes \mathcal{H}_y. \quad (5.11)$$

Moreover, in the notation

$$E_\Lambda := \{\{x, y\} \mid x, y \in \Lambda, x \sim y\}$$

for any  $\Lambda \subseteq_{\text{fin}} L$ , define the vector  $\psi_\Lambda \in \mathcal{H}_\Lambda$  by

$$\psi_\Lambda := \sum_{\omega_\Lambda} \psi_\Lambda(\omega_\Lambda) e_{\omega_\Lambda}, \quad (5.12)$$

$$\psi_\Lambda(\omega_\Lambda) := \prod_{\{x, y\} \in E_\Lambda} \psi_{xy}(\omega_\Lambda(x), \omega_\Lambda(y)). \quad (5.13)$$

**Lemma 5.2.** *If  $\Lambda \subseteq_{\text{fin}, c} L$  enjoys Tree Property then for all  $x \in \vec{\partial}\Lambda$ ,*

$$\|\psi_{\Lambda \cup \{x\}}\|^2 = \|\psi_\Lambda\|^2.$$

*Proof.* Tree Property implies that, for arbitrary  $x \in \vec{\partial}\Lambda$ , there exists a unique point  $y \in \Lambda$  such that  $x \sim y$ . Then

$$\begin{aligned} \|\psi_{\Lambda \cup \{x\}}\|^2 &= \sum_{\omega_\Lambda, \omega_x} |\psi_{\Lambda \cup \{x\}}((\omega_\Lambda, \omega_x))|^2 = \sum_{\omega_{\Lambda \setminus \{y\}}, \omega_y, \omega_x} |\psi_{\Lambda \cup \{x\}}((\omega_{\Lambda \setminus \{y\}}, \omega_y, \omega_x))|^2 \\ &= \sum_{\omega_{\Lambda \setminus \{y\}}, \omega_y} \sum_{\omega_x=1}^d |\psi_\Lambda((\omega_{\Lambda \setminus \{y\}}, \omega_y))|^2 \cdot |\psi_{xy}(\omega_y, \omega_x)|^2 \\ &= \sum_{\omega_\Lambda} |\psi_\Lambda(\omega_\Lambda)|^2 = \|\psi_\Lambda\|^2 \end{aligned}$$

which proves the assertion.  $\square$

**Proposition 5.3.** *Suppose that  $\Lambda$  enjoys Tree Property and let*

$$\Lambda' \subset \subset \Lambda \subseteq_{\text{fin}, c} L.$$

*Then for any  $a \in \mathcal{B}_{\Lambda'}$  and  $x \in \vec{\partial}\Lambda$  one has:*

$$\langle \psi_\Lambda, a\psi_\Lambda \rangle = \langle \psi_{\Lambda \cup \{x\}}, a\psi_{\Lambda \cup \{x\}} \rangle.$$

*Proof.* Because of Tree Property, given  $x \in \vec{\partial}\Lambda$ , there exists a unique point  $y \in \Lambda$  such that  $x \sim y$ . Then we have

$$\begin{aligned} &\langle \psi_{\Lambda \cup \{x\}}, a\psi_{\Lambda \cup \{x\}} \rangle = \\ &= \sum_{\omega_{\Lambda'}, \omega'_{\Lambda'}} \sum_{\omega_{\Lambda \setminus \{\Lambda' \cup \{y\}\}}} \sum_{\omega_x, \omega_y} \psi_{\Lambda \cup \{x\}}((\omega_{\Lambda'}, \omega_{\Lambda \setminus \{\Lambda' \cup \{y\}\}}, \omega_x, \omega_y))^* \\ &\quad \cdot a_{\omega_{\Lambda'}, \omega'_{\Lambda'}} \psi_{\Lambda \cup \{x\}}((\omega'_{\Lambda'}, \omega_{\Lambda \setminus \{\Lambda' \cup \{y\}\}}, \omega_x, \omega_y)) \\ &= \sum_{\omega_{\Lambda'}, \omega'_{\Lambda'}} \sum_{\omega_{\Lambda \setminus \{\Lambda' \cup \{y\}\}}} \sum_{\omega_x, \omega_y} \psi_\Lambda((\omega_{\Lambda'}, \omega_{\Lambda \setminus \{\Lambda' \cup \{y\}\}}, \omega_y))^* \\ &\quad a_{\omega_{\Lambda'}, \omega'_{\Lambda'}} \psi_\Lambda((\omega'_{\Lambda'}, \omega_{\Lambda \setminus \{\Lambda' \cup \{y\}\}}, \omega_y)) |\psi_{xy}(\omega_x, \omega_y)|^2 \\ &= \sum_{\omega_{\Lambda'}, \omega'_{\Lambda'}} \sum_{\omega_{\Lambda \setminus \Lambda'}} \psi_\Lambda((\omega_{\Lambda'}, \omega_{\Lambda \setminus \Lambda'}))^* a_{\omega_{\Lambda'}, \omega'_{\Lambda'}} \psi_\Lambda((\omega'_{\Lambda'}, \omega_{\Lambda \setminus \Lambda'})) \\ &= \langle \psi_\Lambda, a\psi_\Lambda \rangle, \end{aligned}$$

where  $a_{\omega_{\Lambda'}, \omega'_{\Lambda'}} = \langle e_{\omega_{\Lambda'}}, a e_{\omega'_{\Lambda'}} \rangle$ .  $\square$

**Corollary 5.4.** *If  $(L, E)$  is a tree, and the vector  $\psi_\Lambda$  is defined by (5.12), (5.13), then, for any  $\Lambda \subseteq_{\text{fin},c} L$  of cardinality  $\geq 2$ , one has:*

$$\|\psi_\Lambda\|^2 = d \quad (5.14)$$

and the limit

$$\varphi(a) = \frac{1}{d} \lim_{\Lambda \uparrow L} \langle \psi_\Lambda, a \psi_\Lambda \rangle$$

exists for any  $a$  in the local algebra  $\mathcal{B}_{L,loc}$  and defines a state  $\varphi$  on  $\mathcal{B}_L$ .

*Proof.* The first statement follows by induction from Proposition 5.2 and Lemma 5.1 because, if  $\Lambda = \{x, y\}$ , then we get

$$\|\psi_{xy}\|^2 = \sum_{i,j} |\psi_{xy}(i, j)|^2 = d.$$

The second statement follows from the first one and Proposition 5.3.  $\square$

The obtained state in Corollary 5.4 is called *entangled Markov filed* on  $\mathcal{B}_L$ . When  $L = \mathbb{Z}$  such a state was introduced and studied in [2, 14]. We will see that the state  $\varphi$  is a  $d$ -Markov chain and, in special case, it is a generalized quantum Markov state.

For  $\Lambda \subseteq_{\text{fin},c} L$ ,  $x \in \bar{\partial}\Lambda$  and  $z \in \Lambda$ , with  $z \sim x$ , define  $V_{(z|x)} : \mathcal{H}_z \rightarrow \mathcal{H}_z \otimes \mathcal{H}_x$  by

$$V_{(z|x)} e_{i_z} = \sum_{i_x} \psi_{xz}(i_x, i_z) e_{i_x} \otimes e_{i_z}. \quad (5.15)$$

Then  $V_{(z|x)}$  is naturally extended to an operator from  $\mathcal{H} \otimes \mathcal{H}_x$  to  $\mathcal{H} \otimes \mathcal{H}_x \otimes \mathcal{H}_z$  for any Hilbert space  $\mathcal{H}$  by  $I_{\mathcal{H}} \otimes V_{(z|x)}$ . We will also write  $V_{(z|x)}$  for  $I_{\mathcal{H}} \otimes V_{(z|x)}$ .

