Doctoral Dissertation (Shinshu University)

On some properties of solutions to partial differential equations for an incompressible fluid

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Abstract

First of all, we give an alternative proof of a logarithmically improved Beale-Kato-Majda type extension criterion for smooth solutions to the Navier-Stokes equations in the whole space, which was shown by Fan, Jiang, Nakamura and Zhou (J. Math. Fluid Mech. 13:557-571, 2011). By our method, we can also establish a similar criterion to the above in case of the half space, bounded domains and exterior domains.

Next, we show Serrin type extension criteria for smooth solutions to the 3D Navier-Stokes equations. To this end, we use Brezis-Gallouet-Wainger type inequalities.

Finally, we construct time-periodic solutions to the Boussinesq equations in a 3-dimensional exterior domain. To this end, we use Yamazaki's method (Math. Ann. 317:635-675, 2000). He showed existence and uniqueness of time-periodic solutions to the Navier-Stokes equations in a 3-dimensional exterior domain.

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Contents

Chapter 1

Introduction

1.1 Beale-Kato-Majda type extension criteria for smooth solutions to the Navier-Stokes equations

Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary $\partial\Omega$. The motion of a viscous incompressible fluid in Ω is governed by the Navier-Stokes equations:

$$\text{(N-S)} \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi = 0, & x \in \Omega,\ t \in (0, T), \\ \operatorname{div} u = 0, & x \in \Omega,\ t \in (0, T), \\ u|_{\partial\Omega} = 0, u|_{t=0} = u_0, \end{cases}$$

where $u = (u^1(x,t), u^2(x,t), \cdots, u^n(x,t))$ and $\pi = \pi(x,t)$ denote the velocity vector and the pressure, respectively, of the fluid at the point $(x,t) \in \Omega \times (0,T)$ and u_0 is a given initial velocity.

In Chapter 3, we consider Beale-Kato-Majda type extension criteria for smooth solutions to (N-S). Beale-Kato-Majda [1] and Kato-Ponce [24] showed that the L^∞-norm of the vorticity $\omega = \operatorname{curl} u$ controls the breakdown of smooth solutions to the Euler and Navier-Stokes equations. To be precise, if

$$\int_0^T \|\omega(\tau)\|_{L^\infty} d\tau < \infty,$$

then the smooth solution u in $C([0,T); W^{s,p}(\mathbb{R}^n))(s > n/p + 1)$ can be continued beyond $t = T$. Chemin [9] and Kozono-Ogawa-Taniuchi [28] proved similar extension criteria with $\|\omega\|_{L^\infty}$ replaced by $\|u\|_{B^1_{\infty,\infty}}$ and $\|\omega\|_{\dot{B}^0_{\infty,\infty}}$, respectively. Note that Chemin dealt with solutions in $C^\alpha, \alpha > 1$. Chae [8] also proved the same criterion via $\|\omega\|_{\dot{B}^0_{\infty,\infty}}$ for solutions in Triebel-Lizorkin spaces. In case of 3-dimensional bounded domains, for the Euler equations, Ferrari [15] and Shirota-Yanagisawa [55] succeeded in proving the same result of the breakdown as Beale-Kato-Majda holds. See also Zajaczkowski [68]. Ogawa-Taniuchi [48] proved a similar extension criterion with $\|\omega\|_{L^\infty(\Omega)}$ replaced by $\|\omega\|_{bmo(\Omega)}$.

Fan-Jiang-Nakamura-Zhou [14] established a logarithmically improved Beale-Kato-Majda type extension criterion for (N-S) in the whole space:

$$\int_0^T \frac{\|\omega(\tau)\|_{BMO}}{1+\log(1+\|\omega(\tau)\|_{BMO})} d\tau < \infty. \tag{1.1}$$

They showed the criterion (1.1) for H^s solutions to (N-S) by the energy method. In Chapter 3, we will give an alternative proof of this criterion. Moreover, we will show that this extension criterion holds for more general solutions in L^p, $n \le p < \infty$. To this end, we will use the integral equation of (N-S) and the smoothing effect of $e^{t\Delta}$.

An advantage of our approach is that we can also establish a similar type extension criterion to the above in case of the 3-dimensional half space, 3-dimensional bounded domains and 3-dimensional exterior domains with smooth boundary for solutions with the no-slip boundary condition. Concretely, on (1.1) in case of them, we need to replace the BMO-seminorm with the L^∞-norm. Chapter 3 is based on [42].

1.2 Serrin type extension criteria for smooth solutions to the Navier-Stokes equations

In Chapter 4, we consider Serrin type extension criteria for smooth solutions to (N-S) in 3-dimension. Serrin [52] and Giga [18] showed that if a Leray-Hopf weak solution u satisfies

$$(Se) \qquad u \in L^s(0,T;L^r(\Omega)) \text{ for some } 3 < r \le \infty, 2 \le s < \infty \text{ with } \frac{3}{r}+\frac{2}{s} \le 1$$

then u is smooth. Many researchers showed this type regularity criterion, see e.g. [13, 28, 29, 34, 57, 59, 60, 61]. The limiting case $s=\infty, r=3$ was proven in Escauriaza-Seregin-Šverák [12], see also Neustupa [45].

Giga [18] showed that the condition (Se) also guarantees the time-extension of strong L^p solutions, $3 \le p < \infty$. That is, if a strong L^p solution u satisfies (Se), then u can be continued beyond T. In Chapter 4, we will slightly relax condition (Se) in the case $r = \infty$;

$$u \in L^2(0,T;L^\infty(\Omega))$$

by replacing $L^\infty(\Omega)$ with some Banach spaces. Chapter 4 is based on [44].

1.3 Time-periodic solutions to the Boussinesq equations in exterior domains

Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with compact and smooth boundary $\partial\Omega$, and let $(0,0,0) \in \Omega^c$. Heat convection of a viscous incompressible fluid in Ω is governed by the Boussinesq equations:

$$
\text{(B)} \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi = g\theta + f, & x \in \Omega,\ t \in (-\infty, \infty), \\ \partial_t \theta - \Delta\theta + u \cdot \nabla\theta = S, & x \in \Omega,\ t \in (-\infty, \infty), \\ \operatorname{div} u = 0, & x \in \Omega,\ t \in (-\infty, \infty), \\ u|_{\partial\Omega} = 0,\ \frac{\partial\theta}{\partial\eta}|_{\partial\Omega} = 0, & \end{cases}
$$

where $u = (u^1(x,t), u^2(x,t), u^3(x,t))$, $\theta = \theta(x,t)$ and $\pi = \pi(x,t)$ denote the velocity vector, the temperature and the pressure, respectively, of the fluid at the point $(x,t) \in \Omega \times (-\infty, \infty)$. Here $f = (f_1(x,t), f_2(x,t), f_3(x,t))$ and $S = S(x,t)$ are given external forces, and $g = g(x) = -\tilde{g}\frac{x}{|x|^3}$ is the given vector that denotes the acceleration of gravity, where $\tilde{g}$ is a constant. On the temperature, we impose the Neumann boundary condition.

In Chapter 5, we consider time-periodic solutions to (B). Kozono-Nakao [27] showed existence and uniqueness of time-periodic solutions to the Navier-Stokes equations in n-dimensional exterior domains, where $n \geq 4$. However, it was outstanding in case of 3-dimensional exterior domains. Here we recall the L^p-L^q estimate for the gradient of the Stokes semigroup e^{-tA} in exterior domains:

$$
\|\nabla e^{-tA} f\|_q \leq C t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|f\|_p \quad \text{for } \ 1 < p \leq q \leq n,
$$

which Iwashita [22] showed. The restriction $q \leq n$ caused the difficulty in 3-dimensional case.

Yamazaki [66] solved this difficulty by using real interpolation, and he showed existence and uniqueness of time-periodic solutions to the Navier-Stokes equations in n-dimensional exterior domains, where $n \geq 3$. In Chapter 5, we will construct time-periodic solutions to (B) by his method. Chapter 5 is based on [41].

In this paper, we denote by C various constants.

Chapter 2

Preliminaries

In this chapter, we introduce some notations and function spaces.

Let $C_0^\infty(\Omega)$ denote the set of all C^∞ functions with compact support in Ω and $C_{0,\sigma}^\infty(\Omega) = C_{0,\sigma}^\infty := \{\varphi \in (C_0^\infty(\Omega))^n; \operatorname{div}\varphi = 0\}$. Then $L_\sigma^r, 1 < r < \infty$, is the closure of $C_{0,\sigma}^\infty$ with respect to the L^r-norm $\|\cdot\|_r$. Concerning Sobolev spaces we use the notations $W^{k,r}(\Omega)$ and $W_0^{k,r}(\Omega), k \in \mathbb{N}, 1 \le r \le \infty$. Note that very often we will simply write L^r and $W^{k,r}$ instead of $L^r(\Omega)$ and $W^{k,r}(\Omega)$, respectively. The symbol $(\cdot,\cdot)$ denotes the L^2-inner product and the duality pairing between L^r and $L^{r'}$, where $\frac{1}{r} + \frac{1}{r'} = 1$.

For $0 < \theta < 1$ and $1 \le q \le \infty$, let $(\cdot,\cdot)_{\theta,q}$ denote real interpolation. For $1 < r_0 < r < r_1 < \infty$ and $1 \le q \le \infty$, let $L^{r,q}(\Omega) := (L^{r_0}(\Omega), L^{r_1}(\Omega))_{\theta,q}$ denote the Lorentz space, where θ satisfy $\frac{1-\theta}{r_0} + \frac{\theta}{r_1} = \frac{1}{r}$, see [2, 35, 36].

Let us recall the Helmholtz decomposition: $L^r(\Omega) = L_\sigma^r \oplus G_r$ $(1 < r < \infty)$, where $G_r := \{\nabla\pi \in L^r; \pi \in L_{loc}^r(\overline{\Omega})\}$, see Fujiwara-Morimoto [16], Miyakawa [38], Simader-Sohr [56], and Borchers-Miyakawa [3]; P_r denotes the projection operator from L^r onto L_σ^r along G_r. The Stokes operator A_r on L_σ^r is defined by $A_r = -P_r\Delta$ with domain $D(A_r) = W^{2,r} \cap W_0^{1,r} \cap L_\sigma^r$. It is known that $(L_\sigma^r)^*$(the dual space of L_σ^r)$= L_\sigma^{r'}$ and A_r^*(the adjoint operator of A_r)$= A_{r'}$, where $\frac{1}{r} + \frac{1}{r'} = 1$. It is shown by Giga [17], Borchers-Sohr [5], Borchers-Miyakawa [3], and Iwashita [22] that $-A_r$ generates a holomorphic semigroup $\{e^{-tA_r}; t \ge 0\}$ of class C_0 in L_σ^r. Since $P_r u = P_q u$ for all $u \in L^r \cap L^q(1 < r, q < \infty)$ and since $A_r u = A_q u$ for all $u \in D(A_r) \cap D(A_q)$, for simplicity, we shall abbreviate $P_r u, P_q u$ as Pu for $u \in L^r \cap L^q$ and $A_r u, A_q u$ as Au for $u \in D(A_r) \cap D(A_q)$, respectively.

In case of $\Omega = \mathbb{R}^n$, P has the following formula:

$$P = I - \left(\mathcal{F}^{-1}\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}\right)_{ij} \quad (i, j = 1, \cdots, n),$$

and $-P\Delta = -\Delta P$.

Note that we can extend the Helmholtz projection P_r as a projection P in $\sum_{1<r<\infty} L^r$. By the projection P, we have the Helmholtz decomposition in the Lorentz space: $L^{r,q} = L^{r,q}_\sigma \oplus G_{r,q}(1 < r < \infty, 1 \le q \le \infty)$, where $L^{r,q}_\sigma := \{u \in L^{r,q}; \operatorname{div} u = 0 \text{ and } u \cdot \nu|_{\partial\Omega} = 0\}$ and $G_{r,q} := \{\nabla\pi \in L^{r,q}; \pi \in L^{r,q}_{loc}(\overline{\Omega})\}$, see Miyakawa-Yamada [39] and Borchers-Miyakawa [4].

For $1 \le q < \infty$, $C^\infty_{0,\sigma}$ is dense in $L^{r,q}_\sigma$, and $(L^{r,q}_\sigma)^* = L^{r/(r-1),q/(q-1)}_\sigma$. On the other hand, $C^\infty_{0,\sigma}$ is not dense in $L^{r,\infty}$, and $\left(\overline{C^\infty_{0,\sigma}}^{\|\cdot\|_{r,\infty}}\right)^* = L^{r/(r-1),1}_\sigma$.

