

Representations of finite association schemes

Akihide Hanaki

*Faculty of Science, Shinshu University,
Matsumoto, 390-8621, Japan
hanaki@math.shinshu-u.ac.jp*

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1 Introduction

Association schemes, or briefly schemes, are combinatorial objects which are connected to many different mathematical objects, especially to codes and designs; cf. [9] and [34]. Based on their connection to codes and designs commutative schemes (in particular symmetric schemes) have been investigated in numerous articles during the last thirty years. However, independently from combinatorial constraints one does not have to assume schemes to be commutative. In this article, we consider schemes as generalizations of groups and, consequently, we do not assume them to be commutative. Non-commutative schemes were studied by Higman [28], [29] as *homogeneous coherent configurations*.

In scheme theory, one is often interested in a complete description of certain classes of schemes. With the help of computers, one presently knows all schemes of order at most 30; cf. [25]. For larger orders, classifications become increasingly more complicated, since the number of schemes increases too fast. However, the number of different adjacency algebras does not grow with the same speed. For example, one can classify all adjacency algebras of association schemes of order at most 34, whereas the classification of all schemes of this order is out of reach. The classification of adjacency algebras or character tables of schemes gives a rough classification of schemes. Thus, an algebraic approach would help us to attack classification problems.

There exists a well-established representation theory of schemes over the complex number field. The adjacency algebra of a scheme over this field is semisimple, so that the character theory is well understood in this case. Especially, if a scheme is commutative, its complex adjacency algebra is completely determined by its character table. There exist many useful formulas for complex characters. They are immediate consequences of the basic definitions, but they can be obtained from more general results on coalgebras and Frobenius algebras. In Section 2, we will summarize the theory of these algebras, and in Section 4, we will present applications. This enables us to find new formulas on schemes and to generalize formulas for other combinatorial objects.

Imprimitive schemes give rise to subschemes and quotient schemes, so that one can employ inductive arguments. From this point of view it might be useful to understand the relationship between representations of schemes and representations of their subschemes and quotient schemes. We will discuss this relationship in Section 4. Especially, we explain how Clifford theory for finite groups can be generalized to association schemes.

In Section 5, we will give a short introduction to modular representation theory of schemes. Modular representation theory is a relatively new branch in scheme theory and deals with representations (of schemes) over fields of positive characteristic. Its usefulness surfaced recently, when Katsuhiko Uno and the author succeeded in utilizing its techniques in order to prove that association schemes of prime order must be commutative; cf. [26]. In Section 5, we will outline our proof and provide some related problems on modular representations.

2 Algebras and modules

In this section, the reader is assumed to be familiar with basic facts on finite dimensional algebras and their modules. As for coalgebras, we refer to [35]. General introductory textbooks on finite dimensional algebras and Frobenius algebras are [11] and [6].

2.1 Algebras and coalgebras

Let K be a field. In this subsection, tensor products are over K . Though we assume knowledge of basic facts on finite dimensional algebras, we will give

the definition of an algebra to compare it with the one of a coalgebra. Many of our arguments remain valid if K is a commutative ring with unity.

Definition 2.1 (Algebras). Let A be a K -vector space. Given K -linear maps $m : A \otimes A \rightarrow A$ and $u : K \rightarrow A$, the triple (A, m, u) is called a K -algebra if the following diagrams are commutative :

$$\begin{array}{ccc}
 A \otimes A \otimes A & \xrightarrow{m \otimes id} & A \otimes A \\
 \downarrow id \otimes m & & \downarrow m \\
 A \otimes A & \xrightarrow{m} & A
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & & A \otimes A & & \\
 & u \otimes id \nearrow & & \nwarrow id \otimes u & \\
 K \otimes A & & \downarrow m & & A \otimes K \\
 & \searrow & & \swarrow & \\
 & & A & &
 \end{array}$$

(they are called the *associativity* and the *unit*, respectively). Namely,

$$m \circ (m \otimes id) = m \circ (id \otimes m), \quad m \circ (u \otimes id) = id, \quad m \circ (id \otimes u) = id$$

by identifying $K \otimes A = A$. The maps m and u are called the *multiplication* and the *unit*, respectively.

Taking the dual of the definition of K -algebras, we define K -coalgebras.

Definition 2.2 (Coalgebras). Let C be a K -vector space. Given K -linear maps $\Delta : C \rightarrow C \otimes C$ and $\varepsilon : C \rightarrow K$, the triple (C, Δ, ε) is called a K -coalgebra if the following diagrams are commutative :

$$\begin{array}{ccc}
 C & \xrightarrow{\Delta} & C \otimes C \\
 \downarrow \Delta & & \downarrow \Delta \otimes id \\
 C \otimes C & \xrightarrow{id \otimes \Delta} & C \otimes C \otimes C
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & & C & & \\
 & 1 \otimes \nwarrow & & \searrow \otimes 1 & \\
 K \otimes C & & \downarrow \Delta & & C \otimes K \\
 & \swarrow \varepsilon \otimes id & & \nwarrow id \otimes \varepsilon & \\
 & & C \otimes C & &
 \end{array}$$

(they are called the *coassociativity* and the *counit*, respectively). Namely,

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta, \quad 1 \otimes = (\varepsilon \otimes id) \circ \Delta, \quad \otimes 1 = (id \otimes \varepsilon) \circ \Delta.$$

The maps Δ and ε are called the *comultiplication* and the *counit*, respectively.

For each K -vector space V , we set $V^* = \text{Hom}_K(V, K)$. The vector space V^* is called the K -dual of V . Let V, W be K -vector spaces, and $\phi : V \rightarrow W$ a K -linear map. Define $\phi^* : W^* \rightarrow V^*$ by

$$\phi^*(f)(v) = f(\phi(v))$$

for $v \in V$ and $f \in W^*$.

Proposition 2.3 ([35, Lemma 1.2.2, Proposition 1.2.4]). *If (C, Δ, ε) is a K -coalgebra, then $(C^*, \Delta^*, \varepsilon^*)$ is a K -algebra. If (A, m, u) is a finite dimensional K -algebra, then (A^*, m^*, u^*) is a K -coalgebra.*

For a K -coalgebra (C, Δ, ε) , the multiplication of C^* is given by $m = \Delta^*$. So

$$m(f \otimes g)(c) = \Delta^*(f \otimes g)(c) = (f \otimes g) \circ \Delta(c)$$

for $f, g \in C^*$ and $c \in C$.

Definition 2.4 (Group-like elements). Let (C, Δ, ε) be a K -coalgebra. An element $c \in C$ is called a *group-like element* if $\Delta(c) = c \otimes c$ and $\varepsilon(c) = 1$. Let $G(C)$ denote the set of all group-like elements of C .

Proposition 2.5 ([35, p.4]). *Let (C, Δ, ε) be a K -coalgebra. The set $G(C)$ is K -linearly independent.*

Remark. Let (B, m, u) be a K -algebra and (B, Δ, ε) a K -coalgebra. If both Δ and ε are algebra homomorphism, then $(B, m, u, \Delta, \varepsilon)$ is called a *bialgebra*. A bialgebra is called a *Hopf algebra* if it has an *antipode*. Every adjacency algebra of an association scheme over a field of characteristic zero is an algebra and a coalgebra, but not a bialgebra, in general. In the paper [10], Yukio Doi defined *bi-Frobenius algebras* and *group-like algebras*. Adjacency algebras of association schemes over fields of characteristic zero give typical examples of bi-Frobenius algebras and group-like algebras.

