

信州大学審査学位論文

On linear isometries between function spaces

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Abstract

The linear isometries between function spaces have been studied by many mathematicians. In this paper, we consider three kinds of linear isometries; linear isometries on spaces of differentiable functions, backward shifts on uniform algebras and real-linear isometries between complex function spaces.

In Chapter 1, we consider the linear isometries on spaces of differentiable functions. We denote by $C^{(n)}[0, 1]$ the linear space of n -times continuously differentiable functions on the closed unit interval $[0, 1]$. Each of the following norms makes $C^{(n)}[0, 1]$ a Banach space:

$$\begin{aligned} \|f\|_{\sigma} &= \sum_{k=0}^{n-1} |f^{(k)}(0)| + \sup_{x \in [0,1]} |f^{(n)}(x)|, \\ \|f\|_m &= \max \left\{ |f(0)|, |f'(0)|, \dots, |f^{(n-1)}(0)|, \sup_{x \in [0,1]} |f^{(n)}(x)| \right\}, \end{aligned} \quad (f \in C^{(n)}[0, 1]).$$

We characterize the surjective linear isometries on $(C^{(n)}[0, 1], \|\cdot\|_{\sigma})$ and $(C^{(n)}[0, 1], \|\cdot\|_m)$, as follows: Let T be a linear operator on $(C^{(n)}[0, 1], \|\cdot\|_{\sigma})$ or $(C^{(n)}[0, 1], \|\cdot\|_m)$. Then T is a surjective isometry if and only if there exist a homeomorphism φ of $[0, 1]$ onto itself, a unimodular continuous function ω on $[0, 1]$, a permutation $\{\tau(0), \tau(1), \dots, \tau(n-1)\}$ of $\{0, 1, \dots, n-1\}$ and unimodular constants $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ such that

$$(Tf)(x) = \sum_{k=0}^{n-1} \frac{\lambda_k f^{(\tau(k))}(0)}{k!} x^k + \left(S^n(\omega(f^{(n)} \circ \varphi)) \right)(x) \quad (x \in [0, 1], f \in C^{(n)}[0, 1]),$$

where $(Sg)(x) = \int_0^x g(t) dt$ for all $x \in [0, 1]$ and $g \in C([0, 1])$. Also, we prove that every finite codimensional linear isometry on $(C^{(n)}[0, 1], \|\cdot\|_m)$ is surjective. Moreover, we prove a similar statement on the space of Lipschitz continuous functions.

In Chapter 2, we prove that an infinite-dimensional uniform algebra does not admit a backward shift. Also, we introduce a backward quasi-shift as a weak type of backward shift, and show that a uniform algebra A does not admit it, under the assumption that the maximal ideal space of A has at most finitely many isolated points. Moreover, we discuss the existence of shifts on several function spaces.

In Chapter 3, we characterize the surjective real-linear isometries between complex-linear subspaces of continuous functions: Let X and Y be locally compact Hausdorff spaces. Let A and B be complex-linear subspaces of $C_{\mathbb{C},0}(X)$ and $C_{\mathbb{C},0}(Y)$, respectively. Suppose that for each distinct points $x, x', x'' \in X$ there exists $f \in A$ such that $|f(x)| \neq |f(x')|$ and $f(x'') = 0$. Also suppose that for each distinct points $y, y' \in Y$ there exists $g \in B$ such that $|g(y)| \neq |g(y')|$. If T is a real-linear isometry of A onto B , then there exist an open and closed subset E of $\text{Ch}(B)$, a homeomorphism φ of $\text{Ch}(B)$ onto $\text{Ch}(A)$ and a unimodular continuous function ω on $\text{Ch}(B)$ such that $Tf = \omega(f \circ \varphi)$ on E and $Tf = \omega(\overline{f \circ \varphi})$ on $\text{Ch}(B) \setminus E$ for all $f \in A$, where $\text{Ch}(A)$ and $\text{Ch}(B)$ are the Choquet boundaries for A and B , respectively. Moreover, we give an example of the space which indicates the difference between the real-linear case and the complex-linear case.

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Chapter 1

Linear isometries on spaces of differentiable functions

1.1 Introduction

The source of this paper is the classical Banach-Stone theorem, which characterizes the surjective linear isometries on $C(X)$; the Banach space of all continuous functions on a compact Hausdorff space X with the supremum norm. It states that every surjective linear isometry T on $C(X)$ has the form: $Tf = \omega(f \circ \varphi)$ for all $f \in C(X)$, where φ is a homeomorphism of X onto itself and ω is a unimodular continuous function on X . This theorem raised a natural problem: Characterize the surjective linear isometries on other function spaces. In the book [12], we can find the answer to this problem on many function spaces.

On the other hand, we want to remove the surjectivity of linear isometries in the results above. It seems to be difficult to deal with a general linear isometry which is not necessarily surjective. Thus our first step is to investigate the linear isometry which has a finite codimensional range. Such a linear isometry is said to be *finite codimensional*. The finite codimensional linear isometries on various function spaces have been studied by Araujo and Font [2, 13] and many other mathematicians (cf. [15, 18, 20, 23, 46, 47]). Here we note that a surjective linear isometry is finite codimensional, because its range has codimension 0.

As mentioned above, we know much about those isometries on many function spaces. But it is not all. In this chapter, we take up three spaces consisting of differentiable functions; the space of continuously differentiable functions, the space of Lipschitz continuous functions and the Wiener algebra. This chapter is based on [26] and [27].

1.2 The space of continuously differentiable functions

Let n be a positive integer. Let \mathbb{K} denote the real number field \mathbb{R} or the complex number field \mathbb{C} . By $C^{(n)}[0, 1]$ we denote the linear space of all \mathbb{K} -valued n -times continuously differentiable functions on the closed unit interval $[0, 1]$. There exist several norms which make $C^{(n)}[0, 1]$ a Banach space; for example,

$$\begin{aligned}
\|f\|_C &= \max \left\{ \sum_{k=0}^n \frac{|f^{(k)}(x)|}{k!} : x \in [0, 1] \right\}, \\
\|f\|_\Sigma &= \sum_{k=0}^n \frac{\|f^{(k)}\|_\infty}{k!}, \\
\|f\|_M &= \max\{\|f\|_\infty, \|f'\|_\infty, \dots, \|f^{(n)}\|_\infty\}, \\
\|f\|_\sigma &= \sum_{k=0}^{n-1} |f^{(k)}(0)| + \|f^{(n)}\|_\infty, \\
\|f\|_m &= \max\{|f(0)|, |f'(0)|, \dots, |f^{(n-1)}(0)|, \|f^{(n)}\|_\infty\},
\end{aligned} \tag{f \in C^{(n)}[0, 1]},$$

where $\|\cdot\|_\infty$ denotes the supremum norm on $[0, 1]$. These norms are equivalent. In particular, $(C^{(n)}[0, 1], \|\cdot\|_C)$ and $(C^{(n)}[0, 1], \|\cdot\|_\Sigma)$ are unital semisimple commutative Banach algebras.

In [6], Cambern characterized the surjective linear isometries on $(C^{(1)}[0, 1], \|\cdot\|_C)$. Later, Pathak [33] extended this result to $(C^{(n)}[0, 1], \|\cdot\|_C)$. The other extensions may be found in [7] and [29]. On the other hand, Rao and Roy [39] and Jarosz and Pathak [19] characterized the surjective linear isometries on $(C^{(1)}[0, 1], \|\cdot\|_\Sigma)$ and $(C^{(1)}[0, 1], \|\cdot\|_M)$, respectively. All of those results say that every surjective linear isometry T on the designated space has the canonical form; $Tf = \omega(f \circ \varphi)$. In this paper, we show that the surjective linear isometries on $(C^{(n)}[0, 1], \|\cdot\|_\sigma)$ and $(C^{(n)}[0, 1], \|\cdot\|_m)$ have the different form.

To state our theorem, we remark on notations: Put $\mathbb{T} = \{z \in \mathbb{K} : |z| = 1\}$. If $\mathbb{K} = \mathbb{R}$, then $\mathbb{T} = \{1, -1\}$. If $\mathbb{K} = \mathbb{C}$, then \mathbb{T} denotes the unit circle in \mathbb{C} (In Section 1.3 and Chapters 2 and 3, we shall be restricted to the case of $\mathbb{K} = \mathbb{C}$, where \mathbb{T} is the unit circle). A number in \mathbb{T} is said to be *unimodular*. Also, a function f is said to be *unimodular* if the range of f is contained in \mathbb{T} . Next, we introduce an integral operator S : For any $f \in C([0, 1])$, we put $(Sf)(x) = \int_0^x f(t) dt$ for all $x \in [0, 1]$. Then S is a linear operator of $C([0, 1])$ onto $\{f \in C^{(1)}[0, 1] : f(0) = 0\}$, and S^n maps $C([0, 1])$ onto $\{f \in C^{(n)}[0, 1] : f^{(k)}(0) = 0 \text{ for } k = 0, 1, \dots, n-1\}$. This shows that $\{f^{(n)} : f \in C^{(n)}[0, 1]\} = C([0, 1])$. Moreover we have

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + (S^n f^{(n)})(x) \quad (x \in [0, 1], f \in C^{(n)}[0, 1]).$$

Now, let us state our theorem. We characterize the surjective linear isometries on $(C^{(n)}[0, 1], \|\cdot\|_\sigma)$ and $(C^{(n)}[0, 1], \|\cdot\|_m)$, as follows:

Theorem 1.2.1. *Let T be a linear operator on $(C^{(n)}[0, 1], \|\cdot\|_\sigma)$ or $(C^{(n)}[0, 1], \|\cdot\|_m)$. Then T is a surjective isometry if and only if there exist a homeomorphism φ of $[0, 1]$ onto itself, a unimodular continuous function ω on $[0, 1]$, a permutation $\{\tau(0), \tau(1), \dots, \tau(n-1)\}$ of $\{0, 1, \dots, n-1\}$ and unimodular constants $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ such that*

$$(Tf)(x) = \sum_{k=0}^{n-1} \frac{\lambda_k f^{(\tau(k))}(0)}{k!} x^k + \left(S^n (\omega(f^{(n)} \circ \varphi)) \right) (x) \tag{1.1}$$

for all $x \in [0, 1]$ and $f \in C^{(n)}[0, 1]$.

By the theorem above, the linear operator T_1 defined by $(T_1 f)(x) = -f(0) + \int_0^x f'(t) dt$ is a surjective isometry on $(C^{(1)}[0, 1], \|\cdot\|_\sigma)$ and $(C^{(1)}[0, 1], \|\cdot\|_m)$. But we can easily see that T_1 is not of the canonical form.

We also prove the following:

Theorem 1.2.2. *If T is a finite codimensional linear isometry on $(C^{(n)}[0, 1], \|\cdot\|_m)$, then T is surjective.*

Theorems 1.2.1 and 1.2.2 characterize the finite codimensional linear isometries on $(C^{(n)}[0, 1], \|\cdot\|_m)$. We will prove Theorem 1.2.1 in Sections 1.2.2–1.2.4, Theorem 1.2.2 in Section 1.2.5.

1.2.1 Preliminaries

For a normed linear space \mathcal{B} , we put ball $\mathcal{B} = \{\xi \in \mathcal{B} : \|\xi\|_{\mathcal{B}} \leq 1\}$ and denote its dual space by \mathcal{B}^* . For a bounded linear operator T between two normed linear spaces, we denote by T^* the adjoint operator of T . We use these notations throughout this paper.

For any nonnegative integer ℓ , we define $\iota^\ell(x) = x^\ell$ for $x \in [0, 1]$. In particular, we write $\iota^0 = 1$ and $\iota^1 = \iota$. Let $f \in C^{(n)}[0, 1]$ and $\ell = 1, 2, \dots, n$. Then $f = \iota^\ell$ if and only if $f(0) = f'(0) = \dots = f^{(\ell-1)}(0) = 0$ and $f^{(\ell)}(x) = \ell!$ for $x \in [0, 1]$.

Now, we prove two elementary facts which are used later.

Proposition 1.2.3. *Let $\mathcal{B}_1, \dots, \mathcal{B}_\ell$ be normed linear spaces, and let $\mathcal{B} = \mathcal{B}_1 \times \dots \times \mathcal{B}_\ell$ be the product space equipped with the norm*

$$\|(a_1, \dots, a_\ell)\|_{\mathcal{B}} = \max\{\|a_1\|_{\mathcal{B}_1}, \dots, \|a_\ell\|_{\mathcal{B}_\ell}\} \quad ((a_1, \dots, a_\ell) \in \mathcal{B}).$$

Then (a_1, \dots, a_ℓ) is an extreme point of ball \mathcal{B} if and only if a_k is an extreme point of ball \mathcal{B}_k for all $k = 1, \dots, \ell$.

Proof. Suppose that a_k is an extreme point of ball \mathcal{B}_k for all k . To prove that (a_1, \dots, a_ℓ) is an extreme point of ball \mathcal{B} , write $(a_1, \dots, a_\ell) = ((a'_1, \dots, a'_\ell) + (a''_1, \dots, a''_\ell))/2$, where $(a'_1, \dots, a'_\ell), (a''_1, \dots, a''_\ell) \in \text{ball } \mathcal{B}$. Then for each $k = 1, \dots, \ell$ we have

$$a_k = \frac{a'_k + a''_k}{2}.$$

Also, $\|a'_k\|_{\mathcal{B}_k} \leq \max\{\|a'_1\|_{\mathcal{B}_1}, \dots, \|a'_\ell\|_{\mathcal{B}_\ell}\} = \|(a'_1, \dots, a'_\ell)\|_{\mathcal{B}} \leq 1$. Similarly, $\|a''_k\|_{\mathcal{B}_k} \leq 1$. By hypothesis, $a_k = a'_k = a''_k$. Hence $(a_1, \dots, a_\ell) = (a'_1, \dots, a'_\ell) = (a''_1, \dots, a''_\ell)$. Thus (a_1, \dots, a_ℓ) is an extreme point of ball \mathcal{B} .

Conversely, suppose that (a_1, \dots, a_ℓ) is an extreme point of ball \mathcal{B} . Fix $k = 1, \dots, \ell$ and write $a_k = (a'_k + a''_k)/2$, where $a'_k, a''_k \in \text{ball } \mathcal{B}_k$. Then

$$(a_1, \dots, a_k, \dots, a_\ell) = \frac{(a_1, \dots, a'_k, \dots, a_\ell) + (a_1, \dots, a''_k, \dots, a_\ell)}{2}.$$

Also, we have $\|(a_1, \dots, a'_k, \dots, a_\ell)\|_{\mathcal{B}} \leq \max\{\|(a_1, \dots, a_\ell)\|_{\mathcal{B}}, \|a'_k\|_{\mathcal{B}_k}\} \leq 1$. Similarly, $\|(a_1, \dots, a''_k, \dots, a_\ell)\|_{\mathcal{B}} \leq 1$. By hypothesis,

$$(a_1, \dots, a_k, \dots, a_\ell) = (a_1, \dots, a'_k, \dots, a_\ell) = (a_1, \dots, a''_k, \dots, a_\ell),$$

and so $a_k = a'_k = a''_k$. Thus a_k is an extreme point of ball \mathcal{B}_k . □

Proposition 1.2.4. *Let ψ_1 and ψ_2 be injective continuous mappings of $[0, 1]$ into $[0, 1]$, and let $\alpha \in \mathbb{C}$. If $\alpha(g \circ \psi_1) + (g \circ \psi_2)$ is constant on $[0, 1]$ for all real-valued continuous functions g on $[0, 1]$, then $\psi_1 = \psi_2$.*

Proof. Assume $\psi_1 \neq \psi_2$. Then $\psi_1(p) \neq \psi_2(p)$ for some $p \in [0, 1]$. Since ψ_1 is continuous, there exists $q \in [0, 1]$ such that $q \neq p$ and $\psi_1(q) \neq \psi_2(p)$. Then $\psi_2(q) \neq \psi_2(p)$ because ψ_2 is injective. Find a real-valued continuous function g_0 on $[0, 1]$ such that $g_0(\psi_2(p)) = 1$ and $g_0(\psi_1(p)) = g_0(\psi_1(q)) = g_0(\psi_2(q)) = 0$. Then we have $\alpha g_0(\psi_1(p)) + g_0(\psi_2(p)) = 1$ and $\alpha g_0(\psi_1(q)) + g_0(\psi_2(q)) = 0$. This contradicts the hypothesis that $\alpha(g_0 \circ \psi_1) + (g_0 \circ \psi_2)$ is constant. Hence $\psi_1 = \psi_2$. \square

When we consider the finite codimensional linear isometries, we will use the following theorem by Takahasi and Okayasu:

Theorem A (Takahasi and Okayasu [46]). *Let T be a finite codimensional linear isometry on $C(X)$. Then T is surjective if and only if for any continuous mapping ψ of X onto itself which is not injective, the set $\{(x, y) \in X \times X : x \neq y, \psi(x) = \psi(y)\}$ is infinite.*

1.2.2 Proof of Theorem 1.2.1; the “if” part

First, we settle an easy part of Theorem 1.2.1.

Proof of the “if” part of Theorem 1.2.1. Suppose T has the form (1.1). Let $f \in C^{(n)}[0, 1]$. For each $\ell = 0, 1, \dots, n-1$, we have

$$(Tf)^{(\ell)}(x) = \sum_{k=\ell}^{n-1} \frac{\lambda_k f^{(\tau(k))}(0)}{(k-\ell)!} x^{k-\ell} + (S^{n-\ell}(\omega(f^{(n)} \circ \varphi)))(x) \quad (x \in [0, 1]),$$

and so $(Tf)^{(\ell)}(0) = \lambda_\ell f^{(\tau(\ell))}(0)$ because $(Sg)(0) = 0$ for all $g \in C[0, 1]$. Moreover $(Tf)^{(n)} = \omega(f^{(n)} \circ \varphi)$. Therefore

$$\begin{aligned} \|Tf\|_\sigma &= \sum_{\ell=0}^{n-1} |(Tf)^{(\ell)}(0)| + \|(Tf)^{(n)}\|_\infty = \sum_{\ell=0}^{n-1} |\lambda_\ell f^{(\tau(\ell))}(0)| + \|\omega(f^{(n)} \circ \varphi)\|_\infty \\ &= \sum_{\ell=0}^{n-1} |f^{(\tau(\ell))}(0)| + \|f^{(n)} \circ \varphi\|_\infty = \sum_{k=0}^{n-1} |f^{(k)}(0)| + \|f^{(n)}\|_\infty = \|f\|_\sigma. \end{aligned}$$

Similarly, we can show $\|Tf\|_m = \|f\|_m$. Hence T is an isometry.

To see that T is surjective, let $g \in C^{(n)}[0, 1]$. Define $f \in C^{(n)}[0, 1]$ by

$$f(x) = \sum_{k=0}^{n-1} \frac{\overline{\lambda_{\tau^{-1}(k)}} g^{(\tau^{-1}(k))}(0)}{k!} x^k + \left(S^n \left(\frac{g^{(n)} \circ \varphi^{-1}}{\omega \circ \varphi^{-1}} \right) \right) (x) \quad (x \in [0, 1]).$$

Then $f^{(\ell)}(0) = \overline{\lambda_{\tau^{-1}(\ell)}} g^{(\tau^{-1}(\ell))}(0)$ for $\ell = 0, 1, \dots, n-1$ and $f^{(n)} = (g^{(n)} \circ \varphi^{-1})/(\omega \circ \varphi^{-1})$. Hence (1.1) yields

$$\begin{aligned} (Tf)(x) &= \sum_{k=0}^{n-1} \frac{\lambda_k \overline{\lambda_k} g^{(k)}(0)}{k!} x^k + \left(S^n \left(\omega \left(\frac{g^{(n)} \circ \varphi^{-1}}{\omega \circ \varphi^{-1}} \circ \varphi \right) \right) \right) (x) \\ &= \sum_{k=0}^{n-1} \frac{g^{(k)}(0)}{k!} x^k + (S^n g^{(n)})(x) = g(x) \end{aligned} \quad (x \in [0, 1]).$$

Hence T is surjective. □

1.2.3 Proof of Theorem 1.2.1; the “only if” part on $(C^{(n)}[0, 1], \|\cdot\|_\sigma)$

We divide the proof of the “only if” part into two subsections. In this subsection, we deal with only the space $(C^{(n)}[0, 1], \|\cdot\|_\sigma)$. The other space $(C^{(n)}[0, 1], \|\cdot\|_m)$ is considered in the next subsection.

For simplicity, we write $C^{(n)}$ and C for $(C^{(n)}[0, 1], \|\cdot\|_\sigma)$ and $(C([0, 1]), \|\cdot\|_\infty)$, respectively. Let \mathbb{K}^n denote the product space of n copies of \mathbb{K} . The points of \mathbb{K}^n are thus ordered n -tuples $\mathbf{a} = (a_0, a_1, \dots, a_{n-1})$, where $a_0, a_1, \dots, a_{n-1} \in \mathbb{K}$. For instance, we write $\mathbf{b} = (b_0, b_1, \dots, b_{n-1})$, $\mathbf{1} = (1, 1, \dots, 1)$ and so on.

Definition 1.2.5. For each $(\mathbf{a}, c, x) \in \mathbb{T}^n \times \mathbb{T} \times [0, 1]$, we define a functional $\Lambda_{(\mathbf{a}, c, x)}$ on $C^{(n)}$ by

$$\Lambda_{(\mathbf{a}, c, x)}(f) = \sum_{k=0}^{n-1} a_k f^{(k)}(0) + c f^{(n)}(x) \quad (f \in C^{(n)}).$$

It is clear that $\Lambda_{(\mathbf{a}, c, x)} \in \text{ball}(C^{(n)})^*$.

Lemma 1.2.6. *Let $\xi \in (C^{(n)})^*$. Then ξ is an extreme point of $\text{ball}(C^{(n)})^*$ if and only if there exists $(\mathbf{a}, c, x) \in \mathbb{T}^n \times \mathbb{T} \times [0, 1]$ such that $\xi = \Lambda_{(\mathbf{a}, c, x)}$.*

Proof. Suppose that the product spaces $\mathbb{K}^n \times C$ and $\mathbb{K}^n \times C^*$ have the norms

$$\begin{aligned} \|(\mathbf{b}, g)\| &= \sum_{k=0}^{n-1} |b_k| + \|g\|_\infty && ((\mathbf{b}, g) \in \mathbb{K}^n \times C), \\ \|(\mathbf{a}, \eta)\| &= \max\{|a_0|, |a_1|, \dots, |a_{n-1}|, \|\eta\|\} && ((\mathbf{a}, \eta) \in \mathbb{K}^n \times C^*), \end{aligned}$$

respectively. Then $(\mathbb{K}^n \times C)^*$ is linearly isometric to $\mathbb{K}^n \times C^*$. In fact, the linear isometry Q of $\mathbb{K}^n \times C^*$ onto $(\mathbb{K}^n \times C)^*$ is given by

$$(Q(\mathbf{a}, \eta))(\mathbf{b}, g) = \sum_{k=0}^{n-1} a_k b_k + \eta(g) \quad ((\mathbf{a}, \eta) \in \mathbb{K}^n \times C^*, (\mathbf{b}, g) \in \mathbb{K}^n \times C).$$

Now, define a mapping P of $C^{(n)}$ into $\mathbb{K}^n \times C$ by

$$Pf = ((f(0), f'(0), \dots, f^{(n-1)}(0)), f^{(n)}) \quad (f \in C^{(n)}).$$

Clearly, P is a linear isometry of $C^{(n)}$ onto $\mathbb{K}^n \times C$. Hence the adjoint operator P^* is a linear isometry of $(\mathbb{K}^n \times C)^*$ onto $(C^{(n)})^*$, and so P^*Q is a linear isometry of $\mathbb{K}^n \times C^*$ onto $(C^{(n)})^*$. Thus ξ is an extreme point of $\text{ball}(C^{(n)})^*$ if and only if there exists an extreme point (\mathbf{a}, η) of $\text{ball}(\mathbb{K}^n \times C^*)$ such that $\xi = P^*Q(\mathbf{a}, \eta)$. Note that the set of all extreme points of $\text{ball}\mathbb{K}$ is \mathbb{T} . Also, it is known that the set of all extreme points of $\text{ball}C^*$ is $\{ce_x : c \in \mathbb{T}, x \in [0, 1]\}$, where e_x is the evaluation functional at x : $e_x(g) = g(x)$ for $g \in C$ ([8, Theorem V.8.4]). By Proposition 1.2.3, (\mathbf{a}, η) is an extreme point of $\text{ball}(\mathbb{K}^n \times C^*)$ if and only if $\mathbf{a} \in \mathbb{T}^n$ and $\eta = ce_x$, where $c \in \mathbb{T}$, $x \in [0, 1]$. Hence ξ is an extreme point of

$\text{ball}(C^{(n)})^*$ if and only if there exists $(\mathbf{a}, c, x) \in \mathbb{T}^n \times \mathbb{T} \times [0, 1]$ such that $\xi = P^*Q(\mathbf{a}, ce_x)$. Thus the conclusion follows from

$$\begin{aligned} (P^*Q(\mathbf{a}, ce_x))(f) &= (Q(\mathbf{a}, ce_x))(Pf) = (Q(\mathbf{a}, ce_x))((f(0), f'(0), \dots, f^{(n-1)}(0)), f^{(n)}) \\ &= \sum_{k=0}^{n-1} a_k f^{(k)}(0) + (ce_x)(f^{(n)}) = \sum_{k=0}^{n-1} a_k f^{(k)}(0) + cf^{(n)}(x) = \Lambda_{(\mathbf{a}, c, x)}(f) \end{aligned}$$

for all $f \in C^{(n)}$. \square

Let us start the proof of the “only if” part on $C^{(n)}$ of Theorem 1.2.1. For this purpose, let T be a surjective linear isometry on $C^{(n)}$. We complete the proof combining several lemmas.

Lemma 1.2.7. *For any $(\mathbf{a}, c, x) \in \mathbb{T}^n \times \mathbb{T} \times [0, 1]$, there exists a unique $(\mathbf{b}, d, y) \in \mathbb{T}^n \times \mathbb{T} \times [0, 1]$ such that*

$$T^*\Lambda_{(\mathbf{a}, c, x)} = \Lambda_{(\mathbf{b}, d, y)}.$$

Proof. Let $(\mathbf{a}, c, x) \in \mathbb{T}^n \times \mathbb{T} \times [0, 1]$. By Lemma 1.2.6, $\Lambda_{(\mathbf{a}, c, x)}$ is an extreme point of $\text{ball}(C^{(n)})^*$. Since T^* is a surjective linear isometry on $(C^{(n)})^*$, $T^*\Lambda_{(\mathbf{a}, c, x)}$ is an extreme point of $\text{ball}(C^{(n)})^*$. By Lemma 1.2.6, there exists $(\mathbf{b}, d, y) \in \mathbb{T}^n \times \mathbb{T} \times [0, 1]$ such that $T^*\Lambda_{(\mathbf{a}, c, x)} = \Lambda_{(\mathbf{b}, d, y)}$.

For the uniqueness of (\mathbf{b}, d, y) , suppose $T^*\Lambda_{(\mathbf{a}, c, x)} = \Lambda_{(\mathbf{b}', d', y')}$ for some $(\mathbf{b}', d', y') \in \mathbb{T}^n \times \mathbb{T} \times [0, 1]$, where $\mathbf{b}' = (b'_0, b'_1, \dots, b'_{n-1})$. Then $\Lambda_{(\mathbf{b}, d, y)} = \Lambda_{(\mathbf{b}', d', y')}$ and so

$$\sum_{k=0}^{n-1} b_k f^{(k)}(0) + df^{(n)}(y) = \sum_{k=0}^{n-1} b'_k f^{(k)}(0) + d'f^{(n)}(y') \quad (f \in C^{(n)}). \quad (1.2)$$

For each $\ell = 0, 1, \dots, n-1$, we put $f = \iota^\ell$ in (1.2) to get $b_\ell = b'_\ell$. Hence $\mathbf{b} = \mathbf{b}'$. Substituting ι^n and ι^{n+1} for f in (1.2), we obtain $d = d'$ and $y = y'$, respectively. \square

Definition 1.2.8. By Lemma 1.2.7, for each $(\mathbf{a}, x) \in \mathbb{T}^n \times [0, 1]$, there exists a unique $(\mathbf{b}, d, y) \in \mathbb{T}^n \times \mathbb{T} \times [0, 1]$ such that

$$T^*\Lambda_{(\mathbf{a}, 1, x)} = \Lambda_{(\mathbf{b}, d, y)}.$$

Since $\mathbf{b} = (b_0, \dots, b_{n-1})$, d and y depend on (\mathbf{a}, x) , we write

$$b_k = u_k(\mathbf{a}, x) \quad (k = 0, 1, \dots, n-1), \quad d = v(\mathbf{a}, x) \quad \text{and} \quad y = \psi(\mathbf{a}, x).$$

Thus u_k and v are unimodular functions on $\mathbb{T}^n \times [0, 1]$ and ψ is a mapping of $\mathbb{T}^n \times [0, 1]$ into $[0, 1]$.