Let $B_r := -\Delta$ with domain $D(B_r) = \left\{\theta \in W^{2,r}; \frac{\partial\theta}{\partial\nu}|_{\partial\Omega} = 0\right\}$, $1 < r < \infty$. It is known that $-B_r$ generates a holomorphic semigroup $\left\{e^{-tB_r}; t \ge 0\right\}$ of class C_0 in L^r.

Chapter 3

Beale-Kato-Majda type extension criteria for smooth solutions to the Navier-Stokes equations

3.1 Main Results

In this chapter, we consider Beale-Kato-Majda type extension criteria for smooth solutions to (N-S).

Now our main results read as follows.

Theorem 3.1 ([42]). *Let $\Omega = \mathbb{R}^n$, $n \leq p < \infty$, $0 < \alpha < 1$, $0 < T < \infty$, $u_0 \in L^p_\sigma$ and u be a solution to (N-S) on $(0,T)$ in the class*

$$S_p(0,T) := C([0,T); L^p_\sigma) \cap C^1((0,T); L^p_\sigma) \cap C((0,T); W^{2,p}).$$

If

$$\int_s^T \frac{\|\omega(\tau)\|_{BMO}}{\log(e + \|u(\tau)\|_{C^{1+\alpha}})} d\tau < \infty \quad \textit{for some} \quad s \in (0,T), \tag{3.1}$$

then u can be continued to the solution in the class $S_p(0,T')$ for some $T' > T$, where $\omega = \operatorname{curl} u$.

Even in the case where the domain Ω is not the whole space, we have a similar result as below.

Theorem 3.2 ([42]). *Let Ω be the 3-dimensional half space, a 3-dimensional bounded domain or a 3-dimensional exterior domain with smooth boundary,* *and let $3 \le p < \infty$, $0 < \alpha < 1$, $0 < T < \infty$, $u_0 \in L^p_\sigma$ and u be a solution to (N-S) in the class*

$$C_p(0,T) := C([0,T); L^p_\sigma) \cap C^1((0,T); L^p_\sigma) \cap C((0,T); W^{2,p} \cap W^{1,p}_0).$$

If

$$\int_s^T \frac{\|\omega(\tau)\|_{L^\infty(\Omega)}}{\log\left(e + \|u(\tau)\|_{C^{1+\alpha}(\Omega)}\right)} d\tau < \infty \quad \textit{for some} \quad s \in (0,T),$$

then u can be continued to the solution in the class $C_p(0,T')$ for some $T' > T$.

Remark 3.1. (i) Solutions in the class $S_p(0,T)$ or $C_p(0,T)$ are called strong L^p solutions on $(0,T)$. For $p \ge n$, the existence of strong L^p solutions to (N-S) is proven in e.g. [18, 19, 22, 23, 65].
(ii) Note that strong L^p solutions u belong to $C((0,T) : C^m(\Omega))$ for all $m \in \mathbb{N}$.
(iii) Since $\|\omega\|_{BMO} \le 2\|\omega\|_\infty \le 2\|u\|_{C^{1+\alpha}}$, (3.1) can be replaced by

$$\int_s^T \frac{\|\omega(\tau)\|_{BMO}}{\log\left(e + \|\omega(\tau)\|_{BMO}\right)} d\tau < \infty \quad \text{for some} \quad s \in (0,T).$$

(iv) Since

$$\|f\|_{BMO} \cong \|f\|_{\dot{F}^0_{\infty,2}} \le C\|f\|_{\dot{B}^0_{\infty,2}} \le C\left(1 + \|f\|_{\dot{B}^0_{\infty,\infty}} \log^{1/2}\left(e + \|f\|_{\dot{C}^\alpha} + \|f\|_{\dot{B}^{-\alpha}_{\infty,\infty}}\right)\right)$$

for all $f \in \dot{C}^\alpha(\mathbb{R}^n) \cap \dot{B}^{-\alpha}_{\infty,\infty}(\mathbb{R}^n)$, see [26, p. 230] and [28, Theorem 2.1], the condition (3.1) can be replaced by

$$\int_s^T \frac{\|\omega(\tau)\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1 + \log\left(e + \|\omega(\tau)\|_{\dot{B}^0_{\infty,\infty}}\right)}} d\tau < \infty \quad \text{for some} \quad s \in (0,T),$$

which was also given in [14].

3.2 Proof of Theorem 3.1

Proof of Theorem 3.1. For the sake of simplicity, we prove the theorem only in the case where p satisfies

$$\frac{n}{1-\alpha} < p < \infty.$$

Since $u \in C((0,T); W^{2,p})$, without loss of generality, we may assume that $u_0 \in W^{2,p}$. Since the local existence time of strong L^p solutions T_* can be estimated from below as

$$T_* > C(n,p)/\|u_0\|_p^{2p/(p-n)},$$

see e.g. [18, Theorem 1 (ii)], it suffices to show that

$$\sup_{0<t<T} \|u(t)\|_p \le \|u_0\|_p \exp\left(C \exp\left(C \int_0^T \frac{\|\omega(s)\|_{BMO}}{\log(e + \|u(s)\|_{C^{1+\alpha}})} ds \right)\right). \quad (3.2)$$

Recall that u satisfies

$$\text{(I.E.)} \qquad u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta} P(u \cdot \nabla u)(s) ds$$

for all $0 < t < T$ and

$$\|u \cdot \nabla u\|_p \le C\|u\|_p \|\nabla u\|_{BMO} \le C\|u\|_p \|\omega\|_{BMO}, \quad (3.3)$$

see [26, Lemma 3.9]. Since $1 < p < \infty$, P is bounded in L^p.
Then, since

$$\|e^{t\Delta}\|_{L^p \to L^p} \le 1,$$

by (3.3), we have

$$\begin{aligned}
\|u(t)\|_p &\le \|e^{t\Delta}u_0\|_p + \int_0^t \|e^{(t-s)\Delta} P(u \cdot \nabla u(s))\|_p ds \\
&\le \|u_0\|_p + C \int_0^t \|u \cdot \nabla u(s)\|_p ds \\
&\le \|u_0\|_p + C \int_0^t \|u(s)\|_p \|\omega(s)\|_{BMO} ds.
\end{aligned}$$

Therefore, by the Gronwall lemma, we get

$$\|u(s)\|_p \le \|u_0\|_p \exp\left(C \int_0^s \|\omega(\tau)\|_{BMO} d\tau \right),$$

which yields

$$\sup_{0<s<t} \|u(s)\|_p \le \|u_0\|_p \exp\left(C \int_0^t \|\omega(\tau)\|_{BMO} d\tau \right) \quad (3.4)$$

for all $0 < t < T$.

Let

$$M(t) := \frac{\|\omega(t)\|_{BMO}}{\log(e + \|u(t)\|_{C^{1+\alpha}})} \qquad 0 < t < T,$$

and $\delta > 1$ be a sufficiently large number such that

$$\left(\frac{1+\alpha}{2} + \frac{n}{2p} \right) \cdot \left(1 + \frac{1}{\delta} \right) < 1.$$

Then we have

$$
\begin{aligned}
\|\omega(s)\|_{BMO} &= M(s)\log(e+\|u(s)\|_{C^{1+\alpha}}) \\
&\le M(s)\log\left(\left(e+\frac{\|u(s)\|^{1+\delta}_{C^{1+\alpha}}}{M^\delta(s)}\right)\left(e+\frac{M^\delta(s)}{\|u(s)\|^\delta_{C^{1+\alpha}}}\right)\right) \\
&= M(s)\log\left(e+\frac{\|u(s)\|^{1+\delta}_{C^{1+\alpha}}}{M^\delta(s)}\right)+M(s)\log\left(e+\frac{M^\delta(s)}{\|u(s)\|^\delta_{C^{1+\alpha}}}\right) \\
&\le M(s)\log\left(e^\delta+\left(\frac{\|u(s)\|^{1+1/\delta}_{C^{1+\alpha}}}{M(s)}\right)^\delta\right)+M(s)\log\left(e+\left(\frac{\|\omega(s)\|_{BMO}}{\|u(s)\|_{C^{1+\alpha}}}\right)^\delta\right).
\end{aligned}
$$

Since $\frac{\|\omega\|_{BMO}}{\|u\|_{C^{1+\alpha}}} \le \frac{2\|\omega\|_\infty}{\|u\|_{C^{1+\alpha}}} \le 2$, we have

$$
\begin{aligned}
\|\omega(s)\|_{BMO} &\le M(s)\log\left(e+\frac{\|u(s)\|^{1+1/\delta}_{C^{1+\alpha}}}{M(s)}\right)^\delta + M(s)\log(e+C) \\
&\le CM(s)\log\left(e+\frac{\|u(s)\|^{1+1/\delta}_{C^{1+\alpha}}}{M(s)}\right)+M(s)\log(e+C).
\end{aligned}
$$

Let A and $\tilde{M}$ be positive constants and $f(\varepsilon) := A\varepsilon+\tilde{M}\log\left(e+\frac{1}{\varepsilon}\right)$ for $\varepsilon>0$. Then we have

$$
\begin{aligned}
\tilde{M}\log\left(e+\frac{A}{\tilde{M}}\right) &= \tilde{M}\log\left(e+\frac{A\varepsilon}{\tilde{M}}\frac{1}{\varepsilon}\right) \\
&\le \tilde{M}\log\left(\exp\left(\frac{A\varepsilon}{\tilde{M}}\right)e+\exp\left(\frac{A\varepsilon}{\tilde{M}}\right)\frac{1}{\varepsilon}\right) \\
&= \tilde{M}\log\left(\exp\left(\frac{A\varepsilon}{\tilde{M}}\right)\right)+\tilde{M}\log\left(e+\frac{1}{\varepsilon}\right) \\
&= f(\varepsilon).
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
\|\omega(s)\|_{BMO} \le C\varepsilon\|u(s)\|^{1+1/\delta}_{C^{1+\alpha}} + CM(s)\log\left(e+\frac{1}{\varepsilon}\right) \\
+ M(s)\log(e+C)
\end{aligned} \tag{3.5}
$$

for all $\varepsilon>0$.

Let

$$
\begin{aligned}
h(t) &:= \sup_{0<\tau<t}\|u(\tau)\|_p, \\
g(t) &:= \int_0^t \|\omega(s)\|_{BMO}ds
\end{aligned}
$$

for $0 < t < T$. Therefore, from (3.5), for any positive bounded function $\varepsilon(s,t)$ on $(0,T)\times(0,T)$ we see that

$$
\begin{aligned}
g(t) &\le C\int_0^t \varepsilon(s,t)\|u(s)\|_{C^{1+\alpha}}^{1+1/\delta}ds \\
&\quad + C\int_0^t M(s)\log\left(e+\frac{1}{\varepsilon(s,t)}\right)ds + \int_0^t M(s)\log(e+C)ds \\
&=: I_1(t)+I_2(t)+I_3(t).
\end{aligned} \tag{3.6}
$$