**Proposition 5.5.** *For any  $\Lambda \subseteq_{\text{fin},c} L$ ,  $x, y \in \bar{\partial}\Lambda$  and  $z \in \Lambda$  with  $x \sim z$ ,  $y \sim z$ ,  $V_{(z|x)}$  and  $V_{(z|y)}$  are isometries satisfying:*

$$\begin{aligned} V_{(z|x)} \psi_\Lambda &= \psi_{\Lambda \cup \{x\}}, \\ V_{(z|x)} V_{(z|y)} &= V_{(z|y)} V_{(z|x)}. \end{aligned}$$

*Proof.* From a simple calculation, we have

$$\begin{aligned} \langle V_{(z|x)} e_{i_z}, V_{(z|x)} e_{j_z} \rangle &= \delta_{i_z, j_z} \sum_{i_x, j_x} \langle \psi_{xz}(i_x, i_z) e_{i_x}, \psi_{xz}(j_x, i_z) e_{j_x} \rangle \\ &= \delta_{i_z, j_z} \sum_{i_x} |\psi_{xz}(i_x, i_z)|^2 = \delta_{i_z, j_z} = \langle e_{i_z}, e_{j_z} \rangle. \end{aligned}$$

Therefore any  $V_{(z|x)}$  is an isometry. Next, we get  $V_{(z|x)} \psi_\Lambda = \psi_{\Lambda \cup \{x\}}$ . Indeed,

$$\begin{aligned} V_{(z|x)} \psi_\Lambda &= V_{(z|x)} \left( \sum_{\omega_{\Lambda \setminus \{z\}}, i_z} \psi_\Lambda((\omega_{\Lambda \setminus \{z\}}, i_z)) e_{\omega_{\Lambda \setminus \{z\}}} \otimes e_{i_z} \right) \\ &= \sum_{\omega_{\Lambda \setminus \{z\}}, i_z} \psi_\Lambda((\omega_{\Lambda \setminus \{z\}}, i_z)) \left( \sum_{i_x} \psi_{xz}(i_x, i_z) e_{\omega_{\Lambda \setminus \{z\}}} \otimes e_{i_x} \otimes e_{i_z} \right) \\ &= \sum_{\omega_{\Lambda \setminus \{z\}}, i_x, i_z} \psi_{\Lambda \cup \{x\}}((\omega_{\Lambda \setminus \{z\}}, i_x, i_z)) e_{\omega_{\Lambda \setminus \{z\}}} \otimes e_{i_x} \otimes e_{i_z} \\ &= \psi_{\Lambda \cup \{x\}}. \end{aligned}$$

Finally, we obtain the commutation relation:

$$\begin{aligned}
V_{(z|x)}V_{(z|y)}e_{i_z} &= V_{(z|x)}\left(\sum_{i_y} \psi_{yz}(i_y, i_z)e_{i_y} \otimes e_{i_z}\right) \\
&= \sum_{i_x, i_y} \psi_{xz}(i_x, i_z)\psi_{yz}(i_y, i_z)e_{i_x} \otimes e_{i_y} \otimes e_{i_z} \\
&= V_{(z|y)}\left(\sum_{i_x} \psi_{xz}(i_x, i_z)e_{i_x} \otimes e_{i_z}\right) \\
&= V_{(z|y)}V_{(z|x)}e_{i_z}.
\end{aligned}$$

□

For an initial point  $x_1 \in L$ , we define inductively  $\Lambda_1 = \{x_1\}$  and

$$\Lambda_n = \bar{\Lambda}_{n-1}. \quad (5.16)$$

Then we have the following proposition.

**Proposition 5.6.** *Let  $\varphi$  be a state defined in Corollary 5.4, then it is a  $d$ -Markov chain associated to  $\{\Lambda_n\}$ .*

*Proof.* Let  $V_n$  be the isometry defined by

$$V_n = \prod \{V_{(x|y)} : x \in \Lambda_n, y \in \bar{\partial}\Lambda_n, x \sim y\}$$

where the product is well-defined because, due to Proposition 5.5, the factors commute. We define the quasi-conditional expectation with respect to the triple  $\mathcal{B}_{\Lambda_{n-1}} \subset \mathcal{B}_{\Lambda_n} \subset \mathcal{B}_{\Lambda_{n+1}}$  by

$$\mathcal{E}_n(a_{\Lambda_{n+1}}) = V_n^*(a_{\Lambda_{n+1}})V_n$$

for  $a_{\Lambda_{n+1}} \in \mathcal{B}_{\Lambda_{n+1}}$ . Denote

$$\rho = \frac{1}{d} \left\langle \sum_{i_{x_1}=1}^d e_{i_{x_1}}, \cdot \sum_{j_{x_1}=1}^d e_{j_{x_1}} \right\rangle.$$

Then from Proposition 5.5, we have

$$\begin{aligned}
&\rho \circ \mathcal{E}_1 \circ \cdots \circ \mathcal{E}_n(a_{\Lambda_n}) \\
&= \frac{1}{d} \left\langle \sum_{i_{x_1}=1}^d \prod_{x \in \Lambda_2} V_{(x_1|x)} e_{i_{x_1}}, \mathcal{E}_2 \circ \cdots \circ \mathcal{E}_n(a_{\Lambda_n}) \prod_{x \in \Lambda_2} V_{(x_1|x)} \sum_{j_{x_1}=1}^d e_{j_{x_1}} \right\rangle \\
&= \frac{1}{d} \langle \psi_{\Lambda_2}, \mathcal{E}_2 \circ \cdots \circ \mathcal{E}_n(a_{\Lambda_n}) \psi_{\Lambda_2} \rangle \\
&\quad \vdots \\
&= \frac{1}{d} \langle \psi_{\Lambda_{n+1}}, a_{\Lambda_n} \psi_{\Lambda_{n+1}} \rangle \\
&= \varphi(a_{\Lambda_n})
\end{aligned}$$

which implies the assertion. □

We don't know if entangled Markov fields are generalized quantum Markov states or not. But if we assume that

$$|\psi_{xy}(i, j)|^2 = \frac{1}{d}$$

for any  $x \sim y$  and  $1 \leq i, j \leq d$ , we can see the next proposition.

**Proposition 5.7.** *If  $|\psi_{xy}(i, j)|^2 = \frac{1}{d}$  for any  $x \sim y$  and  $1 \leq i, j \leq d$ , a state  $\varphi$  defined in Corollary 5.4 is a generalized quantum Markov state.*

*Proof.* Let  $\Lambda_n$  be as in (5.16). Define an isometry  $V_n$  from  $\mathcal{H}_{\Lambda_{n+2} \setminus \Lambda_n}$  to  $\mathcal{H}_{\Lambda_{n+2} \setminus \Lambda_{n-1}}$  as follows: For  $y \in \Lambda_{n-1}$ , assume  $\Lambda_n \cap \vec{\partial}\{y\} = \{x_1, \dots, x_m\}$ . We define

$$\begin{aligned} & V_n(e_{i_1}(x_1) \otimes e_{i_2}(x_2) \otimes \dots \otimes e_{i_k}(x_m)) \\ &= d^{\frac{m-1}{2}} \sum_{j=1}^d \prod_{l=1}^m \psi_{x_i y}(i_l, j) e_j(y) \otimes e_{i_1}(x_1) \otimes e_{i_2}(x_2) \otimes \dots \otimes e_{i_k}(x_m). \end{aligned}$$

Furthermore, we will extend  $V_n$  naturally, if it is needed.

