Infinitely many shape invariant potentials and new orthogonal polynomials

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Abstract

Three sets of exactly solvable one-dimensional quantum mechanical potentials are presented. These are shape invariant potentials obtained by deforming the radial oscillator and the trigonometric/hyperbolic Pöschl-Teller potentials in terms of their degree ℓ polynomial eigenfunctions. We present the entire eigenfunctions for these Hamiltonians ($\ell = 1, 2, \ldots$) in terms of new orthogonal polynomials. Two recently reported shape invariant potentials of Quesne and Gómez-Ullate et al.'s are the first members of these infinitely many potentials.

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1. Introduction

In this Letter we present two infinite sets and one finite set of exactly solvable one-dimensional quantum mechanical Hamiltonians. As the main part of the eigenfunctions, a new type of orthogonal polynomials is obtained for each Hamiltonian. They are exactly solvable by combining shape invariance [1] with the factorisation method [2, 3] or the so-called supersymmetric quantum mechanics [4]. Then the entire energy spectrum and the corresponding eigenfunctions can be obtained algebraically. However, these new shape invariant Hamiltonians do not possess the exact Heisenberg operator solutions [5], in contrast to most of the known shape invariant Hamiltonians.

Shape invariance is a sufficient condition for exactly solvable quantum mechanical systems. Based on one shape invariant potential, an infinite number of exactly solvable potentials and their eigenfunctions can be constructed by a modification of Crum's method [6, 7]. But these newly derived systems fail to inherit the shape invariance, nor do they possess Heisenberg operator solutions. Although several shape invariant 'discrete' quantum mechanical systems are added to recently [8], the catalogue of the shape invariant potentials was rather short for a long time. In 2008, Quesne [9] reported two new shape invariant potentials based on the Sturm-Liouville problems for the X_1-Laguerre and the X_1-Jacobi polynomials proposed by Gómez-Ullate et al. [10].

Here we present our preliminary results on the three sets of shape invariant potentials and the corresponding new types of orthogonal polynomials, without proof. After brief introduction of notation and the shape invariance method, they are obtained by *deforming* the well-known shape invariant potentials, the radial oscillator and the Darboux-Pöschl-Teller [11, 12] potentials, in terms of the degree ℓ polynomial eigenfunctions, *i.e.* the Laguerre and the Jacobi polynomials. The eigenpolynomials of the new Hamiltonians are orthogonal polynomials starting from degree ℓ, which could be called X_ℓ polynomials. The Quesne-Gómez-Ullate et al. examples [9, 10] correspond to the $\ell = 1$ cases.

2. General setting: shape invariance

The starting point is a generic one-dimensional quantum mechanical system having a square-integrable groundstate together with a finite or infinite number of discrete energy levels: $0 = \mathcal{E}_0 < \mathcal{E}_1 < \mathcal{E}_2 < \cdots$. The groundstate energy $\mathcal{E}_0$ is chosen to be zero, by adjusting the constant part of the Hamiltonian. The *positive semi-definite* Hamiltonian is expressed in a factorised form [2, 3, 4]:

$$\mathcal{H} = \mathcal{A}^\dagger \mathcal{A} = p^2 + U(x), \quad p = -i\partial_x, \tag{1}$$

$$\mathcal{A} \stackrel{\text{def}}{=} \partial_x - w'(x), \quad \mathcal{A}^\dagger = -\partial_x - w'(x), \tag{2}$$

$$U(x) \stackrel{\text{def}}{=} w'(x)^2 + w''(x). \tag{3}$$

For simplicity of presentation we have adopted the unit system in which $\hbar$ and the mass m of the particle are such that $\hbar = 2m = 1$. Here we call a real and smooth function $w(x)$ a *prepotential* and it parametrises the groundstate wavefunction $\phi_0(x)$, which has *no node* and can be chosen real and positive, $\phi_0(x) = e^{w(x)}$. It is trivial to verify $\mathcal{A}\phi_0(x) = 0$ and $\mathcal{H}\phi_0(x) = 0$.