Since $p > n$ and $0 < 1+\alpha < 2-n/p$, the following inequality of Gagliardo-Nirenberg-Sobolev type:

$$
\begin{aligned}
\|F\|_{C^{1+\alpha}} &\le C\left(\|F\|_{\dot W^{2,p}}^{\frac{1+\alpha}{2}+\frac{n}{2p}}\|F\|_p^{1-\frac{1+\alpha}{2}-\frac{n}{2p}}+\|F\|_p\right) \\
&\le C\|F\|_{W^{2,p}}^{\frac{1+\alpha}{2}+\frac{n}{2p}}\|F\|_p^{1-\frac{1+\alpha}{2}-\frac{n}{2p}}
\end{aligned} \tag{3.7}
$$

holds for all $F \in W^{2,p}$, cf. [62, Theorem 3.20, (3.177)]. Then, by (3.7), we obtain

$$
\begin{aligned}
\|e^{t\Delta}f\|_{C^{1+\alpha}} &\le C\|e^{t\Delta}f\|_{W^{2,p}}^{\frac{1+\alpha}{2}+\frac{n}{2p}}\|e^{t\Delta}f\|_p^{1-\frac{1+\alpha}{2}-\frac{n}{2p}} \\
&\le C\|(1-\Delta)e^{t\Delta}f\|_p^{\frac{1+\alpha}{2}+\frac{n}{2p}}\|e^{t\Delta}f\|_p^{1-\frac{1+\alpha}{2}-\frac{n}{2p}} \\
&\le C\left(\|e^{t\Delta}f\|_p+\|(-\Delta)e^{t\Delta}f\|_p^{\frac{1+\alpha}{2}+\frac{n}{2p}}\|e^{t\Delta}f\|_p^{1-\frac{1+\alpha}{2}-\frac{n}{2p}}\right) \\
&\le C\left(1+t^{-\frac{1+\alpha}{2}-\frac{n}{2p}}\right)\|f\|_p
\end{aligned} \tag{3.8}
$$

for all $f \in L^p$. Therefore, from (I.E.), (3.3) and (3.8), we obtain

$$
\begin{aligned}
\|u(s)\|_{C^{1+\alpha}} &\le \|e^{s\Delta}u_0\|_{C^{1+\alpha}} + C\int_0^s\left(1+(s-\tau)^{-\frac{1+\alpha}{2}-\frac{n}{2p}}\right)\|u\cdot\nabla u(\tau)\|_p d\tau \\
&\le C\|e^{s\Delta}u_0\|_{W^{2,p}} + C\int_0^s\left(1+(s-\tau)^{-\frac{1+\alpha}{2}-\frac{n}{2p}}\right)h(\tau)\|\omega(\tau)\|_{BMO}d\tau \\
&\le C\|u_0\|_{W^{2,p}} + Ch(s)\int_0^s\left(1+(s-\tau)^{-\frac{1+\alpha}{2}-\frac{n}{2p}}\right)\|\omega(\tau)\|_{BMO}d\tau,
\end{aligned}
$$

which yields

$$
\|u(s)\|_{C^{1+\alpha}}^{1+1/\delta} \le C\|u_0\|_{W^{2,p}}^{1+1/\delta} + Ch^{1+1/\delta}(s)\left(\int_0^s\left(1+(s-\tau)^{-\frac{1+\alpha}{2}-\frac{n}{2p}}\right)\|\omega(\tau)\|_{BMO}d\tau\right)^{1+1/\delta}.
$$

Hence, for $0 < t < T$ we have

$$
\begin{aligned}
I_1(t) &\le C\|u_0\|_{W^{2,p}}^{1+1/\delta} T \sup_{0<s<T, 0<t<T} \varepsilon(s,t) \\
&\quad + C\int_0^t h^{1+1/\delta}(s)\varepsilon(s,t)\left(\int_0^s \left(1+(s-\tau)^{-\frac{1+\alpha}{2}-\frac{n}{2p}}\right)\|\omega(\tau)\|_{BMO} d\tau\right)^{1+1/\delta} ds.
\end{aligned}
$$

Now we choose $\varepsilon(s,t)$ such as

$$
\varepsilon(s,t) := \frac{\eta}{h^{1+1/\delta}(s)g^{1/\delta}(t)+1}
$$

where $\eta = \eta(T) \in (0,1)$ is a constant to be chosen suitably small later on. Then we have

$$
\begin{aligned}
I_1(t) &\le C\|u_0\|_{W^{2,p}}^{1+1/\delta} T \\
&\quad + C\frac{\eta}{g^{1/\delta}(t)}\int_0^t \left(\int_0^s \|\omega(\tau)\|_{BMO} d\tau + \int_0^s (s-\tau)^{-\frac{1+\alpha}{2}-\frac{n}{2p}}\|\omega(\tau)\|_{BMO} d\tau\right)^{1+1/\delta} ds \\
&\le C(T, \|u_0\|_{W^{2,p}}) + C(T)\eta\int_0^t \|\omega(\tau)\|_{BMO} d\tau \\
&\quad + C\frac{\eta}{g^{1/\delta}(t)}\int_0^t \left(\int_0^s (s-\tau)^{-\frac{1+\alpha}{2}-\frac{n}{2p}}\|\omega(\tau)\|_{BMO} d\tau\right)^{1+1/\delta} ds.
\end{aligned}
$$

Let $\beta := \frac{1+\alpha}{2} + \frac{n}{2p}$ and $\varphi(\tau) := \tau^{-\beta}$. For each $t \in (0,T)$, the Young inequality yields

$$
\begin{aligned}
&\int_0^t \left(\int_0^s (s-\tau)^{-\frac{1+\alpha}{2}-\frac{n}{2p}}\|\omega(\tau)\|_{BMO} d\tau\right)^{1+1/\delta} ds \\
&= \int_0^t \left(\int_0^s \varphi(s-\tau)1_{(0,t)}(s-\tau)\cdot \|\omega(\tau)\|_{BMO}1_{(0,t)}(\tau) d\tau\right)^{1+1/\delta} ds \\
&\le \left\|(\varphi\cdot 1_{(0,t)}) * (\|\omega(\cdot)\|_{BMO}\cdot 1_{(0,t)})\right\|_{L^{1+1/\delta}(\mathbb{R})}^{1+1/\delta} \\
&\le \left\|\|\omega(\cdot)\|_{BMO}1_{(0,t)}\right\|_{L^1(\mathbb{R})}^{1+1/\delta}\left\|\varphi\cdot 1_{(0,t)}\right\|_{L^{1+1/\delta}(\mathbb{R})}^{1+1/\delta} \\
&= C\left(\int_0^t \|\omega(\tau)\|_{BMO} d\tau\right)^{1+1/\delta}\cdot t^{-\beta\left(1+\frac{1}{\delta}\right)+1} \\
&\le C g(t)^{1+1/\delta} T^{-\beta\left(1+\frac{1}{\delta}\right)+1}.
\end{aligned}
$$

Hence, we have

$$
I_1(t) \le C(T, \|u_0\|_{W^{2,p}}) + C_1(T)\eta g(t). \tag{3.9}
$$

Since (3.4) yields

$$
\begin{aligned}
\log\left(e+\frac{1}{\varepsilon(s,t)}\right) &= \log\left(e+\frac{h^{1+1/\delta}(s)g^{1/\delta}(t)+1}{\eta}\right)\\
&\le \log\left(e+\frac{\Big(\|u_0\|_p\exp\big(Cg(s)\big)\Big)^{1+1/\delta}g^{1/\delta}(t)+1}{\eta}\right)\\
&\le \log\left[\Big(\exp\big(Cg(s)\big)\Big)^{1+1/\delta}\right]+\log\left(e+\frac{1}{\eta}+\frac{Cg^{1/\delta}(t)}{\eta}\right)\\
&\le \log\left[\Big(\exp\big(Cg(s)\big)\Big)^{1+1/\delta}\right]+\log\left(e+\frac{1}{\eta}+\frac{C\exp\big(g^{1/\delta}(t)\big)}{\eta}\right)\\
&\le C+Cg(s)+g^{1/\delta}(t)
\end{aligned}
$$

for $0<t<T$ and since $\int_0^T M(s)ds<\infty$, we have

$$
\begin{aligned}
I_2(t) &\le C\int_0^t M(s)ds + C\int_0^t M(s)g(s)ds\\
&\qquad + C\left(\int_0^t M(s)ds\right)\left(g^{1/\delta}(t)\right)\\
&\le C + C\int_0^t M(s)g(s)ds + C\left(g^{1/\delta}(t)\right)\\
&\le C + C\int_0^t M(s)g(s)ds + C + \frac{\left(g^{1/\delta}(t)\right)^{\delta}}{2}\\
&= C + C\int_0^t M(s)g(s)ds + \frac{g(t)}{2}.
\end{aligned}
\tag{3.10}
$$

Clearly, we have

$$
I_3(t) \le \log(e+C)\int_0^T M(s)ds < C. \tag{3.11}
$$

Gathering (3.9), (3.10) and (3.11) with (3.6), we obtain

$$
g(t) \le C + \left(C_1(T)\eta + \frac{1}{2}\right)g(t) + C\int_0^t M(s)g(s)ds.
$$

Therefore, letting $\eta = \frac{1}{4C_1(T)}$, by the Gronwall lemma, we get

$$
g(t) \le C\exp\left(C\int_0^T M(s)ds\right)
$$

for all $0<t<T$. This estimate and (3.4) yield the desired estimate (3.2). $\square$

We can prove Theorem 3.2 in the same way to the proof of Theorem 3.1, by using

$$\|P(u \cdot \nabla u)\|_p = \|P(\omega \times u + (\nabla |u|^2)/2)\|_p = \|P(\omega \times u)\|_p \leq C\|\omega\|_\infty \|u\|_p$$

and

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A} P(\omega \times u)(s) ds,$$

instead of (3.3) and (I.E.) respectively.

Chapter 4

Serrin type extension criteria for smooth solutions to the Navier-Stokes equations

4.1 Function Spaces and Main Results

In this chapter, we consider Serrin type extension criteria for smooth solutions to (N-S) in 3-dimension.

First, we introduce Banach spaces of Morrey type and Besov type which are wider than $L^\infty(\Omega)$. Let $B(x,t) := \{y \in \mathbb{R}^n; |y-x| < t\}$ and

$$L^1_{uloc}(\Omega) := \left\{ f \in L^1_{loc}(\bar{\Omega}); \|f\|_{L^1_{uloc}(\Omega)} := \sup_{x \in \mathbb{R}^n} \int_{B(x,1)\cap\Omega} |f(y)|dy < \infty \right\}.$$

Definition 4.1. Let $\beta > 0$ and $\Omega \subset \mathbb{R}^n$ be a domain.

Then, $M^{\log}_\beta(\Omega) := \left\{ f \in L^1_{uloc}(\Omega); \|f\|_{M^{\log}_\beta(\Omega)} < \infty \right\}$ is introduced by the norm

$$\|f\|_{M^{\log}_\beta(\Omega)} := \sup_{x\in\Omega, 0<t<1} \frac{1}{|B(x,t)| \log^\beta \left(e + \frac{1}{t}\right)} \int_{B(x,t)\cap\Omega} |f(y)|dy.$$

Definition 4.2. Let $\beta > 0$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a spherical symmetric function with $\hat{\psi}(\xi) = 1$ in $B(0,1)$ and $\hat{\psi}(\xi) = 0$ in $B(0,2)^c$.

Then, $V_\beta := \{f \in \mathcal{S}'(\mathbb{R}^n); \|f\|_{V_\beta} < \infty\}$ is introduced by the norm

$$\|f\|_{V_\beta} := \sup_{N=1,2,\cdots} \frac{\|\psi_N * f\|_\infty}{N^\beta}, \quad \text{where} \quad \psi_N(x) := 2^{nN}\psi(2^N x).$$

Note that the space V_β is a modified version of spaces introduced by Vishik [64]. We also note that the following inclusions hold:

$$
\begin{aligned}
&M^{\log}_\beta(\Omega) \supset L^\infty(\Omega),\\
&V_\beta \supset M^{\log}_\beta(\mathbb{R}^n) \supset L^\infty(\mathbb{R}^n).
\end{aligned}
$$

For example, $\sqrt{\log\left(e+\frac{1}{|x|}\right)}$ belongs to $M^{\log}_{1/2}(\Omega)$, but doesn't belong to $L^\infty(\Omega)$.

Let $\dot{W}^{1,2}_{0,\sigma} := \overline{C^\infty_{0,\sigma}}^{\|\nabla\cdot\|_2}$. Now our main results read as follows.

Theorem 4.1 ([44])**.** *Let Ω be $\mathbb{R}^3$, the 3-dimensional half space, a 3-dimensional bounded domain or a 3-dimensional exterior domain with smooth boundary, and let $3 \le p < \infty$, $0 < T < \infty$, $u_0 \in L^p_\sigma \cap \dot{W}^{1,2}_{0,\sigma}$ and u be a solution to (N-S) in the class $C_p(0,T)$. If*

$$
\int_s^T \|u(\tau)\|^2_{M^{\log}_{1/2}(\Omega)} d\tau < \infty \quad \textit{for some } s \in (0,T),
$$

then u can be continued to the solution in the class $C_p(0,T')$ for some $T' > T$.

Theorem 4.2 ([44])**.** *Let $\Omega = \mathbb{R}^3$, $3 \le p < \infty$, $0 < T < \infty$, $u_0 \in L^p_\sigma \cap \dot{W}^{1,2}_{0,\sigma}$ and u be a solution to (N-S) in the class $S_p(0,T)$. If*

$$
\int_s^T \|u(\tau)\|^2_{V_{1/2}} d\tau < \infty \quad \textit{for some } s \in (0,T),
$$

then u can be continued to the solution in the class $S_p(0,T')$ for some $T' > T$.