Moreover, for any $f \in C^{(n)}$, we have

$$\Lambda_{(\mathbf{a}, 1, x)}(Tf) = (T^*\Lambda_{(\mathbf{a}, 1, x)})(f) = \Lambda_{(\mathbf{b}, d, y)}(f) = \Lambda_{((u_0(\mathbf{a}, x), \dots, u_{n-1}(\mathbf{a}, x)), v(\mathbf{a}, x), \psi(\mathbf{a}, x))}(f),$$

and so

$$\sum_{k=0}^{n-1} a_k (Tf)^{(k)}(0) + (Tf)^{(n)}(x) = \sum_{\ell=0}^{n-1} u_\ell(\mathbf{a}, x) f^{(\ell)}(0) + v(\mathbf{a}, x) f^{(n)}(\psi(\mathbf{a}, x)). \quad (1.3)$$

For each $m = 0, 1, \dots, n-1$, we put $f = \iota^m$ in (1.3) to get

$$\sum_{k=0}^{n-1} a_k (T\iota^m)^{(k)}(0) + (T\iota^m)^{(n)}(x) = m!u_m(\mathbf{a}, x). \quad (1.4)$$

Also, we substitute ι^n and ι^{n+1} for f in (1.3) to get

$$\sum_{k=0}^{n-1} a_k (T\iota^n)^{(k)}(0) + (T\iota^n)^{(n)}(x) = n!v(\mathbf{a}, x), \quad (1.5)$$

$$\sum_{k=0}^{n-1} a_k (T\iota^{n+1})^{(k)}(0) + (T\iota^{n+1})^{(n)}(x) = (n+1)!v(\mathbf{a}, x)\psi(\mathbf{a}, x). \quad (1.6)$$

Here we note that the equations (1.3)–(1.6) hold for all $(\mathbf{a}, x) \in \mathbb{T}^n \times [0, 1]$.

Lemma 1.2.9. *For $k = 0, 1, \dots, n-1$, u_k and v are unimodular continuous functions on $\mathbb{T}^n \times [0, 1]$. Also, ψ is a continuous mapping of $\mathbb{T}^n \times [0, 1]$ onto $[0, 1]$.*

Proof. Note that the left hand sides of the equations (1.4), (1.5) and (1.6) are continuous in $(\mathbf{a}, x) \in \mathbb{T}^n \times [0, 1]$. The first two equations show that u_k and v are continuous. Since v is unimodular, (1.6) implies that ψ is also continuous.

To see that $\psi : \mathbb{T}^n \times [0, 1] \rightarrow [0, 1]$ is surjective, pick $y \in [0, 1]$. Since T^* is a surjective linear isometry on $(C^{(n)})^*$, Lemma 1.2.6 guarantees the existence of $(\mathbf{a}, c, x) \in \mathbb{T}^n \times \mathbb{T} \times [0, 1]$ such that $T^*\Lambda_{(\mathbf{a}, c, x)} = \Lambda_{(1, 1, y)}$. Then we have

$$\begin{aligned} (T^*\Lambda_{(\bar{c}\mathbf{a}, 1, x)})(f) &= \Lambda_{(\bar{c}\mathbf{a}, 1, x)}(Tf) = \sum_{k=0}^{n-1} \bar{c}a_k (Tf)^{(k)}(0) + (Tf)^{(n)}(x) \\ &= \bar{c} \left(\sum_{k=0}^{n-1} a_k (Tf)^{(k)}(0) + c(Tf)^{(n)}(x) \right) = \bar{c}\Lambda_{(\mathbf{a}, c, x)}(Tf) \\ &= \bar{c}(T^*\Lambda_{(\mathbf{a}, c, x)})(f) = \bar{c}\Lambda_{(1, 1, y)}(f) = \bar{c} \left(\sum_{k=0}^{n-1} f^{(k)}(0) + f^{(n)}(y) \right) \\ &= \sum_{k=0}^{n-1} \bar{c}f^{(k)}(0) + \bar{c}f^{(n)}(y) = \Lambda_{(\bar{c}1, \bar{c}, y)}(f) \end{aligned}$$

for $f \in C^{(n)}$. By the definition of ψ , we get $\psi(\bar{c}\mathbf{a}, x) = y$. Hence ψ is surjective. \square

Lemma 1.2.10. *For any fixed $x \in [0, 1]$, $\psi(\mathbb{T}^n \times \{x\})$ is a singleton.*

We prove this lemma for two cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

Proof in case of $\mathbb{K} = \mathbb{R}$. Fix $a_1, \dots, a_{n-1} \in \mathbb{T} = \{1, -1\}$. For each $t \in \{1, -1\}$, we write $\mathbf{a}_t = (t, a_1, \dots, a_{n-1})$. By Lemma 1.2.9, $u_k(\mathbf{a}_t, x)$ and $v(\mathbf{a}_t, x)$ are continuous functions in $x \in [0, 1]$ and take values within $\{1, -1\}$. Since the interval $[0, 1]$ is connected, they are constant functions. Thus we can write

$$u_k(\mathbf{a}_t, x) = \alpha_{t,k} \quad \text{and} \quad v(\mathbf{a}_t, x) = \beta_t \quad (x \in [0, 1]),$$

where $\alpha_{t,k}$ and β_t are 1 or -1 . Next, for $t \in \{1, -1\}$, define $\psi_t(x) = \psi(\mathbf{a}_t, x)$ for all $x \in [0, 1]$. Putting $\mathbf{a} = \mathbf{a}_t$ in (1.3), we have

$$t(Tf)(0) + \sum_{k=1}^{n-1} \alpha_k (Tf)^{(k)}(0) + (Tf)^{(n)}(x) = \sum_{\ell=0}^{n-1} \alpha_{t,\ell} f^{(\ell)}(0) + \beta_t f^{(n)}(\psi_t(x)) \quad (1.7)$$

for all $x \in [0, 1]$ and $f \in C^{(n)}$.

Here we check that ψ_t is continuous and injective. By Lemma 1.2.9, ψ_t is continuous. To see that ψ_t is injective, choose $f_0 \in C^{(n)}$ so that $Tf_0 = t^{n+1}/(n+1)!$ because T is surjective. Putting $f = f_0$ in (1.7), we have

$$x = \sum_{\ell=0}^{n-1} \alpha_{t,\ell} f_0^{(\ell)}(0) + \beta_t f_0^{(n)}(\psi_t(x)).$$

Since the left hand side is injective in $x \in [0, 1]$, ψ_t must be injective.

Now the difference of (1.7) with $t = 1$ and (1.7) with $t = -1$ is

$$2(Tf)(0) = \sum_{\ell=0}^{n-1} (\alpha_{1,\ell} - \alpha_{-1,\ell}) f^{(\ell)}(0) + \beta_1 f^{(n)}(\psi_1(x)) - \beta_{-1} f^{(n)}(\psi_{-1}(x))$$

for all $x \in [0, 1]$ and $f \in C^{(n)}$. If $\gamma = -\beta_1/\beta_{-1}$, then the above equation implies that $\gamma(f^{(n)} \circ \psi_1) + (f^{(n)} \circ \psi_{-1})$ is constant on $[0, 1]$ for all $f \in C^{(n)}$. In other words, $\gamma(g \circ \psi_1) + (g \circ \psi_{-1})$ is constant for all $g \in C$. Hence Proposition 1.2.4 gives $\psi_1 = \psi_{-1}$, that is,

$$\psi(1, a_1, \dots, a_{n-1}, x) = \psi(-1, a_1, \dots, a_{n-1}, x) \quad (x \in [0, 1]).$$

If we fix $x \in [0, 1]$, then the set $\psi(\mathbb{T} \times \{a_1\} \times \dots \times \{a_{n-1}\} \times \{x\})$ is a singleton.

By a similar argument, we can show the following assertion for each $\ell = 1, \dots, n-1$: For fixed $a_0, \dots, a_{\ell-1}, a_{\ell+1}, \dots, a_{n-1} \in \mathbb{T}$ and $x \in [0, 1]$, the set

$$\psi(\{a_0\} \times \dots \times \{a_{\ell-1}\} \times \mathbb{T} \times \{a_{\ell+1}\} \times \dots \times \{a_{n-1}\} \times \{x\})$$

is a singleton. Thus we conclude that $\psi(\mathbb{T}^n \times \{x\})$ is a singleton. \square

Proof in case of $\mathbb{K} = \mathbb{C}$. Fix $a_1, \dots, a_{n-1} \in \mathbb{T}$ and $x \in [0, 1]$. Then the set

$$\mathbb{T} \times \{a_1\} \times \dots \times \{a_{n-1}\} \times \{x\}$$

is connected and compact. Since ψ is continuous, $\psi(\mathbb{T} \times \{a_1\} \times \dots \times \{a_{n-1}\} \times \{x\})$ is connected and compact in $[0, 1]$. Hence we can write $\psi(\mathbb{T} \times \{a_1\} \times \dots \times \{a_{n-1}\} \times \{x\}) = [s, t]$, where $s, t \in [0, 1]$ and $s \leq t$. To show that $s = t$, assume the converse; $s < t$. Then we easily find three distinct points $p, q, r \in [s, t]$ and a function $f_0 \in C^{(n)}$ such that $f_0(0) = f_0'(0) = \dots = f_0^{(n-1)}(0) = f_0^{(n)}(p) = f_0^{(n)}(q) = 0$ and $f_0^{(n)}(r) = 1$. Since $p, q, r \in \psi(\mathbb{T} \times \{a_1\} \times \dots \times \{a_{n-1}\} \times \{x\})$, there exist three distinct points $b, c, d \in \mathbb{T}$

such that $\psi(b, a_1, \dots, a_{n-1}, x) = p$, $\psi(c, a_1, \dots, a_{n-1}, x) = q$ and $\psi(d, a_1, \dots, a_{n-1}, x) = r$. Putting $f = f_0$ and $a_0 = b, c, d$ in (1.3), we have

$$b(Tf_0)(0) + \sum_{k=1}^{n-1} a_k(Tf_0)^{(k)}(0) + (Tf_0)^{(n)}(x) = 0, \quad (1.8)$$

$$c(Tf_0)(0) + \sum_{k=1}^{n-1} a_k(Tf_0)^{(k)}(0) + (Tf_0)^{(n)}(x) = 0, \quad (1.9)$$

$$d(Tf_0)(0) + \sum_{k=1}^{n-1} a_k(Tf_0)^{(k)}(0) + (Tf_0)^{(n)}(x) = v(d, a_1, \dots, a_{n-1}, x). \quad (1.10)$$

By (1.8) and (1.9), we have $(Tf_0)(0) = 0$ and $\sum_{k=1}^{n-1} a_k(Tf_0)^{(k)}(0) + (Tf_0)^{(n)}(x) = 0$, because $b \neq c$. Hence (1.10) becomes $0 = v(d, a_1, \dots, a_{n-1}, x)$, which is a contradiction because v is unimodular. Thus we obtain $s = t$, and $\psi(\mathbb{T} \times \{a_1\} \times \dots \times \{a_{n-1}\} \times \{x\})$ is a singleton $\{s\}$.

Repeat the above argument as in the last paragraph of Proof in case of $\mathbb{K} = \mathbb{R}$. Then we conclude that $\psi(\mathbb{T}^n \times \{x\})$ is a singleton. \square

Definition 1.2.11. By Lemma 1.2.10, $\psi(\mathbf{a}, x)$ does not depend on $\mathbf{a} \in \mathbb{T}^n$. Hence we can write

$$\psi(\mathbf{a}, x) = \varphi(x) \quad ((\mathbf{a}, x) \in \mathbb{T}^n \times [0, 1]).$$

Since ψ is a continuous mapping of $\mathbb{T}^n \times [0, 1]$ onto $[0, 1]$, φ is a continuous mapping of $[0, 1]$ onto $[0, 1]$.

Moreover, for any $(\mathbf{a}, x) \in \mathbb{T}^n \times [0, 1]$ and $f \in C^{(n)}$, (1.3) becomes

$$\sum_{k=0}^{n-1} a_k(Tf)^{(k)}(0) + (Tf)^{(n)}(x) = \sum_{\ell=0}^{n-1} u_\ell(\mathbf{a}, x)f^{(\ell)}(0) + v(\mathbf{a}, x)f^{(n)}(\varphi(x)).$$

Using (1.4) and (1.5), we remove u_ℓ and v as follows:

$$\begin{aligned} & \sum_{k=0}^{n-1} a_k(Tf)^{(k)}(0) + (Tf)^{(n)}(x) \\ &= \sum_{\ell=0}^{n-1} \frac{1}{\ell!} \left(\sum_{k=0}^{n-1} a_k(Tt^\ell)^{(k)}(0) + (Tt^\ell)^{(n)}(x) \right) f^{(\ell)}(0) \\ & \quad + \frac{1}{n!} \left(\sum_{k=0}^{n-1} a_k(Tt^n)^{(k)}(0) + (Tt^n)^{(n)}(x) \right) f^{(n)}(\varphi(x)) \\ &= \sum_{k=0}^{n-1} a_k \left(\sum_{\ell=0}^{n-1} \frac{(Tt^\ell)^{(k)}(0)}{\ell!} f^{(\ell)}(0) + \frac{(Tt^n)^{(k)}(0)}{n!} f^{(n)}(\varphi(x)) \right) \\ & \quad + \sum_{\ell=0}^{n-1} \frac{(Tt^\ell)^{(n)}(x)}{\ell!} f^{(\ell)}(0) + \frac{(Tt^n)^{(n)}(x)}{n!} f^{(n)}(\varphi(x)). \end{aligned}$$

Since this holds for all $\mathbf{a} = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{T}^n$, it follows that

$$(Tf)^{(k)}(0) = \sum_{\ell=0}^{n-1} \frac{(T\iota^\ell)^{(k)}(0)}{\ell!} f^{(\ell)}(0) + \frac{(T\iota^n)^{(k)}(0)}{n!} f^{(n)}(\varphi(x)), \quad (1.11)$$

$$(Tf)^{(n)}(x) = \sum_{\ell=0}^{n-1} \frac{(T\iota^\ell)^{(n)}(x)}{\ell!} f^{(\ell)}(0) + \frac{(T\iota^n)^{(n)}(x)}{n!} f^{(n)}(\varphi(x)). \quad (1.12)$$

Lemma 1.2.12. For each $k = 0, 1, \dots, n-1$, $(T\iota^n)^{(k)}(0) = 0$ and

$$(Tf)^{(k)}(0) = \sum_{\ell=0}^{n-1} \frac{(T\iota^\ell)^{(k)}(0)}{\ell!} f^{(\ell)}(0) \quad (f \in C^{(n)}) \quad (1.13)$$

Proof. Fix $k = 0, 1, \dots, n-1$. Putting $f = \iota^{n+1}$ in (1.11), we have

$$(T\iota^{n+1})^{(k)}(0) = (T\iota^n)^{(k)}(0) (n+1)\varphi(x) \quad (x \in [0, 1]).$$

Note that the left hand side is constant while φ maps $[0, 1]$ onto $[0, 1]$. We must have $(T\iota^n)^{(k)}(0) = 0$. Substituting this into (1.11), we obtain (1.13). \square

Definition 1.2.13. Define

$$\omega(x) = \frac{(T\iota^n)^{(n)}(x)}{n!} \quad (x \in [0, 1]).$$

Clearly, ω is a continuous function on $[0, 1]$. Moreover, for $f \in C^{(n)}$, (1.12) becomes

$$(Tf)^{(n)}(x) = \sum_{\ell=0}^{n-1} \frac{(T\iota^\ell)^{(n)}(x)}{\ell!} f^{(\ell)}(0) + \omega(x) f^{(n)}(\varphi(x)) \quad (x \in [0, 1]). \quad (1.14)$$

Lemma 1.2.14. ω is a unimodular continuous function on $[0, 1]$.

Proof. By Lemma 1.2.12 and Equation (1.5), we have

$$|(T\iota^n)^{(n)}(x)| = \left| \sum_{k=0}^{n-1} (T\iota^n)^{(k)}(0) + (T\iota^n)^{(n)}(x) \right| = |n!v(\mathbf{1}, x)| = n!$$

for all $x \in [0, 1]$. Hence $|\omega(x)| = 1$ for $x \in [0, 1]$. \square

Lemma 1.2.15. For each $k \in \{0, 1, \dots, n-1\}$, there exist a unique $m \in \{0, 1, \dots, n-1\}$ and a unique $\alpha \in \mathbb{C}$ such that $T\iota^m = \alpha\iota^k$ and $|\alpha| = m!/k!$.

Proof. Fix $k \in \{0, 1, \dots, n-1\}$. If $(T\iota^\ell)^{(k)}(0) = 0$ for all $\ell \in \{0, 1, \dots, n-1\}$, then (1.13) shows that $(Tf)^{(k)}(0) = 0$ for all $f \in C^{(n)}$, which is a contradiction if we choose f so that $Tf = \iota^k$ because T is surjective. Therefore there exists $m \in \{0, 1, \dots, n-1\}$ such that $(T\iota^m)^{(k)}(0) \neq 0$. By (1.4), we have

$$\begin{aligned} m! &= |m!u_m(\mathbf{a}, x)| = \left| \sum_{\ell=0}^{n-1} a_\ell (T\iota^m)^{(\ell)}(0) + (T\iota^m)^{(n)}(x) \right| \\ &\leq \sum_{\ell=0}^{n-1} |(T\iota^m)^{(\ell)}(0)| + |(T\iota^m)^{(n)}(x)| \leq \|T\iota^m\|_\sigma = \|\iota^m\|_\sigma = m! \end{aligned}$$

for all $(\mathbf{a}, x) \in \mathbb{T}^n \times [0, 1]$. Since the equality holds in the first inequality for all $\mathbf{a} = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{T}^n$ and since $(T\iota^m)^{(k)}(0) \neq 0$, we must have $(T\iota^m)^{(\ell)}(0) = 0$ for all $\ell \in \{0, 1, \dots, n-1\} \setminus \{k\}$ and $(T\iota^m)^{(n)}(x) = 0$ for all $x \in [0, 1]$. Moreover, we have $|(T\iota^m)^{(k)}(0)| = m!$. Put $\alpha = (T\iota^m)^{(k)}(0)/k!$. Then $|\alpha| = m!/k!$ and

$$(T\iota^m)(x) = \sum_{\ell=0}^{n-1} \frac{(T\iota^m)^{(\ell)}(0)}{\ell!} x^\ell + (T\iota^m)^{(n)}(x) = \frac{(T\iota^m)^{(k)}(0)}{k!} x^k = \alpha \iota^k(x) \quad (x \in [0, 1]).$$

For the uniqueness, assume $T\iota^{m'} = \alpha' \iota^k$, where $m' \in \{0, 1, \dots, n-1\}$, $\alpha' \in \mathbb{C}$ and $|\alpha'| = m!/k!$. Then $T(\iota^{m'}/\alpha') = \iota^k = T(\iota^{m'}/\alpha')$. Since T is injective, $\iota^{m'}/\alpha' = \iota^{m'}/\alpha'$. This yields $\alpha = \alpha'$ and $m = m'$. \square

Definition 1.2.16. With each $k \in \{0, 1, \dots, n-1\}$, we associate $m \in \{0, 1, \dots, n-1\}$ and $\alpha \in \mathbb{C}$ such that $T\iota^m = \alpha \iota^k$ and $|\alpha| = m!/k!$, as in Lemma 1.2.15. Since m and α depend on k , we write

$$m = \tau(k) \quad \text{and} \quad \alpha = \frac{m!}{k!} \lambda_k.$$

Then τ is a mapping of $\{0, 1, \dots, n-1\}$ into itself, and we have

$$T\iota^{\tau(k)} = \frac{\tau(k)!}{k!} \lambda_k \iota^k \quad \text{and} \quad |\lambda_k| = 1. \quad (1.15)$$

Lemma 1.2.17. $\{\tau(0), \tau(1), \dots, \tau(n-1)\}$ is a permutation of $\{0, 1, \dots, n-1\}$.

Proof. Since τ is a mapping of $\{0, 1, \dots, n-1\}$ into itself, it suffices to show that τ is injective. Suppose that $\tau(k) = \tau(k')$, where $k, k' \in \{0, 1, \dots, n-1\}$. Then

$$\frac{\tau(k)!}{k!} \lambda_k \iota^k = T\iota^{\tau(k)} = T\iota^{\tau(k')} = \frac{\tau(k')!}{k'!} \lambda_{k'} \iota^{k'}.$$

This implies $k = k'$. Hence τ is injective. \square

Lemma 1.2.18. $(Tf)^{(n)}(x) = \omega(x) f^{(n)}(\varphi(x))$ for $x \in [0, 1]$ and $f \in C^{(n)}[0, 1]$.

Proof. By Lemma 1.2.17, for any $\ell \in \{0, 1, \dots, n-1\}$, there is $k \in \{0, 1, \dots, n-1\}$ such that $\tau(k) = \ell$. Then

$$(T\iota^\ell)^{(n)}(x) = (T\iota^{\tau(k)})^{(n)}(x) = \left(\frac{\tau(k)!}{k!} \lambda_k \iota^k \right)^{(n)}(x) = \frac{\tau(k)! \lambda_k}{k!} (\iota^k)^{(n)}(x) = 0 \quad (x \in [0, 1]),$$

because $k < n$. Hence the desired equation follows from (1.14). \square

Lemma 1.2.19. φ is a homeomorphism of $[0, 1]$ onto itself.

Proof. Since φ is a continuous mapping of $[0, 1]$ onto itself, it suffices to show that φ is injective. Choose $f_0 \in C^{(n)}$ so that $Tf_0 = \iota^{n+1}/(n+1)!$ because T is surjective. Using Lemmas 1.2.14 and 1.2.18, we have

$$|f_0^{(n)}(\varphi(x))| = |\omega(x) f_0^{(n)}(\varphi(x))| = |(Tf_0)^{(n)}(x)| = |\iota(x)| = |x| = x \quad (x \in [0, 1]).$$

Hence, if $\varphi(x') = \varphi(x'')$, then $x' = |f_0^{(n)}(\varphi(x'))| = |f_0^{(n)}(\varphi(x''))| = x''$. Therefore φ is injective. \square

Lemma 1.2.20. T has the form (1.1).

Proof. Let $f \in C^{(n)}$. By Lemma 1.2.17, (1.13) is rewritten as

$$\begin{aligned} (Tf)^{(k)}(0) &= \sum_{\ell=0}^{n-1} \frac{(T_l \tau^{(\ell)})^{(k)}(0)}{\tau^{(\ell)!}} f^{(\tau^{(\ell)})}(0) = \sum_{\ell=0}^{n-1} \frac{1}{\tau^{(\ell)!}} \left(\frac{\tau^{(\ell)!}}{\ell!} \lambda_{\ell} \ell^{\ell} \right)^{(k)}(0) f^{(\tau^{(\ell)})}(0) \\ &= \sum_{\ell=0}^{n-1} \frac{\lambda_{\ell} \ell^{\ell} (k)^{(k)}(0)}{\ell!} f^{(\tau^{(\ell)})}(0) = \lambda_k f^{(\tau(k))}(0) \end{aligned}$$

for $k = 0, 1, \dots, n-1$. This equation and Lemma 1.2.18 yield

$$\begin{aligned} (Tf)(x) &= \sum_{k=0}^{n-1} \frac{(Tf)^{(k)}(0)}{k!} x^k + \left(S^n (Tf)^{(n)} \right) (x) \\ &= \sum_{k=0}^{n-1} \frac{\lambda_k f^{(\tau(k))}(0)}{k!} x^k + \left(S^n (\omega(f^{(n)} \circ \varphi)) \right) (x) \end{aligned} \quad (x \in [0, 1]).$$

□

Noting Lemmas 1.2.14, 1.2.17, 1.2.19, 1.2.20 and Equation (1.15), we establish the “only if” part on $(C^{(n)}[0, 1], \|\cdot\|_{\sigma})$ of Theorem 1.2.1.

1.2.4 Proof of Theorem 1.2.1; the “only if” part on $(C^{(n)}[0, 1], \|\cdot\|_m)$

In this section, we deal with the space $(C^{(n)}[0, 1], \|\cdot\|_m)$. We first see that the space $(C^{(n)}[0, 1], \|\cdot\|_m)$ is linearly isometric to $C(X)$ for some compact Hausdorff space X .

Definition 1.2.21. We put $X_1 = [0, 1] \cup \{p_0, p_1, \dots, p_{n-1}\}$, where p_0, p_1, \dots, p_{n-1} are distinct points in $\mathbb{R} \setminus [0, 1]$. For each $f \in C^{(n)}[0, 1]$, we define a continuous function \tilde{f} on X_1 by

$$\tilde{f}(y) = \begin{cases} f^{(k)}(0) & \text{if } y = p_k \quad (k = 0, 1, \dots, n-1), \\ f^{(n)}(y) & \text{if } y \in [0, 1]. \end{cases}$$

Lemma 1.2.22. The mapping $P_1 : f \mapsto \tilde{f}$ is a linear isometry of $(C^{(n)}[0, 1], \|\cdot\|_m)$ onto $C(X_1)$.

Proof. It is clear that P_1 is linear. For any $f \in C^{(n)}[0, 1]$, we have

$$\begin{aligned} \|f\|_m &= \max\{|f(0)|, |f'(0)|, \dots, |f^{(n-1)}(0)|, \|f^{(n)}\|_{\infty}\} \\ &= \max\{|\tilde{f}(p_0)|, |\tilde{f}(p_1)|, \dots, |\tilde{f}(p_{n-1})|, \|\tilde{f}\|_{\infty}\} \\ &= \sup\{|\tilde{f}(y)| : y \in X_1\}. \end{aligned}$$

Hence P_1 is an isometry of $(C^{(n)}[0, 1], \|\cdot\|_m)$ into $C(X_1)$.

To see that P_1 is surjective, pick $g \in C(X_1)$ arbitrarily. We define $f \in C^{(n)}[0, 1]$ as

$$f(x) = \sum_{k=0}^{n-1} \frac{g(p_k)}{k!} x^k + (S^n g)(x) \quad (x \in [0, 1]).$$

Using the formulae $(Sh)(0) = 0$ and $(Sh)' = h$ for all $h \in C([0, 1])$, we have the following equations:

$$f(0) = g(p_0), f'(0) = g(p_1), \dots, f^{(n-1)}(0) = g(p_{n-1}), f^{(n)}(x) = g(x) \quad (x \in [0, 1]).$$

Hence $P_1 f = \tilde{f} = g$. Thus P_1 is surjective. \square

Proof of the "only if" part on $(C^{(n)}[0, 1], \|\cdot\|_m)$ of Theorem 1.2.1. Let T be a surjective linear isometry on $(C^{(n)}[0, 1], \|\cdot\|_m)$. We associate a linear operator \tilde{T} on $C(X_1)$ such as $\tilde{T} = P_1 T P_1^{-1}$, where P_1 is the isometry from $(C^{(n)}[0, 1], \|\cdot\|_m)$ onto $C(X_1)$ in Lemma 1.2.22. Then \tilde{T} is a surjective linear isometry on $C(X_1)$. By the Banach-Stone theorem, there exist a homeomorphism ρ of X_1 onto X_1 and a unimodular continuous function u on X_1 such that

$$(\tilde{T}h)(y) = u(y)h(\rho(y)) \quad (y \in X_1)$$

for all $h \in C(X_1)$. Then the restriction φ of ρ to $[0, 1]$ becomes a homeomorphism of $[0, 1]$ onto $[0, 1]$, and $\rho(\{p_0, p_1, \dots, p_{n-1}\}) = \{p_0, p_1, \dots, p_{n-1}\}$. For each $k = 0, 1, \dots, n-1$, let $\tau(k)$ be the index ℓ such that $\rho(p_k) = p_\ell$. Then $\{\tau(0), \tau(1), \dots, \tau(n-1)\}$ is a permutation of $\{0, 1, \dots, n-1\}$. Let ω be the restriction of u to $[0, 1]$. Then ω is a unimodular continuous function on $[0, 1]$. For each $k = 0, 1, \dots, n-1$, put $\lambda_k = u(p_k)$. Then each λ_k is a unimodular constant.