Shape invariance, a sufficient condition for exact solvability [1], is realised by specific dependence of the potential, or the prepotential on a set of parameters $\boldsymbol{\lambda} =$

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$(\lambda_1, \lambda_2, \ldots)$, to be denoted by $w(x;\boldsymbol{\lambda})$, $\mathcal{A}(\boldsymbol{\lambda})$, $\mathcal{H}(\boldsymbol{\lambda})$, $\mathcal{E}_n(\boldsymbol{\lambda})$, etc. The shape invariance condition to be discussed in this Letter is

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger = \mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}), \tag{4}$$

$$\begin{aligned} &w'(x;\boldsymbol{\lambda})^2 - w''(x;\boldsymbol{\lambda}) \\ &\quad = w'(x;\boldsymbol{\lambda}+\boldsymbol{\delta})^2 + w''(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}), \end{aligned} \tag{5}$$

in which $\boldsymbol{\delta}$ is a certain shift of the parameters. Then the entire set of discrete eigenvalues and the corresponding eigenfunctions of $\mathcal{H} = \mathcal{H}(\boldsymbol{\lambda})$

$$\mathcal{H}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}) \tag{6}$$

is determined algebraically [1, 4, 8]:

$$\mathcal{E}_n(\boldsymbol{\lambda}) = \sum_{k=0}^{n-1}\mathcal{E}_1(\boldsymbol{\lambda}+k\boldsymbol{\delta}), \tag{7}$$

$$\begin{aligned} \phi_n(x;\boldsymbol{\lambda}) \propto \mathcal{A}(\boldsymbol{\lambda})^\dagger\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\cdots\mathcal{A}(\boldsymbol{\lambda}+(n-1)\boldsymbol{\delta})^\dagger \\ \times e^{w(x;\boldsymbol{\lambda}+n\boldsymbol{\delta})}. \end{aligned} \tag{8}$$

3. The radial oscillator

Here we present an infinite number of shape invariant potentials indexed by a non-negative integer $\ell = 0, 1, 2, \ldots$. For $\ell = 0$, it is the well-known radial oscillator, or the harmonic oscillator with a centrifugal barrier potential, with $\boldsymbol{\lambda} = g > 0$:

$$\mathcal{H}_0(g) = p^2 + x^2 + \frac{g(g-1)}{x^2} - 1 - 2g, \tag{9}$$

$$w_0(x;g) = -\tfrac{1}{2}x^2 + g\log x, \quad 0 < x < \infty. \tag{10}$$

Here we adopt the notation of our previous work [5] §III.A.1. The shape invariance, the Heisenberg operator solution and the creation-annihilation operators of the above Hamiltonian are discussed in some detail there. It is trivial to verify (5) with $\boldsymbol{\delta} = 1$, $\mathcal{E}_1(g) = 4$ and we obtain the equidistant spectrum and the corresponding eigenfunctions $n = 0, 1, 2, \ldots$,

$$\mathcal{E}_n(g) = 4n, \tag{11}$$

$$\phi_n(x;g) = P_n(x^2;g)\,e^{w_0(x;g)},\ P_n(x;g) = L_n^{(g-\frac{1}{2})}(x). \tag{12}$$

The polynomial eigenfunctions are the Laguerre polynomials in x^2, which are orthogonal with respect to the measure $\phi_0(x)^2 = e^{2w_0(x;g)} = e^{-x^2}x^{2g}$.