Here, for definitions of $C_p(0,T)$ and $S_p(0,T)$, see Theorem 3.2 and Theorem 3.1.

Remark 4.1. In [43], we established Beale-Kato-Majda type extension criteria by means of

$$
\int_s^T \|\mathrm{rot}\, u(\tau)\|_{M^{\log}_1(\Omega)} d\tau \quad \text{and} \quad \int_s^T \|\mathrm{rot}\, u(\tau)\|_{V_1} d\tau.
$$

4.2 Brezis-Gallouet-Wainger type inequalities

We introduce logarithmic inequalities for the proof of our theorems.

Lemma 4.1 ([43, 44])**.** (i) *Let $n \ge 3$, and let $\Omega \subset \mathbb{R}^n$ be the whole space, the half space, a bounded domain or an exterior domain with smooth boundary.*

For any $\alpha \in (0,1)$ and $\beta > 0$, there exists a constant $C(\Omega, \alpha, \beta, n) > 0$ such that

$$\|f\|_{L^\infty(\Omega)} \le C \left(1 + \|f\|_{M_\beta^{\log}(\Omega)} \log^\beta \left(e + \|f\|_{\dot{C}^\alpha(\Omega)}\right)\right) \tag{4.1}$$

for all $f \in \dot{C}^\alpha(\Omega) \cap M_\beta^{\log}(\Omega)$.
(ii) *Let $n \ge 3$. For any $\alpha \in (0,1)$ and $\beta > 0$, there exists a constant $C(\alpha, \beta, n) > 0$ such that*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le C \left(1 + \|f\|_{V_\beta} \log^\beta \left(e + \|f\|_{\dot{C}^\alpha(\mathbb{R}^n)}\right)\right) \tag{4.2}$$

for all $f \in \dot{C}^\alpha(\mathbb{R}^n) \cap V_\beta$.

These inequalities are called Brezis-Gallouet-Wainger type inequalities:

$$(BGW)_\beta \quad \|u\|_{L^\infty} \le C(1 + \|f\|_X \log^\beta(e + \|f\|_Y)).$$

When $\Omega = \mathbb{R}^n$, by using the Fourier transform, Brezis-Gallouet-Wainger [6, 7] proved $(BGW)_\beta$ in the case

$$\beta = 1 - 1/p, \quad X = W^{n/p,p}(\mathbb{R}^n), \ Y = W^{n/q+\alpha,q}(\mathbb{R}^n) \left(\subset \dot{C}^\alpha(\mathbb{R}^n)\right) (\alpha > 0).$$

Engler [11] proved the same inequality for general domains Ω without using the Fourier transform. Ozawa [50] proved the Gagliardo-Nirenberg type inequality

$$\|f\|_{L^q(\mathbb{R}^n)} \le C(p,n) q^{1-1/p} \|(-\Delta)^{n/2p} f\|_{L^p(\mathbb{R}^n)}^{1-p/q} \|f\|_{L^p(\mathbb{R}^n)}^{p/q} \ \text{ for all } q \in [p, \infty)$$

with the explicit growth rate with respect to q and that this estimate directly yields $(BGW)_\beta$ with $\beta = 1 - 1/p$. When $\Omega = \mathbb{R}^n$, Chemin [9] proved $(BGW)_\beta$ for $\beta = 1$, $X = B^0_{\infty,\infty}(\mathbb{R}^n)$ and $Y = C^\alpha(\mathbb{R}^n)$. Kozono-Ogawa-Taniuchi [28, 49] proved $(BGW)_\beta$ for $0 \le \beta \le 1$, $X = \dot{B}^0_{\infty,1/(1-\beta)}(\mathbb{R}^n)$ and $Y = \dot{C}^\alpha(\mathbb{R}^n) \cap \dot{B}^{-\alpha}_{\infty,\infty}(\mathbb{R}^n)$. When Ω is a bounded domain, in [47, 48], $(BGW)_\beta$ was proven in the cases

$$\begin{aligned} &\beta = 1, \ X = bmo(\Omega), \quad Y = \dot{C}^\alpha(\Omega), \ \text{or} \\ &\beta = 1, \ X = B(\Omega), \quad Y = \dot{C}^\alpha(\Omega), \end{aligned}$$

where $B(\Omega)$ is introduced by the norm $\|f\|_{B(\Omega)} := \sup_{q \ge 1} \frac{\|f\|_{L^q(\Omega)}}{q}$. Furthermore, in [1, 9, 11, 15, 20, 24, 28, 30, 31, 37, 40, 46, 47, 48, 49, 50, 51, 55, 63, 67, 68] several inequalities of Brezis-Gallouet-Wainger type were established.

We shall show Lemma 4.1.

Proof of Lemma 4.1 (i). We use arguments given in Engler [11] and Ozawa [50]. See also Ogawa-Taniuchi [48]. For the sake of simplicity, we assume $n = 3$. Since $\partial\Omega$ is smooth, we see that $\partial\Omega$ satisfies the interior cone condition. Namely there are $\delta \in (0, 1)$ and $\theta \in (\pi/2, \pi)$ depending only on Ω with the following property: For any point $x \in \Omega$, there exists a spherical sector $C_\delta^\theta(x) = \{x + \xi \in \mathbb{R}^3; 0 < |\xi| < \delta, -|\xi| \le \kappa(x) \cdot \xi < |\xi| \cos\theta\}$ having a vertex at x such that $C_\delta^\theta(x) \subset \Omega$, where $\kappa(x)$ is an appropriate unit vector from x. We note that for each $x \in \Omega$, $C_\delta^\theta(x)$ is congruent to $C_\delta^\theta \equiv \{\xi \in \mathbb{R}^3; 0 < |\xi| < \delta, -|\xi| \le \xi_3 < |\xi| \cos\theta\}$. In particular, for any boundary point $x \in \partial\Omega$, $C_\delta^\theta(x)$ can be expressed as $C_\delta^\theta(x) \equiv \{x+\xi \in \mathbb{R}^3; 0 < |\xi| < \delta, -|\xi| \le \xi \cdot \nu(x) < |\xi| \cos\theta\}$, where $\nu(x)$ denotes the unit outward normal at x.

Let $0 < t \le \delta$ and $C_t^\theta(x) := C_\delta^\theta(x) \cap B(x, t)$. For any fixed $x \in \Omega$ and $y \in C_t^\theta(x) \subset \Omega$,

$$|f(x)| \le |f(x)-f(y)|+|f(y)| \le \|f\|_{\dot{C}^\alpha(\Omega)}|x-y|^\alpha+|f(y)| \le \|f\|_{\dot{C}^\alpha(\Omega)}t^\alpha+|f(y)|.$$

Integrating both sides of the above inequality with respect to y over $C_t^\theta(x)$,

$$\begin{aligned} |f(x)||C_t^\theta(x)| &\le t^\alpha \|f\|_{\dot{C}^\alpha(\Omega)}|C_t^\theta(x)| + \int_{y \in C_t^\theta(x)} |f(y)|dy \\ &\le t^\alpha \|f\|_{\dot{C}^\alpha(\Omega)}|C_t^\theta(x)| + \int_{y \in B(x,t)\cap\Omega} |f(y)|dy \\ &\le t^\alpha \|f\|_{\dot{C}^\alpha(\Omega)}|C_t^\theta(x)| + |B(x,t)| \log^\beta \left(\frac{1}{t} + e\right) \|f\|_{M_\beta^{\log}(\Omega)}. \end{aligned}$$

Since $|B(x,t)|/|C_t^\theta(x)|(=: K_\theta)$ is a constant independent of x and t, we have

$$|f(x)| \le t^\alpha \|f\|_{\dot{C}^\alpha(\Omega)} + K_\theta \log^\beta \left(\frac{1}{t} + e\right) \|f\|_{M_\beta^{\log}(\Omega)}$$

for all $0 < t \le \delta$.

Then we optimize t by letting $t = (1/\|f\|_{\dot{C}^\alpha(\Omega)})^{1/\alpha}$ if $\|f\|_{\dot{C}^\alpha(\Omega)} \ge \delta^{-\alpha}$ and letting $t = \delta$ if $\|f\|_{\dot{C}^\alpha(\Omega)} \le \delta^{-\alpha}$ to obtain (4.1). □

Proof of Lemma 4.1 (ii). We first recall the Littlewood-Paley decomposition. Let ψ be the function given in Definition 4.2 and let $\varphi_j \in \mathcal{S}$ be the functions defined by

$$\hat{\varphi}(\xi) := \hat{\psi}(\xi) - \hat{\psi}(2\xi) \text{ and } \hat{\varphi}_j(\xi) := \hat{\varphi}(\xi/2^j)$$

for $\xi \in \mathbb{R}^n$. Then, $\operatorname{supp} \hat{\varphi}_j \subset \{2^{j-1} \le |\xi| \le 2^{j+1}\}$ and

$$1 = \hat{\psi}(\xi/2^N) + \sum_{j=N+1}^{\infty} \hat{\varphi}(\xi/2^j) = \hat{\psi}_N(\xi) + \sum_{j=N+1}^{\infty} \hat{\varphi}_j(\xi) \tag{4.3}$$

for $\xi \in \mathbb{R}^n, N = 1, 2, \cdots$.

Using (4.3), we decompose f into two parts such as

$$f(x) = \psi_N * f(x) + \sum_{j=N+1}^{\infty} \varphi_j * f(x). \tag{4.4}$$

By Definition 4.2,

$$\|\psi_N * f\|_\infty \leq N^\beta \|f\|_{V_\beta} \tag{4.5}$$

holds. Since $\dot{B}^\alpha_{\infty,\infty}(\mathbb{R}^n) = \dot{C}^\alpha(\mathbb{R}^n)$ for $0 < \alpha < 1$, we have

$$\begin{aligned} \sum_{j=N+1}^{\infty} \|\varphi_j * f\|_\infty &= \sum_{j=N+1}^{\infty} 2^{\alpha j} \|\varphi_j * f\|_\infty 2^{-\alpha j} \\ &\leq \|f\|_{\dot{B}^\alpha_{\infty,\infty}} \sum_{j=N+1}^{\infty} 2^{-\alpha j} \\ &\leq C\|f\|_{\dot{C}^\alpha} 2^{-\alpha N}. \end{aligned} \tag{4.6}$$

Gathering (4.5) and (4.6) with (4.4), we obtain

$$\|f\|_\infty \leq C(2^{-\alpha N}\|f\|_{\dot{C}^\alpha} + N^\beta \|f\|_{V_\beta}).$$

Now we take $N = \left[\frac{\log(\|f\|_{\dot{C}^\alpha} + e)}{\alpha \log 2}\right] + 1$, where $[\cdot]$ denotes Gauss symbol. Then we have the desired estimate (4.2). □

4.3 Proof of Theorem 4.1

Proof of Theorem 4.1. Since $u \in C((0,T); D(A_6) \cap \dot{W}^{1,2}_{0,\sigma})$, without loss of generality, we may assume that $u_0 \in D(A_6) \cap \dot{W}^{1,2}_{0,\sigma}$. Since the local existence time of strong L^p solutions T_* can be estimated from below as

$$T_* > C(\Omega)/\|u_0\|_6^4,$$

see Appendix, it suffices to show that

$$\sup_{0<\tau<T} \|u(\tau)\|_6 \leq C\|\nabla u\|_2 \exp\left(C \exp\left(C \int_0^T \|u(\tau)\|^2_{M^{\log}_{1/2}(\Omega)} d\tau\right)\right). \tag{4.7}$$

Recall that u satisfies

$$\text{(I.E.)}^* \qquad u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A} P\nabla \cdot (u \otimes u)(s) ds$$

for all $0 < t < T$.

From (I.E.)*, the duality argument and the Gronwall lemma we have

$$\begin{aligned}\sup_{0<\tau<t}\|u(\tau)\|_6 &\le C\sup_{0<\tau<t}\|\nabla u(\tau)\|_2 \\ &\le C\|\nabla u_0\|_2\exp\left(C\int_0^t\|u(s)\|_\infty^2 ds\right)\end{aligned} \tag{4.8}$$

for all $0 < t < T$. See Appendix.