Then  $V_n$  is an isometry from the definition. Moreover,  $V_n$  satisfies that , for  $k \geq n+2$ ,

$$V_n(d^{-|\Lambda_{n+1}|/2} \psi_{\Lambda_k \setminus \Lambda_n}) = d^{-|\Lambda_n|/2} \psi_{\Lambda_k \setminus \Lambda_{n-1}},$$

where  $V_0 = \emptyset$  and  $|\cdot|$  is the cardinal number. Indeed, since  $\Lambda_k \setminus \Lambda_n$  is a union of  $|\Lambda_{n+1}|$  connected sets, we have

$$\|\psi_{\Lambda_k \setminus \Lambda_n}\|^2 = d^{|\Lambda_{n+1}|}$$

by Corollary 5.4.

For  $y \in \Lambda_n$  and  $k \geq n+1$ , let  $\mathcal{V}_n(y, k) = \bigcup\{x \in \Lambda_l \mid \text{dist}(x, y) = l-n, n+1 \leq l \leq k\} \cup \{y\}$ , all vertices in  $\Lambda_k \setminus \Lambda_{n-1}$  which connect to  $y$  in  $\Lambda_k \setminus \Lambda_{n-1}$ . Let  $y \in \Lambda_n$  and  $\Lambda_{n+1} \cap \vec{\partial}\{y\} = \{x_1, \dots, x_m\}$ . Then one can see that

$$V_n \left( \bigotimes_{i=1}^m \psi_{\mathcal{V}_{n+1}(x_i, k)} \right) = d^{\frac{m-1}{2}} \psi_{\mathcal{V}_n(y, k)}$$

and

$$\begin{aligned} & V_n \left( d^{-|\Lambda_{n+1}|/2} \psi_{\Lambda_k \setminus \Lambda_n} \right) = V_n \left( d^{-|\Lambda_{n+1}|/2} \bigotimes_{x \in \Lambda_{n+1}} \psi_{\mathcal{V}_{n+1}(x, k)} \right) \\ &= V_n \left( d^{-|\Lambda_n|/2} \bigotimes_{y \in \Lambda_n} \psi_{\mathcal{V}_n(y, k)} \right) = d^{-|\Lambda_n|/2} \psi_{\Lambda_k \setminus \Lambda_{n-1}}. \end{aligned}$$

Therefore, we have

$$V_1 \cdot V_2 \cdots V_n \left( d^{-|\Lambda_{n+1}|/2} \psi_{\Lambda_k \setminus \Lambda_n} \right) = d^{-1/2} \psi_{\Lambda_k}.$$

Now we define that

$$\begin{aligned} \hat{\varphi}_{\Lambda_n^c} &= d^{-|\Lambda_{n+1}|} \langle \psi_{\Lambda_{n+2} \setminus \Lambda_n}, \cdot \psi_{\Lambda_{n+2} \setminus \Lambda_n} \rangle \otimes \varphi|_{\mathcal{B}_{\Lambda_{n+2}^c}} \\ \mathcal{E}_{\Lambda_n^c} &= V_n^* \cdot V_n \end{aligned}$$

Then since

$$\langle \psi_{\Lambda_{n+2} \setminus \Lambda_n}, a \psi_{\Lambda_{n+2} \setminus \Lambda_n} \rangle = \langle \psi_{\Lambda_{n+3} \setminus \Lambda_n}, a \psi_{\Lambda_{n+3} \setminus \Lambda_n} \rangle$$

for all  $a \in \mathcal{B}_{\Lambda_{n+1} \setminus \Lambda_n}$  from a similar proof of Proposition 5.3, we have

$$\hat{\varphi}_{\Lambda_n^c} |_{\mathcal{B}_{\Lambda_{n+1} \setminus \Lambda_n}} = \hat{\varphi}_{\Lambda_{n+1}^c} \circ \mathcal{E}_{\Lambda_{n+1}^c} |_{\mathcal{B}_{\Lambda_{n+1} \setminus \Lambda_n}}$$

and for  $a \in \mathcal{B}_{\Lambda_n}$ ,

$$\begin{aligned} & \hat{\varphi}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_{n-1}^c} \circ \cdots \circ \mathcal{E}_{\Lambda_1^c}(a) \\ &= \langle V_1 \cdot V_2 \cdots V_n \left( d^{-|\Lambda_{n+1}|/2} \psi_{\Lambda_k \setminus \Lambda_n} \right), a V_1 \cdot V_2 \cdots V_n \left( d^{-|\Lambda_{n+1}|/2} \psi_{\Lambda_k \setminus \Lambda_n} \right) \rangle \\ &= d^{-1} \langle \psi_{\Lambda_{n+2}}, a \psi_{\Lambda_{n+2}} \rangle \\ &= \varphi(a). \end{aligned}$$

This says that  $\varphi$  is a generalized quantum Markov state.  $\square$

**Remark.** It is not easy to extend the construction of entangled Markov fields to more general graphs, because Corollary 5.4 does not hold in general. If we want to make a entangled Markov field on a general graph, we need the condition that, for each  $\Lambda \subseteq \vec{\partial}x$ ,

$$\sum_{i_x} \prod_{y \in \Lambda, y \sim x} |\psi_{xy}(i_x, i_y)|^2$$

is constant, i.e. independent of the choice of the  $i_y$ 's, as in Proposition 5.7. Note that the last condition is not true in general.

**Remark.** From the proved Propositions there arises a natural question: would the entangled Markov field be a Markov state. Such a question was not considered in [2, 14]. Now we are going to provide an example of the entangled Markov field, which is not a Markov state.

**Example.** For the sake of simplicity, we consider the simplest tree graph  $\mathbb{Z}$  and  $\mathcal{B}_x = M_2$  for all  $x \in \mathbb{Z}$ .

Before we see the example, we recall some basic notations about Markov states on  $\mathcal{B}_{\mathbb{Z}}$ . A shift  $\gamma$  on  $\mathcal{B}_{\mathbb{Z}}$  is an automorphism on  $\mathcal{B}_{\mathbb{Z}}$  defined by

$$\gamma(X) = I_{M_2} \otimes X$$

for any  $X \in \mathcal{B}_{\Lambda}$  and  $\Lambda \subseteq_{\text{fin},c} \mathbb{Z}$ . A shift-invariant Markov state, i.e.,  $\varphi \circ \gamma = \varphi$ , is generated by a conditional expectation  $\mathcal{E} : M_2 \otimes M_2 \rightarrow M_2$  such that  $\phi \circ \mathcal{E}(A \otimes I) = \phi(A)$  for all  $A \in M_2$  by the formulation

$$\varphi(A_1 \otimes A_2 \otimes \cdots \otimes A_n) = \varphi \circ \mathcal{E}(A_1 \otimes \mathcal{E}(A_2 \otimes \cdots \mathcal{E}(A_{n-1} \otimes A_n) \cdots)).$$

Then there are three possible cases of the range of  $\mathcal{E}$ . Namely,

(i)– case:  $\text{ran} \mathcal{E} = \mathcal{B}_x$ .

In this case,  $\varphi$  is a product state.

(ii)– case:  $\text{ran} \mathcal{E} = \mathbb{C}I$ .

In this case,  $\varphi$  is also a product state.

(iii)– case:  $\text{ran}\mathcal{E} = \mathbb{C} \oplus \mathbb{C}$ .

In this case, we can make a classical shift-invariant Markov chain on  $\otimes \text{ran}\mathcal{E} = \otimes \mathbb{C} \oplus \mathbb{C}$  and  $\varphi$  is a canonical extension of this Markov chain (see [7]).

Now we construct an entangled Markov field which does not belong to the above three cases.