Example 2.6. Let G be a finite group, and let KG be the group algebra of G over a field K . Define $\Delta : KG \rightarrow KG \otimes KG$ and $\varepsilon : KG \rightarrow K$ by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1$$

for every $g \in G$. Then $(KG, \Delta, \varepsilon)$ is a K -coalgebra. (Actually, KG is a Hopf algebra). It is clear that $g \in G$ is a group-like element. By Proposition 2.5 and since G is a K -basis of KG , every group-like element is an element of G .

2.2 Semisimple algebras

In this subsection, K is a field, A a finite dimensional K -algebra with unity, and A -modules are finite dimensional over K and left A -modules.

Definition 2.7 (Jacobson radicals). The intersection of all maximal left ideals of A is called the *Jacobson radical* of A and denoted by $J(A)$.

The Jacobson radical $J(A)$ is a two-sided ideal of A and a nilpotent ideal. Moreover, the Jacobson radical is the largest nilpotent ideal of A .

Definition 2.8 (Semisimple algebras). A K -algebra A is said to be *semisimple* if $J(A) = 0$.

Note that $J(A/J(A)) = 0$ for any A . So $A/J(A)$ is always semisimple. Next we define simple algebras.

Definition 2.9 (Simple algebras). A K -algebra is said to be *simple* if it has no non-trivial ideal.

We are assuming that A is a K -algebra with unity. So A is not a nilpotent ideal. Thus, if A is simple, $J(A) = 0$ and A is semisimple.

Example 2.10. Let D be a division K -algebra. Let $\text{Mat}_n(D)$ denote the full matrix algebra over D and of degree n . Then every non-zero element of $\text{Mat}_n(D)$ generates $\text{Mat}_n(D)$ as a two-sided ideal. So $\text{Mat}_n(D)$ has no non-trivial ideal. Therefore $\text{Mat}_n(D)$ is a simple algebra.

Now we describe the structure of semisimple algebra.

Theorem 2.11 (Wedderburn-Artin [11, Theorem 2.4.3]). *Every semisimple K -algebra A is isomorphic to a direct sum of full matrix algebras over division K -algebras. Moreover, if K is algebraically closed, then A is isomorphic to a direct sum of full matrix algebras over K .*

Let F be an extension field of K , let A be a K -algebra, and set

$$A^F = F \otimes_K A.$$

If $\{x_i\}$ is a K -basis of A , then $\{1 \otimes x_i\}$ is a F -basis of A^F . Usually we identify $1 \otimes x_i$ with x_i and consider $A \subseteq A^F$.

For Jacobson radicals, we see that $F \otimes_K J(A) \subseteq J(A^F)$. So A^F is not necessary semisimple, even if A is semisimple.

Definition 2.12 (Separable algebras). A K -algebra A is called *separable* if A^F is semisimple for any extension field F of K .

A field K is called *perfect* if every finite dimensional semisimple K -algebra is separable. It is known that fields of characteristic zero and finite fields are perfect.

Definition 2.13 (Splitting fields). Let A be a K -algebra. An extension field F of K is called a *splitting field* of A if $A^F/J(A^F)$ is isomorphic to a direct sum of full matrix algebras over F . A K -algebra A is called a *splitting algebra* if K is a splitting field of A .

For any finite dimensional K -algebra A , there exists a finite extension F of K such that F is a splitting field of A . Note that the minimal splitting field is not uniquely determined (see [11, p. 78]).

Suppose A is a semisimple K -algebra. We consider finitely generated A -modules. To simplify our arguments, we suppose K is algebraically closed. By Theorem 2.11, we have

$$A \cong \bigoplus_{i=1}^r \text{Mat}_{n_i}(K).$$

For every direct summand $\text{Mat}_{n_i}(K)$, there is an irreducible left module whose K -dimension is n_i , and it also becomes a left A -module. Conversely, any irreducible left A -module is isomorphic to one of the modules obtained as above. So there are r isomorphism classes of irreducible A -modules. Let $\text{IRR}(A)$ denote the set of representatives of isomorphism classes of irreducible A -modules.

Consider an arbitrary (not necessary semisimple) A . Since every irreducible A -module is annihilated by the Jacobson radical $J(A)$, simple A -modules are obtained as simple $A/J(A)$ -modules.

Definition 2.14 (Completely reducible modules). Let A be a K -algebra. An A -module V is said to be *completely reducible* (or *semisimple*) if it is a sum of irreducible A -submodules of V .

It is known that an A -module V is completely reducible if and only if V is isomorphic to a direct sum of irreducible modules. This is equivalent to the fact that, for any A -submodule W of V , there exists an A -submodule U of V such that $V = W \oplus U$.

We may consider A as a left A -module. This module is called the *left regular A -module*. To distinguish the regular module A from the algebra A , we write ${}_A A$ for the regular module. It is known that the algebra A is semisimple if and only if the left regular A -module ${}_A A$ is completely reducible. Also this condition is equivalent to the fact that every A -module is completely reducible.

2.3 Matrix representations and characters

Let K be a field, A a finite dimensional K -algebra, and V a left A -module with $\dim_K V = n < \infty$. The action of A on V induces a map

$$T : A \rightarrow \text{End}_K(V) \cong \text{Mat}_n(K)$$

by $T(a)(v) = av$ for $a \in A$ and $v \in V$. The map T is a K -algebra homomorphism. In general, a K -algebra homomorphism $T : A \rightarrow \text{Mat}_n(K)$ is called a *matrix representation* of A .

Conversely, let $T : A \rightarrow \text{Mat}_n(K)$ be a matrix representation. Then $V = K^n$ becomes a left A -module by the action $av = T(a)v$ for $a \in A$ and $v \in V$.

For an A -module V , a matrix representation $T : A \rightarrow \text{Mat}_n(K)$ depends on the choice of the basis of V . If we take a different basis of V , we get a *similar* representation, that is $a \mapsto P^{-1}T(a)P$ for some nonsingular matrix P . Conversely, similar representations give isomorphic A -modules.

For a matrix representation $T : A \rightarrow \text{Mat}_n(K)$, the trace function is called the *character* of T . Note that the characters of similar representations are the same. It is easy to see that a character is K -linear, namely, in $A^* = \text{Hom}_K(A, K)$.

Let us now consider a semisimple \mathbb{C} -algebra A and its characters. (Our reasoning remains valid for any finite dimensional splitting algebra over a field of characteristic zero). By Theorem 2.11,

$$A \cong \bigoplus_{i=1}^r \text{Mat}_{n_i}(\mathbb{C}).$$

Let $e_i \in A$ correspond to the identity matrix of $\text{Mat}_{n_i}(\mathbb{C})$. Then every e_i is a central primitive idempotent of A , and we have

$$e_i e_j = \delta_{ij} e_i, \quad \sum_{i=1}^r e_i = 1_A.$$

Recall that isomorphism classes of irreducible A -modules correspond to direct summands of A . So they are indexed by $i = 1, 2, \dots, r$. Let χ_i be the character afforded by the i -th representation. Then it follows readily that

$$\chi_i(ae_j) = \delta_{ij}\chi(a)$$

for $a \in A$. We write $\text{Irr}(A) = \{\chi_i \mid i = 1, \dots, r\}$.

Proposition 2.15. *Let A be a semisimple \mathbb{C} -algebra.*

- (1) *The set $\text{Irr}(A)$ is linearly independent in A^* .*
- (2) *Let T and T' be representations of A . Then T and T' are similar if and only if their characters are the same.*

The following proposition provides a condition under which the hypothesis of Proposition 2.15 is satisfied.