Let

$$\begin{aligned}h(t) &:= \sup_{0<\tau<t}\|u(\tau)\|_6, \\ g(t) &:= \int_0^t\|u(\tau)\|_\infty^2 d\tau\end{aligned}$$

for $0 < t < T$. Then, we have

$$h(t) \le C\|\nabla u_0\|_2\exp(Cg(t)) \tag{4.9}$$

for all $0 < t < T$.

Letting $0 < \alpha < 1/2$ and substituting $f = \frac{u(s)}{\varepsilon\|u(s)\|_{\dot{C}^\alpha}}$ into the Brezis-Gallouet-Wainger type inequality (4.1) with $\beta = 1/2$, we obtain

$$\|u(s)\|_\infty \le C\left(\varepsilon\|u(s)\|_{\dot{C}^\alpha} + \log^{1/2}\left(e+\frac{1}{\varepsilon}\right)\|u(s)\|_{M^{\log}_{1/2}(\Omega)}\right),$$

which means

$$\|u(s)\|_\infty^2 \le C\left(\varepsilon^2\|u(s)\|_{\dot{C}^\alpha}^2 + \log\left(e+\frac{1}{\varepsilon}\right)\|u(s)\|_{M^{\log}_{1/2}(\Omega)}^2\right) \tag{4.10}$$

for all $\varepsilon > 0$, where C is a constant independent of s and ε. Then, by (4.10), for any positive bounded function $\varepsilon(s)$ on $(0, T)$, we have

$$\begin{aligned}g(t) &\le C\int_0^t\varepsilon^2(s)\|u(s)\|_{\dot{C}^\alpha}^2 ds + C\int_0^t\log\left(e+\frac{1}{\varepsilon(s)}\right)\|u(s)\|_{M^{\log}_{1/2}(\Omega)}^2 ds \\ &=: I_1(t) + I_2(t).\end{aligned} \tag{4.11}$$

By the Gagliardo-Nirenberg inequality

$$\|f\|_{\dot{C}^\alpha(\Omega)} \le C\|f\|_{W^{2,6}(\Omega)}^\theta\|f\|_{L^6(\Omega)}^{1-\theta},$$

where $\theta = \frac{1}{4} + \frac{\alpha}{2}$, we have

$$
\begin{aligned}
&\|e^{-tA} P\nabla \cdot f\|_{\dot{C}^\alpha} \\
&\le C\|(1+A)e^{-(t/2)A} e^{-(t/2)A} P\nabla \cdot f\|_6^{\theta} \|e^{-(t/2)A} P\nabla \cdot f\|_6^{1-\theta} \\
&\le C\left((1+(t/2)^{-1})\|e^{-(t/2)A} P\nabla \cdot f\|_6\right)^{\theta} \|e^{-(t/2)A} P\nabla \cdot f\|_6^{1-\theta} \\
&\le C(1+(t/2)^{-\theta})\|e^{-(t/2)A} P\nabla \cdot f\|_6
\end{aligned}
\tag{4.12}
$$

for all $0 < t < T$ and $f \in (L^6(\Omega))^{3\times 3}$. Since the duality argument yields $\|e^{-(t/2)A} P\nabla \cdot f\|_6 \le C(t/2)^{-\frac{1}{2}}\|f\|_6$, by (4.12) we obtain

$$\|e^{-tA} P\nabla \cdot f\|_{\dot{C}^\alpha} \le C(1 + t^{-\frac{3}{4}-\frac{\alpha}{2}})\|f\|_6$$

for all $0 < t < T$ and $f \in (L^6(\Omega))^{3\times 3}$. Thus, from (I.E.)* we obtain

$$
\begin{aligned}
\|u(s)\|_{\dot{C}^\alpha} \le& C\|u_0\|_{D(A_6)} + C\int_0^s (1 + (s-\tau)^{-\frac{3}{4}-\frac{\alpha}{2}})\|u \otimes u(\tau)\|_6 d\tau \\
\le& C\|u_0\|_{D(A_6)} + Ch(s)\int_0^s (1 + (s-\tau)^{-\frac{3}{4}-\frac{\alpha}{2}})\|u(\tau)\|_\infty d\tau,
\end{aligned}
$$

which yields

$$\|u(s)\|_{\dot{C}^\alpha}^2 \le C\|u_0\|_{D(A_6)}^2 + Ch^2(s)\left(\int_0^s (1 + (s-\tau)^{-\frac{3}{4}-\frac{\alpha}{2}})\|u(\tau)\|_\infty d\tau\right)^2.$$

Hence, for $0 < t < T$ we have

$$
\begin{aligned}
I_1(t) \le& C\|u_0\|_{D(A_6)}^2 T \sup_{0<s<T} \varepsilon^2(s) \\
&+ C\int_0^t h^2(s)\varepsilon^2(s)\left(\int_0^s (1 + (s-\tau)^{-\frac{3}{4}-\frac{\alpha}{2}})\|u(\tau)\|_\infty d\tau\right)^2 ds.
\end{aligned}
$$

Now we choose $\varepsilon(s)$ such as

$$\varepsilon(s) := \frac{\delta}{h(s)+1},$$

where $\delta = \delta(T) \in (0,1)$ is a constant to be chosen suitably small later on. Then, since $h^2(s)\varepsilon^2(s) < \delta^2$ and $\varepsilon^2(s) < \delta^2$, we have

$$I_1(t) \le C\|u_0\|_{D(A_6)}^2 T\delta^2 + C\delta^2 \int_0^t \left(\int_0^s (1 + (s-\tau)^{-\frac{3}{4}-\frac{\alpha}{2}})\|u(\tau)\|_\infty d\tau\right)^2 ds.$$

Since $\frac{3}{4}+\frac{\alpha}{2}<1$, the Hardy-Littlewood-Sobolev inequality yields

$$\begin{aligned}
&\int_0^t \left(\int_0^s (s-\tau)^{-\frac{3}{4}-\frac{\alpha}{2}} \|u(\tau)\|_\infty d\tau\right)^2 ds \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left((s-\tau)^{-\frac{3}{4}-\frac{\alpha}{2}} 1_{(0,t)}(s-\tau)\right) \left(\|u(\tau)\|_\infty 1_{(0,t)}(\tau)\right) d\tau\right)^2 ds \\
&\le C \left\| \|u(\cdot)\|_\infty 1_{(0,t)} \right\|^2_{L^{\frac{4}{3-2\alpha}}(\mathbb{R})} \\
&\le C t^{\frac{1}{2}-\alpha} \left\| \|u(\cdot)\|_\infty \right\|^2_{L^2(0,t)} .
\end{aligned}$$

Hence,

$$\begin{aligned}
I_1(t) &\le C\|u_0\|^2_{D(A_6)} T\delta^2 + C\left(t^2 + t^{\frac{1}{2}-\alpha}\right)\delta^2 \int_0^t \|u(\tau)\|^2_\infty d\tau \\
&\le C + C_1(T)\delta^2 g(t).
\end{aligned} \tag{4.13}$$

Since (4.9) yields

$$\log\left(e + \frac{1}{\varepsilon(s)}\right) = \log\left(e + \frac{h(s)+1}{\delta}\right) \le C(1 + \|\nabla u_0\|_2 + g(s)),$$

and since $\int_0^T \|u(s)\|^2_{M^{\log}_{1/2}(\Omega)} ds < \infty$, we have

$$\begin{aligned}
I_2(t) &\le C\int_0^t \|u(s)\|^2_{M^{\log}_{1/2}(\Omega)} (1 + \|\nabla u_0\|_2 + g(s)) ds \\
&\le C(1 + \|\nabla u_0\|_2) \int_0^t \|u(s)\|^2_{M^{\log}_{1/2}(\Omega)} ds \\
&\qquad + C\int_0^t \|u(s)\|^2_{M^{\log}_{1/2}(\Omega)} g(s) ds \\
&\le C + C\int_0^t \|u(s)\|^2_{M^{\log}_{1/2}(\Omega)} g(s) ds.
\end{aligned} \tag{4.14}$$

Gathering (4.13) and (4.14) with (4.11), we obtain

$$g(t) \le C + C_1(T)\delta^2 g(t) + C\int_0^t \|u(s)\|^2_{M^{\log}_{1/2}(\Omega)} g(s) ds.$$

Thus, letting $\delta^2 = \frac{1}{2C_1(T)+1}$, by the Gronwall lemma, we have

$$g(t) \le C\exp\left(C\int_0^T \|u(s)\|^2_{M^{\log}_{1/2}(\Omega)} ds\right)$$

for all $0 < t < T$. Then, this estimate and (4.9) yield the desired estimate (4.7). $\square$

We can prove Theorem 4.2 in the same way to the proof of Theorem 4.1 by using (4.2) instead of (4.1).

4.4 Appendix

In this section, we prove (4.8).

Proposition 4.1. *Let Ω be $\mathbb{R}^3$, the 3-dimensional half space, a 3-dimensional bounded domain or a 3-dimensional exterior domain with smooth boundary.*
(i) *If $a \in \dot{W}^{1,2}_{0,\sigma}$, then $e^{-tA}a \in C([0,\infty); \dot{W}^{1,2}_{0,\sigma})$ and $\|\nabla e^{-tA}a\|_2 \le \|\nabla a\|_2$.*
(ii) *If $v \in C([0,T]; \dot{W}^{1,2}_{0,\sigma})$, then*

$$F(t) := \int_0^t e^{-(t-s)A} P(v \cdot \nabla v)(s) ds \in C([0,T]; \dot{W}^{1,2}_{0,\sigma}).$$

Proof of Proposition 4.1. (i) We first recall $C^\infty_{0,\sigma} \subset D(A_2) \subset \dot{W}^{1,2}_{0,\sigma}$, see Sohr [58, Chap.III, Sect.2.1]. Hence

$$\dot{W}^{1,2}_{0,\sigma} = \overline{D(A_2)}^{\|\nabla \cdot\|_2}.$$

By the definition of $\dot{W}^{1,2}_{0,\sigma}$, there exists $a_n \in C^\infty_{0,\sigma}$ such that $a_n \to a$ in $\dot{W}^{1,2}_{0,\sigma}$. Since $e^{-tA}a_n \in C([0,\infty); D(A_2))$ and since

$$\begin{aligned}
&\sup_{t\ge 0} \|\nabla e^{-tA}a_n - \nabla e^{-tA}a_m\|_2 \\
&= \sup_{t\ge 0} \|A^{1/2}e^{-tA}a_n - A^{1/2}e^{-tA}a_m\|_2 \\
&\le \|A^{1/2}(a_n - a_m)\|_2 = \|\nabla a_m - \nabla a_n\|_2 \to 0
\end{aligned}$$

as $m, n \to \infty$, we see that $e^{-tA}a \in C\left([0,\infty); \overline{D(A_2)}^{\|\nabla\cdot\|_2}\right) = C([0,\infty); \dot{W}^{1,2}_{0,\sigma})$.

Since $\|\nabla e^{-tA}a_n\|_2 = \|e^{-tA}A_2^{1/2}a_n\|_2 \le \|A_2^{1/2}a_n\|_2 = \|\nabla a_n\|_2$, we have $\|\nabla e^{-tA}a\|_2 \le \|\nabla a\|_2$.
(ii) Let $t \in (0,T]$ be fixed and $0 < \varepsilon < t$. Since $\|P(v \cdot \nabla v)(s)\|_{3/2} \le C \sup_{0\le s\le T} \|\nabla v(s)\|_2^2$, it is straightforward to see that $\int_0^{t-\varepsilon} e^{-(t-s)A}P(v\cdot\nabla v)(s)ds \in D(A_2)$. Since

$$\left\| \nabla \int_0^{t-\varepsilon} e^{-(t-s)A} P(v \cdot \nabla v)(s) ds - \nabla F(t) \right\|_2 \to 0$$

as $\varepsilon \downarrow 0$, we have $F(t) \in \dot{W}^{1,2}_{0,\sigma}$. By the direct calculation, we can also show the continuity of $\nabla F(t)$ in L^2 with respect to $t \in [0,T]$, which proves Proposition 4.1.