Put

$$\begin{aligned}\psi_{x,y}(1,1) &= \psi_{x,y}(2,2) = \frac{1}{\sqrt{3}} \\ \psi_{x,y}(1,2) &= \psi_{x,y}(2,1) = \frac{\sqrt{2}}{\sqrt{3}}\end{aligned}$$

for all  $x \sim y$ . Let  $\varphi$  be a entangled Markov field generated by the above  $\psi$  (see Corollary 5.4). Then one can see that  $\varphi$  is shift-invariant. Moreover,  $\varphi$  is not a product state, since

$$\phi(e_{11}) = \frac{1}{2}, \quad \phi(e_{11} \otimes e_{11}) = \frac{1}{6}.$$

Finally,  $\varphi$  is not a canonical extension of classical Markov chain. Indeed, since  $\varphi_{[1,n]}$  is written as a restriction of vector state on  $\mathcal{B}_{[0,n+1]}$ , the density matrix of  $\varphi_{[1,n]}$  is a linear combination of at most 4 one-rank projections. From the direct calculation, one can get that the density matrix of  $\varphi_{[1,2]}$  is a linear combination of just 4 one-rank projections whose vectors are linearly independent. Moreover, let  $\alpha_n$  be a number of combinations of density matrix of a classical Markov chain. Then  $\alpha_n \rightarrow \infty$  or  $\alpha_n = 1$  or  $\alpha_n = 2$ . Therefore,  $\varphi$  is not a canonical extension of classical Markov chain.

**Remark.** Let us first recall a definition of entangled state. Consider  $\mathcal{A}_j$  ( $j \in L$ ),  $C^*$  algebras, here  $L$  is a tree. Denote

$$\begin{aligned}\mathcal{S}_{\text{prod}} &= \overline{\text{Conv}}\left\{\bigotimes_{j \in L} \omega_j ; \omega_j \in \mathcal{S}(\mathcal{A}_j), j \in L\right\}, \\ \mathcal{S}_{\Lambda, \text{prod}} &= \overline{\text{Conv}}\left\{\omega_{\Lambda} \otimes \omega_{\Lambda^c} ; \omega_{\Lambda} \in \mathcal{S}(\otimes_{j \in \Lambda} \mathcal{A}_j), \omega_{\Lambda^c} \in \mathcal{S}(\otimes_{j \in \Lambda^c} \mathcal{A}_j)\right\}, \\ \mathcal{S}_{\mathbb{Z}} &= \bigcup_{\substack{\Lambda \subset L: \\ \Lambda \sim \mathbb{Z}}} \mathcal{S}_{\Lambda, \text{prod}},\end{aligned}$$

here by  $\Lambda \sim \mathbb{Z}$  we mean an isomorphism (i.e. a 1-1 mapping which preserves edges and connected components) of a subgraph  $\Lambda \subset L$  to the integer lattice  $\mathbb{Z}$ .

A state  $\omega \in \mathcal{S}(\otimes_{j \in L} \mathcal{A}_j)$  is said to be *entangled* (see [8] (resp.  $\mathbb{Z}$ -entangled) if  $\omega \notin \mathcal{S}_{\text{prod}}$  (resp.  $\omega \notin \mathcal{S}_{\mathbb{Z}}$ ). One can see that any  $\mathbb{Z}$ -entangled state is entangled, but the converse is not true. In [8] it has been established that entangled quantum Markov states on  $\mathbb{Z}$  are entangled.

From the definition given above we can prove

**Theorem 5.8.** *Let  $\varphi$  be a state on  $\mathcal{B}_L$ . The following assertions hold:*

- (i) *If for some  $\Lambda$  with  $\Lambda \sim \mathbb{Z}$  the restriction of  $\varphi$  to the  $C^*$ -subalgebra  $\mathcal{B}_{\Lambda}$  is entangled, then  $\varphi$  is also entangled on  $\mathcal{B}_L$ ;*
- (ii) *If for any  $\Lambda$  with  $\Lambda \sim \mathbb{Z}$  the restriction of  $\varphi$  to the  $C^*$ -subalgebra  $\mathcal{B}_{\Lambda}$  is  $\mathbb{Z}$ -entangled, then  $\varphi$  is  $\mathbb{Z}$ -entangled on  $\mathcal{B}_L$ .*

## 6 $d$ -Markov chains on Cayley trees

In this section, we consider a particular case of tree, so called Cayley tree. Over such a tree we are going to give a construction of  $d$ -Markov chains.

Recall that a Cayley tree  $\Gamma^k$  of order  $k \geq 1$  is an infinite tree whose each vertices have exactly  $k + 1$  edges. If we cut away an edge  $\{x, y\}$  of the tree  $\Gamma^k$ , then  $\Gamma^k$  splits into connected components, called semi-infinite trees with roots  $x$  and  $y$ , which will be denoted respectively by  $\Gamma^k(x)$  and  $\Gamma^k(y)$ . If we cut away from  $\Gamma^k$  the origin  $O$  together with all  $k + 1$  nearest neighbor vertices, in the result we obtain  $k + 1$  semi-infinite  $\Gamma^k(x)$  trees with  $x \in S_0 = \{y \in \Gamma^k : \text{dist}(O, y) = 1\}$ , where  $\text{dist}$  is a distance of vertices introduced in Sect. 2. Hence we have

$$\Gamma^k = \bigcup_{x \in S_0} \Gamma^k(x) \cup \{O\}.$$

Therefore, in the sequel we will consider semi-infinite Cayley tree  $\Gamma^k(x_0) = (L, E)$  with the root  $x_0$ . Let us set

$$W_n = \{x \in L : \text{dist}(x, x_0) = n\}, \quad \Lambda_n = \bigcup_{k=0}^n W_k, \quad E_n = \{\{x, y\} \in E : x, y \in \Lambda_n\}.$$

In the following, we will construct examples of  $d$ -Markov chains on semi-infinite Cayley trees, that is, we construct a sequence of quasi-conditional expectations  $\mathcal{E}_n$  with respect to  $\mathcal{B}_{\Lambda_{n-1}} \subset \mathcal{B}_{\Lambda_n} \subset \mathcal{B}_{\Lambda_{n+1}}$  and an initial state  $\rho$ , and define

$$\varphi = \lim \rho \circ \mathcal{E}_0 \circ \mathcal{E}_1 \circ \cdots \circ \mathcal{E}_n.$$

For this, we use some operators  $V_n \in \mathcal{B}_{\Lambda_{n+1} \setminus \Lambda_n}$  and define  $\mathcal{E}_n = \text{Tr}_{\Lambda_n}(V_n \cdot V_n^*)$ , where  $\text{Tr}_{\Lambda_n}$  is a normalized trace from  $\mathcal{B}_L$  to  $\mathcal{B}_{\Lambda_n}$ .

Denote

$$S(x) = \{y \in W_{n+1} : x \sim y\}, \quad x \in W_n,$$

this set is called a set of *direct successors* of  $x$ .

From these one can see that

$$\Lambda_m = \Lambda_{m-2} \cup \left( \bigcup_{x \in W_{m-1}} \{x \cup S(x)\} \right) \quad (6.17)$$

$$E_m \setminus E_{m-1} = \bigcup_{x \in W_{m-1}} \bigcup_{y \in S(x)} \{\{x, y\}\} \quad (6.18)$$

Now we are going to introduce a coordinate structure in  $\Gamma^k(x_0)$ . Every vertex  $x$  (except for  $x_0$ ) of  $\Gamma^k(x_0)$  has coordinates  $(i_1, \dots, i_n)$ , here  $i_m \in \{1, \dots, k\}$ ,  $1 \leq m \leq n$  and for the vertex  $x_0$  we put  $\emptyset$ . Namely, the symbol  $\emptyset$  constitutes level 0 and the sites  $(i_1, \dots, i_n)$  form level  $n$  of the lattice. In this notation for  $x \in \Gamma^k(x_0)$ ,  $x = (i_1, \dots, i_n)$  we have

$$S(x) = \{(x, i) : 1 \leq i \leq k\},$$

here  $(x, i)$  means that  $(i_1, \dots, i_n, i)$ .