For each positive integer $\ell \geq 1$, let us introduce a prepotential and a Hamiltonian:

$$\xi_\ell(x;g) \stackrel{\text{def}}{=} L_\ell^{(g+\ell-\frac{3}{2})}(-x), \tag{13}$$

$$w_\ell(x;g) \stackrel{\text{def}}{=} w_0(x;g+\ell) + \log\frac{\xi_\ell(x^2;g+1)}{\xi_\ell(x^2;g)}, \tag{14}$$

$$\mathcal{A}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \partial_x - w'_\ell(x;\boldsymbol{\lambda}),\ \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger = -\partial_x - w'_\ell(x;\boldsymbol{\lambda}), \tag{15}$$

$$\mathcal{H}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\mathcal{A}_\ell(\boldsymbol{\lambda}). \tag{16}$$

Since the polynomial $\xi_\ell(x^2;g)$ has no zero in the domain $0 < x < \infty$, the prepotential and the potential are smooth in the entire domain. It is straightforward to verify the shape invariance condition (5) with $\boldsymbol{\delta} = 1$, $\mathcal{E}_{\ell,1}(g) = 4$. By using (8) as a Rodrigues type formula, we obtain the complete set of eigenfunctions with the equidistant spectrum:

$$\mathcal{H}_\ell(g)\phi_{\ell,n}(x;g) = \mathcal{E}_{\ell,n}(g)\phi_{\ell,n}(x;g), \quad n = 0, 1, \ldots, \tag{17}$$

$$\mathcal{E}_{\ell,n}(g) = \mathcal{E}_n(g+\ell) = 4n, \tag{18}$$

$$\phi_{\ell,n}(x;g) = P_{\ell,n}(x^2;g)\psi_\ell(x),\ \psi_\ell(x) \stackrel{\text{def}}{=} \frac{e^{w_0(x;g+\ell)}}{\xi_\ell(x^2;g)}, \tag{19}$$

$$\begin{aligned} P_{\ell,n}(x;g) \stackrel{\text{def}}{=} &\ \xi_\ell(x;g+1)P_n(x;g+\ell) \\ &- \xi_{\ell-1}(x;g+2)P_{n-1}(x;g+\ell). \end{aligned} \tag{20}$$

Obviously we have $P_{\ell,0}(x;g) = \xi_\ell(x;g+1)$ and $\phi_{\ell,0}(x;g) = e^{w_\ell(x;g)}$. The polynomial eigenfunction $P_{\ell,n}(x^2;g)$ is a degree $\ell+n$ polynomial in x^2 but it has only n zeros in the domain $0 < x < \infty$. These polynomials are orthogonal with respect to the measure $\psi_\ell(x;g)^2$:

$$\begin{aligned} &\int_0^\infty dx\,\psi_\ell(x;g)^2 P_{\ell,n}(x^2;g)P_{\ell,m}(x^2;g) \\ &\quad = \frac{1}{2\,n!}\big(n+g+2\ell-\tfrac{1}{2}\big)\Gamma\big(n+g+\ell-\tfrac{1}{2}\big)\delta_{nm}. \end{aligned} \tag{21}$$

They form a complete basis of the Hilbert space just like the Laguerre polynomials in the $\ell = 0$ case. These new types of polynomials do not satisfy the three term recurrence relation, a characteristic feature of all the ordinary orthogonal polynomials. It should be stressed that all four terms in (20) are the Laguerre polynomials of the same index, $g + \ell - 1/2$. The action of the operators $\mathcal{A}_\ell(g)$ and $\mathcal{A}_\ell(g)^\dagger$ on the eigenfunctions are:

$$\begin{aligned} \mathcal{A}_\ell(g)\phi_{\ell,n}(x;g) &= -2\phi_{\ell,n-1}(x;g+1), \\ \mathcal{A}_\ell(g)^\dagger\phi_{\ell,n-1}(x;g+1) &= -2n\phi_{\ell,n}(x;g). \end{aligned} \tag{22}$$

For $\ell = 1$ the Hamiltonian reads

$$\begin{aligned} \mathcal{H}_1(g) = p^2 + x^2 + \frac{g(g+1)}{x^2} - 3 - 2g \\ + \frac{4}{x^2+g+\frac{1}{2}} - \frac{4(2g+1)}{\big(x^2+g+\frac{1}{2}\big)^2}, \end{aligned}$$

which is equivalent to that of the shape invariant potential of Quesne eq.(8) of [9] with the replacement $\omega \to 2$ and $l \to g$. The formula (20) expressing the polynomial eigenfunctions in terms of the Laguerre polynomials is the generalisation of Gómez-Ullate et al.'s [10] relation eq.(80) between the X_1-Laguerre and the Laguerre polynomials.