To show (1.1), let $f \in C^{(n)}[0, 1]$. Noting that $\tilde{T}f = P_1 T f = \tilde{T}P_1 f = \tilde{T}\tilde{f}$, we have

$$\begin{aligned} (Tf)^{(k)}(0) &= \tilde{T}f(p_k) = (\tilde{T}\tilde{f})(p_k) = u(p_k)\tilde{f}(\rho(p_k)) = \lambda_k \tilde{f}(p_{\tau(k)}) = \lambda_k f^{(\tau(k))}(0), \\ (Tf)^{(n)}(x) &= \tilde{T}f(x) = (\tilde{T}\tilde{f})(x) = u(x)\tilde{f}(\rho(x)) = \omega(x)\tilde{f}(\varphi(x)) = \omega(x)f^{(n)}(\varphi(x)), \end{aligned}$$

for $k = 0, 1, \dots, n-1$ and $x \in [0, 1]$. Therefore, we have

$$\begin{aligned} (Tf)(x) &= \sum_{k=0}^{n-1} \frac{(Tf)^{(k)}(0)}{k!} x^k + \left(S^n(Tf)^{(n)} \right) (x) \\ &= \sum_{k=0}^{n-1} \frac{\lambda_k f^{(\tau(k))}(0)}{k!} x^k + \left(S^n(\omega(f^{(n)} \circ \varphi)) \right) (x) \end{aligned} \quad (x \in [0, 1]).$$

\square

1.2.5 Proof of Theorem 1.2.2

We now turn our attention to Theorem 1.2.2.

Proof of Theorem 1.2.2. Suppose that T is a finite codimensional linear isometry on the space $(C^{(n)}[0, 1], \|\cdot\|_m)$. Let X_1 and P_1 be as in Definition 1.2.21 and Lemma 1.2.22, respectively. Define a linear operator \tilde{T} as $\tilde{T} = P_1 T P_1^{-1}$. Then \tilde{T} is a finite codimensional linear isometry on $C(X_1)$. To complete the proof, it suffices to show that \tilde{T} is surjective.

For this purpose, we will appeal to Theorem A. Suppose that ψ is a continuous mapping of X_1 onto itself which is not injective. Since ψ is continuous and surjective, we see that $\psi([0, 1]) = [0, 1]$. Since ψ is not injective, there are $x_0, y_0 \in [0, 1]$ such that $x_0 < y_0$ and $\psi(x_0) = \psi(y_0)$. Using the intermediate value theorem, we can find infinitely many pairs (x_i, y_i) of points in the interval $[x_0, y_0]$ such that $x_i \neq y_i$ and $\psi(x_i) = \psi(y_i)$. Hence the set $\{(x, y) \in X_1 \times X_1 : x \neq y, \psi(x) = \psi(y)\}$ is infinite. Thus Theorem A shows that \tilde{T} is surjective. \square

1.3 The space of Lipschitz continuous functions

We denote by $\text{Lip}[0, 1]$ the linear space of all \mathbb{K} -valued Lipschitz continuous functions on $[0, 1]$. For each $f \in \text{Lip}[0, 1]$, f has the derivative $f'(x)$ for almost all $x \in [0, 1]$. Then the set $\{f' : f \in \text{Lip}[0, 1]\}$ coincides with $L^\infty[0, 1]$; the Banach algebra of all \mathbb{K} -valued essentially bounded functions on $[0, 1]$ with the essential supremum norm $\|\cdot\|_{L^\infty}$. There exist several norms which make $\text{Lip}[0, 1]$ a Banach space; for example,

$$\begin{aligned} \|f\|_\Sigma &= \|f\|_\infty + \|f'\|_{L^\infty}, \\ \|f\|_M &= \max\{\|f\|_\infty, \|f'\|_{L^\infty}\}, \\ \|f\|_\sigma &= |f(0)| + \|f'\|_{L^\infty}, \\ \|f\|_m &= \max\{|f(0)|, \|f'\|_{L^\infty}\}, \end{aligned} \quad (f \in \text{Lip}[0, 1]).$$

These norms are equivalent. In particular, $(\text{Lip}[0, 1], \|\cdot\|_\Sigma)$ is a unital semisimple commutative Banach algebra.

From [19, 20, 39], we know that every surjective linear isometry T on $(\text{Lip}[0, 1], \|\cdot\|_\Sigma)$ or $(\text{Lip}[0, 1], \|\cdot\|_M)$ has the canonical form; $Tf = \omega(f \circ \varphi)$. In this paper, we characterize the surjective linear isometries on $(\text{Lip}[0, 1], \|\cdot\|_\sigma)$ and $(\text{Lip}[0, 1], \|\cdot\|_m)$, as follows:

Theorem 1.3.1. *Let T be a linear operator on $(\text{Lip}[0, 1], \|\cdot\|_\sigma)$ or $(\text{Lip}[0, 1], \|\cdot\|_m)$. Then T is a surjective isometry if and only if there exist an algebra automorphism Φ of $L^\infty[0, 1]$, a unimodular function $\omega \in L^\infty[0, 1]$ and a unimodular constant λ such that*

$$(Tf)(x) = \lambda f(0) + \int_0^x \omega(t)(\Phi f')(t) dt \quad (1.16)$$

for all $x \in [0, 1]$ and $f \in \text{Lip}[0, 1]$.

It is known that any algebra automorphism Φ of $L^\infty[0, 1]$ has the form; $\Phi h = h \circ \varphi$ for all $h \in L^\infty[0, 1]$, where φ is a function in $L^\infty[0, 1]$ such that $\varphi(x) \in [0, 1]$ a.e. $x \in [0, 1]$. This fact is obtained by the way of the proof of [14, Theorem 1]. Indeed, φ is given by $\varphi = \Phi \iota$. Nevertheless, we easily see that (1.16) is not of the canonical form.

We also prove the following theorem:

Theorem 1.3.2. *If T is a finite codimensional linear isometry on $(\text{Lip}[0, 1], \|\cdot\|_m)$, then T is surjective.*

We will prove Theorem 1.3.1 in Sections 1.3.2–1.3.4, Theorem 1.3.2 in Section 1.3.5.

1.3.1 Preliminaries

If we want to indicate the scalar field \mathbb{K} , we write $L^\infty_{\mathbb{K}}[0, 1]$ instead of $L^\infty[0, 1]$. Let \mathfrak{M} be the maximal ideal space of $L^\infty_{\mathbb{C}}[0, 1]$. Then \mathfrak{M} is a compact Hausdorff space. We know that \mathfrak{M} is totally disconnected ([5, Theorem 1.3.4]). This means that every component of \mathfrak{M} consists of one point. We also know that \mathfrak{M} has no isolated points ([43, Exercise 11.18]). It is easy to see that a totally disconnected compact Hausdorff space is extremally disconnected, that is, if U is open, so is the closure \bar{U} .

We write $C_{\mathbb{K}}(\mathfrak{M})$ or simply $C(\mathfrak{M})$ for the Banach algebra of all \mathbb{K} -valued continuous functions on \mathfrak{M} with the supremum norm $\|\cdot\|_{\mathfrak{M}}$. For any $g \in L_{\mathbb{C}}^{\infty}[0, 1]$, \widehat{g} denotes the Gelfand representation of g . The Gelfand-Naimark theorem says that the Gelfand transformation $\Gamma : g \mapsto \widehat{g}$ is an algebra $*$ -isomorphism of $L_{\mathbb{C}}^{\infty}[0, 1]$ onto $C_{\mathbb{C}}(\mathfrak{M})$ and $\|g\|_{L^{\infty}} = \|\widehat{g}\|_{\mathfrak{M}}$. Also Γ maps $L_{\mathbb{R}}^{\infty}[0, 1]$ onto $C_{\mathbb{R}}(\mathfrak{M})$, and $\{\widehat{f}' : f \in \text{Lip}\} = C(\mathfrak{M})$.

1.3.2 Proof of Theorem 1.3.1; the “if” part

Proof of the “if” part of Theorem 1.3.1. Suppose T has the form (1.16). Put $\Psi = \Gamma\Phi\Gamma^{-1}$. Then Ψ is an algebra automorphism of $C(\mathfrak{M})$. By [21, Theorem 3.4.3], Ψ has the form $\Psi h = h \circ \varphi$ for some homeomorphism φ of \mathfrak{M} onto itself. Hence Ψ is a surjective linear isometry on $C(\mathfrak{M})$, and so Φ is a surjective isometry on $L^{\infty}[0, 1]$. Also, for $f \in \text{Lip}[0, 1]$, we have $(Tf)(0) = \lambda f(0)$ and $(Tf)' = \omega(\Phi f')$. Therefore

$$\begin{aligned} \|Tf\|_{\sigma} &= |(Tf)(0)| + \|(Tf)'\|_{L^{\infty}} \\ &= |\lambda f(0)| + \|\omega(\Phi f')\|_{L^{\infty}} = |f(0)| + \|\Phi f'\|_{L^{\infty}} = |f(0)| + \|f'\|_{L^{\infty}} = \|f\|_{\sigma}. \end{aligned}$$

Similarly, $\|Tf\|_m = \|f\|_m$. Hence T is an isometry.

Next we will see that T is surjective. For any $g \in \text{Lip}[0, 1]$, we define $f \in \text{Lip}[0, 1]$ by

$$f(x) = \bar{\lambda}g(0) + \int_0^x (\Phi^{-1}(\bar{\omega}g'))(t) dt \quad (x \in [0, 1]).$$

Then $f(0) = \bar{\lambda}g(0)$ and $f' = \Phi^{-1}(\bar{\omega}g')$, and so (1.16) implies that

$$(Tf)(x) = \lambda \bar{\lambda}g(0) + \int_0^x \omega(t)(\Phi\Phi^{-1}(\bar{\omega}g'))(t) dt = g(0) + \int_0^x g'(t) dt = g(x) \quad (x \in [0, 1]).$$

□

1.3.3 Proof of Theorem 1.3.1; the “only if” part on $(\text{Lip}[0, 1], \|\cdot\|_{\sigma})$

We divide the proof of the “only if” part into two subsections. We deal with only the space $(\text{Lip}[0, 1], \|\cdot\|_{\sigma})$ in this subsection, and the space $(\text{Lip}[0, 1], \|\cdot\|_m)$ in the next subsection.

For simplicity, we write Lip and L^{∞} for the Banach space $(\text{Lip}[0, 1], \|\cdot\|_{\sigma})$ and the Banach algebra $(L^{\infty}[0, 1], \|\cdot\|_{L^{\infty}})$, respectively.

Definition 1.3.3. For each $(a, c, m) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$, define a functional $\Lambda_{(a,c,m)}$ on Lip by

$$\Lambda_{(a,c,m)}(f) = af(0) + c\widehat{f}'(m) \quad (f \in \text{Lip}).$$

It is clear that $\Lambda_{(a,c,m)} \in \text{ball}(\text{Lip})^*$.

Lemma 1.3.4. *Let $\xi \in (\text{Lip})^*$. Then ξ is an extreme point of $\text{ball}(\text{Lip})^*$ if and only if there exists $(a, c, m) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$ such that $\xi = \Lambda_{(a,c,m)}$.*

Proof. Suppose that the product spaces $\mathbb{K} \times L^{\infty}$ and $\mathbb{K} \times C(\mathfrak{M})^*$ have the norms

$$\begin{aligned} \|(b, g)\| &= |b| + \|g\|_{L^{\infty}} \quad ((b, g) \in \mathbb{K} \times L^{\infty}), \\ \|(a, \eta)\| &= \max\{|a|, \|\eta\|\} \quad ((a, \eta) \in \mathbb{K} \times C(\mathfrak{M})^*), \end{aligned}$$

respectively. Then the next operator Q is a linear isometry of $\mathbb{K} \times C(\mathfrak{M})^*$ onto $(\mathbb{K} \times L^\infty)^*$:

$$(Q(a, \eta))(b, g) = ab + \eta(\widehat{g}) \quad ((a, \eta) \in \mathbb{K} \times C(\mathfrak{M})^*, (b, g) \in \mathbb{K} \times L^\infty).$$

Define a linear isometry P of Lip onto $\mathbb{K} \times L^\infty$ by

$$Pf = (f(0), f') \quad (f \in \text{Lip}).$$

Then P^*Q is a linear isometry of $\mathbb{K} \times C(\mathfrak{M})^*$ onto $(\text{Lip})^*$. Hence ξ is an extreme point of $\text{ball}(\text{Lip})^*$ if and only if there exists an extreme point (a, η) of $\text{ball}(\mathbb{K} \times C(\mathfrak{M})^*)$ such that $\xi = P^*Q(a, \eta)$. By Proposition 1.2.3 and [8, Theorem V.8.4], (a, η) is an extreme point of $\text{ball}(\mathbb{K} \times C(\mathfrak{M})^*)$ if and only if $a \in \mathbb{T}$ and there exist $c \in \mathbb{T}$ and $m \in \mathfrak{M}$ such that $\eta(g) = ce_m(g) = cg(m)$ for $g \in C(\mathfrak{M})$. Hence ξ is an extreme point of $\text{ball}(\text{Lip})^*$ if and only if there exists $(a, c, m) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$ such that $\xi = P^*Q(a, ce_m)$. Thus the conclusion follows from

$$\begin{aligned} P^*(Q(a, ce_m))(f) &= (Q(a, ce_m))(Pf) \\ &= (Q(a, ce_m))(f(0), f') = af(0) + c\widehat{f'}(m) = \Lambda_{(a, c, m)}(f) \end{aligned}$$

for $f \in \text{Lip}$. □

Let us start the proof of the “only if” part on Lip of Theorem 1.3.1. For this purpose, let T be a surjective linear isometry on Lip . We complete the proof combining several lemmas.

Lemma 1.3.5. *For any $(a, c, m) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$, there exists a unique $(b, d, n) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$ such that $T^*\Lambda_{(a, c, m)} = \Lambda_{(b, d, n)}$.*

Proof. Let $(a, c, m) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$. Since T^* is a surjective linear isometry on $(\text{Lip})^*$, Lemma 1.3.4 shows the existence of $(b, d, n) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$ such that $T^*\Lambda_{(a, c, m)} = \Lambda_{(b, d, n)}$.

For the uniqueness of (b, d, n) , suppose $T^*\Lambda_{(a, c, m)} = \Lambda_{(b', d', n')}$ for some $(b', d', n') \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$. Then $\Lambda_{(b, d, n)} = \Lambda_{(b', d', n')}$, that is,

$$bf(0) + d\widehat{f'}(n) = b'f(0) + d'\widehat{f'}(n') \quad (f \in \text{Lip}). \quad (1.17)$$

Substituting 1 and ι for f in (1.17), we get $b = b'$ and $d = d'$, respectively. Hence (1.17) shows $\widehat{f'}(n) = \widehat{f'}(n')$ for all $f \in \text{Lip}$. In other words, $h(n) = h(n')$ for all $h \in C(\mathfrak{M})$. This implies $n = n'$. □

Definition 1.3.6. By Lemma 1.3.5, for each $(a, m) \in \mathbb{T} \times \mathfrak{M}$, there exists a unique $(b, d, n) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$ such that $T^*\Lambda_{(a, 1, m)} = \Lambda_{(b, d, n)}$. Since b, d and n depend on (a, m) , we write

$$b = u(a, m), \quad d = v(a, m) \quad \text{and} \quad n = \psi(a, m).$$

Thus u and v are unimodular functions on $\mathbb{T} \times \mathfrak{M}$ and ψ is a mapping of $\mathbb{T} \times \mathfrak{M}$ into \mathfrak{M} .

Moreover, for any $f \in \text{Lip}$, we have

$$\Lambda_{(a, 1, m)}(Tf) = (T^*\Lambda_{(a, 1, m)})(f) = \Lambda_{(b, d, n)}(f) = \Lambda_{(u(a, m), v(a, m), \psi(a, m))}(f),$$

and so

$$a(Tf)(0) + \widehat{(Tf)'}(m) = u(a, m)f(0) + v(a, m)\widehat{f'}(\psi(a, m)). \quad (1.18)$$

Substituting 1 and ι for f , we have

$$a(T1)(0) + \widehat{(T1)'}(m) = u(a, m), \quad (1.19)$$

$$a(T\iota)(0) + \widehat{(T\iota)'}(m) = v(a, m). \quad (1.20)$$

Here we note that the equations (1.18)–(1.20) hold for all $(a, m) \in \mathbb{T} \times \mathfrak{M}$.

Lemma 1.3.7. ψ is a continuous mapping of $\mathbb{T} \times \mathfrak{M}$ onto \mathfrak{M} .

Proof. By (1.19) and (1.20), we see that u and v are continuous on $\mathbb{T} \times \mathfrak{M}$. Since v is unimodular, (1.18) implies that $\widehat{f}' \circ \psi$ is continuous on $\mathbb{T} \times \mathfrak{M}$ for all $f \in \text{Lip}$. In other words, $h \circ \psi$ is continuous on $\mathbb{T} \times \mathfrak{M}$ for all $h \in C(\mathfrak{M})$. To see that $\psi : \mathbb{T} \times \mathfrak{M} \rightarrow \mathfrak{M}$ is continuous, pick $(a_0, m_0) \in \mathbb{T} \times \mathfrak{M}$, and let V be an open neighborhood of $\psi(a_0, m_0)$ in \mathfrak{M} . By Urysohn's lemma, there exists $h_0 \in C(\mathfrak{M})$ such that $h_0(\psi(a_0, m_0)) = 1$ and $h_0(n) = 0$ for all $n \in \mathfrak{M} \setminus V$. Put $U = \{(a, m) \in \mathbb{T} \times \mathfrak{M} : |(h_0 \circ \psi)(a, m)| > 0\}$. Since $h_0 \circ \psi$ is continuous, U is an open neighborhood of (a_0, m_0) . Moreover we can easily see that $\psi(U) \subset V$. Thus ψ is continuous.

To see that ψ is surjective, let $n \in \mathfrak{M}$. Since T^* is a surjective linear isometry on $(\text{Lip})^*$, Lemma 1.3.4 gives $(a, c, m) \in \mathbb{T} \times \mathbb{T} \times \mathfrak{M}$ such that $T^* \Lambda_{(a,c,m)} = \Lambda_{(1,1,n)}$. Then

$$\begin{aligned} (T^* \Lambda_{(\bar{c}a, 1, m)})(f) &= \Lambda_{(\bar{c}a, 1, m)}(Tf) = \bar{c}a(Tf)(0) + \widehat{(Tf)'}(m) \\ &= \bar{c}(a(Tf)(0) + \widehat{(Tf)'}(m)) = \bar{c} \Lambda_{(a,c,m)}(Tf) = \bar{c}(T^* \Lambda_{(a,c,m)})(f) \\ &= \bar{c} \Lambda_{(1,1,n)}(f) = \bar{c}(f(0) + \widehat{f}'(n)) = \bar{c}f(0) + \bar{c}\widehat{f}'(n) = \Lambda_{(\bar{c}, \bar{c}, n)}(f) \end{aligned}$$

for $f \in \text{Lip}$. By the definition of ψ , $\psi(\bar{c}a, m) = n$. Hence ψ is surjective. \square

Lemma 1.3.8. For any fixed $m \in \mathfrak{M}$, $\psi(\mathbb{T} \times \{m\})$ is a singleton.

Proof in case of $\mathbb{K} = \mathbb{R}$. For $t \in \mathbb{T} = \{1, -1\}$, put $\psi_t(m) = \psi(t, m)$ for all $m \in \mathfrak{M}$. The difference of (1.19) with $a = 1$ and (1.19) with $a = -1$ is $2(T1)(0) = u(1, m) - u(-1, m)$. While the difference of (1.18) with $a = 1$ and (1.18) with $a = -1$ becomes

$$\begin{aligned} 2(Tf)(0) &= (u(1, m) - u(-1, m))f(0) \\ &\quad + v(1, m)\widehat{f}'(\psi(1, m)) - v(-1, m)\widehat{f}'(\psi(-1, m)) \\ &= 2(T1)(0)f(0) + v(1, m)\widehat{f}'(\psi_1(m)) - v(-1, m)\widehat{f}'(\psi_{-1}(m)) \end{aligned} \quad (1.21)$$

for $m \in \mathfrak{M}$ and $f \in \text{Lip}$.

Assume that $\psi_1(m_0) \neq \psi_{-1}(m_0)$ for some $m_0 \in \mathfrak{M}$. Then we find disjoint open sets V_1 and V_{-1} in \mathfrak{M} such that $\psi_1(m_0) \in V_1$ and $\psi_{-1}(m_0) \in V_{-1}$. Since \mathfrak{M} has no isolated points, there exists $n_1 \in V_1 \setminus \{\psi_1(m_0)\}$. Since $\psi : \mathbb{T} \times \mathfrak{M} \rightarrow \mathfrak{M}$ is surjective, there exists $(t_1, m_1) \in \mathbb{T} \times \mathfrak{M}$ such that $\psi(t_1, m_1) = n_1$. Then we have $\psi_{t_1}(m_1) \neq \psi_1(m_0)$ because $n_1 \neq \psi_1(m_0)$. We also have $\psi_{t_1}(m_1) \neq \psi_{-1}(m_0)$ because $n_1 \notin V_{-1}$.

Here we consider the case when $\psi_{-t_1}(m_1) \neq \psi_{-1}(m_0)$. In this case, we can choose $f_0 \in \text{Lip}$ so that $\widehat{f}'_0(\psi_{-1}(m_0)) = 1$ and $\widehat{f}'_0(\psi_1(m_0)) = \widehat{f}'_0(\psi_{t_1}(m_1)) = \widehat{f}'_0(\psi_{-t_1}(m_1)) = 0$, because of $\{\widehat{f}' : f \in \text{Lip}\} = C(\mathfrak{M})$ and Urysohn's lemma. Then we have $\widehat{f}'_0(\psi_1(m_1)) = \widehat{f}'_0(\psi_{-1}(m_1)) = 0$. Therefore, if we put $f = f_0$ and $m = m_0, m_1$ in (1.21), then we get

$$2(Tf_0)(0) = 2(T1)(0)f(0) - v(-1, m_0) \quad \text{and} \quad 2(Tf_0)(0) = 2(T1)(0)f(0).$$

Hence $v(-1, m_0) = 0$. This is a contradiction because v is unimodular.

On the other hand, if $\psi_{-t_1}(m_1) = \psi_{-1}(m_0)$, then we can choose $f_0 \in \text{Lip}$ so that $\widehat{f}'_0(\psi_1(m_0)) = 1$ and $\widehat{f}'_0(\psi_{-1}(m_0)) = \widehat{f}'_0(\psi_{t_1}(m_1)) = \widehat{f}'(\psi_{-t_1}(m_1)) = 0$. A similar argument shows that $v(1, m_0) = 0$, a contradiction.

In any case, we reach a contradiction. Hence $\psi_1(m) = \psi_{-1}(m)$, that is, $\psi(1, m) = \psi(-1, m)$ for all $m \in \mathfrak{M}$. If we fix $m \in \mathfrak{M}$, then the set $\psi(\mathbb{T} \times \{m\})$ is a singleton. \square

Proof in case of $\mathbb{K} = \mathbb{C}$. Fix $m \in \mathfrak{M}$. Then $\mathbb{T} \times \{m\}$ is connected. Since ψ is continuous, $\psi(\mathbb{T} \times \{m\})$ is connected in \mathfrak{M} . Since \mathfrak{M} is totally disconnected, $\psi(\mathbb{T} \times \{m\})$ is a singleton. \square

Definition 1.3.9. By Lemma 1.3.8, $\psi(a, m)$ does not depend on $a \in \mathbb{T}$. Hence we can write

$$\psi(a, m) = \varphi(m) \quad ((a, m) \in \mathbb{T} \times \mathfrak{M}).$$

Since ψ is a continuous mapping of $\mathbb{T} \times \mathfrak{M}$ onto \mathfrak{M} , φ is a continuous mapping of \mathfrak{M} onto itself.

Moreover, for any $(a, m) \in \mathbb{T} \times \mathfrak{M}$ and $f \in \text{Lip}$, (1.18) is written as

$$a(Tf)(0) + \widehat{(Tf)'}(m) = u(a, m)f(0) + v(a, m)\widehat{f}'(\varphi(m)).$$

Use (1.19) and (1.20) to delete u and v in the equation above. The result is

$$\begin{aligned} a(Tf)(0) + \widehat{(Tf)'}(m) \\ = a \left((T1)(0)f(0) + (T\iota)(0)\widehat{f}'(\varphi(m)) \right) + \left(\widehat{(T1)'}(m)f(0) + \widehat{(T\iota)'}(m)\widehat{f}'(\varphi(m)) \right). \end{aligned}$$

Since this holds for all $a \in \mathbb{T}$, it follows that

$$(Tf)(0) = (T1)(0)f(0) + (T\iota)(0)\widehat{f}'(\varphi(m)), \quad (1.22)$$

$$\widehat{(Tf)'}(m) = \widehat{(T1)'}(m)f(0) + \widehat{(T\iota)'}(m)\widehat{f}'(\varphi(m)). \quad (1.23)$$

Definition 1.3.10. Define a constant λ and a function $\omega \in L^\infty$ by

$$\lambda = (T1)(0) \quad \text{and} \quad \omega = (T\iota)'.$$

Lemma 1.3.11. (a) $\widehat{\omega}(m) = v(1, m)$ for all $m \in \mathfrak{M}$.

(b) $(Tf)(0) = \lambda f(0)$ for all $f \in \text{Lip}$.

Proof. Equation (1.22) says that $(T\iota)(0)(\widehat{f}' \circ \varphi)$ is constant on \mathfrak{M} for all $f \in \text{Lip}$. In other words, $(T\iota)(0)(h \circ \varphi)$ is constant for all $h \in C(\mathfrak{M})$. Since φ is surjective, we must have $(T\iota)(0) = 0$. Thus (a) and (b) follow from (1.20) and (1.22), respectively. \square

Lemma 1.3.12. ω is unimodular.

Proof. By Lemma 1.3.11(a), $|\widehat{\omega}(m)| = |v(1, m)| = 1$ for all $m \in \mathfrak{M}$. This implies that $\widehat{\omega\bar{\omega}}$ is a unit of $C(\mathfrak{M})$. Since the transformation $\Gamma : g \mapsto \widehat{g}$ is a $*$ -isomorphism of L^∞ onto $C(\mathfrak{M})$, $\omega\bar{\omega}$ is a unit of L^∞ . Hence we conclude that ω is unimodular. \square

Lemma 1.3.13. (a) $|\lambda| = 1$.

(b) $\widehat{(Tf)'}(m) = \widehat{\omega}(m)\widehat{f}'(\varphi(m))$ for all $m \in \mathfrak{M}$ and $f \in \text{Lip}$.