Then for  $1 \leq i \leq k$ , we define a shift  $\gamma_i$  by

$$\gamma_i(x) = (i, x) = (i, i_1, \dots, i_n).$$

Now we can consider this shift as a shift homomorphism on  $\mathcal{B}_L$ , that is, for any  $a_x \in \mathcal{B}_x$ , we consider  $\gamma_i(a_x) \in \mathcal{B}_{(i,x)}$ .

Let be given a positive operator  $w_0 \in \mathcal{B}_{x_0,+}$  and two family of operators  $\{K_{\langle x,y \rangle} \in \mathcal{B}_{\{x,y\}}\}_{\{x,y\} \in E}$ ,  $\{h_x \in \mathcal{B}_{x,+}\}_{x \in L}$  such that

$$\mathrm{Tr}(w_0 h_0) = 1 \quad (6.19)$$

$$\mathrm{Tr}_x \left( \prod_{i=1}^k K_{\langle x,(x,i) \rangle} \prod_{i=1}^k h_{(x,i)} \prod_{i=1}^k K_{\langle x,(x,k+1-i) \rangle}^* \right) = h_x, \quad \text{for every } x \in L, \quad (6.20)$$

where  $\mathrm{Tr}_\Lambda : \mathcal{B}_L \rightarrow \mathcal{B}_\Lambda$  is a normalized partial trace for any  $\Lambda \subseteq_{\mathrm{fin}} L$  and  $\mathrm{Tr}$  is a normalized trace on  $\mathcal{B}_L$ .

Note that if  $k = 1$  and  $h_x = I$  for all  $x \in V$ , then we get conditional amplitudes introduced by L.Accardi [6].

Denote

$$K_n = w_0^{1/2} \prod_{\{x,y\} \in E_1} K_{\langle x,y \rangle} \prod_{\{x,y\} \in E_2 \setminus E_1} K_{\langle x,y \rangle} \cdots \prod_{\{x,y\} \in E_n \setminus E_{n-1}} K_{\langle x,y \rangle} \prod_{x \in W_n} h_x^{1/2}, \quad (6.21)$$

where by definition we put

$$\prod_{\{x,y\} \in E_m \setminus E_{m-1}} K_{\langle x,y \rangle} := \prod_{x \in W_{m-1}} \prod_{i=1}^k K_{\langle x,(x,i) \rangle} \quad (6.22)$$

Now define

$$\mathcal{W}_n] = K_n K_n^*. \quad (6.23)$$

It is clear that  $\mathcal{W}_n]$  is positive.

Recall that a sequence  $\{\mathcal{W}_n]\}$  is *projective* with respect to  $\mathrm{Tr}_n] = \mathrm{Tr}_{\Lambda_n}$  if

$$\mathrm{Tr}_{n-1]}(\mathcal{W}_n]) = \mathcal{W}_{n-1}] \quad (6.24)$$

is valid for all  $n \in \mathbb{N}$ .

**Theorem 6.1.** *Let (6.20) be satisfied. Then  $\{\mathcal{W}_n]\}$  is a projective sequences of density operators.*

*Proof.* Let us check the equality (6.24). From (6.21) one has

$$\begin{aligned} \mathcal{W}_n] &= w_0^{1/2} \prod_{m=1}^{n-1} \left( \prod_{\{x,y\} \in E_m \setminus E_{m-1}} K_{\langle x,y \rangle} \right) \prod_{\{x,y\} \in E_n \setminus E_{n-1}} K_{\langle x,y \rangle} \prod_{x \in W_n} h_x \\ &\quad \times \left( \prod_{\{x,y\} \in E_n \setminus E_{n-1}} K_{\langle x,y \rangle} \right)^* \prod_{m=1}^{n-1} \left( \prod_{\{x,y\} \in E_{n-m} \setminus E_{n-m-1}} K_{\langle x,y \rangle} \right)^* w_0^{1/2}. \end{aligned}$$

We know that for different  $x$  and  $x'$  taken from  $W_{n-1}$  the algebras  $\mathcal{B}_{x \cup S(x)}$  and  $\mathcal{B}_{x' \cup S(x')}$  commute, therefore from (6.22) one finds

$$\begin{aligned} &\prod_{\{x,y\} \in E_n \setminus E_{n-1}} K_{\langle x,y \rangle} \prod_{x \in W_n} h_x \left( \prod_{\{x,y\} \in E_n \setminus E_{n-1}} K_{\langle x,y \rangle} \right)^* \\ &= \prod_{x \in W_{n-1}} \prod_{i=1}^k K_{\langle x,(x,i) \rangle} \prod_{i=1}^k h_{(x,i)} \prod_{i=1}^k K_{\langle x,(x,k+1-i) \rangle}^*. \end{aligned}$$

Hence from the condition (6.20) we find

$$\begin{aligned}
\mathrm{Tr}_{n-1]}(\mathcal{W}_n) &= w_0^{1/2} \prod_{m=1}^{n-1} \left( \prod_{\{x,y\} \in E_m \setminus E_{m-1}} K_{\langle x,y \rangle} \right) \\
&\times \prod_{x \in W_{n-1}} \mathrm{Tr}_x \left( \prod_{i=1}^k K_{\langle x,(x,i) \rangle} \prod_{i=1}^k h_{(x,i)} \prod_{i=1}^k K_{\langle x,(x,k+1-i) \rangle}^* \right) \\
&\times \prod_{m=1}^{n-1} \left( \prod_{\{x,y\} \in E_{n-m} \setminus E_{n-m-1}} K_{\langle x,y \rangle} \right)^* w_0^{1/2} \\
&= w_0^{1/2} \prod_{m=1}^{n-1} \left( \prod_{\{x,y\} \in E_m \setminus E_{m-1}} K_{\langle x,y \rangle} \right) \\
&\times \prod_{x \in W_{n-1}} h_x \prod_{m=1}^{n-1} \left( \prod_{\{x,y\} \in E_{n-m} \setminus E_{n-m-1}} K_{\langle x,y \rangle} \right)^* w_0^{1/2} \\
&= \mathcal{W}_{n-1]}
\end{aligned}$$

From the above argument and (6.19), one can show that  $\mathcal{W}_n]$  is density operator, i.e.  $\mathrm{Tr}(\mathcal{W}_n]) = 1$ .  $\square$

Define a state on  $\mathcal{B}_{\Lambda_n}$  by

$$\varphi_n(x) = \mathrm{Tr}(\mathcal{W}_{n+1}]x), \quad x \in \mathcal{B}_{\Lambda_n}.$$

Assume that  $h_x$  is invertible for all  $x \in L$  and define

$$\mathcal{E}_n(a) = \mathrm{Tr}_{n]} \left( \prod_{x \in W_n} h_x^{-1/2} \prod_{\{x,y\} \in E_{n+1} \setminus E_n} K_{\langle x,y \rangle} \prod_{x \in W_{n+1}} h_x^{1/2} a \right) \quad (6.25)$$

$$\times \prod_{x \in W_{n+1}} h_x^{1/2} \left( \prod_{\{x,y\} \in E_{n+1} \setminus E_n} K_{\langle x,y \rangle} \right)^* \prod_{x \in W_n} h_x^{-1/2} \quad (6.26)$$

for each  $n \geq 0$  and  $a \in \mathcal{B}_{\Lambda_{n+1}}$ . Similar to the above proof, we get that  $\mathcal{E}_n$  is a quasi-conditional expectation with respect to the triple  $\mathcal{B}_{\Lambda_{n-1}} \subset \mathcal{B}_{\Lambda_n} \subset \mathcal{B}_{\Lambda_{n+1}}$ . One can see that

$$\varphi_n(a) = \mathrm{Tr}(h_0^{1/2} w_0 h_0^{1/2} \mathcal{E}_0 \circ \mathcal{E}_1 \circ \cdots \circ \mathcal{E}_{n-1} \circ \mathcal{E}_n(a)). \quad (6.27)$$

Therefore, according to Theorem 6.1 we can define a  $d$ -Markov chain on  $\mathcal{B}_L$  by  $\varphi = \lim \varphi_n$  in the weak-\* topology. Note that, in classical setting, similar construction were considered in [17].