□

Lemma 4.2. (i) *Let Ω be $\mathbb{R}^3$, the 3-dimensional half space, a 3-dimensional bounded domain or a 3-dimensional exterior domain with smooth boundary, and let $3 \le p < \infty$. If $v_0 \in L^p_\sigma(\Omega) \cap \dot{W}^{1,2}_{0,\sigma}$, then there exist $T_* > 0$ and a unique solution v on $(0, T_*)$ to (N-S) with initial data $v(0) = v_0$ in the class*

$$v \in C_p(0, T_*) \cap C_6(0, T_*) \cap C([0, T_*); \dot{W}^{1,2}_{0,\sigma}).$$

Moreover, it holds that

$$T_* > C/\|v_0\|_6^4,$$

where C is a constant depending only on Ω.
(ii) *Let Ω be $\mathbb{R}^3$, the 3-dimensional half space, a 3-dimensional bounded domain or a 3-dimensional exterior domain with smooth boundary, and let u be a solution to (N-S) in the class*

$$u \in C_6(0, T) \cap C([0, T); \dot{W}^{1,2}_{0,\sigma}).$$

Then, it holds that

$$\sup_{0<\tau<t} \|\nabla u(\tau)\|_2 \le C\|\nabla u_0\|_2 \exp\left(C \int_0^t \|u(s)\|_\infty^2 ds \right) \tag{4.15}$$

for all $0 < t < T$.

Proof of Lemma 4.2. (i) Since Assertion (i) is proven by the standard iteration argument and Proposition 4.1, we omit the proof of (i).
(ii) Let $\varphi \in C^\infty_{0,\sigma}$. Since $w(\tau) = e^{-\tau A}\varphi$ is a solution to the Stokes equation on $(0, \infty)$ with the initial data $w(0) = \varphi$, the energy calculation yields

$$\int_0^t \|\nabla e^{-\tau A}\varphi\|_2^2 d\tau \le \|\varphi\|_2^2$$

for all $t > 0$, which implies

$$\int_0^t \|A^{1/2} e^{-(t-s)A}\varphi\|_2^2 ds \le \|\varphi\|_2^2$$

for all $t > 0$. Since

$$(u(t), \psi) = (e^{-tA}u(0), \psi) + \int_0^t (u \cdot \nabla u(s), e^{-(t-s)A}\psi) ds$$

for all $\psi \in L^{6/5}_\sigma$, letting $\psi = A^{1/2}\varphi$, we have

$$
\begin{aligned}
&|(A^{1/2}u(t), \varphi)| \\
&\le |(e^{-tA}A^{1/2}u(0), \varphi)| + \int_0^t |(u \cdot \nabla u(s), A^{1/2}e^{-(t-s)A}\varphi)|ds \\
&\le \|A^{1/2}u(0)\|_2\|\varphi\|_2 + \int_0^t \|u(s)\|_\infty \|\nabla u(s)\|_2 \|A^{1/2}e^{-(t-s)A}\varphi\|_2 ds \\
&\le \|A^{1/2}u(0)\|_2\|\varphi\|_2 + \left(\int_0^t \|u(s)\|_\infty^2 \|\nabla u(s)\|_2^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \|A^{1/2}e^{-(t-s)A}\varphi\|_2^2 ds\right)^{\frac{1}{2}} \\
&\le \|A^{1/2}u(0)\|_2\|\varphi\|_2 + \left(\int_0^t \|u(s)\|_\infty^2 \|\nabla u(s)\|_2^2 ds\right)^{\frac{1}{2}} \|\varphi\|_2.
\end{aligned}
$$

Thus, the duality yields

$$\|\nabla u(t)\|_2 \le \|\nabla u_0\|_2 + \left(\int_0^t \|u(s)\|_\infty^2 \|\nabla u(s)\|_2^2 ds\right)^{\frac{1}{2}}$$

and consequently

$$\|\nabla u(t)\|_2^2 \le 2\|\nabla u_0\|_2^2 + 2\int_0^t \|u(s)\|_\infty^2 \|\nabla u(s)\|_2^2 ds.$$

Therefore, from the Gronwall lemma, we obtain the desired estimate (4.15). $\square$

By the embedding theorem: $\dot{W}^{1,2}_{0,\sigma} \hookrightarrow L^6_\sigma$ and (4.15), we get (4.8).

Chapter 5

Time-periodic solutions to the Boussinesq equations in exterior domains

5.1 Main Result

In this chapter, we consider time-periodic soutions to (B).

Let $BUC(\mathbb{R}, X)$ denote the set of bounded and uniformly continuous functions on $\mathbb{R}$, equipped with the norm $\|u|X\| = \sup_{t\in\mathbb{R}} \|u(t)\|_X$.

We assume that there exists F such that $f = \operatorname{div} F$. We define the mild solution of (B) as follows (see [66, Definition 1]):

Definition 5.1. A pair of functions (u, θ) is said to be a mild solution of (B) if the identity

$$\begin{aligned}(u(\cdot,t),\varphi) = &\sum_{j,k=1}^{3} \int_0^\infty (u_j(\cdot,t-\tau)u_k(\cdot,t-\tau) - F_{jk}(\cdot,t-\tau), (\partial_j e^{-\tau A}\varphi)_k)d\tau \\ &+ \int_0^\infty (g\theta(\cdot,t-\tau), e^{-\tau A}\varphi)d\tau\end{aligned}$$

holds for all $\varphi \in L^{3/2,1}_\sigma(\Omega)$ and $t \in (-\infty, \infty)$, and the identity

$$(\theta(\cdot,t),\psi) = \int_0^\infty (u(\cdot,t-\tau)\theta(\cdot,t-\tau), \nabla e^{-\tau B}\psi)d\tau + \int_0^\infty (S(\cdot,t-\tau), e^{-\tau B}\psi)d\tau$$

holds for all $\psi \in C_0^\infty(\Omega)$ and $t \in (-\infty, \infty)$.

Now our main result reads as follows:

Theorem 5.1 ([41]). *There exist $C_{\mathrm{force}} > 0$ and $M_\infty > 0$ such that if $\|F|L^{3/2,\infty}\| < C_{\mathrm{force}}$ and $\|S|\dot{H}^{-1}_{6/5,\infty} \cap L^{12/11}\| < C_{\mathrm{force}}$, then there exists one and only one mild solution (u, θ) of (B) such that $u \in \{u \in BUC(\mathbb{R}, L^{3,\infty}_\sigma);$ $\|u|L^{3,\infty}_\sigma\| < M_\infty\}$ and $\theta \in \{\theta \in BUC(\mathbb{R}, L^{3,1}); \|\theta|L^{3,1}\| < M_\infty\}$. Moreover, if f and S are time-periodic, the solution (u, θ) is also time-periodic.*

5.2 Proof of Theorem 5.1

For the proof of our theorem, we prepare some estimates on the Lorentz space. Concretely, we prepare Hölder's inequalities, the Sobolev embedding theorem, and L^p-L^q estimates of the Stokes semigroup and heat semigroups.

Lemma 5.1 ([4, 25, 33]). (i) *Let $1 < p_0, p_1 < \infty, 1 \leq q_0, q_1 \leq \infty, q = \min\{q_0, q_1\}$, and $p > 1$ satisfy $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$. Then there exists a constant C such that*

$$\|fg\|_{p,q} \leq C\|f\|_{p_0,q_0}\|g\|_{p_1,q_1} \tag{5.1}$$

for all $f \in L^{p_0,q_0}$ and $g \in L^{p_1,q_1}$.
(ii) *Let $1 < p_0, p_1 < \infty$, $p \geq 1$ satisfy $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$, and $1 \leq q_0, q_1 \leq \infty$ satisfy $\frac{1}{q_0} + \frac{1}{q_1} \geq 1$. Then there exists a constant C such that*

$$\|fg\|_{p,1} \leq C\|f\|_{p_0,q_0}\|g\|_{p_1,q_1} \tag{5.2}$$

for all $f \in L^{p_0,q_0}$ and $g \in L^{p_1,q_1}$.

Let $\dot{H}^1_{3,1} := \overline{C_0^\infty}^{\|\nabla\cdot\|_{3,1}}$. Then this space satisfies an embedding theorem as follows:

Lemma 5.2 ([32]). *There exists a constant C such that*

$$\|u\|_\infty \leq C\|\nabla u\|_{3,1} \tag{5.3}$$

for all $u \in \dot{H}^1_{3,1}$.

Lemma 5.3 ([66]). *Let $1 < p < q \leq 3$. Then there exists a constant C such that*

$$\int_0^\infty \tau^{\frac{3}{2p}-\frac{3}{2q}-\frac{1}{2}}\|\nabla e^{-\tau A}\varphi\|_{q,1}d\tau \leq C\|\varphi\|_{p,1} \tag{5.4}$$

for all $\varphi \in L^{p,1}_\sigma(\Omega)$.

Lemma 5.4. (i) *Let $1 < p < q < \infty$. Then there exists a constant C such that*

$$\int_0^\infty \tau^{\frac{3}{2p}-\frac{3}{2q}-1}\|e^{-\tau B}\psi\|_{q,1}d\tau \le C\|\psi\|_{p,1} \tag{5.5}$$

for all $\psi \in L^{p,1}(\Omega)$.
(ii) *Let $1 < p < \infty$ and $\max\{2,p\} < q < \infty$. Then there exists a constant C such that*

$$\int_0^\infty \tau^{\frac{3}{2p}-\frac{3}{2q}-\frac{1}{2}}\|\nabla e^{-\tau B}\psi\|_{q,1}d\tau \le C\|\psi\|_{p,1} \tag{5.6}$$

for all $\psi \in L^{p,1}(\Omega)$.

We show Lemma 5.4 in Appendix.

Proof of Theorem 5.1. First, we define functions on integral equations. For $u, v \in BUC(\mathbb{R}, L^{3,\infty}_\sigma)$, $\theta \in BUC(\mathbb{R}, L^{3,1})$ and $F \in BUC(\mathbb{R}, (L^{3/2,\infty})^{3\times 3})$, we define $\Phi[u,v,\theta,F]$ so that the formula

$$\begin{aligned}&(\Phi[u,v,\theta,F](\cdot,t),\varphi)\\&:= \sum_{j,k=1}^{3}\int_0^\infty (u_j(\cdot,t-\tau)v_k(\cdot,t-\tau) - F_{jk}(\cdot,t-\tau),(\partial_j e^{-\tau A}\varphi)_k)d\tau + \int_0^\infty (g\theta, e^{-\tau A}\varphi)d\tau\end{aligned}$$

holds for all $\varphi \in L^{3/2,1}_\sigma(\Omega)$, and that

$$(\Phi[u,v,\theta,F](\cdot,t),\nabla\tilde{\varphi}) = 0$$

for all scalar function $\tilde{\varphi}$ such that $\nabla\tilde{\varphi} \in (L^{3/2,1}(\Omega))^3$.

On the other hand, for $u \in BUC(\mathbb{R}, L^{3,\infty}_\sigma)$, $\theta \in BUC(\mathbb{R}, L^{2,\infty} \cap L^{4,\infty})$ and $S \in BUC(\mathbb{R}, \dot{H}^{-1}_{6/5,\infty}) \cap BUC(\mathbb{R}, L^{12/11})$, we define $\Psi[u,\theta,S]$ so that the formula

$$\begin{aligned}&(\Psi[u,\theta,S](\cdot,t),\psi)\\&:= \int_0^\infty (u(\cdot,t-\tau)\theta(\cdot,t-\tau),\nabla e^{-\tau B}\psi)d\tau + \int_0^\infty (S(\cdot,t-\tau),e^{-\tau B}\psi)d\tau\end{aligned}$$

holds for all $\psi \in C_0^\infty(\Omega)$. Then we get

$$\begin{aligned}&|(\Phi[u,v,\theta,F](\cdot,t),\varphi)|\\&\le \sum\int_0^\infty |(u_j(\cdot,t-\tau)v_k(\cdot,t-\tau),(\partial_j e^{-\tau A}\varphi)_k)|d\tau\\&\quad+\sum\int_0^\infty |(F_{j,k}(\cdot,t-\tau),(\partial_j e^{-\tau A}\varphi)_k)|d\tau\\&\quad+\int_0^\infty |(g\theta(\cdot,t-\tau),e^{-\tau A}\varphi)|d\tau\\&=: I_1(t)+I_2(t)+I_3(t)\end{aligned} \tag{5.7}$$

for all $t \in (-\infty, \infty)$, and

$$
\begin{aligned}
|(\Psi[u,\theta,S](\cdot,t),\psi)| \leq & \int_0^\infty |(u(\cdot,t-\tau)\theta(\cdot,t-\tau), \nabla e^{-\tau B}\psi)| d\tau \\
& \qquad + \int_0^\infty |(S(\cdot,t-\tau), e^{-\tau B}\psi)| d\tau \\
=: & \, II_1(t) + II_2(t)
\end{aligned}
\tag{5.8}
$$

for all $t \in (-\infty, \infty)$.