Proposition 2.16. *Let A be a \mathbb{C} -subalgebra of $\text{Mat}_n(\mathbb{C})$ closed under the transposed complex conjugate $a \mapsto {}^t\bar{a}$. Then A is semisimple.*

2.4 Frobenius algebras

Let K be a field and A a finite dimensional K -algebra. As before, the dual of A is denoted by $A^* = \text{Hom}_K(A, K)$. Then A^* has a structure of an (A, A) -bimodule by

$$(fa)(b) = f(ab), \quad (af)(b) = f(ba)$$

for $f \in A^*$ and $a, b \in A$.

Definition 2.17 (Frobenius algebras). An algebra A is called a *Frobenius algebra* if there exists $\phi \in A^*$ such that ϕ generates A^* as a left A -module. In this case, we also say that (A, ϕ) is a Frobenius algebra.

Let (A, ϕ) be a Frobenius algebra. Then the map

$$\theta : A \rightarrow A^*, \quad \theta(a) = a\phi$$

gives an isomorphism as left A -modules. We define a K -bilinear form $(\ , \)_A$ on A by

$$(a, b)_A = \theta(b)(a) = (b\phi)(a) = \phi(ab).$$

The form $(\ , \)_A$ is non-degenerate.

Definition 2.18 (Dual bases). Let (A, ϕ) be a Frobenius algebra and $\{x_i\}$ a K -basis of A . Since $(\ , \)_A$ is non-degenerate, there exists a K -basis $\{y_i\}$ such that

$$\phi(x_i y_j) = (x_i, y_j)_A = \delta_{ij}.$$

The bases $\{x_i\}$ and $\{y_i\}$ are called *dual bases* of A .

Definition 2.19 (Class functions). For an arbitrary Frobenius K -algebra A , $f \in A^*$ is called a *class function* if $f(ab) = f(ba)$ for any $a, b \in A$. (In [30], a class function is called a *feasible trace*.) We shall denote by $\text{CF}(A)$ the set of all class functions of A .

Now we consider a semisimple \mathbb{C} -algebra A . Then (A, ϕ) is a Frobenius algebra for some ϕ . For example, we can take $\phi = \sum_{\chi \in \text{Irr}(A)} \chi$, but there are many choices of ϕ . Also fix dual bases $\{x_i\}$ and $\{y_i\}$ of (A, ϕ) . It is easy to see that $f \in A^*$ is a class function if and only if f is a linear combination of irreducible characters.

Lemma 2.20. *Let (A, ϕ) be a Frobenius \mathbb{C} -algebra with dual bases $\{x_i\}$ and $\{y_i\}$. A class function $f = \sum_{\chi \in \text{Irr}(A)} f_\chi \chi$ generates A^* as a left A -module if and only if $f_\chi \neq 0$ for all $\chi \in \text{Irr}(A)$.*

Put $\phi = \sum_{\chi \in \text{Irr}(A)} \phi_\chi \chi \in \text{CF}(A)$ for a semisimple Frobenius \mathbb{C} -algebra (A, ϕ) . By e_χ we denote the central primitive idempotent of A corresponding to $\chi \in \text{Irr}(A)$. Note that $\chi(e_\varphi) = \delta_{\chi\varphi} \chi(1)$ for $\chi, \varphi \in \text{Irr}(A)$. Then we have the following formulas.

Proposition 2.21. *Let (A, ϕ) be a Frobenius \mathbb{C} -algebra with dual bases $\{x_i\}$ and $\{y_i\}$. Then $e_\chi = \phi_\chi \sum_i \chi(x_i) y_i$.*

Theorem 2.22 (Orthogonality relations). *Let (A, ϕ) be a Frobenius \mathbb{C} -algebra with dual bases $\{x_i\}$ and $\{y_i\}$. Assume (A, ϕ) to be semisimple, and set $\phi = \sum_{\chi \in \text{Irr}(A)} \phi_\chi \chi \in \text{CF}(A)$. Let $\chi, \varphi \in \text{Irr}(A)$. Then*

$$\frac{\phi_\chi}{\chi(1)} \sum_i \chi(x_i) \varphi(y_i) = \delta_{\chi\varphi}.$$

Orthogonality relations for non-semisimple Frobenius algebras were studied in [13].

3 Association schemes and their adjacency algebras

In this section, we will give some definitions and notation for subsequent sections. Basic facts will be stated without proofs. The reader is referred to textbooks such as [2], [38], and [39].

3.1 Association schemes

Definition 3.1 (Association schemes). Let X be a finite set and S a partition of $X \times X$. The pair (X, S) is called an *association scheme*, or in short a *scheme*, if the following conditions are satisfied.

- (1) $\{(x, x) \mid x \in X\} \in S$ (this relation will be denoted by 1_X or simply 1).
- (2) If $s \in S$, then $\{(y, x) \mid (x, y) \in s\} \in S$ (this relation will be denoted by s^*).
- (3) For $s, t, u \in S$, there exists an integer p_{st}^u such that

$$\#\{z \in X \mid (x, z) \in s, (z, y) \in t\} = p_{st}^u$$

whenever $(x, y) \in u$ (p_{st}^u is called *intersection number* or *structure constant*).

We say that the scheme is *commutative* if $p_{st}^u = p_{ts}^u$ for all $s, t, u \in S$. We say that the scheme is *symmetric* if $s^* = s$ for all $s \in S$. Symmetric schemes are commutative. An association scheme is also called a *homogeneous coherent configuration*.

Let (X, S) be a scheme. The cardinality of X will be called the *order* of (X, S) . The *class* of the scheme is $|S| - 1$. For $x \in X$ and $s \in S$, put $xs = \{y \in X \mid (x, y) \in s\}$ and $sx = \{z \in X \mid (z, x) \in s\}$. Then, for any $x \in X$, $p_{ss^*}^1 = |xs| = |sx| = p_{s^*s}^1$. This number will be called the *valency* of $s \in S$ and denoted by n_s . For a subset T of S , we set $n_T = \sum_{t \in T} n_t$. Note that $n_S = |X|$.

Definition 3.2 (Complex products). For $s, t \in S$, we put $st = \{u \in S \mid p_{st}^u > 0\}$ and call this the *complex product* of s and t . For subsets T and U of S , we also define the complex product by $TU = \bigcup_{t \in T} \bigcup_{u \in U} tu$. When

$U = \{u\}$, $T\{u\}$ and $\{u\}T$ will be denoted by Tu and uT , respectively. Complex multiplication of both elements and subsets are associative.

We also set $xT = \bigcup_{t \in T} xt$.

Definition 3.3 (Closed subsets). Let (X, S) be a scheme. A nonempty subset T of S is called *closed* if $TT \subseteq T$. A closed subset T of S is said to be *normal* if $sT = Ts$ for all $s \in S$. A closed subset T of S is said to be *strongly normal* if $sTs^* = T$ for all $s \in S$. Strongly normal closed subsets are normal, but the converse is not true, in general.

Let (X, S) be a scheme, T a closed subset of S . For $x \in X$, put

$$S_{xT \times xT} = \{s \cap (xT \times xT) \mid s \in S, s \cap (xT \times xT) \neq \emptyset\}.$$

Then $(xT, S_{xT \times xT})$ is a scheme. We call this scheme a *subscheme* of (X, S) by T with respect to $x \in X$. (Note that *subschemes* are defined differently in [2].) For $s, t, u \in S$, put $s' = s \cap (xT \times xT)$, $t' = t \cap (xT \times xT)$, and $u' = u \cap (xT \times xT)$. Suppose $s' \neq \emptyset$, $t' \neq \emptyset$, and $u' \neq \emptyset$. Then $p_{st}^u = p_{s't'}^{u'}$ holds. Also, for valencies, we have $n_s = n_{s'}$.