If  $h_x = h$  and  $K_{\langle x,y \rangle} = K$ , for all  $x \in L$  and  $\{x,y\} \in E$ , and  $w_0$  satisfies the initial condition

$$\mathrm{Tr}_{(i)} \left( w_0 \prod_{j=1}^k K_{\langle 0,j \rangle} \prod_{j=1}^k h_j \left( \prod_{j=1}^k K_{\langle 0,j \rangle} \right)^* \right) = h_i^{1/2} w_0 h_i^{1/2}, \quad (6.28)$$

where  $K_{\langle 0,j \rangle}$  means  $K_{\langle x_0,(j) \rangle}$ ,  $\varphi$  is shift-invariant for  $\gamma_i$ . Indeed, since (6.28) means

$$\mathrm{Tr}(h_0^{1/2} w_0 h_0^{1/2} \mathcal{E}_0(\cdot)) = \mathrm{Tr}(h_i^{1/2} w_0 h_i^{1/2} \cdot)$$

on  $\mathcal{B}_i$ , we have

$$\begin{aligned}\varphi_n(\gamma^i(a)) &= \text{Tr}(h_0^{1/2}w_0h_0^{1/2}\mathcal{E}_0 \circ \mathcal{E}_1 \circ \cdots \circ \mathcal{E}_{n-1} \circ \mathcal{E}_n(\gamma^i(a))) \\ &= \text{Tr}(h_i^{1/2}w_0h_i^{1/2}\mathcal{E}_1 \circ \mathcal{E}_2 \circ \cdots \circ \mathcal{E}_{n-1} \circ \mathcal{E}_n(\gamma^j(a))) \\ &= \text{Tr}(h_0^{1/2}w_0h_0^{1/2}\mathcal{E}_0 \circ \mathcal{E}_1 \circ \cdots \circ \mathcal{E}_{n-2} \circ \mathcal{E}_{n-1}(a)) = \varphi(a)\end{aligned}$$

for all  $a \in \mathcal{B}_{\Lambda_{n-1}}$ . In the third equation, we use  $h_0 = h_i = h$  and  $K_{\langle x,y \rangle} = K$ .

## 7 Example of $d$ -Markov chain on Cayley tree

In this and next sections, we provide more concrete examples of  $d$ -Markov chains on Cayley tree. For the sake of simplicity we consider a semi-infinite Cayley tree  $\Gamma^2(x_0) = (L, E)$  of order 2 so that  $d = 2$ . Our starting  $C^*$ -algebra is the same  $\mathcal{B}_L$  but with  $\mathcal{B}_x = M_2(\mathbb{C})$  for  $x \in L$ . By  $e_{ij}^{(x)}$  we denote the standard matrix units of  $\mathcal{B}_x = M_2(\mathbb{C})$ .

For every edge  $\{x, y\} \in E$  put

$$K_{\langle x,y \rangle} = \exp\{\beta H_{\langle x,y \rangle}\}, \quad \beta \in \mathbb{R} \quad (7.29)$$

where

$$H_{\langle x,y \rangle} = e_{12}^{(x)} \otimes e_{21}^{(y)} + e_{21}^{(x)} \otimes e_{12}^{(y)}. \quad (7.30)$$

Now we are going to find a solution  $\{h_x\}$  and  $w_0$  of equations (6.19), (6.20) for the defined  $\{K_{\langle x,y \rangle}\}$ . Note that from (7.29),(7.30) for every  $K_{\langle x,y \rangle}$  one can see that

$$K_{\langle x,y \rangle} = K_{\langle x,y \rangle}^* \quad (7.31)$$

for all  $\{x, y\} \in E$ .

Assume that  $h_x = \alpha I$  for every  $x \in V$ . Hence, thanks to (7.31), the equations (6.19),(6.20) can be rewritten as follows

$$\alpha \text{Tr}_0(w_0) = 1 \quad (7.32)$$

$$\alpha^2 \text{Tr}_x \left( K_{\langle x,(x,1) \rangle} K_{\langle x,(x,2) \rangle} K_{\langle x,(x,1) \rangle} \right) = \alpha I, \quad \text{for every } x \in L. \quad (7.33)$$

One can see that

$$H_{\langle x,y \rangle}^{2n} = H_{\langle x,y \rangle}^2 = e_{11}^{(x)} \otimes e_{22}^{(y)} + e_{22}^{(x)} \otimes e_{11}^{(y)} \quad (7.34)$$

$$H_{\langle x,y \rangle}^{2n-1} = H_{\langle x,y \rangle} \quad (7.35)$$

for every  $n \in \mathbb{N}$ . Then we get

$$\begin{aligned}K_{\langle x,y \rangle}^2 &= I + (\sinh 2\beta)H_{\langle x,y \rangle} + (\cosh 2\beta - 1)H_{\langle x,y \rangle}^2 \\ \text{Tr}_x(K_{\langle x,y \rangle}^2) &= \frac{\cosh 2\beta + 1}{2}I = \cosh^2 \beta I\end{aligned} \quad (7.36)$$

for every  $\{x, y\} \in E$ . Hence, for  $x \in L$  and  $y, z \in S(x)$ , one finds

$$\begin{aligned}\text{Tr}_x (K_{\langle x,y \rangle} K_{\langle x,z \rangle}^2 K_{\langle x,y \rangle}) &= \text{Tr}_x (K_{\langle x,y \rangle} \text{Tr}_{xy}(K_{\langle x,z \rangle}^2) K_{\langle x,y \rangle}) \\ &= (\cosh^2 \beta) \text{Tr}_x (K_{\langle x,y \rangle}^2) \\ &= (\cosh^4 \beta) I.\end{aligned}$$

Therefore we obtain  $\alpha = \cosh^{-4} \beta$  and  $\text{Tr}(w_0) = \cosh^4 \beta$ .

Next, consider the initial condition (6.28). For convenience, we will write  $K_{\langle 0,1 \rangle}$  for  $K_{\langle x_0, (1) \rangle}$ , for example. Since

$$\begin{aligned} \text{Tr}_1 (w_0 K_{\langle 0,1 \rangle} K_{\langle 0,2 \rangle}^2 K_{\langle 0,1 \rangle}) &= \text{Tr}_1 (w_0 K_{\langle 0,1 \rangle} \text{Tr}_{0,1}(K_{\langle 0,2 \rangle}^2) K_{\langle 0,1 \rangle}) \\ &= (\cosh^2 \beta) \text{Tr}_1 (w_0 K_{\langle 0,1 \rangle}^2), \end{aligned}$$

by putting  $w_0 = \sum_{i,j=1,2} a_{ij} e_{ij}^0$ , thanks to (7.36) we have

$$\begin{aligned} &\text{Tr}_1 (w_0 K_{\langle 0,1 \rangle} K_{\langle 0,2 \rangle}^2 K_{\langle 0,1 \rangle}) \\ &= \frac{\cosh^2 \beta}{2} ((a_{11} + a_{22})I + (\cosh 2\beta - 1)(a_{11}e_{22} + a_{22}e_{11}) + (\sinh 2\beta)(a_{12}e_{12} + a_{21}e_{21})). \end{aligned}$$

This is equal to  $(\cosh^4 \beta)w_0$  from (6.28). Therefore we have the solution  $w_0 = I$ . Hence,  $\varphi$  generated by the above notations is  $\gamma_1$ -invariant  $d$ -Markov chain. Similarly, it is easily seen that  $\varphi$  is also  $\gamma_2$ -invariant.