By (5.1), (5.2) and (5.4), we have

$$
\begin{aligned}
I_1(t) &\leq C \sum \int_0^\infty \|u_j(\cdot,t-\tau)v_k(\cdot,t-\tau)\|_{3/2,\infty} \|\nabla e^{-\tau A}\varphi\|_{3,1} d\tau \\
&\leq C \sup_{t\in\mathbb{R}} \|u(t)\|_{3,\infty} \sup_{t\in\mathbb{R}} \|v(t)\|_{3,\infty} \int_0^\infty \|\nabla e^{-\tau A}\varphi\|_{3,1} d\tau \\
&\leq C \left(\sup_{t\in\mathbb{R}} \|u(t)\|_{3,\infty} \sup_{t\in\mathbb{R}} \|v(t)\|_{3,\infty} \right) \|\varphi\|_{3/2,1},
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
I_2(t) &\leq C \int_0^\infty \|F(\cdot,t-\tau)\|_{3/2,\infty} \|\nabla e^{-\tau A}\varphi\|_{3,1} d\tau \\
&\leq C \sup_{t\in\mathbb{R}} \|F(t)\|_{3/2,\infty} \int_0^\infty \|\nabla e^{-\tau A}\varphi\|_{3,1} d\tau \\
&\leq C \left(\sup_{t\in\mathbb{R}} \|F(t)\|_{3/2,\infty} \right) \|\varphi\|_{3/2,1}.
\end{aligned}
\tag{5.10}
$$

By (5.1), (5.2), (5.3) and (5.4), we have

$$
\begin{aligned}
I_3(t) &\leq \int_0^\infty \|g\theta(\cdot,t-\tau)\|_1 \|e^{-\tau A}\varphi\|_\infty d\tau \\
&\leq \int_0^\infty \|g\|_{3/2,\infty} \|\theta(\cdot,t-\tau)\|_{3,1} \|\nabla e^{-\tau A}\varphi\|_{3,1} d\tau \\
&\leq C(g) \left(\sup_{t\in\mathbb{R}} \|\theta(t)\|_{2,\infty} + \sup_{t\in\mathbb{R}} \|\theta(t)\|_{4,\infty} \right) \int_0^\infty \|\nabla e^{-\tau A}\varphi\|_{3,1} d\tau \\
&\leq C(g) \left(\sup_{t\in\mathbb{R}} \|\theta(t)\|_{2,\infty} + \sup_{t\in\mathbb{R}} \|\theta(t)\|_{4,\infty} \right) \|\varphi\|_{3/2,1}.
\end{aligned}
\tag{5.11}
$$

Here, by the definition of the acceleration of gravity $g(x) = -\tilde{g}\frac{x}{|x|^3}$, we have $g \in L^{3/2,\infty}$.

By (5.1), (5.2), and (5.6), we have

$$\begin{aligned} II_1(t) &\le C\int_0^\infty \|u(\cdot,t-\tau)\theta(\cdot,t-\tau)\|_{6/5,\infty}\|\nabla e^{-\tau B}\psi\|_{6,1}d\tau \\ &\le C\sup_{t\in\mathbb{R}}\|u(t)\|_{3,\infty}\sup_{t\in\mathbb{R}}\|\theta(t)\|_{2,\infty}\int_0^\infty \|\nabla e^{-\tau B}\psi\|_{6,1}d\tau \\ &\le C\left(\sup_{t\in\mathbb{R}}\|u(t)\|_{3,\infty}\sup_{t\in\mathbb{R}}\|\theta(t)\|_{2,\infty}\right)\|\psi\|_{2,1}, \end{aligned} \tag{5.12}$$

$$\begin{aligned} II_1(t) &\le C\int_0^\infty \|u(\cdot,t-\tau)\theta(\cdot,t-\tau)\|_{12/7,\infty}\|\nabla e^{-\tau B}\psi\|_{12/5,1}d\tau \\ &\le C\sup_{t\in\mathbb{R}}\|u(t)\|_{3,\infty}\sup_{t\in\mathbb{R}}\|\theta(t)\|_{4,\infty}\int_0^\infty \|\nabla e^{-\tau B}\psi\|_{12/5,1}d\tau \\ &\le C\left(\sup_{t\in\mathbb{R}}\|u(t)\|_{3,\infty}\sup_{t\in\mathbb{R}}\|\theta(t)\|_{4,\infty}\right)\|\psi\|_{4/3,1}. \end{aligned} \tag{5.13}$$

By (5.2) and (5.6), we have

$$\begin{aligned} II_2(t) &\le \int_0^\infty \|S(\cdot,t-\tau)\|_{\dot H^{-1}_{6/5,\infty}}\|\nabla e^{-\tau B}\psi\|_{6,1}d\tau \\ &\le \sup_{t\in\mathbb{R}}\|S(t)\|_{\dot H^{-1}_{6/5,\infty}}\int_0^\infty \|\nabla e^{-\tau B}\psi\|_{6,1}d\tau \\ &\le C\left(\sup_{t\in\mathbb{R}}\|S(t)\|_{\dot H^{-1}_{6/5,\infty}}\right)\|\psi\|_{2,1}, \end{aligned} \tag{5.14}$$

Since $L^{12} = L^{12,12} \supset L^{12,1}$ and (5.5), we have

$$\begin{aligned} II_2(t) &\le \int_0^\infty \|S(\cdot,t-\tau)\|_{12/11}\|e^{-\tau B}\psi\|_{12}d\tau \\ &\le \sup_{t\in\mathbb{R}}\|S(t)\|_{12/11}\int_0^\infty \|e^{-\tau B}\psi\|_{12,1}d\tau \\ &\le C\left(\sup_{t\in\mathbb{R}}\|S(t)\|_{12/11}\right)\|\psi\|_{4/3,1}. \end{aligned} \tag{5.15}$$

Gathering (5.9)-(5.15) with (5.7) and (5.8), we get

$$\begin{aligned} &\|\Phi[u,v,\theta,F]|L^{3,\infty}_\sigma\| \\ &\le C\|F|(L^{3/2,\infty})^{3\times 3}\| + C\|u|L^{3,\infty}_\sigma\|\|v|L^{3,\infty}_\sigma\| \\ &\qquad\qquad + C\|\theta|L^{2,\infty}\| + C\|\theta|L^{4,\infty}\|, \end{aligned} \tag{5.16}$$

$$\begin{aligned} &\|\Psi[u,\theta,S]|L^{2,\infty}\cap L^{4,\infty}\| \\ &\le C(\|S|\dot H^{-1}_{6/5,\infty}\| + \|S|L^{12/11}\|) + C\|u|L^{3,\infty}_\sigma\|(\|\theta|L^{2,\infty}\| + \|\theta|L^{4,\infty}\|). \end{aligned} \tag{5.17}$$

By (5.16) and (5.17), we can show continuity of Φ and Ψ (see [66, pp.652-653]).

We construct sequences

$$\begin{aligned}\left\{u^{(j)}(x,t)\right\}_0^\infty &= \left\{\left\{u_l^{(j)}(x,t)\right\}_{l=1}^3\right\}_{j=0}^\infty \\ \left\{\theta^{(j)}(x,t)\right\}_0^\infty &= \left\{\theta^{(j)}(x,t)\right\}_{j=0}^\infty\end{aligned}$$

inductively by $\theta^{(0)}(x,t) := 0$, $u^{(0)}(x,t) := 0$, $\theta^{(j+1)}(x,t) := \Psi[u^{(j)}, \theta^{(j)}, S]$ and $u^{(j+1)}(x,t) := \Phi[u^{(j)}, u^{(j)}, \theta^{(j+1)}, F]$.

We set

$$\begin{aligned}&A_j := \|u^{(j)}|L_\sigma^{3,\infty}\|, B_{2,j} := \|\theta^{(j)}|L^{2,\infty}\|, B_{4,j} := \|\theta^{(j)}|L^{4,\infty}\|, \\ &D_j := \max\{B_{2,j}, B_{4,j}\}.\end{aligned}$$

By (5.16) and (5.17), we have

$$\begin{aligned}A_{j+1} &\leq C_2 A_j^2 + C_1 D_{j+1} + C(F), \\ D_{j+1} &\leq C_3 A_j D_j + C(S)\end{aligned}$$

Furthermore we set $M_j := \max\{A_j, D_j\}$. Then we have

$$\begin{aligned}M_{j+1} &\leq (C_1C_3 + C_2 + C_3)M_j^2 + (C_1C(S) + C(F) + C(S)) \\ &=: C_{\max} M_j^2 + (C_1C(S) + C(F) + C(S))\end{aligned}$$

Therefore, if F, S are sufficiently small i.e.

$$C_1C(S) + C(F) + C(S) < \frac{5}{36C_{\max}},$$

then we have

$$M_j \leq \frac{1 - \sqrt{1 - 4C_{\max}(C_1C(S) + C(F) + C(S))}}{2C_{\max}} =: M_\infty.$$

Here we get

$$2C_{\max} M_\infty < \frac{1}{3}. \tag{5.18}$$

We set

$\tilde{A}_j = \|u^{(j+1)} - u^{(j)}|L_\sigma^{3,\infty}\|$, $\tilde{B}_{2,j} := \|\theta^{(j+1)} - \theta^{(j)}|L^{2,\infty}\|$, $\tilde{B}_{4,j} := \|\theta^{(j+1)} - \theta^{(j)}|L^{4,\infty}\|$,
$\tilde{M}_j := \max\{\tilde{A}_j, \tilde{B}_{2,j}, \tilde{B}_{4,j}\}$.

Since

$$
\begin{aligned}
&u^{(j+2)} - u^{(j+1)} \\
&= \Phi[u^{(j+1)}, u^{(j+1)}, \theta^{(j+2)}, F] - \Phi[u^{(j)}, u^{(j)}, \theta^{(j+1)}, F] \\
&= \Phi[u^{(j+1)}, u^{(j+1)} - u^{(j)}, 0, 0] + \Phi[u^{(j+1)} - u^{(j)}, u^{(j)}, 0, 0] + \Phi[0, 0, \theta^{(j+2)} - \theta^{(j+1)}, 0]
\end{aligned}
$$

and

$$
\begin{aligned}
\theta^{(j+2)} - \theta^{(j+1)} &= \Psi[u^{(j+1)}, \theta^{(j+1)}, S] - \Psi[u^{(j)}, \theta^{(j)}, S] \\
&= \Psi[u^{(j+1)}, \theta^{(j+1)} - \theta^{(j)}, 0] + \Psi[u^{(j+1)} - u^{(j)}, \theta^{(j)}, 0],
\end{aligned}
$$

we have

$$
\begin{aligned}
\tilde{B}_{2,j+1} &\leq C_3 \|u^{(j+1)} | L^{3,\infty}_\sigma\| \|\theta^{(j+1)} - \theta^{(j)} | L^{2,\infty}\| + C_3 \|u^{(j+1)} - u^{(j)} | L^{3,\infty}_\sigma\| \|\theta^{(j)} | L^{2,\infty}\| \\
&\leq 2C_3 M_\infty \tilde{M}_j (\leq 6C_{\max} M_\infty \tilde{M}_j), \\
\tilde{B}_{4,j+1} &\leq 2C_3 M_\infty \tilde{M}_j (\leq 6C_{\max} M_\infty \tilde{M}_j),
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{A}_{j+1} &\leq C_2 \|u^{(j+1)} | L^{3,\infty}_\sigma\| \|u^{(j+1)} - u^{(j)} | L^{3,\infty}_\sigma\| + C_2 \|u^{(j+1)} - u^{(j)} | L^{3,\infty}_\sigma\| \|u^{(j)} | L^{3,\infty}_\sigma\| \\
&\quad + C_1 \|\theta^{(j+2)} - \theta^{(j+1)} | L^{2,\infty}\| + C_1 \|\theta^{(j+2)} - \theta^{(j+1)} | L^{4,\infty}\| \\
&\leq 2C_2 M_\infty \tilde{A}_j + C_1 \tilde{B}_{2,j+1} + C_1 \tilde{B}_{4,j+1} \\
&\leq 2C_2 M_\infty \tilde{A}_j + 2C_1 C_3 M_\infty \tilde{M}_j + 2C_1 C_3 M_\infty \tilde{M}_j \leq 6C_{\max} M_\infty \tilde{M}_j.
\end{aligned}
$$

Namely, we have

$$
\tilde{M}_{j+1} \leq 6C_{\max} M_\infty \tilde{M}_j.
$$

Then we get

$$
\begin{aligned}
\|u^{(k)} - u^{(j)} | L^{3,\infty}_\sigma\| &\leq \sum_{l=j}^{k-1} \tilde{A}_l \\
&\leq \sum_{l=j}^{k-1} \tilde{M}_l \leq \sum_{l=j}^{k-1} \tilde{M}_0 (6C_{\max} M_\infty)^l \leq \frac{\tilde{M}_0 (6C_{\max} M_\infty)^j}{1 - 6C_{\max} M_\infty}
\end{aligned}
$$

for all $j, k > 0$ such that $j < k$. Similarly we get

$$
\|\theta^{(k)} - \theta^{(j)} | L^{2,\infty}\| \leq \frac{\tilde{M}_0 (6C_{\max} M_\infty)^j}{1 - 6C_{\max} M_\infty}, \|\theta^{(k)} - \theta^{(j)} | L^{4,\infty}\| \leq \frac{\tilde{M}_0 (6C_{\max} M_\infty)^j}{1 - 6C_{\max} M_\infty}.
$$

for all $j, k > 0$ such that $j < k$. Since

$$
6C_{\max} M_\infty = 3 \left(1 - \sqrt{1 - 4C_{\max}(C_1 C(S) + C(F) + C(S))}\right) < 1,
$$

we have $\|u^{(k)}-u^{(j)}|L^{3,\infty}_\sigma\| \to 0$, $\|\theta^{(k)}-\theta^{(j)}|L^{2,\infty}\| \to 0$ and $\|\theta^{(k)}-\theta^{(j)}|L^{4,\infty}\| \to 0$ as $j,k \to \infty$.