Again, let (X, S) be a scheme, T a closed subset of S . Put $X/T = \{xT \mid x \in X\}$. For $s \in S$, we define relation s^T on X/T by

$$s^T = \{(xT, yT) \mid (x', y') \in s \text{ for some } x' \in xT \text{ and } y' \in yT\}$$

and put $S//T = \{s^T \mid s \in S\}$. Then $(X/T, S//T)$ is a scheme. We call this scheme the *quotient scheme* of (X, S) by T . For the intersection numbers, we have

$$p_{s^T t^T}^{u^T} = \frac{1}{n_T} \sum_{s' \in TsT} \sum_{t' \in TtT} p_{s't'}^{u^T}$$

and $n_{s^T} = n_{TsT}/n_T$.

Definition 3.4 (Thin elements and thin subsets). Let (X, S) be a scheme. An element s of S is said to be *thin* if $n_s = 1$. A subset U of S is said to be *thin* if every $s \in U$ is thin. A thin closed subset is essentially a finite group. It is known that a closed subset T is strongly normal in S if and only if the quotient scheme $(X/T, S//T)$ is thin.

Definition 3.5 (Isomorphisms and algebraic isomorphisms). Let (X, S) and (Y, T) be schemes. An *isomorphism* from (X, S) to (Y, T) is defined to be a pair (ϕ, ψ) such that both $\phi : X \rightarrow Y$ and $\psi : S \rightarrow T$ are bijections

and $(x, y) \in s$ if and only if $(\phi(x), \phi(y)) \in \psi(s)$ for $s \in S$ and $x, y \in X$. An *algebraic isomorphism* from (X, S) to (Y, T) is a map $\psi : S \rightarrow T$ such that $p_{st}^u = p_{\psi(s)\psi(t)}^{\psi(u)}$ for $s, t, u \in S$. Each isomorphism induces an algebraic isomorphism, but the converse does not hold, in general.

3.2 Adjacency algebras

Let R be a commutative ring with unity. Let $\text{Mat}_n(R)$ denote the full matrix ring over R and of degree n . For a finite set X , let $\text{Mat}_X(R)$ denote the full matrix ring over R whose both rows and columns are indexed by X . Obviously, one has $\text{Mat}_X(R) \cong \text{Mat}_{|X|}(R)$.

Let (X, S) be an association scheme. For $s \in S$, we define the *adjacency matrix* $\sigma_s \in \text{Mat}_X(R)$ by

$$(\sigma_s)_{xy} = \begin{cases} 1, & \text{if } (x, y) \in s, \\ 0, & \text{otherwise.} \end{cases}$$

It follows right from the definition of association schemes that the set of R -linear combinations of $\{\sigma_s \mid s \in S\}$ is an R -algebra. Especially, we have $\sigma_{s^*} = {}^t\sigma_s$ (the transposed matrix), $\sigma_1 = I$ (the identity matrix), and $\sigma_s\sigma_t = \sum_{u \in S} p_{st}^u \sigma_u$. We call the R -algebra the *adjacency algebra* of (X, S) over R and denote it by RS . Note that adjacency algebras of algebraically isomorphic schemes are isomorphic as algebras.

It is easy to see that the map $RS \rightarrow R$ ($\sigma_s \mapsto n_s 1_R$) is a representation of RS of degree 1. We call this the *trivial representation* of RS and denote it by 1_{RS} or 1_S . The corresponding character is called the *trivial character* and also denoted by 1_{RS} or 1_S .

Let RX denote the R -free R -module with a formal basis X . Then RX is a left $\text{Mat}_X(R)$ -module with respect to the natural multiplication. Since RS is defined as a subalgebra of $\text{Mat}_X(R)$, RX is also a left RS -module. We call RX the *standard module* of (X, S) over R . The corresponding representation and character are called the *standard representation* and the *standard character* of (X, S) . They are denoted by Γ_S and γ_S , respectively. It is easy to see that

$$\gamma_S(\sigma_s) = \begin{cases} |X| 1_R, & \text{if } s = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.6. *There exists an RS -monomorphism $\iota : RS \rightarrow RX$. Indeed, the map $\sigma_s \mapsto \sigma_s x$ is an RS -monomorphism for any $x \in X$.*

Let (X, S) be a scheme and T a closed subset of S . Then, by the definition, $RT = \sum_{t \in T} R\sigma_t$ is an R -subalgebra of RS , and the adjacency algebra of every subscheme of S by T is isomorphic to RT . This allows us to consider representations of closed subsets as representations of the corresponding subschemes.

4 Ordinary representations

In this section, (X, S) is an association scheme. By a representation of (X, S) , we mean a representation of the adjacency algebra of (X, S) over a commutative ring with unity. The theory will be divided into two cases, the case where the underlying ring is a field of characteristic zero and the other cases, especially the case where the underlying ring is a field of positive characteristic or an integral domain. In the former case, one speaks about *ordinary representation theory*, in and latter one about *modular representation theory*. In this section, we compile facts on ordinary representations. The theory of modular representations will be discussed in the next section.

4.1 Ordinary adjacency algebras and characters

By Proposition 2.16, the adjacency algebra $\mathbb{C}S$ of (X, S) over \mathbb{C} is semisimple. The set $\text{Irr}(\mathbb{C}S)$ of irreducible characters of $\mathbb{C}S$ will be denoted by $\text{Irr}(S)$. We consider the irreducible decomposition of the standard character γ_S :

$$\gamma_S = \sum_{\chi \in \text{Irr}(S)} m_\chi \chi.$$

The multiplicity m_χ of χ in γ_S is called the *multiplicity* of $\chi \in \text{Irr}(S)$. The multiplicity of χ in the regular character is $\chi(1)$. So by Lemma 3.6, we have the following easy but important fact. This is proved in [12].

Theorem 4.1. *We have $m_\chi \geq \chi(1) > 0$ for every $\chi \in \text{Irr}(S)$.*

It is easy to see that $m_{1_S} = 1_S(1) = 1$.

Theorem 4.2. *For a scheme (X, S) , $(\mathbb{C}S, n_S^{-1}\gamma_S)$ is a semisimple Frobenius algebra. Moreover, $\{\sigma_s \mid s \in S\}$ and $\{n_s^{-1}\sigma_{s^*} \mid s \in S\}$ are dual bases of this algebra.*

Applying this theorem to Proposition 2.21 and Theorem 2.22, we have the following.

Theorem 4.3. *For each $\chi \in \text{Irr}(S)$, the central primitive idempotent corresponding to χ has the representation*

$$e_\chi = \frac{m_\chi}{n_S} \sum_{s \in S} \frac{1}{n_s} \chi(\sigma_{s^*}) \sigma_s.$$

Theorem 4.4 (Orthogonality relations). *For $\chi, \varphi \in \text{Irr}(S)$, we have*

$$\frac{m_\chi}{n_S \chi(1)} \sum_{s \in S} \frac{1}{n_s} \chi(\sigma_{s^*}) \varphi(\sigma_s) = \delta_{\chi\varphi}.$$

Theorem 4.4 can be generalized to (generalized) table algebras and to (non-homogeneous) coherent configurations.

As for character values, one has the following.

Proposition 4.5. *Let χ be a character of a scheme (X, S) afforded by a representation Φ over \mathbb{C} . For $s \in S$, we have $\chi(\sigma_{s^*}) = \overline{\chi(\sigma_s)}$ (where $\overline{\chi(\sigma_s)}$ is the complex conjugate). Moreover, there exists a representation Φ' which is similar to Φ such that $\Phi'(\sigma_{s^*}) = {}^t\Phi'(\sigma_s)$ for all $s \in S$.*

Let Φ be a matrix representation of $\mathbb{C}S$. For $s \in S$, every eigenvalue of $\Phi(\sigma_s)$ is an eigenvalue of σ_s . Since σ_s is a matrix over rational integers, its eigenvalue is an algebraic integer. Thus, character values are algebraic integers. Note also that the valency n_s is the Perron-Frobenius root of every connected component of σ_s . So $|\xi| \leq n_s$ for every eigenvalue ξ of σ_s .