Finally we show the clustering property. Recall that a state  $\varphi$  on  $\mathcal{B}_L$  satisfies *the clustering property* w.r.t.  $\gamma_i$  if and only if

$$\lim_{n \rightarrow \infty} \varphi(\gamma_i^n(a)b) = \varphi(a)\varphi(b)$$

.

**Theorem 7.1.** *A state  $\varphi$  generated by the above notations is  $\gamma_1$  and  $\gamma_2$ -invariant and satisfies clustering property w.r.t.  $\gamma_i$ ,  $i = 1, 2$ .*

*Proof.* The first assertion is already proven in above.

To show the clustering property, it is enough to prove for any  $a \in \mathcal{B}_0 = M_2(\mathbb{C})$

$$\lim_{n \rightarrow \infty} \mathcal{E}_0 \circ \mathcal{E}_1 \circ \cdots \circ \mathcal{E}_{n-1} \circ \mathcal{E}_n(\gamma_1^{n+1}(a)) = \varphi(a)I.$$

Indeed, for  $a, b \in \mathcal{B}_0$ , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \varphi(\gamma_1^n(a)b) &= \lim_{n \rightarrow \infty} \text{Tr} \left( h_0^{1/2} w_0 h_0^{1/2} \mathcal{E}_0(\mathcal{E}_1 \circ \cdots \circ \mathcal{E}_{n-1} \circ \mathcal{E}_n(\gamma_1^{n+1}(a)b)) \right) \\ &= \varphi(a) \text{Tr} \left( h_0^{1/2} w_0 h_0^{1/2} \mathcal{E}_0(b) \right) = \varphi(a)\varphi(b). \end{aligned}$$

Assume  $\gamma_1^{n+1}(a) \in \mathcal{B}_y$  and  $y, z \in S(x)$ , then essentially, we can restrict  $\mathcal{E}_n$  to  $\mathcal{E}_n|_{\mathcal{B}_{x,y,z}}$ . From a simple calculation, we have

$$\begin{aligned} \text{Tr}_x(K_{\langle x,y \rangle} K_{\langle x,z \rangle} e_{11}^{(y)} K_{\langle x,z \rangle} K_{\langle x,y \rangle}) &= \text{Tr}_x(K_{\langle x,y \rangle} e_{11}^{(y)} \text{Tr}_{xy}(K_{\langle x,z \rangle}^2) K_{\langle x,y \rangle}) \\ &= (\cosh^2 \beta) \text{Tr}_x(K_{\langle x,y \rangle} e_{11}^{(y)} K_{\langle x,y \rangle}) \\ &= \frac{\cosh^4 \beta}{2} I. \end{aligned}$$

Similarly, we get

$$\begin{aligned} \text{Tr}_x(K_{\langle x,y \rangle} K_{\langle x,z \rangle} e_{22}^{(y)} K_{\langle x,z \rangle} K_{\langle x,y \rangle}) &= \frac{\cosh^4 \beta}{2} I, \\ \text{Tr}_x(K_{\langle x,y \rangle} K_{\langle x,z \rangle} e_{12}^{(y)} K_{\langle x,z \rangle} K_{\langle x,y \rangle}) &= \cosh^2 \beta \sinh 2\beta e_{12}^{(x)}, \\ \text{Tr}_x(K_{\langle x,y \rangle} K_{\langle x,z \rangle} e_{21}^{(y)} K_{\langle x,z \rangle} K_{\langle x,y \rangle}) &= \cosh^2 \beta \sinh 2\beta e_{21}^{(x)}. \end{aligned}$$

Therefore, we obtain that

$$\lim_{n \rightarrow \infty} \mathcal{E}_0 \circ \mathcal{E}_1 \circ \cdots \circ \mathcal{E}_{n-1} \circ \mathcal{E}_n(\gamma_1^{n+1}(a)) = \text{Tr}(a) = \varphi(a)I$$

which implies the assertion.

Similarly, one can prove that  $\varphi$  satisfies clustering property w.r.t.  $\gamma_2$ .  $\square$

## 8 Another example of $d$ -Markov chain on Cayley tree

Now consider the next example. For every edge  $\{x, y\} \in E$  put

$$K_{\langle x, y \rangle} = \exp\{\beta P_{\langle x, y \rangle}\}, \quad \beta \in \mathbb{C}$$

where

$$P_{\langle x, y \rangle} = e_{11}^{(x)} \otimes e_{11}^{(y)} + e_{22}^{(x)} \otimes e_{22}^{(y)}.$$

Explicitly, we can write

$$K_{\langle x, y \rangle} = I + (e^\beta - 1)P_{\langle x, y \rangle}. \quad (8.37)$$

Now we are going to find a solution  $\{h_x\}$  and  $w_0$  of equations (6.19), (6.20) for the defined  $\{K_{\langle x, y \rangle}\}$ . Note that for every  $K_{\langle x, y \rangle}$  and  $K_{\langle x, z \rangle}$ , one can see that

$$\begin{aligned} K_{\langle x, y \rangle} &= K_{\langle x, y \rangle}^*, \\ K_{\langle x, y \rangle} K_{\langle x, z \rangle} &= K_{\langle x, z \rangle} K_{\langle x, y \rangle}. \end{aligned}$$

Assume that  $h_x = \alpha I$  for every  $x \in V$ . Hence, thanks to the above equations, the equations (6.19), (6.20) can be rewritten as follows

$$\alpha \text{Tr}_0(w_0) = 1 \quad (8.38)$$

$$\alpha^2 \text{Tr}_x \left( K_{\langle x, (x,1) \rangle}^2 K_{\langle x, (x,2) \rangle}^2 \right) = \alpha I, \quad \text{for every } x \in L. \quad (8.39)$$

From  $\text{Tr}_{xy}(K_{\langle x, z \rangle}^2) = \frac{e^{2\beta} + 1}{2} I$ , we have

$$\text{Tr}_x \left( K_{\langle x, (x,1) \rangle}^2 K_{\langle x, (x,2) \rangle}^2 \right) = \frac{(e^{2\beta} + 1)^2}{4} I.$$

Hence we obtain

$$\alpha = \frac{4}{(e^{2\beta} + 1)^2}$$

and  $\text{Tr}(w_0) = (e^{2\beta} + 1)^2/4$ .

Next, consider the initial condition (6.28). Since

$$\begin{aligned} \text{Tr}_1(w_0 K_{\langle 0,1 \rangle}^2 K_{\langle 0,2 \rangle}^2) &= \text{Tr}_1(w_0 K_{\langle 0,1 \rangle}^2 \text{Tr}_{0,1}(K_{\langle 0,2 \rangle}^2)) \\ &= \frac{e^{2\beta} + 1}{2} \text{Tr}_1(w_0 K_{\langle 0,1 \rangle}^2), \end{aligned}$$

by putting  $w_0 = \sum_{i,j=1,2} a_{ij} e_{ij}^0$ , thanks to (8.37) we have

$$\begin{aligned} &\text{Tr}_1(w_0 K_{\langle 0,1 \rangle}^2 K_{\langle 0,2 \rangle}^2) \\ &= \frac{e^{2\beta} + 1}{4} \left( (e^{2\beta} a_{11} + a_{22}) e_{11} + (a_{11} e^{2\beta} + a_{22}) e_{22} \right) \end{aligned}$$

This is equal to  $\frac{(e^{2\beta}+1)^2}{4}w_0$  from (6.28). Therefore we have the solution  $w_0 = I$ . Therefore,  $\varphi$  generated by the above notations is  $\gamma_1$ -invariant  $d$ -Markov chain. Similarly, it is easily seen that  $\varphi$  is also  $\gamma_2$ -invariant.