Proof. We first note that $\lambda \neq 0$. Indeed, if $\lambda = 0$, Lemma 1.3.11(b) yields $(Tf)(0) = 0$ for all $f \in \text{Lip}$, which is a contradiction because T is surjective. Now, we use (1.19) to get

$$\begin{aligned} 1 = |u(a, m)| &= |a(T1)(0) + \widehat{(T1)'}(m)| = |a\lambda + \widehat{(T1)'}(m)| \leq |\lambda| + |\widehat{(T1)'}(m)| \\ &\leq |\lambda| + \|\widehat{(T1)'}\|_{\mathfrak{M}} = |(T1)(0)| + \|(T1)'\|_{L^\infty} = \|T1\|_\sigma = \|1\|_\sigma = 1 \end{aligned}$$

for all $(a, m) \in \mathbb{T} \times \mathfrak{M}$. Note that the equality holds in the first inequality for all $a \in \mathbb{T}$. Since $\lambda \neq 0$, we must have $|\lambda| = 1$ and $\widehat{(T1)'}(m) = 0$. Hence (1.23) becomes (b). \square

Lemma 1.3.14. φ is a homeomorphism of \mathfrak{M} onto itself.

Proof. Since φ is a continuous mapping of a compact Hausdorff space onto itself, it suffices to show that φ is injective. Assume $m' \neq m''$ and $\varphi(m') = \varphi(m'')$. Then we can choose $f_1 \in \text{Lip}$ such that $\widehat{f}'_1(m') = 1$ and $\widehat{f}'_1(m'') = 0$, because of $\{\widehat{f}' : f \in \text{Lip}\} = C(\mathfrak{M})$ and Urysohn's lemma. Since T is surjective, there exists $f_0 \in \text{Lip}$ such that $Tf_0 = f_1$. By Lemmas 1.3.12 and 1.3.13(b), we have

$$|\widehat{f}'_0(\varphi(m))| = |\widehat{\omega}(m)\widehat{f}'_0(\varphi(m))| = |\widehat{(Tf_0)'}(m)| = |\widehat{f}'_1(m)| \quad (m \in \mathfrak{M}),$$

and so $1 = |\widehat{f}'_1(m')| = |\widehat{f}'_0(\varphi(m'))| = |\widehat{f}'_0(\varphi(m''))| = |\widehat{f}'_1(m'')| = 0$, a contradiction. Therefore φ is injective. \square

Definition 1.3.15. For each $h \in C(\mathfrak{M})$, we define a function Ψh on \mathfrak{M} by

$$(\Psi h)(m) = h(\varphi(m)) \quad (m \in \mathfrak{M}).$$

Since φ is a homeomorphism of \mathfrak{M} onto itself, Ψ is an algebra automorphism of $C(\mathfrak{M})$. Put $\Phi = \Gamma^{-1}\Psi\Gamma$. Since the Gelfand transformation Γ is an algebra isomorphism of L^∞ onto $C(\mathfrak{M})$, Φ is an algebra automorphism of L^∞ .

Lemma 1.3.16. T has the form (1.16).

Proof. Let $f \in \text{Lip}$. By Lemma 1.3.13(b), we have

$$\begin{aligned} \widehat{(Tf)'}(m) &= \widehat{\omega}(m)\widehat{f}'(\varphi(m)) = \widehat{\omega}(m)(\Psi\widehat{f}')(m) = \widehat{\omega}(m)(\Psi\Gamma f')(m) \\ &= \widehat{\omega}(m)(\Gamma\Phi f')(m) = \widehat{\omega}(m)\widehat{\Phi f}'(m) = \widehat{\omega \cdot (\Phi f')}(m) \end{aligned} \quad (m \in \mathfrak{M}).$$

Hence $(Tf)' = \omega \cdot (\Phi f')$. Together with Lemma 1.3.11(b), we obtain

$$(Tf)(x) = (Tf)(0) + \int_0^x (Tf)'(t) dt = \lambda f(0) + \int_0^x \omega(t)(\Phi f')(t) dt \quad (x \in [0, 1]).$$

\square

Noting Lemmas 1.3.12, 1.3.13(a), 1.3.16 and Definition 1.3.15, we establish the “only if” part on $(\text{Lip}[0, 1], \|\cdot\|_\sigma)$ of Theorem 1.3.1.

1.3.4 Proof of Theorem 1.3.1; the “only if” part on $(\text{Lip}[0, 1], \|\cdot\|_m)$

In this subsection, we deal with the space $(\text{Lip}[0, 1], \|\cdot\|_m)$. We first see that the space $(\text{Lip}[0, 1], \|\cdot\|_m)$ is linearly isometric to $C(X)$ for some compact Hausdorff space X .

Definition 1.3.17. Put $X_2 = \mathfrak{M} \cup \{p\}$, where p is another point. We assume that X_2 is equipped with the topology consisting of all open sets of \mathfrak{M} and all sets of the form $(\mathfrak{M} \setminus K) \cup \{p\}$ where K is a compact subset of \mathfrak{M} . Clearly X_2 is a compact Hausdorff space. For each $f \in \text{Lip}[0, 1]$, we define a continuous function \tilde{f} on X_2 by

$$\tilde{f}(y) = \begin{cases} f(0) & \text{if } y = p, \\ \hat{f}'(y) & \text{if } y \in \mathfrak{M}. \end{cases}$$

Lemma 1.3.18. *The mapping $P_2 : f \mapsto \tilde{f}$ is a linear isometry of $(\text{Lip}[0, 1], \|\cdot\|_m)$ onto $C(X_2)$.*

Proof. For each $f \in \text{Lip}[0, 1]$, we have

$$\begin{aligned} \|f\|_m &= \max\{|f(0)|, \|f'\|_{L^\infty}\} = \max\{|f(0)|, \|\hat{f}'\|_{\mathfrak{M}}\} \\ &= \max\{|\tilde{f}(p)|, \|\tilde{f}\|_{\mathfrak{M}}\} = \sup\{|\tilde{f}(y)| : y \in X_2\}. \end{aligned}$$

Hence P_2 is an isometry.

To see that P_2 is surjective, pick $g \in C(X_2)$ arbitrarily. Then there exists $h \in L^\infty[0, 1]$ such that $\hat{h} = g|_{\mathfrak{M}}$; the restriction on \mathfrak{M} . We define $f \in \text{Lip}[0, 1]$ by

$$f(x) = g(p) + \int_0^x h(t) dt \quad (x \in [0, 1]).$$

Then $\tilde{f}(p) = f(0) = g(p)$. Also, we have $f' = h$ a.e., and so $\tilde{f}(m) = \hat{f}'(m) = \hat{h}(m) = g(m)$ for $m \in \mathfrak{M}$. Hence $P_2 f = \tilde{f} = g$. Thus P_2 is surjective. \square

Proof of the “only if” part on $(\text{Lip}[0, 1], \|\cdot\|_m)$ of Theorem 1.3.1. Let T be a surjective linear isometry on $(\text{Lip}[0, 1], \|\cdot\|_m)$. We associate a linear operator \tilde{T} on $C(X_2)$ such as $\tilde{T} = P_2 T P_2^{-1}$, where P_2 is the linear isometry of $(\text{Lip}[0, 1], \|\cdot\|_m)$ onto $C(X_2)$ in Lemma 1.3.18. Then \tilde{T} is a surjective linear isometry on $C(X_2)$. By the Banach-Stone theorem, there exist a homeomorphism ρ of X_2 onto X_2 and a unimodular continuous function u on X_2 such that

$$(\tilde{T}h)(y) = u(y)h(\rho(y)) \quad (y \in X_2)$$

for all $h \in C(X_2)$. Then the restriction of ρ to \mathfrak{M} becomes a homeomorphism of \mathfrak{M} onto \mathfrak{M} and $\rho(\{p\}) = \{p\}$. Therefore, ρ induces the surjective automorphism $\tilde{\Phi}$ of $C(\mathfrak{M})$ in the following manner:

$$(\tilde{\Phi}h)(m) = h(\rho(m)) \quad (m \in \mathfrak{M})$$

for all $h \in C(\mathfrak{M})$. Put $\Phi = \Gamma^{-1}\tilde{\Phi}\Gamma$. Then Φ is a surjective automorphism of $L^\infty[0, 1]$, and we have $\widehat{\Phi}g = \Gamma\tilde{\Phi}g = \tilde{\Phi}\Gamma g = \tilde{\Phi}\hat{g}$ for all $g \in L^\infty[0, 1]$. On the other hand, since the restriction $u|_{\mathfrak{M}}$ belongs to $C(\mathfrak{M})$, there exists $\omega \in L^\infty[0, 1]$ such that $\hat{\omega} = u|_{\mathfrak{M}}$. Then $\widehat{\omega\bar{\omega}} = \widehat{\omega}\hat{\bar{\omega}} = \widehat{u\bar{u}} = 1$ on \mathfrak{M} , it follows that $\omega\bar{\omega} = 1$ a.e. on $[0, 1]$. Hence we may assume that ω is unimodular. Moreover, we put $\lambda = u(p)$. Of course, λ is a unimodular constant.

To show (1.16), let $f \in \text{Lip}[0, 1]$. Since $\widetilde{T}f = P_2Tf = \widetilde{T}P_2f = \widetilde{T}\widetilde{f}$, we have

$$(Tf)(0) = (\widetilde{T}f)(0) = (\widetilde{T}\widetilde{f})(0) = u(0)\widetilde{f}(\rho(0)) = \lambda\widetilde{f}(0) = \lambda f(0).$$

Moreover, if $m \in \mathfrak{M}$, then

$$\begin{aligned} \widehat{(\widetilde{T}f)'(m)} &= \widehat{(\widetilde{T}f)(m)} = \widehat{(\widetilde{T}\widetilde{f})(m)} = u(m)\widetilde{f}(\rho(m)) = \widehat{\omega(m)\widetilde{f}(\rho(m))} \\ &= \widehat{\omega(m)(\widetilde{\Phi}\widetilde{f}')}(m) = \widehat{\omega(m)(\widehat{\Phi}f')}(m) = \widehat{\omega \cdot (\Phi f')}(m). \end{aligned}$$

Hence $(Tf)' = \omega \cdot (\Phi f')$. Therefore, we have

$$(Tf)(x) = (Tf)(0) + \int_0^x (Tf)'(t) dt = \lambda f(0) + \int_0^x \omega(t)(\Phi f')(t) dt \quad (x \in [0, 1]).$$

□

1.3.5 Proof of Theorem 1.3.2

Recall from Section 1.3.1 that \mathfrak{M} is a compact Hausdorff space which is extremally disconnected and has no isolated points. We first investigate the property of such a space.

Lemma 1.3.19. *Suppose that a compact Hausdorff space X is extremally disconnected and has no isolated points. If ψ is a continuous mapping of X onto itself which is not injective, then the set $\{(x, y) \in X \times X : x \neq y, \psi(x) = \psi(y)\}$ is infinite.*

Proof. Suppose that ψ is a continuous mapping of X onto itself which is not injective. We define an equivalence relation $x \sim y$ on X by $\psi(x) = \psi(y)$. Consider the quotient space X/\sim , and denote by q the quotient mapping of X onto X/\sim . Now, define a mapping π of X/\sim to X by $\pi(q(x)) = \psi(x)$ for all $x \in \mathfrak{M}$. We easily see that X/\sim is compact and π is bijective and continuous. Hence π is a homeomorphism, and so X/\sim is extremally disconnected and has no isolated points.

Let Y be the set of all points $x \in X$ such that $\psi(x) = \psi(y)$ for some $y \in X \setminus \{x\}$. To complete the proof, it suffices to show that Y is infinite. Assume, to reach a contradiction, that Y is finite. Pick two distinct points x_0 and y_0 in Y so that $\psi(x_0) = \psi(y_0)$. Since Y is finite, we find open sets U and V in X so that

$$x_0 \in U, \quad y_0 \in V, \quad \overline{U} \cap Y = \{x_0\}, \quad \overline{V} \cap Y = \{y_0\} \quad \text{and} \quad \overline{U} \cap \overline{V} = \emptyset.$$

Then we have

$$\{q(x_0)\} = q(\overline{U}) \cap q(\overline{V}). \tag{1.24}$$

Let us verify that $q(\overline{U})$ is open in X/\sim . Since X has no isolated points, $\overline{U \setminus \{x_0\}} = \overline{U}$. Since q is continuous and X is compact, we see that $\overline{q(U \setminus \{x_0\})} = q(\overline{U \setminus \{x_0\}})$. Hence $\overline{q(U \setminus \{x_0\})} = q(\overline{U})$. Noting that $\overline{U} \cap Y = \{x_0\}$, we see that $q^{-1}(q(U \setminus \{x_0\})) = U \setminus \{x_0\}$. This implies that $q(U \setminus \{x_0\})$ is open in X/\sim . Since X/\sim is extremally disconnected, $\overline{q(U \setminus \{x_0\})}$ is open. Namely, $q(\overline{U})$ is open. Similarly, we can show that $q(\overline{V})$ is open. Hence (1.24) implies that $q(x_0)$ is an isolated point in X/\sim . This is a contradiction and the proof is completed. □

This lemma can be extended as follows:

Lemma 1.3.20. *Suppose that a compact Hausdorff space X is extremally disconnected and has at most finitely many isolated points. If ψ is a continuous mapping of X onto itself which is not injective, then the set $\{(x, y) \in X \times X : x \neq y, \psi(x) = \psi(y)\}$ is infinite.*

Proof. Let ψ be a continuous mapping of X onto itself which is not injective. If X has no isolated points, the conclusion follows at once from Lemma 1.3.19. Suppose that there exists a isolated point in X , say p_1, p_2, \dots, p_n . Put $X_0 = X \setminus \{p_1, p_2, \dots, p_n\}$. Then X_0 has no isolated points and is a compact Hausdorff space. Since X_0 is open in X , X_0 is extremally disconnected. Since ψ is continuous and surjective, we see that $\psi(\{p_1, p_2, \dots, p_n\}) = \{p_1, p_2, \dots, p_n\}$, and so $\psi(X_0) = X_0$. Since ψ is not injective on X_0 , Lemma 1.3.19 implies that the set $\{(x, y) \in X_0 \times X_0 : x \neq y, \psi(x) = \psi(y)\}$ is infinite. This implies the desired conclusion. \square

Together with Theorem A, we obtain the following theorem:

Theorem 1.3.21. *Suppose that a compact Hausdorff space X is extremally disconnected and has at most finitely many isolated points. If T is a finite codimensional linear isometry on $C(X)$, then T is surjective.*

Theorem 1.3.2 is a corollary to Theorem 1.3.21.

Proof of Theorem 1.3.2. Let X_2 be as in Definition 1.3.17. We can easily check that $X_2 = \mathfrak{M} \cup \{p\}$ is extremally disconnected and that p is the only isolated point in X_2 . Hence Theorem 1.3.21 implies that if \tilde{T} is a finite codimensional linear isometry on $C(X_2)$, then \tilde{T} is surjective. Recall from Lemma 1.3.18 that $(\text{Lip}[0, 1], \|\cdot\|_m)$ is linearly isometric to $C(X_2)$. The conclusion follows immediately. \square

Finally, we apply Theorem 1.3.21 to the L^∞ -spaces. Let $(\Omega, \mathfrak{B}, \mu)$ be a positive measure space. We denote by $L^\infty_{\mathbb{K}}(\Omega, \mathfrak{B}, \mu)$ or simply $L^\infty(\Omega, \mathfrak{B}, \mu)$ the Banach algebra of all equivalence classes of \mathbb{K} -valued essentially bounded μ -measurable functions on Ω , equipped with the essential supremum norm.

Corollary 1.3.22. *Let $(\Omega, \mathfrak{B}, \mu)$ be a positive measure space. Suppose that μ has at most finitely many atoms. If T is a finite codimensional linear isometry on $L^\infty(\Omega, \mathfrak{B}, \mu)$, then T is surjective.*

Proof. In the same way as Section 1.3.1, we consider the space $L^\infty(\Omega, \mathfrak{B}, \mu)$. Since $L^\infty_{\mathbb{C}}(\Omega, \mathfrak{B}, \mu)$ is a unital commutative C^* -algebra, the Gelfand-Naimark theorem says that $L^\infty_{\mathbb{C}}(\Omega, \mathfrak{B}, \mu)$ is isometrically $*$ -isomorphic to $C_{\mathbb{C}}(\mathfrak{M}_{L^\infty})$, where \mathfrak{M}_{L^∞} is the maximal ideal space of $L^\infty_{\mathbb{C}}(\Omega, \mathfrak{B}, \mu)$. Thus we see that $L^\infty(\Omega, \mathfrak{B}, \mu)$ is linearly isometric to $C(\mathfrak{M}_{L^\infty})$. Also, it is known that \mathfrak{M}_{L^∞} is extremally disconnected ([5, Theorem 1.3.4]).

Now, suppose that μ has at most finitely many atoms. Then we easily see that \mathfrak{M}_{L^∞} has an equal number of isolated points. Thus Theorem 1.3.21 implies that if \tilde{T} is a finite codimensional linear isometry on $C(\mathfrak{M}_{L^\infty})$, then \tilde{T} is surjective. This fact leads to the corollary. \square

1.4 The Wiener algebra

Let W denote the space of all complex-valued continuous functions on the unit circle \mathbb{T} whose Fourier series is absolutely convergent. For each $f \in W$, we denote by $c_n(f)$ the n -th Fourier coefficient. With respect to the norm

$$\|f\|_W = \sum_{n=-\infty}^{\infty} |c_n(f)| \quad (f \in W),$$

W is a unital semisimple commutative Banach algebra. The algebra W is sometimes called the *Wiener algebra*. We characterize the surjective linear isometries on W , as follows:

Theorem 1.4.1. *Let T be a linear operator on W . Then T is a surjective isometry if and only if there exist a bijection φ of \mathbb{Z} onto itself and a unimodular function ω on \mathbb{Z} such that*

$$(Tf)(z) = \sum_{n=-\infty}^{\infty} \omega(n)c_{\varphi(n)}(f) z^n \quad (1.25)$$

for all $z \in \mathbb{T}$ and $f \in W$.

Proof. Let $\ell^1(\mathbb{Z})$ denote the Banach space of all doubly infinite sequences $\{x_n\}_{n=-\infty}^{\infty}$ of complex numbers satisfying $\|\{x_n\}\|_1 := \sum_{n=-\infty}^{\infty} |x_n| < \infty$. Define a mapping P of W into $\ell^1(\mathbb{Z})$ by $Pf = \{c_n(f)\}_{n=-\infty}^{\infty}$ for all $f \in W$. Then it is easy to see that P is a surjective linear isometry of W onto $\ell^1(\mathbb{Z})$ (cf. [22, Example 1.1.5]). Let T be a surjective linear isometry on W . We associate a linear operator \tilde{T} on $\ell^1(\mathbb{Z})$ such as $\tilde{T} = PTP^{-1}$. Then \tilde{T} is a surjective linear isometry on $\ell^1(\mathbb{Z})$. By the characterization of the surjective linear isometries on $\ell^1(\mathbb{Z})$ (cf. [4, Theorem 11.5.2]), there exist a bijection φ of \mathbb{Z} onto itself and a unimodular function ω on \mathbb{Z} such that

$$\tilde{T}(\{c_n(f)\}) = \{\omega(n)c_{\varphi(n)}(f)\}_{n=-\infty}^{\infty}$$

for all $f \in W$. Hence we have

$$(Tf)(z) = \sum_{n=-\infty}^{\infty} c_n(Tf) z^n = \sum_{n=-\infty}^{\infty} \omega(n)c_{\varphi(n)}(f) z^n$$

for all $z \in \mathbb{T}$ and $f \in W$.

Conversely, suppose that T has the form (1.25). Then we see that

$$c_n(Tf) = \omega(n)c_{\varphi(n)}(f)$$

for all $n \in \mathbb{Z}$ and $f \in W$. Hence we have

$$\begin{aligned} \|Tf\|_W &= \|\{c_n(Tf)\}\|_1 = \sum_{n=-\infty}^{\infty} |c_n(Tf)| \\ &= \sum_{n=-\infty}^{\infty} |\omega(n)c_{\varphi(n)}(f)| = \sum_{n=-\infty}^{\infty} |c_n(f)| = \|\{c_n(f)\}\|_1 = \|f\|_W. \end{aligned}$$

and so T is an isometry. To see that T is surjective, let $g \in W$. We put

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{c_{\varphi^{-1}(n)}(g)}{\omega(\varphi^{-1}(n))} z^n \quad (z \in \mathbb{T}).$$

By (1.25), we have

$$(Tf)(z) = \sum_{n=-\infty}^{\infty} \omega(n) c_{\varphi(n)}(f) z^n = \sum_{n=-\infty}^{\infty} \omega(n) \frac{c_{\varphi^{-1}(\varphi(n))}(g)}{\omega(\varphi^{-1}(\varphi(n)))} z^n = \sum_{n=-\infty}^{\infty} c_n(g) z^n = g(z)$$

for all $z \in \mathbb{T}$. Hence $Tf = g$ and this completes the proof. \square

The form (1.25) is not of the canonical form. Indeed, let ω be the constant 1 and put $\varphi(0) = 1$, $\varphi(1) = 2$ and $\varphi(2) = 0$ and $\varphi(n) = n$ for $n \neq 0, 1, 2$,

The Banach-Stone theorem has been extended to function algebras by Nagasawa [31] and deLeeuw, Rudin and Wermer [10], that is, every surjective linear isometries on a function algebra has the canonical form. However, Theorem 1.4.1 suggests to us that this result does not hold over unital semisimple commutative Banach algebras any longer.

Finally, we note that W admits a finite codimensional linear isometry which is not surjective.

Example 1.4.2. We define a linear operator T on W by

$$(Tf)(z) = \sum_{n=-\infty}^{-1} c_n(f) z^n + \sum_{n=1}^{\infty} c_{n-1}(f) z^n \quad (z \in \mathbb{T}, f \in W).$$

Then we see that T is a linear isometry and the range of T has codimension 1.

Chapter 2

Backward shifts on uniform algebras

2.1 Introduction

In this chapter, we discuss a special linear isometry; a shift. We take up two types, isometric shift and backward shift.

The shifts on Hilbert spaces have played an important role in branches of mathematics; for example, invariant subspaces, isometries, composition operators and so on. They are defined as follows: Let \mathcal{H} be an infinite-dimensional separable Hilbert space and T a bounded linear operator on \mathcal{H} . We call T an *isometric shift* (or a *forward shift*) on \mathcal{H} if there is a complete orthonormal system $\{\phi_n\}_{n=1}^{\infty}$ in \mathcal{H} such that $T\phi_n = \phi_{n+1}$ for $n = 1, 2, \dots$. Also, we call T a *backward shift* on \mathcal{H} if there is a complete orthonormal system $\{\phi_n\}_{n=1}^{\infty}$ in \mathcal{H} such that $T\phi_1 = 0$ and $T\phi_n = \phi_{n-1}$ for $n = 2, 3, \dots$.

In [9], Crownover extended the definition of an isometric shift on \mathcal{H} to a Banach space without using a basis:

Definition. Let \mathcal{B} be a Banach space and T a bounded linear operator on \mathcal{B} . We call T an *isometric shift* on \mathcal{B} if T satisfies the following conditions:

- (i) T is an isometry.
- (ii) The codimension of the range of T in \mathcal{B} is 1.
- (iii) $\bigcap_{n=1}^{\infty} T^n(\mathcal{B}) = \{0\}$.

We say that T is an *isometric quasi-shift* (or a *codimension 1 linear isometry*) on \mathcal{B} if T satisfies (i) and (ii) only.

In [17], Holub gave a similar extension for a backward shift:

Definition. Let \mathcal{B} be a Banach space and T a bounded linear operator on \mathcal{B} . We write $\ker T$ to denote the kernel $\{f \in \mathcal{B} : Tf = 0\}$. We call T a *backward shift* on \mathcal{B} if T satisfies the following conditions:

- (i)' The induced operator $\widehat{T} : f + \ker T \mapsto Tf$ of the quotient space $\mathcal{B}/\ker T$ into \mathcal{B} is an isometry.

(ii)' The dimension of $\ker T$ is 1.

(iii)' $\bigcup_{n=1}^{\infty} \ker T^n$ is dense in \mathcal{B} .

According to [36, Proposition 1.2], every backward shift on an infinite-dimensional space is surjective. We say that T is a *backward quasi-shift* if T satisfies (i)' and (ii)', and if T is surjective. Also, we know that the adjoint operator of a backward shift on \mathcal{B} is an isometric shift on \mathcal{B}^* ([41, 44]).

In [17], Holub posed the problem whether a concrete function space admits an isometric shift or a backward shift. In other words:

Does there exist an isometric shift or a backward shift on a concrete function space?

The shifts on $C(X)$ have been well studied. Here $C(X)$ denotes the Banach space of all \mathbb{K} -valued continuous functions on a compact Hausdorff space X , equipped with the supremum norm. In [15], Gutek, Hart, Jamison and Rajagopalan studied isometric shifts on $C(X)$ and classified them using the Holsztyński theorem [16]. On the other hand, Rajagopalan and Sundaresan studied backward shifts on $C(X)$ and proved the following theorem:

Theorem B (Rajagopalan and Sundaresan [36, 37]). *If $C(X)$ is infinite-dimensional, then $C(X)$ does not admit a backward shift.*

This theorem was proved in case of $\mathbb{K} = \mathbb{R}$ in [36] and in case of $\mathbb{K} = \mathbb{C}$ in [37]. Later, Rajagopalan, Rassias and Sundaresan [35] extended this theorem to the Banach space of E -valued continuous functions on X , where E is a Banach space with E^* strictly convex.

In this paper, we are concerned with the case of $\mathbb{K} = \mathbb{C}$. We denote by $C_{\mathbb{C}}(X)$ the *Banach algebra* of all *complex-valued* continuous functions on X . As a generalization of $C_{\mathbb{C}}(X)$, we consider a uniform algebra. A *uniform algebra* A on X is a closed subalgebra of $C_{\mathbb{C}}(X)$ which contains the constants and separates the points of X , that is, for each pair of distinct points $x_1, x_2 \in X$, there exists $f \in A$ such that $f(x_1) \neq f(x_2)$.

The main result in this chapter is the following two theorems:

Theorem 2.1.1. *An infinite-dimensional uniform algebra does not admit a backward shift.*

Theorem 2.1.2. *Let A be a uniform algebra. Suppose that the maximal ideal space of A has at most finitely many isolated points. Then A does not admit a backward quasi-shift.*

Clearly, Theorem 2.1.1 is a generalization of Theorem B. Here the adjective “infinite-dimensional” is crucially necessary because a finite-dimensional space always admits a backward shift. In Theorem 2.1.2, the same adjective is unnecessary because backward quasi-shifts on finite-dimensional spaces are not surjective.

The essential part of Theorem 2.1.1 was obtained in the master’s thesis by Ariizumi [3]. The author refined its proof partly and prove it together with Theorem 2.1.2 (see [45]). We give their proofs in Section 2.2. After the proof, we discussed the shifts on concrete function spaces.

2.2 Backward shifts on uniform algebras

In this section, we prove Theorems 2.1.1 and 2.1.2. Throughout this section, X is a compact Hausdorff space and A is a uniform algebra on X .

2.2.1 Preliminaries

As a preliminary, we explain some facts concerning a uniform algebra.

For each $x \in X$, the *evaluation functional* e_x on A is defined by $e_x(f) = f(x)$ for all $f \in A$. It is clear that $e_x \in A^*$ and $\|e_x\| = e_x(1) = 1$, where A^* denotes the dual space of A . The *Choquet boundary* $\text{Ch}(A)$ for A is defined as

$$\text{Ch}(A) = \{x \in X : e_x \text{ is an extreme point of ball } A^*\}.$$

It is known that $\text{Ch}(A)$ is a boundary for A , that is, for any $f \in A$ there exists $x \in \text{Ch}(A)$ such that $|f(x)| = \|f\|$ ([12, Theorem 2.3.8]). The next fact also seems to be known:

$$\begin{aligned} \text{Let } \xi \in \text{ball } A^*. \text{ Then } \xi \text{ is an extreme point of ball } A^* \text{ if and only if} \\ \text{there exist } x \in \text{Ch}(A) \text{ and } \alpha \in \mathbb{T} \text{ such that } \xi = \alpha e_x. \end{aligned} \quad (2.1)$$

The “if” part follows immediately from the definition of $\text{Ch}(A)$. The proof of the “only if” part may be found in [12, Corollary 2.3.6].