4. Darboux-Pöschl-Teller potential

Here we present another infinite number of shape invariant potentials indexed by a non-negative integer $\ell =$

$0, 1, 2, \ldots$. For $\ell = 0$, it is known as the Pöschl-Teller potential [12], with two positive parameters $\boldsymbol{\lambda} = (g, h)$:

$$\mathcal{H}_0(\boldsymbol{\lambda}) = p^2 + \frac{g(g-1)}{\sin^2 x} + \frac{h(h-1)}{\cos^2 x} - (g+h)^2, \tag{23}$$
$$w_0(x;\boldsymbol{\lambda}) = g\log\sin x + h\log\cos x, \quad 0 < x < \frac{\pi}{2}. \tag{24}$$

We again follow the notation of our previous work [5] §III.A.2. The shape invariance, the Heisenberg operator solution and the creation-annihilation operators of the above Hamiltonian are discussed in some detail there. Apparently, it was Darboux [11] who first introduced this potential, although the coupling constants were restricted to positive integers only. It is trivial to verify the shape invariance condition (5) with $\boldsymbol{\delta} = (1,1)$, $\mathcal{E}_1(\boldsymbol{\lambda}) = 4(1+g+h)$ and we obtain the quadratic energy spectrum and the corresponding eigenfunctions $n = 0, 1, \ldots$,

$$\mathcal{E}_n(\boldsymbol{\lambda}) = 4n(n+g+h), \tag{25}$$
$$\phi_n(x;\boldsymbol{\lambda}) = P_n(\cos 2x;\boldsymbol{\lambda})\, e^{w_0(x;\boldsymbol{\lambda})},$$
$$P_n(x;\boldsymbol{\lambda}) = P_n^{(g-\frac12, h-\frac12)}(x). \tag{26}$$

The polynomial eigenfunctions are the Jacobi polynomials in $\cos 2x$, which are orthogonal with respect to the measure $\phi_0(x;\boldsymbol{\lambda})^2 = e^{2w_0(x;\boldsymbol{\lambda})} = (\sin x)^{2g}(\cos x)^{2h}$.

For each positive integer $\ell \geq 1$, let us introduce a prepotential and a Hamiltonian (15)–(16) together with the restriction on the parameters, $h > g > 0$:

$$\xi_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_\ell^{(-g-\ell-\frac12, h+\ell-\frac32)}(x), \tag{27}$$
$$w_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} w_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}) + \log\frac{\xi_\ell(\cos 2x;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\xi_\ell(\cos 2x;\boldsymbol{\lambda})}. \tag{28}$$

Since the polynomial $\xi_\ell(\cos 2x;\boldsymbol{\lambda})$ has no zero in the domain $0 < x < \frac{\pi}{2}$, the prepotential and the potential are smooth in the entire domain. It is straightforward to verify (5) with $\boldsymbol{\delta} = (1,1)$, $\mathcal{E}_{\ell,1}(\boldsymbol{\lambda}) = 4(1+2\ell+g+h)$. The eigenvalues and the eigenfunctions of $\mathcal{H}_\ell(\boldsymbol{\lambda})$ have the following form:

$$\mathcal{H}_\ell(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}) = \mathcal{E}_{\ell,n}(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}),\ n = 0, 1, \ldots, \tag{29}$$
$$\mathcal{E}_{\ell,n}(\boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}) = 4n(n+2\ell+g+h), \tag{30}$$
$$\phi_{\ell,n}(x;\boldsymbol{\lambda}) = P_{\ell,n}(\cos 2x;\boldsymbol{\lambda})\psi_\ell(x;\boldsymbol{\lambda}), \tag{31}$$
$$\psi_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{e^{w_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}}{\xi_\ell(\cos 2x;\boldsymbol{\lambda})}, \tag{32}$$
$$P_{\ell,n}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} a_{\ell,n}(x;\boldsymbol{\lambda})P_n(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}) + b_{\ell,n}(x;\boldsymbol{\lambda})P_{n-1}(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}), \tag{33}$$
$$a_{\ell,n}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \xi_\ell(x;g+1,h+1) + \frac{2n(-g+h+\ell-1)\,\xi_{\ell-1}(x;g,h+2)}{(-g+h+2\ell-2)(g+h+2n+2\ell-1)} - \frac{n(2h+4\ell-3)\,\xi_{\ell-2}(x;g+1,h+3)}{(2g+2n+1)(-g+h+2\ell-2)}, \tag{34}$$
$$b_{\ell,n}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{(-g+h+\ell-1)(2g+2n+2\ell-1)}{(2g+2n+1)(g+h+2n+2\ell-1)} \times \xi_{\ell-1}(x;g,h+2). \tag{35}$$

The polynomial eigenfunction $P_{\ell,n}(x;\boldsymbol{\lambda})$ is a degree $\ell+n$ polynomial in x and we have $P_{\ell,0}(x;\boldsymbol{\lambda}) = \xi_\ell(x;\boldsymbol{\lambda}+\boldsymbol{\delta})$ and $\phi_{\ell,0}(x;\boldsymbol{\lambda}) = e^{w_\ell(x;\boldsymbol{\lambda})}$. Again $P_{\ell,n}(\cos 2x;\boldsymbol{\lambda})$ has only n zeros in the domain $0 < x < \frac{\pi}{2}$. It should be stressed that $P_{\ell,n}(x;\boldsymbol{\lambda})$ are polynomials in the coupling constants g, h. They are orthogonal with respect to the measure $\psi_\ell(x;\boldsymbol{\lambda})^2$,

$$\int_0^{\frac{\pi}{2}} dx\, \psi_\ell(x;\boldsymbol{\lambda})^2 P_{\ell,n}\big(\cos 2x;\boldsymbol{\lambda}\big)P_{\ell,m}\big(\cos 2x;\boldsymbol{\lambda}\big) = \frac{\Gamma(n+g+\ell+\frac12)\Gamma(n+h+\ell+\frac12)}{2\,n!\,(2n+g+h+2\ell)\Gamma(n+g+h+2\ell)} \times \frac{(n+g+\ell+\frac12)(n+h+2\ell-\frac12)}{(n+g+\frac12)(n+h+\ell-\frac12)}\,\delta_{nm}, \tag{36}$$

and they form a complete basis of the Hilbert space just like the Jacobi polynomials in the $\ell = 0$ case. The action of the operators $\mathcal{A}_\ell(\boldsymbol{\lambda})$ and $\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger$ on the eigenfunctions are:

$$\mathcal{A}_\ell(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}) = -2(n+2\ell+g+h)\phi_{\ell,n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}),$$
$$\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\phi_{\ell,n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) = -2n\phi_{\ell,n}(x;\boldsymbol{\lambda}). \tag{37}$$

For $\ell = 1$ the Hamiltonian reads explicitly as

$$\mathcal{H}_1(\boldsymbol{\lambda}) = p^2 + \frac{g(g+1)}{\sin^2 x} + \frac{h(h+1)}{\cos^2 x} - (2+g+h)^2 + \frac{8(g+h+1)}{1+g+h+(g-h)\cos 2x} - \frac{8(2g+1)(2h+1)}{(1+g+h+(g-h)\cos 2x)^2},$$

which is equivalent to that of the shape invariant potential of Quesne eq.(11) of [9] with the replacement $A \to \frac12(g+h)+1$, $B \to \frac12(h-g)$ and $x \to 2(\frac{\pi}{4}-x)$. The formula (33) expressing the polynomial eigenfunctions in terms of the Jacobi polynomials is the generalisation of Gómez-Ullate et al.'s [10] relation eq.(56) between the X_1-Jacobi and the Jacobi polynomials.