Finally, we show the clustering property.

**Theorem 8.1.** *The above  $\varphi$  is  $\gamma_1$  and  $\gamma_2$ -invariant and satisfies clustering property w.r.t.  $\gamma_i$ ,  $i = 1, 2$ .*

*Proof.* The first assertion is already proven in above.

This proof is similar to Theorem 7.1, and we need to show

$$\lim_{n \rightarrow \infty} \mathcal{E}_0 \circ \mathcal{E}_1 \circ \cdots \circ \mathcal{E}_{n-1} \circ \mathcal{E}_n(\gamma_1^{n+1}(a)) = \varphi(a)I.$$

for  $a \in \mathcal{B}_0$ . To see this, we make following lists for  $x \in L$  and  $y, z \in S(x)$ :

$$\begin{aligned} \text{Tr}_x(K_{\langle x,y \rangle} K_{\langle x,z \rangle} e_{11}^{(z)} K_{\langle x,z \rangle} K_{\langle x,y \rangle}) &= \frac{e^{2\beta}(e^{2\beta}+1)}{4} e_{11}^{(x)} + \frac{e^{2\beta}+1}{4} e_{22}^{(x)}, \\ \text{Tr}_x(K_{\langle x,y \rangle} K_{\langle x,z \rangle} e_{22}^{(z)} K_{\langle x,z \rangle} K_{\langle x,y \rangle}) &= \frac{e^{2\beta}+1}{4} e_{11}^{(x)} + \frac{e^{2\beta}(e^{2\beta}+1)}{4} e_{22}^{(x)}. \\ \text{Tr}_x(K_{\langle x,y \rangle} K_{\langle x,z \rangle} e_{12}^{(z)} K_{\langle x,z \rangle} K_{\langle x,y \rangle}) &= 0 \\ \text{Tr}_x(K_{\langle x,y \rangle} K_{\langle x,z \rangle} e_{21}^{(z)} K_{\langle x,z \rangle} K_{\langle x,y \rangle}) &= 0. \end{aligned}$$

As in the classical Markov chain case, we can prove that

$$\lim_{n \rightarrow \infty} \mathcal{E}_0 \circ \mathcal{E}_1 \circ \cdots \circ \mathcal{E}_{n-1} \circ \mathcal{E}_n(\gamma_1^{n+1}(a)) = \text{Tr}(a) = \varphi(a)I.$$

which proves the theorem.

Similarly, we can prove that  $\varphi$  satisfies clustering property w.r.t.  $\gamma_2$ . □

## 9 Conclusions

Let us note that a first attempt of consideration of quantum Markov fields began in [3, 4] for the regular lattices (namely for  $\mathbb{Z}$ ). But there, concrete examples of such fields were not given. In the present paper we have extended a notion of generalized quantum Markov states to fields, i.e. to graphs with an hierarchy property. Here such states have been considered on discrete infinite tensor products of  $C^*$ -algebras over trees. A tree structure of graphs allowed us to give a construction an entangled Markov field, which generalizes the construction of [2] to trees. It has been shown that such states are  $d$ -Markov chains and, in special cases, they are generalized quantum Markov states.

As well as, we have considered a particular case of tree, so called Cayley tree. Over such a tree we gave a construction of  $d$ -Markov chains, and some more concrete examples of such chains were provided, which are shift invariant and have the clustering property. Note that  $d$ -Markov chains describe ground states of quantum systems over trees. Certain particular examples of such systems were considered in [9],[11]. As well as, such shift invariant  $d$ -Markov chains can be also considered as an extension of  $C^*$ -finitely correlated states defined in [12] to the Cayley trees.

## Acknowledgement

The second named author (H.O.) is partially supported by Grant-in-Aid for JSPS Fellows 19-2166. The third named author (F.M.) thanks the grant FRGS0308-91 of MOHE. Finally, the authors also would like to thank to a referee for his useful suggestions which allowed us to improve the content of the paper.

## References

- [1] Accardi L.: On the noncommutative Markov property, *Func. Anal. Appl.*, **9** (1975) 1–8.
- [2] Accardi L., Fidaleo F.: Entangled Markov chains. *Annali di Matematica Pura e Applicata*, **184**(2005), 327–346.
- [3] Accardi L., Fidaleo F.: Quantum Markov fields, *Inf. Dim. Analysis, Quantum Probab. Related Topics* **6** (2003) 123–138.
- [4] Accardi L., Fidaleo F.: On the structure of quantum Markov fields, Proceedings Burg Conference 15–20 March 2001, W. Freudenberg (ed.), World Scientific, QP–PQ Series 15 (2003) 1–20
- [5] Accardi L., Fidaleo F. Mukhamedov, F.: Markov states and chains on the CAR algebra, *Inf. Dim. Analysis, Quantum Probab. Related Topics* **10** (2007), 165–183.
- [6] Accardi L., Frigerio A.: Markovian cocycles, *Proc. Royal Irish Acad.* **83A** (1983) 251–263.
- [7] Accardi L., Liebscher V.: Markovian KMS-states for one-dimensional spin chains, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **2**(1999) 645–661.
- [8] Accardi L., Matsuoka T., Ohya M.: Entangled Markov chains are indeed entangled, *Inf. Dim. Analysis, Quantum Probab. Related Topics* **9** (2006), 379–390.
- [9] Affleck L, Kennedy E., Lieb E.H., Tasaki H.: Valence bond ground states in isotropic quantum antiferromagnets, *Commun. Math. Phys.* **115** (1988), 477–528.
- [10] Dobrushin R.L.: Description of Gibbsian Random Fields by means of conditional probabilities, *Probability Theory and Applications* **13**(1968) 201–229
- [11] Fannes M., Nachtergaele B. Werner R. F.: Ground states of VBS models on Cayley trees, *J. Stat. Phys.* **66** (1992) 939–973.
- [12] Fannes M., Nachtergaele B. Werner R. F.: Finitely correlated states on quantum spin chains, *Commun. Math. Phys.* **144** (1992) 443–490.
- [13] Liebscher V.: Markovianity of quantum random fields, Proceedings Burg Conference 15–20 March 2001, W. Freudenberg (ed.), World Scientific, QP–PQ Series 15 (2003) 151–159
- [14] Miyadera T.: Entangled Markov Chains generated by symmetric channels, *Inf. Dim. Analysis, Quantum Probab. Related Topics* **8** (2005) 497–504.

- [15] Ohno H.: Extendability of generalized quantum Markov chains on gauge invariant  $C^*$ -algebras, *Inf. Dim. Analysis, Quantum Probab. Related Topics*, **8**(2005) 141–152.
- [16] Preston C.: Gibbs States on Countable Sets (Cambridge University Press, London, 1974).
- [17] Spataru A.: Construction of a Markov Field on an infinite Tree, *Advances in Math* **81**(1990), 105–116.
- [18] Spitzer F.: Markov random Fields on an infinite tree, *Ann. Prob.* **3** (1975) 387-398.
- [19] Zachary S.: Countable state space Markov random fields and Markov chains on trees, *Ann. Prob.* **11** (1983) 894–903.
- [20] Zachary S.: Bounded attractive and repulsive Markov specifications on trees and on the one-dimensional lattice, *Stochastic Process. Appl.* **20** (1985) 247–256.