Theorem 4.6. *For a character χ of S and $s \in S$, we have $|\chi(\sigma_s)| \leq n_s \chi(1)$.*

Let us consider the case where one has equality in Theorem 4.6. For $\chi \in \text{Irr}(S)$, put

$$\begin{aligned} K(\chi) &= \{s \in S \mid \chi(\sigma_s) = n_s \chi(1)\}, \\ Z(\chi) &= \{s \in S \mid |\chi(\sigma_s)| = n_s \chi(1)\}. \end{aligned}$$

For a character $\eta = \sum_{\chi \in \text{Irr}(S)} a_\chi \chi$, put

$$K(\eta) = \bigcap_{a_\chi > 0} K(\chi) \quad \text{and} \quad Z(\eta) = \bigcap_{a_\chi > 0} Z(\chi).$$

Theorem 4.7. *Let η be the character of S afforded by a representation Φ . Then $K(\eta)$ and $Z(\eta)$ are closed subsets of S . Moreover, $K(\eta) = \{s \in S \mid \Phi(\sigma_s) = n_s I\}$ where I is the identity matrix. If $\eta \in \text{Irr}(S)$, then $Z(\eta) = \{s \in S \mid \Phi(\sigma_s) = \varepsilon n_s I \text{ for some } \varepsilon \in \mathbb{C} \text{ such that } |\varepsilon| = 1\}$.*

We remark that $K(\eta)$ and $Z(\eta)$ are not necessary normal in S .

The matrix whose rows are indexed by elements of $\text{Irr}(S)$ and whose columns are indexed by elements of S with the (χ, s) -entry $\chi(\sigma_s)$ is called the *character table* of (X, S) . We state a question.

Question 4.8. Which properties of a scheme can be read from its character table?

We give one answer to this question.

Theorem 4.9 ([16, §3]). *All (strongly) normal closed subsets of a given scheme can be read from its character table.*

We summarize the proof of this theorem.

Proposition 4.10 ([16, Theorem 3.4]). *Let η be a character of (X, S) . Put $I(\eta) = \{\chi \in \text{Irr}(S) \mid \chi(\sigma_s) = n_s \chi(1) \text{ for any } s \in K(\eta)\}$. Then $K(\eta)$ is normal in S if and only if*

$$\sum_{\chi \in I(\eta)} m_\chi \chi(1) = \frac{n_S}{n_{K(\eta)}}.$$

One sees easily that, for a normal closed subset T of S , there exists a character η of (X, S) such that $K(\eta) = T$. Thus, Proposition 4.10 provides us with all normal closed subsets of S . In order to obtain the strongly normal closed subsets, we may apply following criterion.

Proposition 4.11 ([31, Theorem 2.8]). *For $\chi \in \text{Irr}(S)$, $K(\chi)$ is strongly normal in S if and only if $m_\chi = \chi(1)$ (in which case χ is essentially a group character).*

Every strongly normal closed subset is an intersection of some $K(\chi)$'s for $\chi \in \text{Irr}(S)$. This way one obtains all strongly normal closed subsets by inspection of the character table.

Another answer to Problem 4.8 was given in [20].

Let us now consider a closed subset of a scheme (X, S) . For a subset U of S , put $e_U = n_U^{-1} \sigma_U$. Then we have the following.

Theorem 4.12 ([17, Proposition 3.3]). *For a subset U of S , U is a closed subset if and only if e_U is an idempotent. Moreover, U is a normal closed subset if and only if e_U is a central idempotent of $\mathbb{C}S$.*

4.2 Representations of quotient schemes

In this subsection, we will consider representations of quotient schemes. Let A be a finite dimensional semisimple \mathbb{C} -algebra and e an idempotent of A . Then eAe is a \mathbb{C} -algebra with unity e . Since A is semisimple,

$$A \cong \bigoplus_{i=1}^r \text{Mat}_{n_i}(\mathbb{C}).$$

Let $\pi_i : A \rightarrow \text{Mat}_{n_i}(\mathbb{C})$ be the projection. Without loss of generality, we may assume that $\pi_i(e)$ is of the form $\text{diag}(1, \dots, 1, 0 \dots, 0)$. Let m_i be the rank of $\pi_i(e)$. Then one verifies easily that

$$eAe \cong \bigoplus_{i=1}^r \text{Mat}_{m_i}(\mathbb{C}).$$

Let χ_i be the character corresponding to the i -th direct summand. Then $m_i = \chi_i(e)$. Note that m_i can be zero. So we have the following.

Proposition 4.13. *Let A be a finite dimensional semisimple \mathbb{C} -algebra and e an idempotent of A . Then there exists a natural bijection between $\text{Irr}(eAe)$ and $\{\chi \in \text{Irr}(A) \mid \chi(e) \neq 0\}$.*

Let (X, S) be a scheme and T a closed subset of S . The idempotent $e_T = n_T^{-1}\sigma_T$ establishes a relation between $\text{Irr}(S)$ and $\text{Irr}(S//T)$.

Proposition 4.14. *The map*

$$\rho : \mathbb{C}(S//T) \rightarrow e_T \mathbb{C}S e_T, \quad \rho(\sigma_{sT}) = \frac{n_{sT}}{n_s} e_T \sigma_s e_T$$

is an algebra isomorphism.

So we can apply Proposition 4.13. Furthermore, by [23, Theorem 3.10], we have the following.

Theorem 4.15. *Let (X, S) be a scheme and T a closed subset of S . Then there exists a natural bijection between $\text{Irr}(S//T)$ and $\{\chi \in \text{Irr}(S) \mid \chi(e_T) \neq 0\}$. Moreover, this map preserves the multiplicities of irreducible characters.*

Remark. Let (X, S) be a (non-homogeneous) coherent configuration, and let Y be a fiber of (X, S) . Define $e \in \text{Mat}_X(\mathbb{C})$ by $|Y|^{-1}$ times the characteristic function of $Y \times Y$. Then e is an idempotent, and one obtains a similar relation between the irreducible characters of (X, S) and the irreducible characters of the homogeneous component of (X, S) defined by Y .

In general, there is no canonical algebra epimorphism from $\mathbb{C}S$ to $\mathbb{C}(S//T)$. But if T is normal, then there exists a such epimorphism. Suppose T is normal. Then e_T is a central idempotent of $\mathbb{C}S$. So the map $\sigma_s \mapsto e_T \sigma_s e_T$ is an algebra epimorphism. Especially, the map in Theorem 4.15 preserves the degree of a character. Thus, one may view $\text{Irr}(S//T)$ as a subset of $\text{Irr}(S)$.

Let $\chi \in \text{Irr}(S)$, and consider $K(\chi)$. Suppose a closed subset T of S is contained in $K(\chi)$. Then one obtains easily that $\chi(e_T) = \chi(1)$. So, in this case, χ can be considered as a character of $S//T$ though we do not assume that T is normal.

Question 4.16. Let K be an algebraically closed field of positive characteristic, (X, S) a scheme, and T a normal closed subset of S . Then there is a natural algebra homomorphism $\pi : KS \rightarrow K(S//T)$ which, in general, does not need to be an epimorphism. Is it possible to describe the relationship between representations of KS and representations of $K(S//T)$? Especially, can we describe the relationship between the irreducible representations of KS and irreducible representations of $K(S//T)$?

4.3 Character products

In this subsection, we will consider products of characters. To do this, we need a coalgebra structure of adjacency algebras. The tensor products in this subsection are over \mathbb{C} .