We here describe the characterizations of the point of $\text{Ch}(A)$.

Proposition 2.2.1. *Let $p \in X$. Then the following are equivalent:*

- (i) $p \in \text{Ch}(A)$.
- (ii) e_p is an extreme point of the set $\{\xi \in A^* : \|\xi\| = \xi(1) = 1\}$.
- (iii) For each neighborhood U of p and for each $\varepsilon > 0$, there exists $f \in \text{ball } A$ such that $f(p) > 1 - \varepsilon$ and $|f(x)| < \varepsilon$ for all $x \in X \setminus U$.

Proof. It is easy to see that (i) implies (ii). Let us show the converse. Assume that (ii) holds. To see (i), it suffices to show that e_p is an extreme point of ball A^* . For this purpose, write $e_p = t\xi + (1-t)\eta$, where $\xi, \eta \in \text{ball } A^*$ and $0 < t < 1$. Then we have

$$\begin{aligned} 1 = |e_p(1)| &= |t\xi(1) + (1-t)\eta(1)| \leq t|\xi(1)| + (1-t)|\eta(1)| \\ &\leq t\|\xi\| + (1-t)\|\eta\| \leq t + (1-t) = 1. \end{aligned}$$

Hence $\|\xi\| = |\xi(1)| = 1$ and $\|\eta\| = |\eta(1)| = 1$. Since $1 = e_p(1) = t\xi(1) + (1-t)\eta(1)$, it follows that $1 = \xi(1) = \eta(1)$. Therefore, (ii) implies that $e_p = \xi = \eta$. Thus we see that e_p is an extreme point of ball A^* , that is, $p \in \text{Ch}(A)$.

The equivalence of (ii) and (iii) is known as the Bishop-deLeeuw theorem [34, page 39]. \square

Proposition 2.2.2. *Let p and q be distinct points in $\text{Ch}(A)$, and let $\alpha, \beta \in \mathbb{T}$. Then for each neighborhood U of $\{p, q\}$ and each $\varepsilon > 0$, there exists $f \in \text{ball } A$ such that $|f(p) - \alpha| < \varepsilon$, $|f(q) - \beta| < \varepsilon$ and $|f(x)| < \varepsilon$ for all $x \in X \setminus U$.*

Proof. Choose disjoint open sets G_1 and G_2 so that $p \in G_1 \subset U$, $q \in G_2 \subset U$. By Proposition 2.2.1, there exist $g, h \in \text{ball } A$ such that

$$\begin{aligned} g(p) > 1 - \varepsilon \quad \text{and} \quad |g(x)| < \varepsilon \quad \text{for } x \in X \setminus G_1, \\ h(q) > 1 - \varepsilon \quad \text{and} \quad |h(x)| < \varepsilon \quad \text{for } x \in X \setminus G_2. \end{aligned}$$

Then we have

$$|\alpha g(x) + \beta h(x)| \leq \begin{cases} \|g\| + |h(x)| \leq 1 + \varepsilon & \text{if } x \in G_1, \\ |g(x)| + \|h\| \leq \varepsilon + 1 & \text{if } x \in X \setminus G_1. \end{cases}$$

Now, we define a function $f \in \text{ball } A$ by $f = (\alpha g + \beta h)/(1 + \varepsilon)$. Then we have

$$|f(p) - \alpha| = \left| \frac{(\alpha g(p) + \beta h(p)) - \alpha(1 + \varepsilon)}{1 + \varepsilon} \right| \leq \frac{|\alpha| |g(p) - 1| + |\beta| |h(p)| + |\alpha| \varepsilon}{1 + \varepsilon} < \frac{3\varepsilon}{1 + \varepsilon}.$$

Similarly, we obtain $|f(q) - \beta| < 3\varepsilon/(1 + \varepsilon)$. Furthermore, if $x \in X \setminus U$, then $|g(x)| < \varepsilon$ and $|h(x)| < \varepsilon$, so that $|f(x)| < 2\varepsilon/(1 + \varepsilon)$. Finally, we only have to arrange a positive number ε to find the desired function f . \square

Proposition 2.2.3. *Let p be an isolated point of $\text{Ch}(A)$. Then there exists $f \in A$ such that $f(p) = 1$ and $f(x) = 0$ for all $x \in \text{Ch}(A) \setminus \{p\}$.*

Proof. Since p is an isolated point of $\text{Ch}(A)$, we find a neighborhood U of p in X so that $U \cap \text{Ch}(A) = \{p\}$. Then Proposition 2.2.1 gives a sequence of functions $\{f_n\} \subset \text{ball } A$ such that $f_n(p) > 1 - 1/2^n$ and $|f_n(x)| < 1/2^n$ for all $x \in X \setminus U$. This sequence satisfies $\sup\{|f_m(x) - f_n(x)| : x \in \text{Ch}(A)\} \leq 1/2^{n-1}$ whenever $m > n$. Since $\|f\| = \sup\{|f(x)| : x \in \text{Ch}(A)\}$ for all $f \in A$, it follows that $\{f_n\}$ is a Cauchy sequence in A . By the completeness of A , there exists $f \in A$ such that $\|f_n - f\| \rightarrow 0$. This function f must have the desired properties. \square

Proposition 2.2.4. *Let \mathfrak{M} be the maximal ideal space of A and let $\xi \in \mathfrak{M}$. Then ξ is an isolated point of \mathfrak{M} if and only if there exists an isolated point p of $\text{Ch}(A)$ such that $\xi = e_p$.*

Proof. Suppose that ξ is an isolated point of \mathfrak{M} . Then $\{\xi\}$ and $\mathfrak{M} \setminus \{\xi\}$ are disjoint open subsets of \mathfrak{M} . Hence there exists $f \in A$ such that $\widehat{f}(\xi) = 1$ and $\widehat{f}(\eta) = 0$ for all $\eta \in \mathfrak{M} \setminus \{\xi\}$ by Shilov's idempotent theorem [22, Proposition 3.5.3], where \widehat{f} is the Gelfand representation of f . Recalling that $\text{Ch}(A)$ is a boundary for A , we can find a point $p \in \text{Ch}(A)$ such that $|f(p)| = \|f\|$. Then we have

$$|\widehat{f}(e_p)| = |f(p)| = \|f\| = \|\widehat{f}\| = 1 \neq 0.$$

Hence our choice of f implies that $e_p = \xi$. By hypothesis, e_p is an isolated point of \mathfrak{M} . Since $x \mapsto e_x$ is a homeomorphism of X into \mathfrak{M} , p is an isolated point of $\text{Ch}(A)$.

Conversely, suppose that p is an isolated point of $\text{Ch}(A)$. By Proposition 2.2.3, there exists $f \in A$ such that $f(p) = 1$ and $f(x) = 0$ for all $x \in \text{Ch}(A) \setminus \{p\}$. Put

$$U = \{\eta \in \mathfrak{M} : \widehat{f}(\eta) = 1\}.$$

Since $\widehat{f}(e_p) = f(p) = 1$, we have $e_p \in U$. Next, note that $f^2 = f$ on $\text{Ch}(A)$. This implies $(\widehat{f})^2 = \widehat{f}$ and so \widehat{f} takes values in $\{0, 1\}$ on \mathfrak{M} . Hence U is open in \mathfrak{M} . To see that e_p is an isolated point, let us show that $U = \{e_p\}$. Assume, to reach a contradiction, that there exists $\xi \in U \setminus \{e_p\}$. Then we can find $g \in A$ so that $\widehat{g}(e_p) = 1$ and $\widehat{g}(\xi) = 2$. Define $h \in A$ by $h = fg$. Then we have $h(p) = f(p)g(p) = 1 = \widehat{f}(p)$ and $h(x) = f(x)g(x) = 0 = \widehat{f}(x)$ for all $x \in \text{Ch}(A) \setminus \{p\}$, and hence $h = f$. However, $h(\xi) = \widehat{f}(\xi)\widehat{g}(\xi) = 2 \neq 1 = \widehat{f}(\xi)$, and so $\widehat{h} \neq \widehat{f}$. This is a contradiction. Thus we see that e_p is an isolated point of \mathfrak{M} . \square

Next, we remark on the measure. Let $M(X)$ denote the Banach space of all complex regular Borel measures on X , with the total variation norm. By the Hahn-Banach theorem and the Riesz representation theorem, we see the following fact: For each $\xi \in A^*$, there exists a measure $\mu \in M(X)$ such that

$$\xi(f) = \int_X f d\mu \quad (f \in A) \quad \text{and} \quad \|\xi\| = \|\mu\|.$$

Such a μ is called a *representing measure* for ξ . We should note that a representing measure for ξ is not always determined uniquely.

A simple example of a measure in $M(X)$ is the *point mass* δ_x concentrated at $x \in X$. We know that $\int f d\delta_x = f(x)$ for all $f \in A$ and $\|\delta_x\| = 1$. Thus δ_x is one of the representing measures for the evaluation functional e_x .

2.2.2 Lemmas

For the proofs of Theorems 2.1.1 and 2.1.2, we prepare several lemmas.

Definition 2.2.5. Let $u \in C(X)$ and put $S(u) = \{x \in X : u(x) \neq 0\}$. For any distinct points $p, q \in S(u)$, we put

$$k_{upq} = \frac{u(q)}{|u(p)| + |u(q)|},$$

and define a measure λ_{upq} on X by

$$\lambda_{upq} = k_{upq}\delta_p - k_{uqp}\delta_q.$$

Since $|k_{upq}| + |k_{uqp}| = 1$, it follows that

$$\|\lambda_{upq}\| \leq |k_{upq}| \|\delta_p\| + |k_{uqp}| \|\delta_q\| = 1.$$

We characterize the measure λ_{upq} , as follows:

Lemma 2.2.6. Let $\mu \in M(X)$ and $u \in C(X)$. Suppose that p and q are distinct points in $S(u)$. Then $\mu = \lambda_{upq}$ if and only if μ satisfies the following conditions:

$$\mu(\{p\}) = k_{upq}, \quad \mu(\{q\}) = -k_{uqp} \quad \text{and} \quad \|\mu\| \leq 1. \quad (2.2)$$

Moreover, $\|\lambda_{upq}\| = 1$ and $|\lambda_{upq}|(X \setminus \{p, q\}) = 0$.

Proof. It is clear that $\mu = \lambda_{upq}$ satisfies (2.2). For the “if” part, suppose that μ satisfies (2.2). Then we have

$$\begin{aligned} 0 \leq |\mu|(X \setminus \{p, q\}) &= |\mu|(X) - |\mu|(\{p\}) - |\mu|(\{q\}) \\ &= \|\mu\| - |\mu(\{p\})| - |\mu(\{q\})| = \|\mu\| - |k_{upq}| - |k_{uqp}| = \|\mu\| - 1 \leq 0. \end{aligned}$$

Thus we obtain

$$\|\mu\| = 1 \quad \text{and} \quad |\mu|(X \setminus \{p, q\}) = 0.$$

Now let us show $\mu = \lambda_{upq}$. Let E be an arbitrary Borel set in X . If $p, q \notin E$, then $|\mu(E)| \leq |\mu|(E) \leq |\mu|(X \setminus \{p, q\}) = 0$, and hence $\mu(E) = 0 = \lambda_{upq}(E)$. If $p \in E$ and $q \notin E$, then $\mu(E \setminus \{p\}) = 0$, and so

$$\mu(E) = \mu(E \setminus \{p\}) + \mu(\{p\}) = k_{upq} = \lambda_{upq}(E).$$

If $p \notin E$ and $q \in E$, we can see $\mu(E) = \lambda_{upq}(E)$ similarly. Finally, if $p, q \in E$, then $\mu(E \setminus \{p, q\}) = 0$, and so

$$\mu(E) = \mu(E \setminus \{p, q\}) + \mu(\{p\}) + \mu(\{q\}) = k_{upq} - k_{uqp} = \lambda_{upq}(E).$$

In any case, we obtain $\mu(E) = \lambda_{upq}(E)$. \square

Definition 2.2.7. For $u \in C(X)$, we define a subspace $M([u]^\perp)$ of $M(X)$ by

$$M([u]^\perp) = \left\{ \mu \in M(X) : \int_X u d\mu = 0 \right\}.$$

Lemma 2.2.8. If $u \in C(X)$, and if p and q are distinct points in $S(u)$, then λ_{upq} is an extreme point of ball $M([u]^\perp)$.

Proof. By Lemma 2.2.6, $|\lambda_{upq}|(X \setminus \{p, q\}) = 0$, and so

$$\begin{aligned} \int_X u d\lambda_{upq} &= \int_{\{p, q\}} u d\lambda_{upq} = u(p)\lambda_{upq}(\{p\}) + u(q)\lambda_{upq}(\{q\}) \\ &= u(p)k_{upq} - u(q)k_{uqp} = \frac{u(p)u(q)}{|u(p)| + |u(q)|} - \frac{u(q)u(p)}{|u(q)| + |u(p)|} = 0. \end{aligned}$$

Hence $\lambda_{upq} \in M([u]^\perp)$. Since $\|\lambda_{upq}\| \leq 1$, we get $\lambda_{upq} \in \text{ball } M([u]^\perp)$.

Let us show that λ_{upq} is an extreme point of ball $M([u]^\perp)$. Assume that

$$\lambda_{upq} = t\mu + (1-t)\nu, \tag{2.3}$$

where $\mu, \nu \in \text{ball } M([u]^\perp)$ and $0 < t < 1$. We first observe the equations:

$$|\mu(\{p\})| + |\mu(\{q\})| = |\nu(\{p\})| + |\nu(\{q\})| = 1, \tag{2.4}$$

$$\arg \mu(\{p\}) = \arg \nu(\{p\}) \quad \text{and} \quad \arg \mu(\{q\}) = \arg \nu(\{q\}). \tag{2.5}$$

Indeed, we have

$$\begin{aligned} 1 &= |k_{upq}| + |k_{uqp}| \\ &= |\lambda_{upq}(\{p\})| + |\lambda_{upq}(\{q\})| \\ &= |t\mu(\{p\}) + (1-t)\nu(\{p\})| + |t\mu(\{q\}) + (1-t)\nu(\{q\})| \\ &\leq t|\mu(\{p\})| + (1-t)|\nu(\{p\})| + t|\mu(\{q\})| + (1-t)|\nu(\{q\})| \\ &= t(|\mu(\{p\})| + |\mu(\{q\})|) + (1-t)(|\nu(\{p\})| + |\nu(\{q\})|) \\ &\leq t\|\mu\| + (1-t)\|\nu\| \\ &\leq t + (1-t) = 1. \end{aligned}$$

Thus all above inequalities become equalities. Note that the inequality in the fourth line follows from the triangle inequality; $|\alpha + \beta| \leq |\alpha| + |\beta|$, where equality holds if and only if $\arg \alpha = \arg \beta$ or $\alpha\beta = 0$. Hence we obtain (2.5). Moreover the instance of equality in the last three lines implies (2.4).

Next, we show that

$$u(p)\mu(\{p\}) + u(q)\mu(\{q\}) = u(p)\nu(\{p\}) + u(q)\nu(\{q\}) = 0. \tag{2.6}$$

By (2.4), we have $|\mu|(X \setminus \{p, q\}) = |\mu|(X) - |\mu|(\{p\}) - |\mu|(\{q\}) = \|\mu\| - 1 \leq 0$, and so

$$0 = \int_X u d\mu = \int_{\{p, q\}} u d\mu = u(p)\mu(\{p\}) + u(q)\mu(\{q\}).$$

Similarly, we get $u(p)\nu(\{p\}) + u(q)\nu(\{q\}) = 0$.

By (2.6), $\mu(\{q\}) = -(u(p)/u(q))\mu(\{p\})$. Inserting this into (2.4) gives

$$|\mu(\{p\})| = \frac{|u(q)|}{|u(p)| + |u(q)|} = |k_{upq}|.$$

In the same way, we get $|\nu(\{p\})| = |k_{upq}|$. Hence $|\mu(\{p\})| = |\nu(\{p\})|$. Combining with the first equation in (2.5), we obtain $\mu(\{p\}) = \nu(\{p\})$. Hence (2.3) leads to $\mu(\{p\}) = \nu(\{p\}) = \lambda_{upq}(\{p\}) = k_{upq}$. By a similar argument, we can see that $\mu(\{q\}) = \nu(\{q\}) = \lambda_{upq}(\{q\}) = -k_{uqp}$. Here we recall that $\|\mu\| \leq 1$ and $\|\nu\| \leq 1$. By Lemma 2.2.6, we obtain $\mu = \nu = \lambda_{upq}$. Thus (2.3) implies $\lambda_{upq} = \mu = \nu$, and hence λ_{upq} is an extreme point. \square

Let us consider the functional on A that is represented by the measure λ_{upq} .

Definition 2.2.9. For each $u \in A$ and for each pair of distinct points $p, q \in S(u)$, we define a bounded linear functional θ_{upq} on A by

$$\theta_{upq} = k_{upq}e_p - k_{uqp}e_q,$$

where the constants k_{upq}, k_{uqp} are defined in Definition 2.2.5.

Lemma 2.2.10. Let $u \in A$, and let p and q be distinct points in $S(u) \cap \text{Ch}(A)$. Then

- (i) For each neighborhood U of $\{p, q\}$ and each $\varepsilon > 0$, there exists $f \in \text{ball } A$ such that $|\theta_{upq}(f)| > 1 - \varepsilon$ and $|f(x)| < \varepsilon$ for all $x \in X \setminus U$.
- (ii) $\|\theta_{upq}\| = 1$.

Proof. To see (i), pick $\alpha = |u(q)|/u(q)$ and $\beta = -|u(p)|/u(p)$ in Proposition 2.2.2. Then the resulting function f in ball A satisfies $|f(x)| < \varepsilon$ for all $x \in X \setminus U$. Moreover, f satisfies $|f(p) - \alpha| < \varepsilon$ and $|f(q) - \beta| < \varepsilon$, so that

$$\begin{aligned} 1 - |\theta_{upq}(f)| &\leq |\theta_{upq}(f) - 1| = |k_{upq}f(p) - k_{uqp}f(q) - (|k_{upq}| + |k_{uqp}|)| \\ &= |k_{upq}f(p) - k_{uqp}f(q) - k_{upq}\alpha + k_{uqp}\beta| \\ &\leq |k_{upq}| |f(p) - \alpha| + |k_{uqp}| |f(q) - \beta| \\ &< |k_{upq}|\varepsilon + |k_{uqp}|\varepsilon = \varepsilon. \end{aligned}$$

Thus (i) is proved.

For (ii), note that $\|\theta_{upq}\| \leq |k_{upq}|\|e_p\| + |k_{uqp}|\|e_q\| = |k_{upq}| + |k_{uqp}| = 1$. Also, the function f in (i) satisfies $\|\theta_{upq}\| \geq |\theta_{upq}(f)| > 1 - \varepsilon$. Since ε is arbitrary, we get $\|\theta_{upq}\| \geq 1$. \square

Lemma 2.2.11. Let $u \in A$, and let p and q be distinct points in $S(u) \cap \text{Ch}(A)$. Then λ_{upq} is the only representing measure for θ_{upq} .

Proof. For any $f \in A$, we have

$$\theta_{upq}(f) = k_{upq}e_p(f) - k_{uqp}e_q(f) = k_{upq} \int_X f d\delta_p - k_{uqp} \int_X f d\delta_q = \int_X f d\lambda_{upq}.$$

Also, Lemma 2.2.10(ii) and Lemma 2.2.6 yield $\|\theta_{upq}\| = 1 = \|\lambda_{upq}\|$. Therefore, λ_{upq} is a representing measure for θ_{upq} .

Let us show the uniqueness of λ_{upq} . Let μ be another representing measure for θ_{upq} . For each neighborhood U of $\{p, q\}$ and each $\varepsilon > 0$, Lemma 2.2.10(i) gives a function $f \in \text{ball } A$ such that $|\theta_{upq}(f)| > 1 - \varepsilon$ and $|f(x)| < \varepsilon$ for all $x \in X \setminus U$. Then we have

$$\begin{aligned} 1 - \varepsilon < |\theta_{upq}(f)| &= \left| \int_X f d\mu \right| \leq \left| \int_U f d\mu \right| + \left| \int_{X \setminus U} f d\mu \right| \\ &\leq \|f\| |\mu|(U) + \varepsilon |\mu|(X \setminus U) \leq |\mu|(U) + \varepsilon(1 - |\mu|(U)) = (1 - \varepsilon)|\mu|(U) + \varepsilon, \end{aligned}$$

so that

$$|\mu|(U) \geq \frac{1 - 2\varepsilon}{1 - \varepsilon}.$$

Letting $\varepsilon \rightarrow 0$, we get $|\mu|(U) \geq 1$, and the regularity of μ forces $|\mu|(\{p, q\}) \geq 1$. Since $|\mu|(X) = \|\mu\| = \|\theta_{upq}\| = 1$, it follows that $|\mu|(X \setminus \{p, q\}) = 0$. Hence, for each $f \in A$, we have

$$k_{upq}f(p) - k_{uqp}f(q) = \theta_{upq}(f) = \int_X f d\mu = \int_{\{p, q\}} f d\mu = f(p)\mu(\{p\}) + f(q)\mu(\{q\}).$$

Taking $f \in A$ so that $f(p) = 1$ and $f(q) = 0$, we obtain $k_{upq} = \mu(\{p\})$. While, taking f so that $f(p) = 0$ and $f(q) = 1$ yields $-k_{uqp} = \mu(\{q\})$. Moreover, we know $\|\mu\| = 1$. Finally, we appeal to Lemma 2.2.6 to get $\mu = \lambda_{upq}$. \square

We show the functional version of Lemma 2.2.8.

Definition 2.2.12. For $u \in A$, we put

$$[u] = \{\alpha u : \alpha \in \mathbb{C}\} \quad \text{and} \quad [u]^\perp = \{\xi \in A^* : \xi(u) = 0\}.$$

Lemma 2.2.13. *If $u \in A$, and if p and q are distinct points in $S(u) \cap \text{Ch}(A)$, then θ_{upq} is an extreme point of $\text{ball}[u]^\perp$.*

Proof. Since

$$\theta_{upq}(u) = k_{upq}e_p(u) - k_{uqp}e_q(u) = \frac{u(q)u(p)}{|u(p)| + |u(q)|} - \frac{u(p)u(q)}{|u(q)| + |u(p)|} = 0,$$

it follows $\theta_{upq} \in [u]^\perp$. Combining with Lemma 2.2.10(ii), we get $\theta_{upq} \in \text{ball}[u]^\perp$.

Next, we show that θ_{upq} is an extreme point of $\text{ball}[u]^\perp$. Assume that

$$\theta_{upq} = t\xi + (1 - t)\eta,$$

where $\xi, \eta \in \text{ball}[u]^\perp$ and $0 < t < 1$. Let μ and ν be representing measures for ξ and η , respectively. Put $\lambda = t\mu + (1 - t)\nu$. Then for any $f \in A$, we have

$$\int_X f d\lambda = t \int_X f d\mu + (1 - t) \int_X f d\nu = t\xi(f) + (1 - t)\eta(f) = \theta_{upq}(f).$$

This implies

$$|\theta_{upq}(f)| = \left| \int_X f d\lambda \right| \leq \int_X |f| d|\lambda| \leq \|f\| \|\lambda\|,$$

and so $\|\theta_{upq}\| \leq \|\lambda\|$. Also, $\|\mu\| = \|\xi\| \leq 1$ and $\|\nu\| = \|\eta\| \leq 1$, and hence

$$\|\lambda\| \leq t\|\mu\| + (1-t)\|\nu\| \leq 1 = \|\theta_{upq}\|.$$

Therefore, $\|\theta_{upq}\| = \|\lambda\|$. As a consequence, λ is a representing measure for θ_{upq} , and Lemma 2.2.11 shows that $\lambda = \lambda_{upq}$. Thus we obtain

$$\lambda_{upq} = t\mu + (1-t)\nu. \quad (2.7)$$

Since ξ and η belong to $[u]^\perp$, it follows that

$$\int_X u d\mu = \xi(u) = 0 \quad \text{and} \quad \int_X u d\nu = \eta(u) = 0.$$

Hence $\mu, \nu \in \text{ball } M([u]^\perp)$. Recall from Lemma 2.2.8 that λ_{upq} is an extreme point of $\text{ball } M([u]^\perp)$. Then (2.7) leads to $\lambda_{upq} = \mu = \nu$. Thus we have

$$\theta_{upq}(f) = \int_X f d\lambda_{upq} = \int_X f d\mu = \xi(f)$$

for all $f \in A$, that is, $\theta_{upq} = \xi$. Similarly, we get $\theta_{upq} = \eta$. We reach the desired equation $\theta_{upq} = \xi = \eta$. \square

We investigate the distance $\|\xi - \eta\|$ for $\xi, \eta \in \text{ball } A^*$.

Lemma 2.2.14. *If p and q are distinct points in $\text{Ch}(A)$ and if $\alpha, \beta \in \mathbb{T}$, then*

$$\|\alpha e_p - \beta e_q\| = 2.$$

Proof. It is clear that $\|\alpha e_p - \beta e_q\| \leq 2$. For the reverse inequality, let $\varepsilon > 0$. Proposition 2.2.2 gives a function $f \in \text{ball } A$ such that $|f(p) - \bar{\alpha}| < \varepsilon$ and $|f(q) + \bar{\beta}| < \varepsilon$. Then we have

$$\begin{aligned} 2 - |\alpha e_p(f) - \beta e_q(f)| &\leq |\alpha e_p(f) - \beta e_q(f) - 2| = |\alpha(f(p) - \bar{\alpha}) - \beta(f(q) + \bar{\beta})| \\ &\leq |\alpha| |f(p) - \bar{\alpha}| + |\beta| |f(q) + \bar{\beta}| < \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Therefore, $\|\alpha e_p - \beta e_q\| \geq |\alpha e_p(f) - \beta e_q(f)| > 2 - 2\varepsilon$. Since ε is arbitrary, we obtain $\|\alpha e_p - \beta e_q\| \geq 2$. \square

Lemma 2.2.15. *Let $u \in A$. If the set $S(u) \cap \text{Ch}(A)$ contains at least three distinct points, then there exist extreme points ξ and η of $\text{ball}[u]^\perp$ such that*

- (i) $\|\xi - \eta\| < 2$, and
- (ii) ξ and η are linearly independent.

Proof. By hypothesis, we find three distinct points p, q and r in $S(u) \cap \text{Ch}(A)$. Then we may assume that

$$\arg u(p) \neq \arg(-u(q)). \quad (2.8)$$

For, if there exist no such points p and q , then three equations

$$\arg u(p) = \arg(-u(q)), \arg u(q) = \arg(-u(r)) \text{ and } \arg u(r) = \arg(-u(p))$$

hold simultaneously, which is impossible. Now, put $\xi = \theta_{upr}$ and $\eta = \theta_{uqr}$. By Lemma 2.2.13, ξ and η are extreme points of $\text{ball}[u]^\perp$.