Therefore, we see that there exist functions $u \in BUC(\mathbb{R}, L^{3,\infty}_\sigma)$ and $\theta \in BUC(\mathbb{R}, L^{3,1})$ such that

$$\begin{aligned}
u^{(j)} &\to u \quad \text{in } BUC(\mathbb{R}, L^{3,\infty}_\sigma) \quad \text{as } j \to \infty,\\
\theta^{(j)} &\to \theta \quad \text{in } BUC(\mathbb{R}, L^{3,1}) \quad \text{as } j \to \infty.
\end{aligned}$$

By the direct calculation, we can show that (u,θ) satisfies integral equations and time-periodicity (see [66, pp.656-657, Corollary 1.2]).

Similarly [27, p.42] and [66, p.658], we can show the uniqueness of the solution. Concretely, we suppose that (u,θ) and (v,ξ) are mild solutions of (B) with $\|u|L^{3,\infty}\| < M_\infty$, $\|\theta|L^{3,1}\| < M_\infty$, $\|v|L^{3,\infty}\| < M_\infty$ and $\|\xi|L^{3,1}\| < M_\infty$. Then we have $u = \Phi[u,u,\theta,F]$, $\theta = \Psi[u,\theta,S]$, $v = \Phi[v,v,\xi,F]$ and $\xi = \Psi[v,\xi,S]$. Then we obtain

$$\begin{aligned}
&\|\theta-\xi|L^{2,\infty}\cap L^{4,\infty}\|\\
&= \|\Psi[u,\theta,S]-\Psi[v,\xi,S]|L^{2,\infty}\cap L^{4,\infty}\|\\
&= \|\Psi[u,\theta-\xi,0]+\Psi[u-v,\xi,0]|L^{2,\infty}\cap L^{4,\infty}\|\\
&\le \|\Psi[u,\theta-\xi,0]|L^{2,\infty}\cap L^{4,\infty}\|+\\
&\qquad\qquad \|\Psi[u-v,\xi,0]|L^{2,\infty}\cap L^{4,\infty}\|\\
&= \|\Psi[u,\theta-\xi,0]|L^{2,\infty}\| + \|\Psi[u,\theta-\xi,0]|L^{4,\infty}\|+\\
&\qquad\qquad \|\Psi[u-v,\xi,0]|L^{2,\infty}\| + \|\Psi[u-v,\xi,0]|L^{4,\infty}\|\\
&\le C_3\|u|L^{3,\infty}_\sigma\|(\|\theta-\xi|L^{2,\infty}\| + \|\theta-\xi|L^{4,\infty}\|)\\
&\qquad + C_3\|u-v|L^{3,\infty}_\sigma\|(\|\xi|L^{2,\infty}\| + \|\xi|L^{4,\infty})\\
&\le C_3M_\infty(\|\theta-\xi|L^{2,\infty}\| + \|\theta-\xi|L^{4,\infty}\|) + 2C_3M_\infty\|u-v|L^{3,\infty}_\sigma\|,
\end{aligned}$$

and

$$\begin{aligned}
\|u-v|L^{3,\infty}\| &= \|\Phi[u,u,\theta,F]-\Phi[v,v,\xi,F]|L^{3,\infty}\|\\
&= \|\Phi[u,u-v,0,0]\\
&\qquad + \Phi[u-v,v,0,0] + \Phi[0,0,\theta-\xi,0]|L^{3,\infty}\|\\
&\le C_2(\|u|L^{3,\infty}\| + \|v|L^{3,\infty}\|)\|u-v|L^{3,\infty}\|\\
&\qquad + C_1(\|\theta-\xi|L^{2,\infty}\| + \|\theta-\xi|L^{4,\infty}\|)\\
&\le 2C_2M_\infty\|u-v|L^{3,\infty}_\sigma\|\\
&\qquad + C_1(\|\theta-\xi|L^{2,\infty}\| + \|\theta-\xi|L^{4,\infty}\|)\\
&\le 2C_2M_\infty\|u-v|L^{3,\infty}_\sigma\|\\
&\qquad + C_1C_3M_\infty(\|\theta-\xi|L^{2,\infty}\| + \|\theta-\xi|L^{4,\infty}\|)\\
&\qquad\quad + 2C_1C_3M_\infty\|u-v|L^{3,\infty}_\sigma\|.
\end{aligned}$$

Then we have

$$
\begin{aligned}
&\|u-v|L^{3,\infty}_\sigma\| + \|\theta-\xi|L^{2,\infty}\cap L^{4,\infty}\| \\
&\le 2(C_1C_3+C_2+C_3)M_\infty(\|u-v|L^{3,\infty}_\sigma\| + \|\theta-\xi|L^{2,\infty}\cap L^{4,\infty}\|) \\
&= 2C_{\max}M_\infty(\|u-v|L^{3,\infty}_\sigma\| + \|\theta-\xi|L^{2,\infty}\cap L^{4,\infty}\|),
\end{aligned}
$$

where $C_{\max} = C_1C_3+C_2+C_3$. Noting that $2C_{\max}M_\infty < \frac{1}{3} < 1$ (see (5.18)), we get $u = v$ and $\theta = \xi$. □

5.3 Appendix

In this section, we show Lemma 5.4. We consider the heat equation as follows:

$$
\text{(H)} \begin{cases} \partial_t v - \Delta v = 0, x \in \Omega, \ t \in (0,\infty), \\ v|_{t=0} = \psi, \ \frac{\partial v}{\partial \eta}|_{\partial\Omega} = 0. \end{cases}
$$

Then $v = e^{-tB}\psi$ is the solution of (H).

Lemma 5.5 ([21])**.** (i) *Let $1 \le p \le q \le \infty$. Then there exists a constant C such that*

$$
\|e^{-tB}\psi\|_q \le Ct^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\psi\|_p \tag{5.19}
$$

for all $t > 0$ and $\psi \in L^p$.
(ii) *Let $1 \le p \le \infty$. Then there exists a constant C such that*

$$
\|\nabla e^{-tB}\psi\|_\infty \le Ct^{-\frac{3}{2p}-\frac{1}{2}}\|\psi\|_p \tag{5.20}
$$

for all $t > 0$ and $\psi \in L^p$.

Lemma 5.6. *There exists a constant C such that*

$$
\|\nabla e^{-tB}\psi\|_2 \le Ct^{-\frac{1}{2}}\|\psi\|_2 \tag{5.21}
$$

for all $t > 0$ and $\psi \in L^2$.

Corollary 5.1. *Let $1 \le p \le \infty$ and $\max\{2,p\} \le q \le \infty$. Then there exists a constant C such that*

$$
\|\nabla e^{-tB}\psi\|_q \le Ct^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\|\psi\|_p \tag{5.22}
$$

for all $t > 0$ and $\psi \in L^p$.

Proof of Lemma 5.6. We use a method by Dan-Shibata [10, pp.205-206]. Multiplying the first equation of (H) by v, we obtain the energy equality.

$$\frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\nabla v(s)\|^2 ds = \frac{1}{2}\|\psi\|_2^2. \tag{5.23}$$

By the equation of (H), we have

$$\begin{aligned}\frac{d}{dt}\left(t\|\nabla v(t)\|_2^2\right) &= \|\nabla v(t)\|_2^2 + 2t(\nabla v(t), \nabla\partial_t v(t)) \\ &= \|\nabla v(t)\|_2^2 - 2t(\Delta v(t), \partial_t v(t)) \\ &= \|\nabla v(t)\|_2^2 - 2t(\partial_t v(t), \partial_t v(t)) \\ &\le \|\nabla v(t)\|_2^2.\end{aligned} \tag{5.24}$$

Therefore, by (5.23) and (5.24), we get the desired estimate. □

Proof of Corollary 5.1. By the Riesz-Thorin theorem on ∇e^{-tB} (see e.g. [2, Theorem 1.1.1]) with (5.20) and (5.21), we have

$$\|\nabla e^{-tB}\psi\|_q \le Ct^{-\frac{1}{2}}\|\psi\|_q \tag{5.25}$$

for $t > 0$ and $\psi \in L^q$, where $q \ge 2$. Therefore, by (5.19) and (5.25), we get the desired estimate. □

Proof of Lemma 5.4. We prove only (5.6). We can prove (5.5) in the same way. We use a method by Yamazaki [66, pp.649-650].

Let p_0 and p_1 satisfy $1 < p_0 < p < p_1$, $\frac{1}{p} - \frac{1}{p_1} < \frac{2}{3}$ and $p_1 < q$. By real interpolation on ∇e^{-tB} with (5.22), we have

$$\|\nabla e^{-tB}\psi\|_{q,1} \le Ct^{-\frac{3}{2}\left(\frac{1}{p_i}-\frac{1}{q}\right)-\frac{1}{2}}\|\psi\|_{p_i,1} \tag{5.26}$$

for $t > 0$ and $\psi \in L^{p_i,1}$, where $i = 0, 1$. (On L^p-L^q estimates of the Stokes semigroup in the Lorentz space, see [53, 54, 66].)

Here we set $\rho = \frac{3}{2p} - \frac{3}{2q} - \frac{1}{2}$ and we define an operator T which maps $u \in L^{p_0,1} + L^{p_1,1}$ to a function $v(t)$ on $(0, \infty)$, where $v(t) := t^\rho\|\nabla e^{-tB}u\|_{q,1}$. By (5.26), we have

$$v(t) \le Ct^{\frac{3}{2p}-\frac{3}{2p_i}-1}\|u\|_{p_i,1}. \tag{5.27}$$

for $u \in L^{p_i,1}$. There exist numbers s_0 and s_1 such that $\frac{1}{s_i} = 1 - \frac{3}{2p} + \frac{3}{2p_i}$. By (5.27), we have $v(t) \in L^{s_i,\infty}(0, \infty)$ and $\|v(\cdot)\|_{s_i,\infty} \le C\|u\|_{p_i,1}$, that is, the operator T maps $L^{p_0,1}(\Omega) + L^{p_1,1}(\Omega)$ to $L^{s_0,\infty}(0, \infty) + L^{s_1,\infty}(0, \infty)$.

Here, when we choose $0 < \theta < 1$ satisfying $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, we have $1 = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}$. Then we obtain as follows:

$$(L^{p_0,1}(\Omega), L^{p_1,1}(\Omega))_{\theta,1} = L^{p,1}(\Omega) \tag{5.28}$$

and

$$(L^{s_0,\infty}(0,\infty), L^{s_1,\infty}(0,\infty))_{\theta,1} = L^1(0,\infty). \tag{5.29}$$

Therefore by real interpolation on the operator T with (5.28) and (5.29), we have

$$\int_0^\infty v(\tau) d\tau \le C\|u\|_{p,1},$$

that is, we get the desired estimate. □

Additional information

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