Let (X, S) be a scheme. Define $\Delta : \mathbb{C}S \rightarrow \mathbb{C}S \otimes \mathbb{C}S$ and $\varepsilon : \mathbb{C}S \rightarrow \mathbb{C}$ by

$$\Delta(\sigma_s) = \frac{1}{n_s} \sigma_s \otimes \sigma_s, \quad \varepsilon(\sigma_s) = n_s.$$

Then $(\mathbb{C}S, \Delta, \varepsilon)$ is a (cocommutative) coalgebra. Note that ε is an algebra homomorphism but Δ is not, and so $\mathbb{C}S$ is not a bialgebra. By Proposition

2.3, $(\mathbb{C}S)^* = \text{Hom}_{\mathbb{C}}(\mathbb{C}S, \mathbb{C})$ is an algebra with respect to the multiplication

$$(fg)(\sigma_s) = (f \otimes g) \circ \Delta(\sigma_s) = \frac{1}{n_s} f(\sigma_s)g(\sigma_s)$$

for $f, g \in (\mathbb{C}S)^*$ and $s \in S$.

For every $s \in S$, $n_s^{-1}\sigma_s$ is a group-like element of the coalgebra $(\mathbb{C}S, \Delta, \varepsilon)$. Also the set $\{n_s^{-1}\sigma_s \mid s \in S\}$ is a basis of $\mathbb{C}S$. So by Proposition 2.5, $a \in \mathbb{C}S$ is group-like if and only if there exists $s \in S$ such that $a = n_s^{-1}\sigma_s$. So the coalgebra structure determines the distinguished basis $\{\sigma_s \mid s \in S\}$ of $\mathbb{C}S$.

Recall that $\text{CF}(\mathbb{C}S)$ is the set of class functions on $\mathbb{C}S$ and consists of all linear combinations of irreducible characters of $\mathbb{C}S$. Since $\text{CF}(\mathbb{C}S) \subseteq (\mathbb{C}S)^*$, we can define the product fg for $f, g \in \text{CF}(\mathbb{C}S)$. But fg is not necessary in $\text{CF}(\mathbb{C}S)$.

Definition 4.17 (Group-like schemes [16]). Let (X, S) be a scheme. We say that (X, S) is a *group-like scheme* if $\text{CF}(\mathbb{C}S)$ is closed with respect to the above multiplication (we note that there is no relation between group-like elements and group-like schemes).

Note that every commutative scheme (X, S) is group-like since $\text{CF}(\mathbb{C}S) = (\mathbb{C}S)^*$ in this case. Group-like schemes have many good properties. Especially, if a scheme (X, S) is group-like, then there is some fusion (X, \tilde{S}) of (X, S) such that the adjacency algebra $\mathbb{C}\tilde{S}$ is the center of $\mathbb{C}S$ (see [16]).

We consider one problem on character products.

Question 4.18. Let (X, S) be a (non-group-like) scheme, and let $\chi, \varphi \in \text{Irr}(S)$. When is $\chi\varphi \in \text{CF}(\mathbb{C}S)$?

We give a partial answer to this question.

Proposition 4.19 ([18, Theorem 3.3, Theorem 3.4]). *Let (X, S) be a scheme and $\chi, \varphi \in \text{Irr}(S)$. If $m_\chi = \chi(1)$, then $\chi\varphi \in \text{CF}(\mathbb{C}S)$. Moreover, if $m_\chi = 1$, then $\chi\varphi \in \text{Irr}(S)$ and $m_\varphi = m_{\chi\varphi}$.*

Now we suppose (X, S) is a commutative scheme. Then (X, S) is group-like. So, for $\chi, \varphi, \xi \in \text{Irr}(S)$, there exists $r_{\chi\varphi}^\xi \in \mathbb{C}$ such that

$$\chi\varphi = \sum_{\xi \in \text{Irr}(S)} r_{\chi\varphi}^\xi \xi.$$

This product is essentially the same as the *Hadamard product* of primitive idempotents [2, p. 64]. Namely, let

$$e_\chi \circ e_\varphi = \frac{1}{n_S} \sum_{\xi \in \text{Irr}(S)} q_{\chi\varphi}^\xi e_\xi$$

where \circ is the entry-wise product of matrices. Then we have

$$q_{\chi\varphi}^\xi = \frac{m_\chi m_\varphi}{m_\xi} r_{\chi\varphi}^\xi.$$

The number $q_{\chi\varphi}^\xi$ is known as the *Krein parameter* and it must be a non-negative real number (Krein condition).

4.4 Inductions and restrictions

In this subsection, we define inductions and restrictions of modules. Induced and restricted modules give rise to induced and restricted representations and to induced and restricted characters.

Let R be a commutative ring with unity, (X, S) a scheme, and T a closed subset of S . Then RT is an R -subalgebra of RS . Let M be a left RS -module. Then M can be considered as an RT -module. We call this module the *restriction* of M to RT and denote it by $M \downarrow_{RT}$. For an RT -module L , we define an RS -module

$$L \uparrow^{RS} = RS \otimes_{RT} M$$

and call this the *induction* of L to RS .

Induced and restricted representations and characters are denoted correspondingly. Especially, when we consider characters over the complex number field, we will write $\chi \downarrow_T$ and $\varphi \uparrow^S$ instead of $\chi \downarrow_{\mathbb{C}T}$ and $\varphi \uparrow^{\mathbb{C}S}$, respectively.

Let us now consider complex characters of schemes. Let (X, S) be a scheme and T a closed subset of S . Obviously $\chi(1) = \chi \downarrow_T(1)$ holds for each character χ of S . But there is no such formula for induced characters. There is a formula on multiplicities. To state the result, we extend the definition of multiplicities. Usually, the multiplicity m_χ (in the standard character) is defined only for an irreducible character χ . Let $\eta = \sum_{\chi \in \text{Irr}(S)} a_\chi \chi$ be a character. Define the *multiplicity* m_η of η by

$$m_\eta = \sum_{\chi \in \text{Irr}(S)} a_\chi m_\chi.$$

Then we have the following.

Theorem 4.20. *Let (X, S) be a scheme and T a closed subset of S . Let χ and φ be complex characters of S and T , respectively. Then we have*

$$\chi(1) = \chi \downarrow_T (1), \quad m_{\varphi \uparrow^S} = \frac{n_S}{n_T} m_\varphi.$$

4.5 Clifford theory

In Subsection 4.2, we saw how representations of schemes are related to representations of their quotient schemes. Now we want to see how representations of schemes are related to representations of their subschemes. But this is difficult, even for representations of finite groups. In group representation theory, one of the most important results in this direction is provided by the so-called Clifford theory. We will try to mimic Clifford theory for schemes. We can apply Dade's results in [8]. For details, see [21].

In the following, the letter T stands for a strongly normal closed subset of a scheme (X, S) . Then the quotient $S//T$ can be regarded as a finite group. If S is thin, then the group $S//T$ acts on the adjacency algebra $\mathbb{C}T$, and so we can define $S//T$ -conjugates of a $\mathbb{C}T$ -module. But, in general, the group $S//T$ does not act on $\mathbb{C}T$. We define $S//T$ -conjugates of an irreducible $\mathbb{C}T$ -module as follows.

For each $s \in S$, $\mathbb{C}(TsT)$ is a $(\mathbb{C}T, \mathbb{C}T)$ -bimodule. So, for a left $\mathbb{C}T$ -module L , $\mathbb{C}(TsT) \otimes_{\mathbb{C}T} L$ is also a left $\mathbb{C}T$ -module. The next proposition is crucial.