Let us show (i). By (2.8),

$$\arg k_{urp} \neq \arg(-k_{urq}).$$

Therefore, the triangle inequality $|k_{urp} - k_{urq}| < |k_{urp}| + |k_{urq}|$ holds strictly. Hence we have

$$\begin{aligned} \|\xi - \eta\| &= \|\theta_{upr} - \theta_{uqr}\| = \|(k_{upr}e_p - k_{urp}e_r) - (k_{uqr}e_q - k_{urq}e_r)\| \\ &= \|k_{upr}e_p - (k_{urp} - k_{urq})e_r - k_{uqr}e_q\| \\ &\leq |k_{upr}| + |k_{urp} - k_{urq}| + |k_{uqr}| \\ &< |k_{upr}| + |k_{urp}| + |k_{urq}| + |k_{uqr}| = 2. \end{aligned}$$

To verify (ii), assume $\alpha\xi + \beta\eta = 0$ and $\alpha, \beta \in \mathbb{C}$. Then, for any $f \in A$, we have

$$\begin{aligned} 0 &= \alpha\xi(f) + \beta\eta(f) = \alpha(k_{upr}e_p(f) - k_{urp}e_r(f)) + \beta(k_{uqr}e_q(f) - k_{urq}e_r(f)) \\ &= \alpha k_{upr} f(p) - (\alpha k_{urp} + \beta k_{urq}) f(r) + \beta k_{uqr} f(q). \end{aligned}$$

Taking $f \in A$ so that $f(p) = 1$ and $f(q) = f(r) = 0$, we have $0 = \alpha k_{upr}$. Noting $k_{upr} \neq 0$, we get $\alpha = 0$. On the other hand, if we take $f \in A$ so that $f(q) = 1$ and $f(p) = f(r) = 0$, then we get $\beta = 0$. Thus ξ and η are linearly independent. \square

The preceding two lemmas yield the following lemma:

Lemma 2.2.16. *Let $u \in A$. If the set $S(u) \cap \text{Ch}(A)$ contains at least three distinct points, then $[u]^\perp$ is not linearly isometric to A^* .*

Proof. Assume that $[u]^\perp$ is linearly isometric to A^* . Then there is a linear isometry T of $[u]^\perp$ onto A^* . Consider extreme points ξ and η of $\text{ball}[u]^\perp$ described in Lemma 2.2.15. Then $T\xi$ and $T\eta$ are extreme points of $\text{ball}A^*$. By (2.1), there exist $p, q \in \text{Ch}(A)$ and $\alpha, \beta \in \mathbb{T}$ such that $T\xi = \alpha e_p$ and $T\eta = \beta e_q$.

If $p \neq q$, Lemma 2.2.14 implies that $\|T\xi - T\eta\| = \|\alpha e_p - \beta e_q\| = 2$. Since T is an isometry, $\|\xi - \eta\| = 2$, which contradicts Lemma 2.2.15(i).

On the other hand, if $p = q$, then we have

$$T(\beta\xi - \alpha\eta) = \beta T\xi - \alpha T\eta = \beta\alpha e_p - \alpha\beta e_q = \alpha\beta(e_p - e_p) = 0.$$

Since T is injective, it follows that $\beta\xi - \alpha\eta = 0$. Note that $\alpha, \beta \neq 0$. This contradicts the linear independence of ξ and η from Lemma 2.2.15(ii). Consequently, $[u]^\perp$ is not linearly isometric to A^* . \square

Let us consider a backward quasi-shift on A .

Lemma 2.2.17. *Let T be a backward quasi-shift on A . If $f \in \bigcup_{n=1}^{\infty} \ker T^n$, then $S(f) \cap \text{Ch}(A)$ is a finite set. In particular, if $\ker T = [u]$, then $S(u) \cap \text{Ch}(A)$ is finite.*

Proof. Since $\ker T$ is one-dimensional, we can write $\ker T = [u]$, where $u \in A$ and $u \neq 0$. Since the induced operator $\widehat{T} : f + [u] \mapsto Tf$ is a linear isometry of $A/[u]$ onto A , the adjoint operator \widehat{T}^* is a linear isometry of A^* onto $(A/[u])^*$. Note that $(A/[u])^*$ is linearly isometric to $[u]^\perp$, via the linear isometry $P : (A/[u])^* \rightarrow [u]^\perp$ defined by $(P\Phi)(f) = \Phi(f + [u])$ for all $f \in A$ and $\Phi \in (A/[u])^*$. Thus we have

$$((P\widehat{T}^*)\xi)(f) = (P(\widehat{T}^*\xi))(f) = (\widehat{T}^*\xi)(f + [u]) = \xi(\widehat{T}(f + [u])) = \xi(Tf) = (T^*\xi)(f)$$

for all $f \in A$ and $\xi \in A^*$. Hence $P\widehat{T}^* = T^*$, and so T^* is a linear isometry of A^* onto $[u]^\perp$.

Once we have seen that $[u]^\perp$ is linearly isometric to A^* , Lemma 2.2.16 says that the number of elements of $S(u) \cap \text{Ch}(A)$ is less than 2. Of course, $S(u) \cap \text{Ch}(A)$ is finite.

To prove the lemma, we show the following assertion for all $n = 1, 2, \dots$:

$$\text{If } f \in \ker T^n, \text{ then } S(f) \cap \text{Ch}(A) \text{ is a finite set.} \quad (2.9)$$

We adopt an induction on n .

First, consider the case $n = 1$. If $f \in \ker T = [u]$, then $f = \alpha u$ for some $\alpha \in \mathbb{C}$. Hence

$$S(f) \cap \text{Ch}(A) = S(\alpha u) \cap \text{Ch}(A) \subset S(u) \cap \text{Ch}(A).$$

Since $S(u) \cap \text{Ch}(A)$ is finite, so is $S(f) \cap \text{Ch}(A)$. Thus (2.9) is true when $n = 1$.

For the inductive step, assume that (2.9) is valid for some n . We must show that if $f \in \ker T^{n+1}$, then $S(f) \cap \text{Ch}(A)$ is finite. Put $g = Tf$. Then $g \in \ker T^n$, and the assumption (2.9) implies that $S(g) \cap \text{Ch}(A)$ is finite.

Consider the set Z of all $p \in \text{Ch}(A)$ such that there exist $q \in S(g) \cap \text{Ch}(A)$ and $\alpha \in \mathbb{T}$ satisfying $T^*(\alpha e_q) = e_p$. We know that for each $p \in Z$, the pair (q, α) as above is uniquely determined, because T^* is injective. Thus we can define the map $\pi : Z \rightarrow S(g) \cap \text{Ch}(A)$ by $\pi(p) = q$, where $p \in Z$, $q \in S(g) \cap \text{Ch}(A)$, $\alpha \in \mathbb{T}$ and $T^*(\alpha e_q) = e_p$. Let us show that π is injective. If not, there exist $p, p' \in Z$ such that $\pi(p) = \pi(p') (= q)$. Then $T^*(\alpha e_q) = e_p$ and $T^*(\alpha' e_q) = e_{p'}$ for some $\alpha, \alpha' \in \mathbb{T}$. Choose a function f so that $f(p) = 1$ and $f(p') = 0$. Then we have

$$1 = f(p) = e_p(f) = (T^*(\alpha e_q))(f) = \frac{\alpha}{\alpha'} (T^*(\alpha' e_q))(f) = \frac{\alpha}{\alpha'} e_{p'}(f) = \frac{\alpha}{\alpha'} f(p') = 0,$$

which is a contradiction. Hence $\pi : Z \rightarrow S(g) \cap \text{Ch}(A)$ is injective, and so the number of the elements of Z is less than that of the elements of $S(g) \cap \text{Ch}(A)$. Since $S(g) \cap \text{Ch}(A)$ is finite, so is Z .

Next, we show the inclusion:

$$S(f) \cap \text{Ch}(A) \subset (S(u) \cap \text{Ch}(A)) \cup Z. \quad (2.10)$$

For this, it suffices to show that if $p \in S(f) \cap \text{Ch}(A)$ and if $p \notin S(u)$, then $p \in Z$. Since $p \notin S(u)$, $e_p(u) = u(p) = 0$, and so $e_p \in [u]^\perp$. It is easy to see that e_p is an extreme point of ball $[u]^\perp$. Since T^* is a linear isometry of A^* onto $[u]^\perp$, we find an extreme point ξ of

ball A^* such that $T^*\xi = e_p$, and (2.1) gives the form $\xi = \alpha e_q$, where $q \in \text{Ch}(A)$ and $\alpha \in \mathbb{T}$. Thus $T^*(\alpha e_q) = e_p$. Also, $p \in S(f)$ implies

$$\alpha g(q) = \alpha e_q(g) = (\alpha e_q)(Tf) = (T^*(\alpha e_q))(f) = e_p(f) = f(p) \neq 0,$$

and so $q \in S(g)$. Thus we arrive at $p \in Z$, and the inclusion (2.10) is established.

We now know that both $S(u) \cap \text{Ch}(A)$ and Z are finite. Therefore, (2.10) implies that $S(f) \cap \text{Ch}(A)$ is finite. This accomplishes the inductive step and completes the proof. \square

2.2.3 Proofs of Theorems 2.1.1 and 2.1.2

We are now in a position to prove Theorems 2.1.1 and 2.1.2.

Proof of Theorem 2.1.1. Let A be an infinite-dimensional uniform algebra on a compact Hausdorff space X . The linear space $\{f|_{\text{Ch}(A)} : f \in A\}$ is isomorphic to A , and it is also infinite-dimensional. Hence $\text{Ch}(A)$ must have infinitely many points. Thus the compact set X contains an accumulation point p of $\text{Ch}(A)$. In other words, there exists a net $\{p_i\}$ consisting of infinitely many points of $\text{Ch}(A)$ such that $\{p_i\}$ converges to p .

Now, assume that there exists a backward shift T on A . From the comment in the definition of backward shift, we know that T is a backward quasi-shift on A . Let $f \in \bigcup_{n=1}^{\infty} \ker T^n$. By Lemma 2.2.17, the set $S(f) \cap \text{Ch}(A)$ is finite. So, we may assume that $\{p_i\} \subset \text{Ch}(A) \setminus S(f)$. Then, for each i , we have $f(p_i) = 0$, and the continuity of f shows that $f(p) = 0$. Thus we have

$$\|1 - f\| \geq |1 - f(p)| = 1.$$

Since this holds for all $f \in \bigcup_{n=1}^{\infty} \ker T^n$, the constant function 1 cannot lie in the closure of $\bigcup_{n=1}^{\infty} \ker T^n$. Hence, $\bigcup_{n=1}^{\infty} \ker T^n$ is not dense in A . This contradicts the fact that T is a backward shift, and the theorem is proved. \square

Proof of Theorem 2.1.2. Assume that there exists a backward quasi-shift T on A . Since $\ker T$ is one-dimensional, we can write $\ker T = [u]$, where $u \in A$ and $u \neq 0$. Note that $S(u)$ is open in X and that $S(u) \cap \text{Ch}(A)$ is finite by Lemma 2.2.17. We see that all points in $S(u) \cap \text{Ch}(A)$ are isolated points of $\text{Ch}(A)$. While, $u \neq 0$ implies that $S(u) \cap \text{Ch}(A)$ is non-empty. As a consequence, there exists at least one isolated point of $\text{Ch}(A)$. By Proposition 2.2.4, the maximal ideal space of A has at least one isolated point.

Now, let m be the number of isolated points of the maximal ideal space of A . We show that the dimension of $\ker T^{m+1}$ is less than m . By Proposition 2.2.4, the number of isolated points of $\text{Ch}(A)$ is m exactly. Write down all isolated points of $\text{Ch}(A)$ as p_1, \dots, p_m . For each $j = 1, \dots, m$, Proposition 2.2.3 gives us a function $f_j \in A$ such that $f_j(p_j) = 1$ and $f_j(x) = 0$ for all $x \in \text{Ch}(A) \setminus \{p_j\}$. Pick $f \in \ker T^{m+1}$ arbitrarily. By Lemma 2.2.17, $S(f) \cap \text{Ch}(A)$ is finite, and so we again see that all points in $S(f) \cap \text{Ch}(A)$ are isolated points of $\text{Ch}(A)$, that is, $S(f) \cap \text{Ch}(A) \subset \{p_1, \dots, p_m\}$. Hence, if we put $\alpha_j = f(p_j)$ for each $j = 1, \dots, m$, then

$$f|_{\text{Ch}(A)} = \alpha_1 f_1|_{\text{Ch}(A)} + \dots + \alpha_m f_m|_{\text{Ch}(A)} = (\alpha_1 f_1 + \dots + \alpha_m f_m)|_{\text{Ch}(A)},$$

which implies $f = \alpha_1 f_1 + \dots + \alpha_m f_m$. Thus every $f \in \ker T^{m+1}$ is written as a linear combination of f_1, \dots, f_m , and we conclude that the dimension of $\ker T^{m+1}$ is less than m .

Now note that

$$[u] = \ker T \subset \ker T^2 \subset \dots \subset \ker T^m \subset \ker T^{m+1}.$$

As a consequence of the preceding paragraph, we must have $\ker T^N = \ker T^{N+1}$ for some $N \in \{0, 1, \dots, m\}$. Since T^N , like T , is surjective, we find $h \in A$ with $T^N h = u$. Then $T^{N+1} h = T(T^N h) = Tu = 0$ and so $h \in \ker T^{N+1} = \ker T^N$. Hence $u = T^N h = 0$, a contradiction. \square

2.3 Examples

We examine the existence of isometric (quasi-)shift and backward (quasi-)shift in some concrete spaces. The first example gives a uniform algebra which admits an isometric shift and no backward quasi-shifts.

Example 2.3.1. Let $A(\mathbb{D})$ be the disc algebra, that is, the uniform algebra of all continuous functions on the closed unit disc \mathbb{D} which are analytic in the open unit disc. The isometric shifts on $A(\mathbb{D})$ are characterized by Takayama and Wada [47]. A typical example of it is the multiplication operator S :

$$(Sf)(z) = zf(z) \quad (z \in \mathbb{D}, f \in A(\mathbb{D})).$$

This example suggests to us that the following operator T may be a backward shift on $A(\mathbb{D})$:

$$(Tf)(z) = \begin{cases} \frac{f(z)-f(0)}{z} & \text{if } z \neq 0, \\ f'(0) & \text{if } z = 0, \end{cases} \quad (f \in A(\mathbb{D})).$$

It is easy to see that T is surjective and satisfies the conditions (ii)' and (iii)' in the definition of backward shift. However, T does not satisfy (i)'. Indeed, $\ker T$ is the subspace of constant functions, and the function $f(z) = z^2 + z$ satisfies that

$$\inf\{\|f + g\| : g \in \ker T\} \leq \left\| f - \frac{1}{2} \right\| = \sqrt{\frac{27}{8}} < 2 = \|Tf\|.$$

Hence T is not a backward shift. Moreover, Theorem 2.1.2 implies that $A(\mathbb{D})$ does not admit a backward quasi-shift, because the maximal ideal space of $A(\mathbb{D})$ is homeomorphic to \mathbb{D} and it has no isolated points.

The second example gives a uniform algebra which admits neither isometric quasi-shifts nor backward quasi-shifts.

Example 2.3.2. It is known that $C([0, 1])$ and $C(\mathbb{T})$ admit no isometric quasi-shifts, where \mathbb{T} is the unit circle in \mathbb{C} ([15, Theorem 2.2] and [46, Corollary 1]). Hence $C([0, 1])$ and $C(\mathbb{T})$ admit no isometric shifts. Moreover, Theorem 2.1.2 implies that $C([0, 1])$ and $C(\mathbb{T})$ admit no backward quasi-shifts, because $[0, 1]$ and \mathbb{T} have no isolated points.

Next, we consider the spaces of differentiable functions on $[0, 1]$. Holub proved that $C^{(n)}[0, 1]$ and $\text{Lip}[0, 1]$ admit no isometric shifts if $C^{(n)}[0, 1]$ and $\text{Lip}[0, 1]$ consist of *real-valued* functions and have the norm $\|f\| = \|f\|_\infty + \alpha(f)$, where α is a semi-norm and $\alpha(1) = 0$ ([17, Theorem 2.3]). Thus the spaces $(C^{(n)}[0, 1], \|\cdot\|_\Sigma)$ and $(\text{Lip}[0, 1], \|\cdot\|_\Sigma)$ with the *real-valued* functions admit no isometric shifts. Here we take up the spaces $(C^{(n)}[0, 1], \|\cdot\|_m)$ and $(\text{Lip}[0, 1], \|\cdot\|_m)$.

Example 2.3.3. The Banach spaces $(C^{(n)}[0, 1], \|\cdot\|_m)$ and $(\text{Lip}[0, 1], \|\cdot\|_m)$ also admit neither isometric quasi-shifts nor backward quasi-shifts. Indeed, write \mathcal{B} for $(C^{(n)}[0, 1], \|\cdot\|_m)$ or $(\text{Lip}[0, 1], \|\cdot\|_m)$. By Theorems 1.2.2 and 1.3.2, every finite codimensional linear isometry on \mathcal{B} is surjective. Thus there are no isometric quasi-shifts on \mathcal{B} . Moreover, by Lemmas 1.2.22 and 1.3.18, \mathcal{B} is linearly isometric to $C(X)$ for some compact Hausdorff space X with at most finitely many isolated points. Hence, by Theorem 2.1.2, \mathcal{B} admits no backward quasi-shifts.

Let $(\Omega, \mathfrak{B}, \mu)$ be a positive measure space. It is known that if μ has no atoms, then $L^p(\Omega, \mathfrak{B}, \mu)$ admits neither isometric quasi-shifts nor backward shifts, where $1 \leq p < \infty, p \neq 2$ ([15, Corollary 4.1] and [38, Theorems 2.1 and 2.3]). In the next example, we deal with the space $L^\infty(\Omega, \mathfrak{B}, \mu)$.

Example 2.3.4. Let $(\Omega, \mathfrak{B}, \mu)$ be a positive measure space. By Corollary 1.3.22, every finite codimensional linear isometry on $L^\infty(\Omega, \mathfrak{B}, \mu)$ is surjective. Thus there are no isometric quasi-shifts on $L^\infty(\Omega, \mathfrak{B}, \mu)$. Hence there are no isometric shifts on $L^\infty(\Omega, \mathfrak{B}, \mu)$.

Recall from the proof of Corollary 1.3.22 that $L^\infty(\Omega, \mathfrak{B}, \mu)$ is linearly isometric to $C(\mathfrak{M}_{L^\infty})$. Applying Theorem B to $C(\mathfrak{M}_{L^\infty})$, we see that if $L^\infty(\Omega, \mathfrak{B}, \mu)$ is infinite-dimensional, then $L^\infty(\Omega, \mathfrak{B}, \mu)$ does not admit a backward shift. In particular, if the measure μ has at most finitely many atoms, then \mathfrak{M}_{L^∞} has at most finitely many isolated points, and so Theorem 2.1.2 shows that $L^\infty(\Omega, \mathfrak{B}, \mu)$ does not admit a backward quasi-shift. For example, $L^\infty[0, 1]$ does not admit a backward quasi-shift.

Next, we consider the sequence spaces $\ell^\infty(\mathbb{N})$, c , c_0 and $\ell^p(\mathbb{N})$ ($1 \leq p < \infty$). The space $\ell^\infty(\mathbb{N})$ is an example of the case that μ has infinitely many atoms in Example 2.3.4. We will see that $\ell^\infty(\mathbb{N})$ and c admit a backward quasi-shift but not a backward shift.

Example 2.3.5. By $\ell^\infty(\mathbb{N})$, c and c_0 , we denote the space of all sequences that are bounded, converge and converge to zero, respectively. They are Banach spaces with respect to the supremum norm. For $1 \leq p < \infty$, $\ell^p(\mathbb{N})$ denotes the Banach space of all sequences $x = \{x_n\}$ such that $\sum_{n=1}^\infty |x_n|^p < \infty$, equipped with the norm $\|x\| = (\sum_{n=1}^\infty |x_n|^p)^{1/p}$. We know that $\ell^\infty(\mathbb{N})$, c , c_0 and $\ell^p(\mathbb{N})$ admit an isometric shift ([44, Example 4.1]).

Note that $\ell^\infty(\mathbb{N})$ and c are unital commutative C^* -algebras in the complex case. Using the Gelfand-Naimark theorem, we see that $\ell^\infty(\mathbb{N})$ and c are linearly isometric to $C(X_1)$ and $C(X_2)$ for some compact Hausdorff spaces X_1 and X_2 , respectively. Since $\ell^\infty(\mathbb{N})$ and c are infinite-dimensional, Theorem B implies that $\ell^\infty(\mathbb{N})$ and c admit no backward shifts.

Next, consider the operator $T : (x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$. We can easily check that T is a backward quasi-shift on each space $\ell^\infty(\mathbb{N})$, c , c_0 and $\ell^p(\mathbb{N})$. However, T cannot be a backward shift on $\ell^\infty(\mathbb{N})$ and c , as we saw above. Indeed, T does not satisfy (iii)'. On the other hand, the other spaces c_0 and $\ell^p(\mathbb{N})$ exhibit a different aspect. It is easily seen that T is a backward shift on c_0 and $\ell^p(\mathbb{N})$ ([44, Example 4.2]). Hence c_0 and $\ell^p(\mathbb{N})$ admit a backward quasi-shift.

Here, we ask whether there exists a backward shift on a unital commutative Banach algebra which is not a uniform algebra. The next example answers "Yes".

Example 2.3.6. Recall the Wiener algebra W . It is known that W is a unital commutative Banach algebra which is not a uniform algebra. As we saw in the proof of Theorem 1.4.1, W is linearly isometric to $\ell^1(\mathbb{Z})$. Consider the operators T_1 and T_2 on $\ell^1(\mathbb{Z})$:

$$T_1 : (\dots, x_{-3}, x_{-2}, x_{-1}, \underset{\hat{0}}{x_0}, x_1, x_2, x_3, \dots) \mapsto (\dots, x_3, x_2, x_1, \underset{\hat{0}}{0}, x_0, x_{-1}, x_{-2}, \dots),$$

$$T_2 : (\dots, x_{-3}, x_{-2}, x_{-1}, \underset{\hat{0}}{x_0}, x_1, x_2, x_3, \dots) \mapsto (\dots, x_4, x_3, x_2, \underset{\hat{0}}{x_1}, x_{-1}, x_{-2}, x_{-3} \dots).$$

We see that T_1 and T_2 are an isometric shift and a backward shift on $\ell^1(\mathbb{Z})$, respectively. Hence W admits an isometric shift and a backward shift.

Finally, we summarize these observations in the following table.

Does there exist an isometric (quasi-)shift or a backward (quasi-)shift on the space?

Space	Isometric shift	Isometric quasi-shift	Backward shift	Backward quasi-shift
Uniform algebra	Yes or No	Yes or No	No	Yes or No
Disc algebra	Yes	Yes	No	No
$C([0, 1]), C(\mathbb{T})$	No	No	No	No
$(C^{(n)}[0, 1], \ \cdot\ _m)$	No	No	No	No
$(\text{Lip}[0, 1], \ \cdot\ _m)$	No	No	No	No
$L^\infty[0, 1]$	No	No	No	No
ℓ^∞	Yes	Yes	No	Yes
c	Yes	Yes	No	Yes
c_0	Yes	Yes	Yes	Yes
ℓ^p ($1 \leq p < \infty$)	Yes	Yes	Yes	Yes
Wiener algebra	Yes	Yes	Yes	Yes

Chapter 3

Real-linear isometries between complex function spaces

3.1 Introduction

Our final purpose is a characterization of general isometries which are not necessarily linear. For this purpose, the Mazur-Ulam theorem is our good tool. It states: *If T is a surjective isometry between normed linear spaces, then $T - T0$ is real-linear.* Thus we turn our attention to the *real-linear* isometries. So far, we have dealt with the \mathbb{K} -linear isometries between \mathbb{K} -linear spaces. In this chapter, we consider the *real-linear* isometries between *complex-linear* spaces.

Let X be a locally compact Hausdorff space. We denote by $C_{\mathbb{C},0}(X)$ the Banach space of all complex-valued continuous functions on X which vanish at infinity, equipped with the supremum norm. If X is compact, we write $C_{\mathbb{C}}(X)$ instead of $C_{\mathbb{C},0}(X)$. A *complex function space* on X means a nonzero complex-linear subspace of $C_{\mathbb{C},0}(X)$, which is understood to be equipped the supremum norm and need not be closed in its topology.

Let A be a complex function space on X . For each $x \in X$, the *evaluation functional* e_x is defined by $e_x(f) = f(x)$ for all $f \in A$. We define the *Choquet boundary* $\text{Ch}(A)$ for A as

$$\text{Ch}(A) = \{x \in X : e_x \text{ is an extreme point of ball } A^*\}.$$

By definition, we see that for each $x \in \text{Ch}(A)$ there exists $f \in A$ such that $f(x) \neq 0$. Also, we know that $\text{Ch}(A)$ is a boundary for A , that is, given $f \in A$, there exists $x \in \text{Ch}(A)$ such that $|f(x)| = \|f\|$ ([12, Theorem 2.3.8]).

We say that A is *strongly separating* if for each pair of distinct points $x, y \in X$ there exists $f \in A$ such that $|f(x)| \neq |f(y)|$. Also, we introduce a more restricted separation: We say that A is *strongly triple-separating* if for each triple of distinct points $x, y, z \in \text{Ch}(A)$ there exists $f \in A$ such that $|f(x)| \neq |f(y)|$ and $f(z) = 0$.

In this chapter, we give a characterization of surjective real-linear isometries between complex function spaces, as follows:

Theorem 3.1.1. *Let A and B be complex function spaces on locally compact Hausdorff spaces X and Y , respectively. Suppose that A and B are strongly separating and that A is strongly triple-separating. If T is a real-linear isometry of A onto B , then there exist a*

(possibly empty) open and closed subset E of $\text{Ch}(B)$, a homeomorphism φ of $\text{Ch}(B)$ onto $\text{Ch}(A)$ and a unimodular continuous function ω on $\text{Ch}(B)$ such that

$$(Tf)(y) = \begin{cases} \omega(y)f(\varphi(y)) & \text{if } y \in E, \\ \omega(y)\overline{f(\varphi(y))} & \text{if } y \in \text{Ch}(B) \setminus E, \end{cases}$$

for all $f \in A$.

In [11], Ellis proved the similar statement on the setting where X and Y are compact and A is a uniform algebra on X . In general, a closed subalgebra of $C_{\mathbb{C},0}(X)$ which separates the points of X is called a *function algebra* on X . In [30], Miura investigated the case where A and B are strongly separating function algebras. In any case, we can verify that A is strongly triple-separating (see Proposition 3.4.2 in §3.4). Thus the above theorem is a generalization of the theorems in [11] and [30]. (In [30], the definition of $\text{Ch}(A)$ is different, but it agrees with our definition ([40, Theorem 2.1])).

In Section 3.2, we investigate the property of the strongly triple-separating complex function space. By using it, we give a new statement of Theorem 3.1.1 in order to prove it conveniently. This is stated in Theorem 3.2.2 and proved in Section 3.3. We will see from Example 3.4.5 that Theorem 3.2.2 is a refinement of Theorem 3.1.1. In Section 3.4, we obtain many complex function spaces which Theorems 3.1.1 and 3.2.2 are applied to. In Section 3.5, we use Theorem 3.2.2 to characterize the complex-linear isometries (Corollary 3.5.1). Unfortunately, our corollary does not lead to the result by Araujo and Font [1]. This curious situation will be explained in Example 3.5.2. In Section 3.6, we give a characterization of surjective isometries between complex function spaces.

3.2 General theorem

We begin with the property of the strongly triple-separating complex function space. Put $\mathbb{T} = \{\alpha \in \mathbb{C} : |\alpha| = 1\}$.

Proposition 3.2.1. *Let A be a strongly triple-separating complex function space on a locally compact Hausdorff space X . If $x, x' \in \text{Ch}(A)$, if $\alpha, \beta \in \mathbb{T}$ and if $(s\alpha)e_x + (t\beta)e_{x'}$ is an extreme point of ball A^* for all $s, t \in \mathbb{R}$ with $s^2 + t^2 = 1$, then $\beta e_{x'} = \pm i\alpha e_x$.*

For the proof, we need the following known fact:

Let A be a complex function space on a locally compact Hausdorff space X and let $\xi \in A^$. Then ξ is an extreme point of ball A^* if and only if $\xi = \alpha e_x$ for some $x \in \text{Ch}(A)$ and $\alpha \in \mathbb{T}$.* (3.1)

The “if” part follows immediately from the definition of $\text{Ch}(A)$. The proof of the “only if” part may be found in [12, Corollary 2.3.6].

Proof. Let $x, x' \in \text{Ch}(A)$ and $\alpha, \beta \in \mathbb{T}$. Suppose that $(s\alpha)e_x + (t\beta)e_{x'}$ is an extreme point of ball A^* for all $s, t \in \mathbb{R}$ with $s^2 + t^2 = 1$. Take $s = t = 1/\sqrt{2}$. Since $(1/\sqrt{2})(\alpha e_x + \beta e_{x'})$ is an extreme point of ball A^* , it follows from (3.1) that

$$\alpha e_x + \beta e_{x'} = \sqrt{2}\gamma e_{x''}. \quad (3.2)$$

for some $x'' \in \text{Ch}(A)$ and $\gamma \in \mathbb{T}$.