5. Hyperbolic -Pöschl-Teller potential

The next example provides only a finite number of shape invariant potentials, as many as the existing bound states of the starting Hamiltonian with the hyperbolic Pöschl-Teller potential with $\boldsymbol{\lambda} = (g, h)$, $h > g > 0$:

$$\mathcal{H}_0(\boldsymbol{\lambda}) = p^2 + \frac{g(g-1)}{\sinh^2 x} - \frac{h(h+1)}{\cosh^2 x} + (h-g)^2, \tag{38}$$
$$w_0(x;\boldsymbol{\lambda}) = g\log\sinh x - h\log\cosh x,\ 0 < x < \infty. \tag{39}$$

As the name suggests, it is the hyperbolic function version of the Darboux-Pöschl-Teller model discussed in the preceding section. It is trivial to verify (5) with $\boldsymbol{\delta} = (1,-1)$, $\mathcal{E}_1(\boldsymbol{\lambda}) = 4(h-g-1)$. We obtain the quadratic energy spectrum and the corresponding eigenfunctions $n =$

$0,1,\ldots,n_B \stackrel{\text{def}}{=} [(h-g)/2]'$, expressed in terms of the Jacobi polynomials:

$$\mathcal{E}_n(\boldsymbol{\lambda}) = 4n(h-g-n), \quad n=0,1,2,\ldots,n_B, \tag{40}$$

$$\phi_n(x;\boldsymbol{\lambda}) = P_n(\cosh 2x;\boldsymbol{\lambda})\, e^{w_0(x;\boldsymbol{\lambda})}, \quad P_n(x;\boldsymbol{\lambda}) = P_n^{(g-\frac{1}{2},-h-\frac{1}{2})}(x). \tag{41}$$

Here $[x]'$ denotes the greatest integer not equal or exceeding x. These finite number of polynomials in $\cosh 2x$ are square integrable and are orthogonal with respect to the measure $\phi_0(x;\boldsymbol{\lambda})^2 = e^{2w_0(x;\boldsymbol{\lambda})} = (\sinh x)^{2g}(\cosh x)^{-2h}$.

For each positive integer $1 \le \ell < n_B$, let us introduce a prepotential and a Hamiltonian (15)–(16):

$$\xi_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_\ell^{(-g-\ell-\frac{1}{2},-h+\ell-\frac{3}{2})}(x), \tag{42}$$

$$w_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} w_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}) + \log\frac{\xi_\ell(\cosh 2x;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\xi_\ell(\cosh 2x;\boldsymbol{\lambda})}. \tag{43}$$

Since the polynomial $\xi_\ell(\cosh 2x;\boldsymbol{\lambda})$, $1 \le \ell < n_B$, has no zero in the domain $0 < x < \infty$, the prepotential and the potential are smooth in the entire domain. It is straightforward to verify (5) with $\boldsymbol{\delta} = (1,-1)$, $\mathcal{E}_{\ell,1}(\boldsymbol{\lambda}) = 4(h-g-2\ell-1)$. The repetition of the shape invariant transformation $w_\ell(x;\boldsymbol{\lambda}) \to w_\ell(x;\boldsymbol{\lambda}+\boldsymbol{\delta})$ is unlimited for the previous two examples. For the hyperbolic Pöschl-Teller Hamiltonian $\mathcal{H}_\ell(\boldsymbol{\lambda})$, the maximal repetition is $n_B - \ell$. Beyond that point, the transformed Hamiltonian no longer possesses a bound state. The maximal ℓ case, $\mathcal{H}_{n_B}$ has only one bound state, which is exactly calculable, and the transformed one has none. The energy spectrum is

$$\mathcal{E}_{\ell,n}(\boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda}+\ell\boldsymbol{\delta}) = 4n(h-g-2\ell-n), \quad n=0,\ldots,n_B-\ell. \tag{44}$$