Proposition 4.21. *Let L be an irreducible left $\mathbb{C}T$ -module and $s \in S$. Then $\mathbb{C}(TsT) \otimes_{\mathbb{C}T} L$ is an irreducible left $\mathbb{C}T$ -module or 0.*

Let L be an irreducible left $\mathbb{C}T$ -module. When $\mathbb{C}(TsT) \otimes_{\mathbb{C}T} L \neq 0$, we say that L and $\mathbb{C}(TsT) \otimes_{\mathbb{C}T} L$ are $S//T$ -conjugate. Being $S//T$ -conjugate is an equivalence relation on the set of representatives of isomorphism classes of irreducible $\mathbb{C}T$ -modules. Actually, irreducible $\mathbb{C}T$ -modules L and L' are $S//T$ -conjugate if and only if there exists an irreducible $\mathbb{C}S$ -module M such that both L and L' are irreducible constituents of $M \downarrow_{\mathbb{C}T}$. Now we can state our first result on Clifford theory.

Theorem 4.22. *Let M be an irreducible $\mathbb{C}S$ -module and L an irreducible constituent of $M \downarrow_{\mathbb{C}T}$. Then there exists a positive integer m such that*

$$M \downarrow_{\mathbb{C}T} \cong m \left(\bigoplus_{L'} L' \right),$$

where L' runs over all $S//T$ -conjugates of L .

We note that the dimensions of $S//T$ -conjugate irreducible $\mathbb{C}S$ -modules are not necessary equal. But their multiplicities are the same.

Now we consider relations between representations of a scheme and its subschemes or quotient schemes. Again, let L be an irreducible left $\mathbb{C}T$ -module. Let $\text{IRR}(\mathbb{C}S \mid L)$ denote the set of all representatives of isomorphism classes of irreducible $\mathbb{C}S$ -modules whose restrictions to $\mathbb{C}T$ contain L as an irreducible constituent. Put

$$I_S(L) = \{s \in S \mid \mathbb{C}(TsT) \otimes_{\mathbb{C}T} L \cong L\}.$$

Then $I_S(L)$ is a closed subset of S containing T . We have the following theorem.

Theorem 4.23. *The map $\text{IRR}(\mathbb{C}I_S(L) \mid L) \rightarrow \text{IRR}(\mathbb{C}S \mid L)$ defined by $M \mapsto M \uparrow^{\mathbb{C}S}$ is a bijection.*

Since the correspondence is very easy, many problems on S can be reduced to $I_S(L)$. So, if $I_S(L)$ is a proper closed subset of S , then questions about S are transferred to questions about smaller schemes. If $I_S(L) = S$, we have the following theorem.

Theorem 4.24. *Suppose $I_S(L) = S$. Then there exists a factor set α of $S//T$ such that there exists a bijection between $\text{IRR}(\mathbb{C}S \mid L)$ and the generalized group algebra $\text{IRR}(\mathbb{C}^{(\alpha)}(S//T))$. If $M \in \text{IRR}(\mathbb{C}S \mid L)$ corresponds to $N \in \text{IRR}(\mathbb{C}^{(\alpha)}(S//T))$, then we have $\dim_{\mathbb{C}} M = (\dim_{\mathbb{C}} L)(\dim_{\mathbb{C}} N)$.*

As for generalized group algebras, the reader is referred to [36, II. §8].

Let us add the following remarks. If $S//T$ is a cyclic group, then the second cohomology group $H^2(S//T, \mathbb{C}^\times) = 1$ and so the factor set α can be assumed to be trivial. If L is extendible to S , in other words, if there exists $M \in \text{IRR}(\mathbb{C}S)$ with $M \downarrow_{\mathbb{C}T} \cong L$, then we may suppose that $\alpha = 1$. If $\alpha = 1$, then the generalized group algebra is the usual group algebra.

Let us now consider a commutative scheme (X, S) and a strongly normal closed subset T of S . Let L be an irreducible $\mathbb{C}T$ -module, and put $U = I_S(L)$. In this case, we may assume that $\alpha = 1$. Moreover, there exists an irreducible $\mathbb{C}U$ -module M such that

$$\text{IRR}(\mathbb{C}U \mid L) = \{M \otimes_{\mathbb{C}} N \mid N \in \text{IRR}(\mathbb{C}(U//T))\}.$$

Here $M \otimes_{\mathbb{C}} N$ is irreducible by Proposition 4.19. We have

$$\text{IRR}(\mathbb{C}S \mid L) = \{\mathbb{C}S \otimes_{\mathbb{C}U} (M \otimes_{\mathbb{C}} N) \mid N \in \text{IRR}(\mathbb{C}(U//T))\}.$$

Especially, we have $|\text{IRR}(\mathbb{C}S \mid L)| = n_S/n_U$.

So far we have seen that Clifford theory works fine for schemes with non-trivial strongly normal closed subsets. We do not know if Clifford theory also works for normal closed subsets that are not strongly normal. It is desirable to have a result controlling this case at least for commutative schemes.

5 Modular representations

In this article, modular representations mean representations over positive characteristic fields or integral domains. Modular representation theory is more complicated (but also more interesting) than ordinary representation theory, since the adjacency algebras over positive characteristic fields are not necessary semisimple anymore.

5.1 Preliminaries

Definition 5.1 (*p*-Modular systems [36, III. §6]). Let R be a complete discrete valuation ring with maximal ideal πR , where $\pi \in R$. Let K be the quotient field of R , and let F be the residue class field $R/\pi R$. Suppose that K and F have characteristic 0 and p (> 0), respectively. Then we call (K, R, F) a *p*-modular system.

There is a natural way to construct a *p*-modular system from algebraic number fields which we wish to present now. Fix a rational prime number p . Let K be an algebraic number field, and denote by R the ring of integers of K . Choose a prime ideal \mathfrak{P} of R lying above $p\mathbb{Z}$. Denote by $K_{\mathfrak{P}}$ the \mathfrak{P} -adic completion of K , and let $R_{\mathfrak{P}}$ denote the ring of \mathfrak{P} -integers in $K_{\mathfrak{P}}$. Then $(K_{\mathfrak{P}}, R_{\mathfrak{P}}, F)$ is a *p*-modular system, where $F = R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}} \cong R/\mathfrak{P}R$. For details, see [36].

Let (K, R, F) be a *p*-modular system, and let πR be the maximal ideal of R . Let A be an R -free R -algebra with finite R -rank. We can define a K -algebra $A^K = K \otimes_R A$ and an F -algebra $A^F = A/\pi A$. Let M be an R -free A -module with finite R -rank. Then we can define an A^K -module $M^K = K \otimes_R M$ and an A^F -module $M^F = M/\pi M$. Let \bar{a} denote the image of $a \in A$ by the natural epimorphism $A \rightarrow A/\pi A$.

Definition 5.2 (*R*-Forms [36, II. Theorem 1.6]). Let N be an A^K -module. Then there exists an A -module \tilde{N} such that $\tilde{N}^K \cong N$. We call \tilde{N} an *R-form* of N .

Note that an *R-form* \tilde{N} is not uniquely determined by an A^K -module N . But it is known that the multiplicities of simple A^F -modules in \tilde{N}^F as simple components are determined only by N .

Let e be an idempotent of A . Then e and \bar{e} are idempotents of A^K and A^F , respectively. Moreover, if e is central, then so is \bar{e} in A^F . Conversely the following proposition holds.