Assume that x, x' and x'' are distinct. Since A is strongly triple-separating, there exists $f \in A$ such that $|f(x)| \neq |f(x')|$ and $f(x'') = 0$. By (3.2), we have

$$\alpha f(x) + \beta f(x') = (\alpha e_x + \beta e_{x'})(f) = \sqrt{2}\gamma e_{x''}(f) = \sqrt{2}\gamma f(x'') = 0$$

and so

$$|f(x)| = |\alpha f(x)| = |-\beta f(x')| = |f(x')|,$$

which is a contradiction. Hence x, x' and x'' are not distinct. Consequently, we have at least one of $x = x', x = x''$ and $x' = x''$.

Assume that $x = x'$. Then (3.2) becomes $(\alpha + \beta)e_x = \sqrt{2}\gamma e_{x''}$. Hence

$$2 = \|\sqrt{2}\gamma e_{x''}\|^2 = \|(\alpha + \beta)e_x\|^2 = |\alpha + \beta|^2 = 1 + 2\text{Re}(\bar{\alpha}\beta) + 1,$$

and so $\text{Re}(\bar{\alpha}\beta) = 0$. Since $|\bar{\alpha}\beta| = 1$, we obtain $\bar{\alpha}\beta = \pm i$, namely $\beta = \pm i\alpha$. Hence $\beta e_{x'} = \pm i\alpha e_{x'} = \pm i\alpha e_x$.

Next, we assume that $x = x''$. Then (3.2) becomes $\beta e_{x'} = (\sqrt{2}\gamma - \alpha)e_x$. Hence

$$1 = \|\beta e_{x'}\|^2 = \|(\sqrt{2}\gamma - \alpha)e_x\|^2 = |\sqrt{2}\gamma - \alpha|^2 = 2 - 2\sqrt{2}\text{Re}(\gamma\bar{\alpha}) + 1,$$

and so $\text{Re}(\gamma\bar{\alpha}) = 1/\sqrt{2}$. Since $|\gamma\bar{\alpha}| = 1$, we obtain $\gamma\bar{\alpha} = (1 \pm i)/\sqrt{2}$, namely $\sqrt{2}\gamma = (1 \pm i)\alpha$. Therefore,

$$\beta e_{x'} = (\sqrt{2}\gamma - \alpha)e_x = \pm i\alpha e_x.$$

For the case of $x' = x''$, we can show that $\beta e_{x'} = \pm i\alpha e_x$ similarly. \square

Remark. In the third paragraph of the above proof, we saw the fact that if $|\alpha| = |\beta| = 1$ and $|\alpha + \beta| = \sqrt{2}$, then $\beta = \pm i\alpha$.

Thus Theorem 3.1.1 becomes a special case of the next general theorem:

Theorem 3.2.2. *Let A and B be complex function spaces on locally compact Hausdorff spaces X and Y , respectively. Suppose that A is strongly separating and satisfies the following condition:*

$$\text{If } x, x' \in \text{Ch}(A), \text{ if } \alpha, \beta \in \mathbb{T} \text{ and if } (s\alpha)e_x + (t\beta)e_{x'} \text{ is an extreme point of ball } A^* \text{ for all } s, t \in \mathbb{R} \text{ with } s^2 + t^2 = 1, \text{ then } \beta e_{x'} = \pm i\alpha e_x. \quad (3.3)$$

If T is a real-linear isometry of A onto B , then there exist a (possibly empty) open and closed subset E of $\text{Ch}(B)$, a continuous mapping φ of $\text{Ch}(B)$ onto $\text{Ch}(A)$ and a unimodular continuous function ω on $\text{Ch}(B)$ such that

$$(Tf)(y) = \begin{cases} \omega(y)f(\varphi(y)) & \text{if } y \in E, \\ \omega(y)\overline{f(\varphi(y))} & \text{if } y \in \text{Ch}(B) \setminus E, \end{cases} \quad (3.4)$$

for all $f \in A$. If, in addition, B is strongly separating, then the mapping φ becomes a homeomorphism of $\text{Ch}(B)$ onto $\text{Ch}(A)$.

3.3 Proof of Theorem 3.2.2

We begin with the general lemma:

Lemma 3.3.1. *Let A and B be complex normed linear spaces with (complex) dual spaces A^* and B^* , respectively. If T is a real-linear isometry of A onto B , then there exists a real-linear isometry T_* of B^* onto A^* such that*

$$\operatorname{Re}(T_*\eta)(f) = \operatorname{Re}\eta(Tf) \quad (f \in A, \eta \in B^*). \quad (3.5)$$

Proof. When we regard A as a real-linear space, we denote the (real) dual space of A by A_r^* . For each $\xi \in A^*$, $\operatorname{Re}\xi$ belongs to A_r^* . This correspondence induces a real-linear isometry $P_A : \xi \mapsto \operatorname{Re}\xi$ of A^* onto A_r^* (cf. [42, Proposition 5.17]). For B , we will use the similar notations B_r^* and P_B .

With each $\rho \in B_r^*$, we associate a functional $T_r\rho \in A_r^*$ defined by

$$(T_r\rho)(f) = \rho(Tf) \quad (f \in A).$$

Since T is a real-linear isometry of A onto B , it is seen that T_r is a real-linear isometry of B_r^* onto A_r^* . Now, put $T_* = P_A^{-1}T_rP_B$. Then T_* is a real-linear isometry of B^* onto A^* and $P_AT_* = T_rP_B$. Hence we have

$$\operatorname{Re}(T_*\eta)(f) = (P_AT_*\eta)(f) = (T_rP_B\eta)(f) = (P_B\eta)(Tf) = \operatorname{Re}\eta(Tf)$$

for all $f \in A$ and $\eta \in B^*$. □

The rest of this section is devoted to the proof of Theorem 3.2.2. Throughout the proof, A , B , X and Y are as in Theorem 3.2.2. Suppose that T is a real-linear isometry of A onto B . Let T_* be the corresponding real-linear isometry of B^* onto A^* described in Lemma 3.3.1.

Lemma 3.3.2. *For each $y \in \operatorname{Ch}(B)$, there exist a unique $x \in \operatorname{Ch}(A)$ and a unique $\alpha \in \mathbb{T}$ such that $T_*e_y = \alpha e_x$*

Proof. Let $y \in \operatorname{Ch}(B)$. Then e_y is an extreme point of ball B^* . Since T_* is a real-linear isometry of B^* onto A^* , T_*e_y is an extreme point of ball A^* . By (3.1), there exist $x \in \operatorname{Ch}(A)$ and $\alpha \in \mathbb{T}$ such that $T_*e_y = \alpha e_x$.

Let us show the uniqueness of x and α . Suppose that $T_*e_y = \alpha'e_{x'}$ for some $x' \in \operatorname{Ch}(A)$ and $\alpha' \in \mathbb{T}$. Then for each $f \in A$, we have

$$\alpha f(x) = \alpha e_x(f) = (T_*e_y)(f) = \alpha'e_{x'}(f) = \alpha'f(x'), \quad (3.6)$$

and so $|f(x)| = |f(x')|$. Since A is strongly separating, this implies that $x = x'$. Thus (3.6) becomes $\alpha f(x) = \alpha'f(x)$ for all $f \in A$. Taking $f \in A$ so that $f(x) \neq 0$, we get $\alpha = \alpha'$. □

Lemma 3.3.3. *For each $y \in \operatorname{Ch}(B)$, $T_*(ie_y) = iT_*e_y$ or $T_*(ie_y) = -iT_*e_y$.*

Proof. Let $y \in \text{Ch}(B)$. In Lemma 3.3.2, we have just found $x \in \text{Ch}(A)$ and $\alpha \in \mathbb{T}$ such that $T_*e_y = \alpha e_x$. Next, note that ie_y is also an extreme point of ball B^* . A similar argument gives $x' \in \text{Ch}(A)$ and $\beta \in \mathbb{T}$ such that $T_*(ie_y) = \beta e_{x'}$. Choose $s, t \in \mathbb{R}$ so that $s^2 + t^2 = 1$. Since T_* is real-linear, it follows that

$$(s\alpha)e_x + (t\beta)e_{x'} = sT_*e_y + tT_*(ie_y) = T_*((s + it)e_y).$$

Here $s + it \in \mathbb{T}$, and so $(s + it)e_y$ is an extreme point of ball B^* . Therefore we see that $(s\alpha)e_x + (t\beta)e_{x'}$ is an extreme point of ball A^* . By the assumption (3.3), we obtain $\beta e_{x'} = \pm i\alpha e_x$, that is, $T_*(ie_y) = \pm iT_*e_y$. \square

Definition 3.3.4. Put

$$E = \{y \in \text{Ch}(B) : T_*(ie_y) = iT_*e_y\}.$$

By Lemma 3.3.3, $\text{Ch}(B) \setminus E = \{y \in \text{Ch}(B) : T_*(ie_y) = -iT_*e_y\}$.

Lemma 3.3.5. *For each $x \in \text{Ch}(A)$, there exist $y \in \text{Ch}(B)$ and $\alpha \in \mathbb{T}$ such that $T_*e_y = \alpha e_x$.*

Proof. Let $x \in \text{Ch}(A)$. Then e_x is an extreme point of ball A^* . Since the inverse T_*^{-1} is a real-linear isometry of A^* onto B^* , $T_*^{-1}e_x$ is an extreme point of ball B^* . Hence (3.1) gives $y \in \text{Ch}(B)$ and $\beta = a + ib \in \mathbb{T}$ such that $T_*^{-1}e_x = \beta e_y$. Assume that $y \in \text{Ch}(B) \setminus E$. Using the real-linearity of T_* and the equation $T_*(ie_y) = -iT_*e_y$, we see that

$$e_x = T_*T_*^{-1}e_x = T_*(\beta e_y) = T_*(ae_y + ibe_y) = aT_*e_y - ibT_*e_y = \bar{\beta}T_*e_y.$$

Hence $T_*e_y = \alpha e_x$ with $\alpha = \beta$. For the case of $y \in E$, we put $\alpha = \bar{\beta}$ to get $T_*e_y = \alpha e_x$. \square

Lemma 3.3.6. *For each $y \in \text{Ch}(B)$, let $x \in \text{Ch}(A)$ and $\alpha \in \mathbb{T}$ be as in Lemma 3.3.2. Then*

$$(Tf)(y) = \begin{cases} \alpha f(x) & \text{if } y \in E, \\ \frac{\alpha f(x)}{\alpha} & \text{if } y \in \text{Ch}(B) \setminus E, \end{cases} \quad (3.7)$$

for all $f \in A$.

Proof. Pick $f \in A$. Using (3.5), we have

$$\begin{aligned} \text{Re}(Tf)(y) &= \text{Re } e_y(Tf) = \text{Re}(T_*e_y)(f). \\ \text{Im}(Tf)(y) &= -\text{Re } i(Tf)(y) = -\text{Re}(ie_y(Tf)) = -\text{Re}(T_*(ie_y))(f) \\ &= \begin{cases} -\text{Re}(iT_*e_y)(f) = \text{Im}(T_*e_y)(f) & \text{if } y \in E, \\ -\text{Re}(-iT_*e_y)(f) = -\text{Im}(T_*e_y)(f) & \text{if } y \in \text{Ch}(B) \setminus E. \end{cases} \end{aligned}$$

Hence

$$(Tf)(y) = \begin{cases} (T_*e_y)(f) & \text{if } y \in E, \\ \frac{(T_*e_y)(f)}{\alpha} & \text{if } y \in \text{Ch}(B) \setminus E. \end{cases}$$

Since $(T_*e_y)(f) = (\alpha e_x)(f) = \alpha f(x)$, we arrive at (3.7). \square

Definition 3.3.7. To each $y \in \text{Ch}(B)$, we associate a unique $x \in \text{Ch}(A)$ and a unique $\alpha \in \mathbb{T}$, as in Lemma 3.3.2, and write

$$x = \varphi(y) \quad \text{and} \quad \alpha = \begin{cases} \omega(y) & \text{if } y \in E, \\ \overline{\omega(y)} & \text{if } y \in \text{Ch}(B) \setminus E. \end{cases}$$

Then φ is a mapping of $\text{Ch}(B)$ into $\text{Ch}(A)$ and ω is a unimodular function on $\text{Ch}(B)$. By Lemma 3.3.5, φ is surjective. Also, Lemma 3.3.6 says that for each $f \in A$,

$$(Tf)(y) = \begin{cases} \omega(y)f(\varphi(y)) & \text{if } y \in E, \\ \overline{\omega(y)f(\varphi(y))} & \text{if } y \in \text{Ch}(B) \setminus E. \end{cases} \quad (3.8)$$

Hence

$$|(Tf)(y)| = |f(\varphi(y))| \quad (y \in \text{Ch}(B)). \quad (3.9)$$

Lemma 3.3.8. E is open and closed in $\text{Ch}(B)$.

Proof. We first observe that

$$E = \bigcap_{f \in A} \{y \in \text{Ch}(B) : (T(if))(y) = i(Tf)(y)\}, \quad (3.10)$$

$$\text{Ch}(B) \setminus E = \bigcap_{f \in A} \{y \in \text{Ch}(B) : (T(if))(y) = -i(Tf)(y)\}. \quad (3.11)$$

If $y \in E$, then (3.8) implies $(T(if))(y) = \omega(y)(if)(\varphi(y)) = i\omega(y)f(\varphi(y)) = i(Tf)(y)$ for all $f \in A$. On the other hand, if $y \in \text{Ch}(B) \setminus E$, then $(T(if))(y) = \overline{\omega(y)(if)(\varphi(y))} = -i\overline{\omega(y)f(\varphi(y))} = -i(Tf)(y)$ for all $f \in A$. Hence E and $\text{Ch}(B) \setminus E$ are contained in the sets on the right sides of (3.10) and (3.11), respectively. Moreover, for each $y \in \text{Ch}(B)$, there exists $f \in A$ such that $(Tf)(y) \neq 0$, because T is surjective. This implies that two intersections in (3.10) and (3.11) are disjoint. Thus we obtain (3.10) and (3.11).

For each $f \in A$, the set $\{y \in \text{Ch}(B) : (T(if))(y) = i(Tf)(y)\}$ is closed in $\text{Ch}(B)$, because $T(if)$ and Tf are continuous on Y . Hence (3.10) implies that E is closed in $\text{Ch}(B)$. Similarly, (3.11) implies that $\text{Ch}(B) \setminus E$ is closed in $\text{Ch}(B)$. Thus E is open and closed in $\text{Ch}(B)$. \square

Lemma 3.3.9. φ is continuous on $\text{Ch}(B)$.

Proof. Assume, to get a contradiction, that φ is not continuous at some $y \in \text{Ch}(B)$. Then there exist a net $\{y_\mu\} \subset \text{Ch}(B)$ and an open neighborhood U of $\varphi(y)$ in X such that $y_\mu \rightarrow y$ and $\varphi(y_\mu) \notin U$ for all μ . Regard $\{\varphi(y_\mu)\}$ as a net in the one-point compactification $X_\infty = X \cup \{x_\infty\}$ of X . Since $X_\infty \setminus U$ is compact, there exist a subnet $\{\varphi(y_\nu)\}$ of $\{\varphi(y_\mu)\}$ and $z \in X_\infty \setminus U$ such that $\varphi(y_\nu) \rightarrow z$ ([24, Theorem 5.2]). Then $\varphi(y) \neq z$. Thus we can find $f \in A$ such that

$$|f(\varphi(y))| \neq |f(z)|. \quad (3.12)$$

For, if $z = x_\infty$, then there exists $f \in A$ such that $f(\varphi(y)) \neq 0 = f(z)$, while if $z \in X$, then the strong separation of A gives $f \in A$ satisfying (3.12). On the other hand, we use (3.9) and the continuity of Tf and f to see that

$$|(Tf)(y_\nu)| \rightarrow |(Tf)(y)| = |f(\varphi(y))| \quad \text{and} \quad |(Tf)(y_\nu)| = |f(\varphi(y_\nu))| \rightarrow |f(z)|.$$

Hence $|f(\varphi(y))| = |f(z)|$, in contradiction to (3.12). Thus φ is continuous on $\text{Ch}(B)$. \square

Lemma 3.3.10. ω is continuous on $\text{Ch}(B)$.

Proof. Let $y \in \text{Ch}(B)$. Choose $f \in A$ so that $f(\varphi(y)) \neq 0$, and put $U = \{x \in \text{Ch}(A) : f(x) \neq 0\}$. We first consider the case of $y \in E$. Since φ is continuous and E is open, the set $\varphi^{-1}(U) \cap E$ is an open neighborhood of y . By (3.8), we have $\omega = Tf/(f \circ \varphi)$ on $\varphi^{-1}(U) \cap E$. Since Tf , f and φ are continuous, ω is also continuous on $\varphi^{-1}(U) \cap E$, that is, around the point y . For the case of $y \in \text{Ch}(B) \setminus E$, we can show that ω is continuous on $\varphi^{-1}(U) \setminus E$, that is, around the point y . Thus ω is continuous on $\text{Ch}(B)$. \square

Lemma 3.3.11. If B is strongly separating, then φ is a homeomorphism.

Proof. We first observe that φ is injective. To do this, suppose that $y, y' \in \text{Ch}(B)$ and $\varphi(y) = \varphi(y')$. For each $g \in B$, there exists $f \in A$ such that $Tf = g$, because T is surjective. Using (3.9), we have

$$|g(y)| = |(Tf)(y)| = |f(\varphi(y))| = |f(\varphi(y'))| = |(Tf)(y')| = |g(y')|.$$

Since this holds for all $g \in B$, the strong separation of B implies that $y = y'$. Hence φ is injective.

Once φ is bijective, we have the inverse $\varphi^{-1} : \text{Ch}(A) \rightarrow \text{Ch}(B)$. To finish this lemma, it suffices to show that φ^{-1} is continuous on $\text{Ch}(A)$. Conversely, assume that φ^{-1} is not continuous at some $x \in \text{Ch}(A)$. As in the proof of Lemma 3.3.9, we find a net $\{x_\nu\} \subset \text{Ch}(A)$ and a point z in the one-point compactification $Y_\infty = Y \cup \{\infty\}$ of Y such that $x_\nu \rightarrow x$, $\varphi^{-1}(x_\nu) \rightarrow z$ and $\varphi^{-1}(x) \neq z$. Moreover, we use the strong separation of B to find $g \in B$ such that

$$|g(\varphi^{-1}(x))| \neq |g(z)|. \quad (3.13)$$

If we take $f \in A$ so that $Tf = g$, then (3.9) and the continuity of f and g imply

$$\begin{aligned} |f(x_\nu)| &\rightarrow |f(x)| = |f(\varphi(\varphi^{-1}(x)))| = |(Tf)(\varphi^{-1}(x))| = |g(\varphi^{-1}(x))|, \\ |f(x_\nu)| &= |f(\varphi(\varphi^{-1}(x_\nu)))| = |(Tf)(\varphi^{-1}(x_\nu))| = |g(\varphi^{-1}(x_\nu))| \rightarrow |g(z)|. \end{aligned}$$

Hence $|g(\varphi^{-1}(x))| = |g(z)|$, which contradicts (3.13). Thus φ^{-1} is continuous on $\text{Ch}(A)$. \square

Noting Lemmas 3.3.8–3.3.11 and Equation (3.8), we establish the theorem.

3.4 Examples

In this section, we exhibit some examples of complex function spaces which Theorem 3.1.1 or 3.2.2 can be applied to.

Let A be a complex function space on a locally compact Hausdorff space X . A point $x \in X$ is called a *strong boundary point* of A , if for each neighborhood U of x and for each $\varepsilon > 0$, there exists $f \in A$ such that $f(x) = \|f\| = 1$ and $|f(y)| < \varepsilon$ for all $y \in X \setminus U$. We denote by $\sigma(A)$ the set of all strong boundary points of A . We say that A is *C-regular* if $\text{Ch}(A) \subset \sigma(A)$ (cf. [12, Definition 2.3.9]). This concept is a sufficient condition for A to be strongly triple-separating:

Proposition 3.4.1. Every C-regular complex function space is strongly triple-separating.

Proof. Let x, x' and x'' be distinct points in $\text{Ch}(A)$. If A is \mathbb{C} -regular, we find $f \in A$ such that $f(x) = \|f\| = 1$, $|f(x')| < 1/2$ and $|f(x'')| < 1/2$. We also find $g \in A$ such that $g(x'') = \|g\| = 1$, $|g(x)| < 1/2$ and $|g(x')| < 1/2$. Define $h \in A$ by $h = f - f(x'')g$. Then we easily check that $|h(x)| > 3/4 > |h(x')|$ and $h(x'') = 0$. Hence A is strongly triple-separating. \square

The next proposition was mentioned in Introduction 3.1:

Proposition 3.4.2. *Every function algebra is strongly triple-separating. In particular, a uniform algebra is strongly triple-separating.*

Though we can prove this directly, we here appeal to Proposition 3.4.1.

Proof. For any function algebra A , we have $\sigma(A) = \text{Ch}(A)$. (This is well known if A is a uniform algebra, and the general case is dealt with in [40, Theorem 2.1].) Hence A is \mathbb{C} -regular. Thus the proposition follows from Proposition 3.4.1. \square

Here is another example of strongly triple-separating complex function space.

Example 3.4.3. Let K be a compact subset of \mathbb{C} and let A be a complex function space on K . The letter z is also used to denote the identity function on \mathbb{C} . If A contains the polynomials in z of degree 2, then A is strongly triple-separating. In particular, the complex function space

$$\{f \in C_{\mathbb{C}}(K) : f(z) = az^2 + bz + c \ (z \in K), a, b, c \in \mathbb{C}\}$$

is strongly triple-separating.

Proof. For any distinct points $w, w', w'' \in K$, put $f(z) = (z - w')(z - w'')$ for all $z \in K$. Clearly, $f \in A$. Also, we have $|f(w)| \neq 0 = |f(w')|$ and $f(w'') = 0$. \square

Next we consider a complex function space which satisfies (3.3) in Theorem 3.2.2.

Proposition 3.4.4. *Let A be a complex function space on a locally compact Hausdorff space X . Suppose that there exists $\lambda \in \mathbb{T} \setminus \{\pm 1, \pm i\}$ satisfying the following condition:*

$$\begin{aligned} &\text{For any pair of distinct points } x, x' \in \text{Ch}(A), \text{ there exists } f \in A \text{ such} \\ &\text{that } f(x) = \|f\| = 1 \text{ and } f(x') = \lambda. \end{aligned} \tag{3.14}$$

Then A satisfies (3.3).

Proof. Let $x, x' \in \text{Ch}(A)$ and $\alpha, \beta \in \mathbb{T}$. Assume that $(s\alpha)e_x + (t\beta)e_{x'}$ is an extreme point of ball A^* for all $s, t \in \mathbb{R}$ with $s^2 + t^2 = 1$.

To see that $x = x'$, we assume the converse; $x \neq x'$. Then, by (3.14), we find $f \in A$ such that $f(x) = \|f\| = 1$ and $f(x') = \lambda$. Hence we have

$$|s\alpha + t\beta\lambda| = |(s\alpha)f(x) + (t\beta)f(x')| \leq \|(s\alpha)e_x + (t\beta)e_{x'}\| = 1.$$

This gives $s^2 + 2st \text{Re}(\alpha\bar{\beta}\lambda) + t^2 \leq 1$, and so

$$st \text{Re}(\alpha\bar{\beta}\lambda) \leq 0.$$

Since this holds when $st > 0$ and when $st < 0$, we must have $\operatorname{Re}(\alpha\bar{\beta}\bar{\lambda}) = 0$. Hence $\alpha\bar{\beta}\bar{\lambda} = \pm i$, and so $\bar{\lambda} = i\bar{\alpha}\beta$ or $\bar{\lambda} = -i\bar{\alpha}\beta$. Alternatively, if $f \in A$ is chosen so that $f(x') = \|f\| = 1$ and $f(x) = \lambda$, then the similar argument shows that $\lambda = i\bar{\alpha}\beta$ or $\lambda = -i\bar{\alpha}\beta$. Thus we have

$$\bar{\lambda} = \lambda \quad \text{or} \quad \bar{\lambda} = -\lambda,$$

namely $\operatorname{Im} \lambda = 0$ or $\operatorname{Re} \lambda = 0$. This contradicts the hypothesis $\lambda \neq \pm 1, \pm i$. Hence $x = x'$.

Once we have established $x = x'$, the first assumption says that $(s\alpha + t\beta)e_x$ is an extreme point of ball A^* . Hence $|s\alpha + t\beta| = 1$. Taking $s = t = 1/\sqrt{2}$, we have $|\alpha + \beta| = \sqrt{2}$. By Remark after the proof of Proposition 3.2.1, we get $\beta = \pm i\alpha$. Hence $\beta e_{x'} = \pm i\alpha e_x$, and A satisfies (3.3). \square

We give an example of a complex function space which satisfies (3.3) but is not strongly triple-separating.

Example 3.4.5. Let \mathbb{S} be an arc in the unit circle \mathbb{T} , that is,

$$\mathbb{S} = \{z \in \mathbb{T} : \arg \sigma \leq \arg z \leq \arg \tau\},$$

where $\sigma, \tau \in \mathbb{T}$ and $0 < \arg \tau - \arg \sigma < 2\pi$. Define a complex function space $A_{\mathbb{S}}$ on \mathbb{S} by

$$A_{\mathbb{S}} = \{f \in C_{\mathbb{C}}(\mathbb{S}) : f(z) = az + b \ (z \in \mathbb{S}), \ a, b \in \mathbb{C}\}.$$

Then $A_{\mathbb{S}}$ is strongly separating and satisfies (3.3). But $A_{\mathbb{S}}$ is not strongly triple-separating. Moreover, T is a real-linear isometry of $A_{\mathbb{S}}$ onto $A_{\mathbb{S}}$ if and only if there exists $\lambda \in \mathbb{T}$ such that T has one of four forms:

$$\begin{aligned} T(az + b) &= \lambda(az + b), & T(az + b) &= \lambda(bz + \sigma\tau a), \\ T(az + b) &= \lambda(\bar{a}z + \sigma\tau \bar{b}), & T(az + b) &= \lambda(\bar{b}z + \bar{a}). \end{aligned} \quad (3.15)$$

Let us determine $\operatorname{Ch}(A_{\mathbb{S}})$: For each $w \in \mathbb{S}$, define $f \in A$ by $f(z) = (\bar{w}z + 1)/2$. Then $f(w) = 1$ and $|f(z)| < 1$ for all $z \in \mathbb{S} \setminus \{w\}$. Since the point z with $|f(z)| = \|f\|$ is nothing but $z = w$ and $\operatorname{Ch}(A_{\mathbb{S}})$ is a boundary for $A_{\mathbb{S}}$, it follows that $w \in \operatorname{Ch}(A_{\mathbb{S}})$. Thus we establish

$$\operatorname{Ch}(A_{\mathbb{S}}) = \mathbb{S}.$$

Let us prove the statements of Example 3.4.5.

Proof. We first observe that $A_{\mathbb{S}}$ is strongly separating. For any distinct points $w, w' \in \mathbb{S}$, put $f(z) = z - w$ for $z \in \mathbb{S}$. Then $f \in A_{\mathbb{S}}$ and $|f(w)| = 0 \neq |f(w')|$.

Next we show that $A_{\mathbb{S}}$ is not strongly triple-separating. Choose three points $w, w', w'' \in \mathbb{S}$ so that $|w - w''| = |w' - w''|$. If $f(z) = az + b \in A_{\mathbb{S}}$ satisfies $f(w'') = 0$, then $b = -aw''$, and so

$$|f(w)| = |aw + b| = |aw - aw''| = |a||w - w''| = |a||w' - w''| = |f(w')|.$$

Hence there is no $f \in A_{\mathbb{S}}$ such that $|f(w)| \neq |f(w')|$ and $f(w'') = 0$. In other words, $A_{\mathbb{S}}$ is not strongly triple-separating.