Using the same notation as (29) and (31)–(33) with the replacement $\cos 2x \to \cosh 2x$, the eigenfunctions are

$$a_{\ell,n}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \xi_\ell(x;g+1,h-1) + \frac{2n(-g-h+\ell-1)\,\xi_{\ell-1}(x;g,h-2)}{(-g-h+2\ell-2)(g-h+2n+2\ell-1)} - \frac{n(-2h+4\ell-3)\,\xi_{\ell-2}(x;g+1,h-3)}{(2g+2n+1)(-g-h+2\ell-2)}, \tag{45}$$

$$b_{\ell,n}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{(-g-h+\ell-1)(2g+2n+2\ell-1)}{(2g+2n+1)(g-h+2n+2\ell-1)} \times \xi_{\ell-1}(x;g,h-2). \tag{46}$$

The polynomials $\{P_{\ell,n}(x;\boldsymbol{\lambda})\}$ are orthogonal with respect to the measure $\psi_\ell(x;\boldsymbol{\lambda})^2$. The action of the operators $\mathcal{A}_\ell(\boldsymbol{\lambda})$ and $\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger$ on the eigenfunctions are:

$$\mathcal{A}_\ell(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}) = -2(h-g-2l-n)\phi_{\ell,n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}),$$

$$\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\phi_{\ell,n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) = -2n\phi_{\ell,n}(x;\boldsymbol{\lambda}). \tag{47}$$

The $\ell=1$ Hamiltonian reads

$$\mathcal{H}_1(\boldsymbol{\lambda}) = p^2 + \frac{g(g+1)}{\sinh^2 x} - \frac{h(h-1)}{\cosh^2 x} + (h-g-2)^2 + \frac{8(h-g-1)}{1+g-h+(g+h)\cosh 2x} - \frac{8(2g+1)(2h-1)}{(1+g-h+(g+h)\cosh 2x)^2}. \tag{48}$$

6. Summary and Comments

By deforming two well-known shape invariant potentials in terms of their eigenpolynomials, two infinite sets of shape invariant potentials are obtained. Their eigenpolynomials form new types of orthogonal polynomials starting with degree ℓ, which is the degree of the polynomial used for deformation. It would be interesting to try to deform other shape invariant potentials in a similar way. It is a good challenge to clarify various properties of these new polynomials, *e.g.* generating functions, the Gram-Schmidt construction, substitutes of the three term recurrence relations, etc., and to pursue possible physical applications.

The forward shift operator $\mathcal{F}_\ell$ and the backward shift operator $\mathcal{B}_\ell$ are defined by

$$\mathcal{F}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \psi_\ell(x;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1} \circ \mathcal{A}_\ell(\boldsymbol{\lambda}) \circ \psi_\ell(x;\boldsymbol{\lambda}),$$

$$\mathcal{B}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \psi_\ell(x;\boldsymbol{\lambda})^{-1} \circ \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger \circ \psi_\ell(x;\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{49}$$

and their action on the eigenpolynomials $P_{\ell,n}$ can be read from (22), (37) and (47).

The $\ell=1$ Hamiltonian $\mathcal{H}_1(\boldsymbol{\lambda})$ (48) for the deformed hyperbolic Pöschl-Teller potential is equivalent to the 'new extended potential' (9) of Bagchi et al.'s paper [13] with the replacement $A \to \frac{1}{2}(h-g)-1$, $B \to \frac{1}{2}(h+g)$ and $x \to 2x$. We thank C. Quesne for pointing this out. After submitting the present Letter for publication, a new paper appeared [14], which discussed the $\ell=2$ deformed Hamiltonian $\mathcal{H}_2(\boldsymbol{\lambda})$ (15)–(16) with (13)–(14) or (27)–(28), and other potentials related to the X_2-Laguerre or X_2-Jacobi polynomials.

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