Our next task is to show that $A_{\mathbb{S}}$ satisfies (3.3). Let $w, w' \in \mathbb{S}$ and $\alpha, \beta \in \mathbb{T}$. Suppose that for each $s, t \in \mathbb{R}$ with $s^2 + t^2 = 1$, $(s\alpha)e_w + (t\beta)e_{w'}$ is an extreme point of ball $A_{\mathbb{S}}^*$. By

(3.1), there exist $w_{s,t} \in \mathbb{S}$ and $\gamma_{s,t} \in \mathbb{T}$ such that $(s\alpha)e_w + (t\beta)e_{w'} = \gamma_{s,t}e_{w_{s,t}}$. Applying the identity function z and the constant function 1, we have

$$(s\alpha)w + (t\beta)w' = \gamma_{s,t}w_{s,t} \quad \text{and} \quad s\alpha + t\beta = \gamma_{s,t}, \quad (3.16)$$

respectively. When $s = t = 1/\sqrt{2}$, the first equation implies $|\alpha w + \beta w'| = \sqrt{2}$, and so Remark after Proposition 3.2.1 shows that $\beta w' = i\alpha w$ or $\beta w' = -i\alpha w$. Similarly, the second equation yields $\beta = i\alpha$ or $\beta = -i\alpha$. Therefore,

$$w' = w \quad \text{or} \quad w' = -w. \quad (3.17)$$

Now, assume that $w' = -w$. If $\beta = i\alpha$, (3.16) becomes

$$(s - it)\alpha w = \gamma_{s,t}w_{s,t} \quad \text{and} \quad (s + it)\alpha = \gamma_{s,t},$$

and so $w_{s,t} = (s - it)^2 w$, for each $s, t \in \mathbb{R}$ with $s^2 + t^2 = 1$. Take $\zeta \in \mathbb{T} \setminus \mathbb{S}$ and choose s, t so that $(s - it)^2 = \zeta \bar{w}$. Then we have $w_{s,t} = \zeta \notin \mathbb{S}$, which is a contradiction. On the other hand, if $\beta = -i\alpha$, we choose s, t so that $(s + it)^2 = \zeta \bar{w}$, to reach a contradiction. Thus the latter case in (3.17) is impossible and we get $w' = w$. Hence $\beta e_{w'} = \pm i\alpha e_w$. Thus we proved that $A_{\mathbb{S}}$ satisfies (3.3).

Finally, we discuss the real-linear isometries. Let T be a real-linear isometry of $A_{\mathbb{S}}$ onto $A_{\mathbb{S}}$. Since $A_{\mathbb{S}}$ is strongly separating and satisfies (3.3), we can apply Theorem 3.2.2. Note that $E = \mathbb{S}$ or $E = \emptyset$ in Theorem 3.2.2, because \mathbb{S} is connected. Hence there exist a homeomorphism φ of \mathbb{S} onto \mathbb{S} and a unimodular continuous function ω on \mathbb{S} such that

$$(Tf)(z) = \omega(z)f(\varphi(z)) \quad (z \in \mathbb{S}, f \in A_{\mathbb{S}}) \quad (3.18)$$

or

$$(Tf)(z) = \omega(z)\overline{f(\varphi(z))} \quad (z \in \mathbb{S}, f \in A_{\mathbb{S}}). \quad (3.19)$$

Taking f as the constant 1 in (3.18) and (3.19), we have $\omega = T1 \in A_{\mathbb{S}}$, and so we can write $\omega(z) = uz + v$ for some $u, v \in \mathbb{C}$. Since ω is unimodular on \mathbb{S} , we have only two cases: $\omega(z) = uz$ where $u \in \mathbb{T}$, or $\omega(z) = v$ where $v \in \mathbb{T}$.

Now assume that (3.18) holds and $\omega(z) = uz$. Putting $f(z) = z$ in (3.18), we have $\omega\varphi = Tz \in A_{\mathbb{S}}$. Hence we write $uz\varphi(z) = \omega(z)\varphi(z) = pz + q$ for some $p, q \in \mathbb{C}$. So $\varphi(z) = \bar{u}(p + q\bar{z})$. Since φ is a homeomorphism of \mathbb{S} onto \mathbb{S} , we must have $\bar{u}p = 0$, $\bar{u}q = \sigma\tau$ and $\varphi(z) = \sigma\tau\bar{z}$. Hence

$$T(az + b) = \omega(z)(a\varphi(z) + b) = uz(a(\sigma\tau\bar{z}) + b) = u(bz + \sigma\tau a),$$

which is the second equation in (3.15) with $u = \lambda$. If (3.18) holds and $\omega(z) = v$, then we similarly obtain the first equation in (3.15). While, if (3.19) holds, the case $\omega(z) = uz$ and the case $\omega(z) = v$ yield the fourth and the third equations in (3.15), respectively.

Conversely, if T has the form in (3.15), then it is easily shown that T is a real-linear isometry of $A_{\mathbb{S}}$ onto $A_{\mathbb{S}}$. \square

Finally, we consider another function space.

Example 3.4.6. We denote by $(C_{\mathbb{C}}^{(1)}[0, 1], \|\cdot\|_m)$ the Banach space of complex-valued continuously differentiable functions on $[0, 1]$, with the norm $\|f\|_m = \max\{|f(0)|, \|f'\|_{\infty}\}$. Then T is a real-linear isometry of $(C_{\mathbb{C}}^{(1)}[0, 1], \|\cdot\|_m)$ onto itself if and only if there exist a homeomorphism φ of $[0, 1]$ onto $[0, 1]$, a unimodular continuous function ω on $[0, 1]$ and a unimodular constant λ such that T has one of four forms:

$$\begin{aligned} (Tf)(x) &= \lambda f(0) + \int_0^x \omega(t) f'(\varphi(t)) dt, & (Tf)(x) &= \lambda \overline{f(0)} + \int_0^x \omega(t) f'(\varphi(t)) dt, \\ (Tf)(x) &= \lambda f(0) + \int_0^x \omega(t) \overline{f'(\varphi(t))} dt, & (Tf)(x) &= \lambda \overline{f(0)} + \int_0^x \omega(t) \overline{f'(\varphi(t))} dt \end{aligned} \quad (3.20)$$

for all $x \in [0, 1]$ and $f \in C_{\mathbb{C}}^{(1)}[0, 1]$.

Proof. Let T be a real-linear isometry of $(C_{\mathbb{C}}^{(1)}[0, 1], \|\cdot\|_m)$ onto itself. Put $X_1 = [0, 1] \cup \{p\}$. For each $f \in C_{\mathbb{C}}^{(1)}[0, 1]$, we define a continuous function \tilde{f} on X_1 by

$$\tilde{f}(y) = \begin{cases} f(0) & \text{if } y = p, \\ f'(y) & \text{if } y \in [0, 1]. \end{cases}$$

Then Lemma 1.2.22 implies that $P : f \mapsto \tilde{f}$ is a complex-linear isometry of $(C_{\mathbb{C}}^{(1)}[0, 1], \|\cdot\|_m)$ onto $C_{\mathbb{C}}(X_1)$. Of course, P is a real-linear isometry. Hence $\tilde{T} = PTP^{-1}$ is a real-linear isometry of $C_{\mathbb{C}}(X_1)$ onto itself. It is clear that $\text{Ch}(C_{\mathbb{C}}(X_1)) = X_1$ and $C_{\mathbb{C}}(X_1)$ is strongly triple-separating. Hence we can apply Theorem 3.1.1. In this case, we have $E = X_1$, $E = [0, 1]$, $E = \{p\}$ or $E = \emptyset$ in Theorem 3.1.1. Hence there exist a homeomorphism ρ of X_1 onto X_1 , a unimodular continuous function u on X_1 such that

$$(\tilde{T}h)(y) = u(y)h(\rho(y)) \quad (y \in X_1), \quad (3.21)$$

$$(\tilde{T}h)(y) = \begin{cases} u(y)h(\rho(y)) & \text{if } y \in [0, 1] \\ u(y)\overline{h(\rho(y))} & \text{if } y = p \end{cases} \quad (3.22)$$

$$(\tilde{T}h)(y) = \begin{cases} u(y)h(\rho(y)) & \text{if } y = p \\ u(y)\overline{h(\rho(y))} & \text{if } y \in [0, 1] \end{cases} \quad (3.23)$$

$$(\tilde{T}h)(y) = u(y)\overline{h(\rho(y))} \quad (y \in X_1) \quad (3.24)$$

for all $h \in C_{\mathbb{C}}(X_1)$. Then we have $\rho([0, 1]) = [0, 1]$ and $\rho(p) = p$. Now, put $\varphi = \rho|_{[0, 1]}$, $\omega = u|_{[0, 1]}$ and $\lambda = u(p)$. We easily check that φ , ω and λ have the desired properties. If (3.22) holds, then we have

$$\begin{aligned} (Tf)(0) &= \tilde{T}\tilde{f}(p) = (\tilde{T}\tilde{f})(p) = u(p)\overline{\tilde{f}(\rho(p))} = u(p)\overline{\tilde{f}(p)} = \lambda \overline{f(0)}, \\ (Tf)'(x) &= \tilde{T}\tilde{f}'(x) = (\tilde{T}\tilde{f}')(x) = u(x)\tilde{f}'(\rho(x)) = u(x)f'(\varphi(x)) \quad (x \in [0, 1]), \end{aligned}$$

for all $f \in C_{\mathbb{C}}^{(1)}[0, 1]$. Hence $(Tf)(x) = \lambda \overline{f(0)} + \int_0^x \omega(t) f'(\varphi(t)) dt$. Similarly, by (3.21), (3.23) and (3.24), we obtain the rest of the forms in (3.20).

Conversely, if T has the form in (3.20), then it is easily shown that T is a real-linear isometry of $(C_{\mathbb{C}}^{(1)}[0, 1], \|\cdot\|_m)$ onto itself. \square

3.5 The complex-linear case

Note that a complex-linear isometry is the real-linear isometry T satisfying $T(if) = iTf$. Hence a complex-linear isometry T does not admit the identity $(Tf)(y) = \omega(y)f(\varphi(y))$ in (3.4) of Theorem 3.2.2. Thus we obtain the next corollary:

Corollary 3.5.1. *Let A and B be complex function spaces on locally compact Hausdorff spaces X and Y , respectively. Suppose that A is strongly separating and satisfies the condition (3.3). If T is a complex-linear isometry of A onto B , then there exist a continuous mapping φ of $\text{Ch}(B)$ onto $\text{Ch}(A)$ and a unimodular continuous function ω on $\text{Ch}(B)$ such that*

$$(Tf)(y) = \omega(y)f(\varphi(y)) \quad (y \in \text{Ch}(B)) \quad (3.25)$$

for all $f \in A$. If, in addition, B is strongly separating, then the mapping φ becomes a homeomorphism of $\text{Ch}(B)$ onto $\text{Ch}(A)$.

This corollary is included in Theorems 3.1, 4.1 and Corollaries 3.2, 4.2 in [1]. Moreover, we know from them that the hypothesis (3.3) on A is unnecessary in Corollary 3.5.1. However, in Theorem 3.2.2, we cannot remove the condition (3.3). This is seen from the following example:

Example 3.5.2. Define a complex function space $A_{\mathbb{T}}$ on \mathbb{T} by

$$A_{\mathbb{T}} = \{f \in C_{\mathbb{C}}(\mathbb{T}) : f(z) = az + b \ (z \in \mathbb{T}), \ a, b \in \mathbb{C}\}.$$

Then $A_{\mathbb{T}}$ is strongly separating but does not satisfy (3.3). Also, T is a real-linear isometry of $A_{\mathbb{T}}$ onto $A_{\mathbb{T}}$ if and only if there exist $\kappa, \lambda \in \mathbb{T}$ such that T has one of eight forms:

$$\begin{aligned} T(az + b) &= \kappa az + \lambda b, & T(az + b) &= \kappa bz + \lambda a, \\ T(az + b) &= \kappa az + \lambda \bar{b}, & T(az + b) &= \kappa bz + \lambda \bar{a}, \\ T(az + b) &= \kappa \bar{a}z + \lambda b, & T(az + b) &= \kappa \bar{b}z + \lambda a, \\ T(az + b) &= \kappa \bar{a}z + \lambda \bar{b}, & T(az + b) &= \kappa \bar{b}z + \lambda \bar{a}. \end{aligned} \quad (3.26)$$

Some of these equations cannot be written in the form (3.4) in Theorem 3.2.2. Therefore, we cannot remove the condition (3.3) in Theorem 3.2.2. On the other hand, T is a complex-linear isometry of $A_{\mathbb{T}}$ onto $A_{\mathbb{T}}$ if and only if there exist $\kappa, \lambda \in \mathbb{T}$ such that T has one of two forms in the first line of (3.26). These forms are represented in the form (3.25) in Corollary 3.5.1.

Proof. It is clear that $A_{\mathbb{T}}$ is a strongly separating complex function space on \mathbb{T} . Also, the equipped (supremum) norm is given by

$$\|az + b\| = |a| + |b|. \quad (3.27)$$

Moreover, we see that $\text{Ch}(A_{\mathbb{T}}) = \mathbb{T}$, similarly to Example 3.4.5.

Let us consider the real-linear isometries. If T has the form in (3.26), then it follows from (3.27) that T is a real-linear isometry of $A_{\mathbb{T}}$ onto $A_{\mathbb{T}}$. In particular,

$$T_1(az + b) = az + \bar{b}$$

is a real-linear isometry of $A_{\mathbb{T}}$ onto $A_{\mathbb{T}}$. Let us see that T_1 is out of the form (3.4) in Theorem 3.2.2. To do this, we assume the converse. Note that $E = \mathbb{T}$ or $E = \emptyset$ in Theorem 3.2.2, because \mathbb{T} is connected. Then our assumption says that there exist a homeomorphism φ of \mathbb{T} onto \mathbb{T} and a unimodular continuous function ω on \mathbb{T} such that

$$az + \bar{b} = T_1(az + b) = \omega(z)(a\varphi(z) + b) \quad (3.28)$$

or

$$az + \bar{b} = T_1(az + b) = \omega(z)\overline{(a\varphi(z) + b)}. \quad (3.29)$$

Taking $a = 0$ and $b = 1$ in (3.28) and (3.29), we obtain $\omega(z) = 1$. If (3.28) holds, we take $a = 1$ and $b = 0$ in (3.28) to see that $z = \omega(z)\varphi(z) = \varphi(z)$ and $az + \bar{b} = az + b$. This is impossible when $b = i$. On the other hand, if (3.29) holds, then $\varphi(z) = \bar{z}$ and $az + \bar{b} = \bar{a}z + \bar{b}$. This is also impossible when $a = i$. In any case, we reach a contradiction. Thus T_1 does not have the form (3.4).

Since we have just found a real-linear isometry of $A_{\mathbb{T}}$ onto $A_{\mathbb{T}}$ out of the form (3.4), Theorem 3.2.2 tells us that $A_{\mathbb{T}}$ does not satisfy (3.3).

Next, we start with an arbitrary real-linear isometry T of $A_{\mathbb{T}}$ onto $A_{\mathbb{T}}$. Write

$$T(z) = c_1z + d_1, \quad T(iz) = c_2z + d_2, \quad T(1) = c_3z + d_3, \quad T(i) = c_4z + d_4, \quad (3.30)$$

for some $c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4 \in \mathbb{C}$. By (3.27), the first equation gives

$$|c_1| + |d_1| = \|c_1z + d_1\| = \|T(z)\| = \|z\| = 1, \quad (3.31)$$

and the remaining ones give

$$|c_2| + |d_2| = 1, \quad |c_3| + |d_3| = 1, \quad |c_4| + |d_4| = 1. \quad (3.32)$$

Next, we consider the function $z \pm 1$, and use the above equations to see

$$\begin{aligned} 2 &= \|z \pm 1\| = \|T(z \pm 1)\| = \|T(z) \pm T(1)\| = \|(c_1z + d_1) \pm (c_3z + d_3)\| \\ &= \|(c_1 \pm c_3)z + (d_1 \pm d_3)\| = |c_1 \pm c_3| + |d_1 \pm d_3| \leq |c_1| + |c_3| + |d_1| + |d_3| = 2, \end{aligned}$$

and so

$$|c_1 + c_3| = |c_1| + |c_3| = |c_1 - c_3| \quad \text{and} \quad |d_1 + d_3| = |d_1| + |d_3| = |d_1 - d_3|.$$

By the equality condition for the triangle inequality, these imply

$$\text{"}c_1 = 0 \text{ or } c_3 = 0\text{"} \quad \text{and} \quad \text{"}d_1 = 0 \text{ or } d_3 = 0\text{"}. \quad (3.33)$$

Here we replace the function $z \pm 1$ by $z \pm i$ or $iz \pm 1$. Then we similarly get

$$\text{"}c_1 = 0 \text{ or } c_4 = 0\text{"} \quad \text{and} \quad \text{"}d_1 = 0 \text{ or } d_4 = 0\text{"}, \quad (3.34)$$

$$\text{"}c_2 = 0 \text{ or } c_3 = 0\text{"} \quad \text{and} \quad \text{"}d_2 = 0 \text{ or } d_3 = 0\text{"}. \quad (3.35)$$

Now assume that $c_1 \neq 0$. Then the first fact in (3.33) says that $c_3 = 0$, and so $|d_3| = 1$ by the second equation in (3.32). Here if we use (3.34) instead of (3.33), we get $c_4 = 0$

and $|d_4| = 1$. Since $d_3 \neq 0$, the second fact in (3.33) says that $d_1 = 0$, and so $|c_1| = 1$ by (3.31). If we use (3.35) instead of (3.33), we get $d_2 = 0$ and $|c_2| = 1$. Thus we obtain

$$|c_1| = |c_2| = |d_3| = |d_4| = 1 \quad \text{and} \quad c_3 = c_4 = d_1 = d_2 = 0. \quad (3.36)$$

Hence (3.30) becomes

$$T(z) = c_1 z, \quad T(iz) = c_2 z, \quad T(1) = d_3, \quad T(i) = d_4,$$

and the real-linearity of T implies that

$$\begin{aligned} T(az + b) &= (\operatorname{Re} a)T(z) + (\operatorname{Im} a)T(iz) + (\operatorname{Re} b)T(1) + (\operatorname{Im} b)T(i) \\ &= ((\operatorname{Re} a)c_1 + (\operatorname{Im} a)c_2)z + ((\operatorname{Re} b)d_3 + (\operatorname{Im} b)d_4). \end{aligned} \quad (3.37)$$

Putting $a = 1 + i$ and $b = 0$ in (3.37), we have $T((1 + i)z) = (c_1 + c_2)z$, and so

$$|c_1 + c_2| = \|T((1 + i)z)\| = \|(1 + i)z\| = |1 + i| = \sqrt{2}.$$

Applying Remark after Proposition 3.2.1, we get $c_2 = \pm ic_1$. If we put $a = 0$ and $b = 1 + i$ in (3.37), we obtain $d_4 = \pm id_3$. Thus (3.37) becomes

$$T(az + b) = c_1(\operatorname{Re} a \pm i \operatorname{Im} a)z + d_3(\operatorname{Re} b \pm i \operatorname{Im} b).$$

Putting $\kappa = c_1$ and $\lambda = d_3$, we arrive at the left four equations in (3.26).

On the other hand, if $c_1 = 0$, then we can prove the right four equations in (3.26) in the same way.

Finally, we consider the complex-linear case. Note that the only two equations in the first line in (3.26) satisfy $T(if) = iTf$. So, every complex-linear isometry T of $A_{\mathbb{T}}$ onto $A_{\mathbb{T}}$ has one of such forms. If $T(az + b) = \kappa az + \lambda b$, then we put $\varphi(z) = \kappa \bar{\lambda} z$ and $\omega(z) = \lambda$. In this case, we have

$$T(az + b) = \lambda(a(\kappa \bar{\lambda} z) + b) = \omega(z)(a\varphi(z) + b).$$

On the other hand, if $T(az + b) = \kappa bz + \lambda a$, then we put $\varphi(z) = \bar{\kappa} \lambda \bar{z}$ and $\omega(z) = \kappa z$ to see that $T(az + b) = \omega(z)(a\varphi(z) + b)$. Thus both equations are represented in the form (3.25) in Corollary 3.5.1. \square

3.6 Isometries between complex function spaces

From Theorem 3.2.2 and the Mazur-Ulam theorem, we obtain the following characterization of general isometries.

Corollary 3.6.1. *Let A and B be complex function spaces on locally compact Hausdorff spaces X and Y , respectively. Suppose that A and B are strongly separating and that A satisfies the condition (3.3). If T is an isometry of A onto B , then there exist a (possibly empty) open and closed subset E of $\operatorname{Ch}(B)$, a homeomorphism φ of $\operatorname{Ch}(B)$ onto $\operatorname{Ch}(A)$ and a unimodular continuous function ω on $\operatorname{Ch}(B)$ such that*

$$(Tf)(y) - (T0)(y) = \begin{cases} \omega(y)f(\varphi(y)) & \text{if } y \in E, \\ \omega(y)\overline{f(\varphi(y))} & \text{if } y \in \operatorname{Ch}(B) \setminus E, \end{cases}$$

for all $f \in A$.

Bibliography

- [1] J. Araujo and J. J. Font, *Linear isometries between subspaces of continuous functions*, Trans. Amer. Math. Soc., **349** (1997), 413–428.
- [2] J. Araujo and J. J. Font, *Codimension 1 linear isometries on function algebras*, Proc. Amer. Math. Soc., **127** (1999), 2273–2281.
- [3] H. Ariizumi, *On backward shift operators on function algebras (Japanese)*, Master's thesis, Shinshu Univ., 1998.
- [4] S. Banach, "Theory of Linear Operations", North-Holland, Amsterdam, 1987.
- [5] A. Browder, "Introduction to Function Algebras", Benjamin, 1969.
- [6] M. Cambern, *Isometries of certain Banach algebras*, Studia Math., **25** (1965), 217–225.
- [7] M. Cambern and V. D. Pathak, *Isometries of spaces of differentiable functions*, Math. Japon., **26** (1981), 253–260.
- [8] J. B. Conway. "A Course in Functional Analysis" 2nd ed., Springer, 1990.
- [9] R. M. Crownover, *Commutants of shifts on Banach spaces*, Michigan Math. J., **19** (1972), 233–247.
- [10] K. deLeeuw, W. Rudin and J. Wermer, *The isometries of some function spaces*, Proc. Amer. Math. Soc., **11** (1960), 694–698.
- [11] A. J. Ellis, *Real characterization to function algebras amongst function spaces*, Bull. London Math. Soc., **22** (1990), 381–385.
- [12] R. J. Fleming and J. E. Jamison, "Isometries on Banach Spaces: Function Spaces", Chapman & Hall/CRC, 2003.
- [13] J. J. Font, *Isometries between function algebras with finite codimensional range*, Manuscripta Math., **100** (1999), 13–21.
- [14] J. J. Font, *On weighted composition operators between spaces of measurable functions*, Quaest. Math., **22** (1999), 143–148.
- [15] A. Gutek, D. Hart, J. Jamison and M. Rajagopalan, *Shift operators on Banach spaces*, J. Funct. Anal., **101** (1991), 97–119.

- [16] K. Holsztyński, *Continuous mappings induced by isometries of spaces of continuous functions*, Studia. Math. **26** (1966), 133–136.
- [17] J. R. Holub, *On shift operators*, Canad. Math. Bull., **31** (1988), 85–94.
- [18] K. Izuchi, *Douglas algebras which admit codimension 1 linear isometries*, Proc. Amer. Math. Soc., **129** (2001), 2069–2074.
- [19] K. Jarosz and V. D. Pathak, *Isometries between function spaces*, Trans. Amer. Math. Soc., **305** (1988), 193–206.
- [20] A. Jiménez-Vargas and M. Villegas-Vallecillos, *Into linear isometries between spaces of Lipschitz functions*, Houston J. Math., **34** (2008), 1165–1184.
- [21] R. V. Kadison and J. R. Ringrose, “Fundamentals of the Theory of Operator Algebras” Vol. I, Academic Press, New York, 1983.
- [22] E. Kaniuth, “A Course in Commutative Banach Algebras”, Springer, 2009.
- [23] K. Kasuga, *There are no codimension 1 linear isometries on the ball and polydisk algebras*, Sci. Math. Jpn., **54** (2001), 387–390. **56** (2002), 69–70.
- [24] J. L. Kelley, “General Topology”, Springer, 1975.
- [25] H. Koshimizu, *Shift operators on $C^1[0, 1]$ (Japanese)*, Master’s thesis, Shinshu Univ., 2009.
- [26] H. Koshimizu, *Finite codimensional linear isometries on spaces of differentiable and Lipschitz functions*, Cent. Eur. J. Math., **9** (2011), 139–146.
- [27] H. Koshimizu, *Linear isometries on spaces of continuously differentiable and Lipschitz continuous functions*, to appear in Nihonkai Math. J.
- [28] H. Koshimizu, T. Miura, H. Takagi and S. -E. Takahasi, *Real-linear isometries between subspaces of continuous functions*, preprint.
- [29] T. Matsumoto and S. Watanabe, *Surjective linear isometries of the domain of a *-derivation equipped with the Cambern norm*, Math. Z., **230** (1999), 185–200.
- [30] T. Miura, *Real linear isometries between function algebras*, Cent. Eur. J. Math., **9** (2011), 778–788.
- [31] M. Nagasawa, *Isomorphism between commutative Banach algebras with an application to rings of analytic functions*, Kodai Math. Sep. Rep., **11** (1959), 182–188.
- [32] W. Novinger, *Linear isometries of subspaces of continuous functions*, Studia Math., **53** (1975), 273–276.
- [33] V. D. Pathak, *Isometries of $C^{(n)}[0, 1]$* , Pacific J. Math., **94** (1981), 211–222.
- [34] R. R. Phelps, “Lectures on Choquet’s Theorem” 2nd ed., Springer, Berlin, 2001.

- [35] M. Rajagopalan, T. M. Rassias and K. Sundaresan, *Generalized backward shifts on Banach spaces $C(X, E)$* , Bull. Sci. Math., **124** (2000), 685–693.
- [36] M. Rajagopalan and K. Sundaresan, *Backward shifts on Banach spaces $C(X)$* , J. Math. Anal. Appl., **202** (1996), 485–491.
- [37] M. Rajagopalan and K. Sundaresan, *Backward shifts on Banach spaces $C(X)$, II*, In: Proceedings of the Tennessee Topology Conference, World Scientific, 1997, 199–205.
- [38] M. Rajagopalan and K. Sundaresan, *Generalized backward and forward shift on function space*, J. Anal., **7** (1999), 75–81.
- [39] N. V. Rao and A. K. Roy, *Linear isometries of some function spaces*, Pacific J. Math., **38** (1971), 177–192.
- [40] N. V. Rao and A. K. Roy, *Multiplicatively spectrum-preserving maps of function algebras, II*, Proc. Edinb. Math. Soc., **48** (2005), 219–229.
- [41] T. M. Rassias and K. Sundaresan, *Generalized backward shifts on Banach spaces*, J. Math. Anal. Appl., **260** (2001), 36–45.
- [42] W. Rudin, “Real and Complex Analysis” 3rd ed., McGraw-Hill, 1986.
- [43] W. Rudin, “Functional Analysis” 2nd ed., McGraw-Hill, 1991.
- [44] Y. Shinke, *Shift operators on Banach spaces (Japanese)*, Master’s thesis, Shinshu Univ., 2008.
- [45] H. Takagi, H. Koshimizu and H. Ariizumi, *Backward shifts on function algebras*, J. Math. Anal. Appl., **377** (2011), 135–144.
- [46] S. -E. Takahasi and T. Okayasu, *A note on finite codimensional linear isometries of $C(X)$ into $C(Y)$* , Internat. J. Math. Math. Sci., **18** (1995), 677–680.
- [47] T. Takayama and J. Wada, *Isometric shift operators on the disc algebra*, Tokyo J. Math., **21** (1998), 115–120.