Proposition 5.3 ([36, Theorem I.14.2], [7, Proposition 1.12]). *Let f be an idempotent of A^F . Then there exists an idempotent e of A such that $\bar{e} = f$. Moreover the following statements hold.*

- (1) *f is a primitive idempotent if and only if so is e .*
- (2) *If f is a central (primitive) idempotent, then there exists a unique central (primitive) idempotent e of A such that $\bar{e} = f$.*

Let $1 = e_1 + \cdots + e_\ell$ be the central idempotent decomposition of 1 in A . Then $\bar{1} = \bar{e}_1 + \cdots + \bar{e}_\ell$ is the central idempotent decomposition of $\bar{1}$ in A^F by Proposition 5.3. These decompositions yield indecomposable direct sum decompositions

$$A = \bigoplus_{i=1}^{\ell} e_i A, \quad A^F = \bigoplus_{i=1}^{\ell} \bar{e}_i A^F$$

as two-sided ideals. We call $e_i A$ (or $\bar{e}_i A^F$) a *block* of A (or A^F), and e_i (or \bar{e}_i) a *block idempotent*. For an indecomposable A -module or A^K -module M , there is a unique block idempotent e_i such that $e_i M \neq 0$. Then we say that M belongs to the block $e_i A$. Similarly, for an indecomposable A^F -module L , there is a unique block idempotent \bar{e}_i such that $\bar{e}_i L \neq 0$ and we say that L belongs to the block $e_i A$ or $\bar{e}_i A^F$.

5.2 Semisimplicity

Let (K, R, F) be a p -modular system. For $\mathcal{O} \in \{K, R, F\}$, let A be an \mathcal{O} -free \mathcal{O} -algebra of finite rank, and let M be an \mathcal{O} -free A -module of finite rank n . Let $\Phi : A \rightarrow \text{Mat}_n(\mathcal{O})$ be the corresponding matrix representation, and

χ its character. Choose an \mathcal{O} -basis $\{v_i\}$ of A . We define the *discriminant* $D_{M,\{v_i\}}(A)$ of M with respect to the basis $\{v_i\}$ by

$$D_{M,\{v_i\}}(A) = \det(\chi(v_i v_j))_{ij}.$$

When \mathcal{O} is a field, it is known that the algebra A is separable if and only if there exist an extension field \mathcal{O}' of \mathcal{O} and an $A^{\mathcal{O}'}$ -module M such that its discriminant is nonzero.

Let us now discuss semisimplicity of adjacency algebras. For each $L \in \text{IRR}(KS)$, choose its R -form \tilde{L} . Put $M = \bigoplus_{L \in \text{IRR}(KS)} L$ (*reduced regular module* defined in [6, §59]). Then $\tilde{M} = \bigoplus_{L \in \text{IRR}(KS)} \tilde{L}$ is an R -form of M . Compute the discriminant of \tilde{M} with respect to the basis $\{\sigma_s \mid s \in S\}$. Then we have

$$|D_{\tilde{M},\{\sigma_s\}}(RS)| = n_S^{|S|} \frac{\prod_{s \in S} n_s}{\prod_{\chi \in \text{Irr}(S)} m_\chi^{(\chi(1)^2)}}.$$

The number $|D_{\tilde{M},\{\sigma_s\}}(RS)|$ is a rational integer and depends only on the scheme (X, S) . This number is called the *Frame number* of the scheme and denoted by $\mathcal{F}(X, S)$. It is easy to see that $D_{\tilde{M}^F, \{\bar{\sigma}_s\}}(FS) = \varepsilon \mathcal{F}(X, S)$, where $\varepsilon \in \{-1, 1\}$. The following theorem holds.

Theorem 5.4 ([1, Theorem 1.1], [14, Theorem 4.2]). *The adjacency algebra FS is semisimple (separable) if and only if the characteristic of F is not a divisor of the Frame number $\mathcal{F}(X, S)$.*

Problem 5.5. Let G be a finite group, F a field of positive characteristic p . Then, by Maschke's theorem, the group algebra FG is semisimple if and only if $p \nmid |G|$. Consider the thin scheme defined by G , then the Frame number is bigger than the group order $|G|$. Find a good invariant of a scheme which generalizes the group order.

5.3 Schemes of prime order

One of the major achievements in modular representation theory of schemes is the theorem that every scheme of prime order is commutative. This theorem is based on the following fundamental observation.

Theorem 5.6 ([15, Theorem 3.4]). *Let p be a prime number, F a field of characteristic p , and (X, S) a scheme of p -power order. Then the adjacency algebra FG is local.*

Theorem 5.6 together with a variation of a famous argument of R. Brauer [4] is the key in the proof of the following main result of this subsection.

Theorem 5.7 ([26, Theorem 3.3, Theorem 5.3]). *Let p be a prime number, and (X, S) a scheme of order p . Then*

- (1) (X, S) is commutative.
- (2) There exists a positive integer k such that $n_s = k$ for any $s \in S - \{1\}$ and $m_\chi = k$ for any $\chi \in \text{Irr}(S) - \{1_S\}$. Moreover all members of $\text{Irr}(S) - \{1_S\}$ are algebraically conjugate to each other.
- (3) If there is an abelian number field which is a splitting field of (X, S) , then (X, S) is algebraically isomorphic to a cyclotomic scheme.

In [2, §2.7], it was asked whether the minimal splitting field of a commutative scheme is abelian. Although we were able to get much information about minimal splitting fields of schemes of prime order, we were not able to show that these fields are abelian. Toru Komatsu constructed examples of such fields [33]. After that Eiichi Bannai and Komatsu constructed integral table algebras whose minimal splitting fields are not abelian. However, whether these table algebras come from schemes seems to be still an open question.

In a certain sense, Theorem 5.7 generalizes the fact that groups of prime order are commutative. Since groups of prime square order are commutative, it is natural to ask the following.

Question 5.8. Are schemes of prime square order necessarily commutative?

The following partial answers have been achieved.

Theorem 5.9 ([19], [24]). *Let p be a prime number, and let (X, S) be a scheme of order p^2 . Then (X, S) is commutative if one of the following conditions holds.*

- (1) (X, S) is Schurian.
- (2) There exists a thin closed subset T with $n_T \geq p$.
- (3) There exists a strongly normal closed subset T with $n_T \leq p$.

5.4 Blocks of modular adjacency algebras

Let (K, R, F) be a p -modular system, and let (X, S) be a scheme. Suppose KS and FS are splitting algebras. In this subsection, we consider some invariants of blocks of RS .

Let $\text{Bl}(S)$ denote the set of all blocks of RS . For $B \in \text{Bl}(S)$, e_B will denote the central primitive idempotent corresponding to B . For $\chi \in \text{Irr}(S)$, there is a unique $B \in \text{Bl}(S)$ such that $\chi(e_B) \neq 0$. We say that χ belongs to B . We define $\text{Irr}(B)$ to be the set of all irreducible characters of S belonging to B . Similarly, for an irreducible FS -module M , there is the unique $B \in \text{Bl}(S)$ such that $\overline{e_B}M \neq 0$. In this case, we say that M belongs to B .

Let us first clarify what it means for two irreducible characters to be in the same block. For $\chi \in \text{Irr}(S)$ and $\alpha \in Z(KS)$, put

$$\omega_\chi(\alpha) = \frac{\chi(\alpha)}{\chi(1)}.$$

Then ω_χ is an irreducible character of $Z(KS)$ and $\text{Irr}(Z(KS)) = \{\omega_\chi \mid \chi \in \text{Irr}(S)\}$. If $\alpha \in Z(RS) = RS \cap Z(KS)$, then $\omega_\chi(\alpha) \in R$. So we can define $\overline{\omega_\chi} : Z(RS)^F \rightarrow F$ by $\overline{\omega_\chi}(\overline{\alpha}) = \overline{\omega_\chi(\alpha)}$. Here we remark that, in general, $Z(RS)^F = Z(RS)/\pi Z(RS) \neq Z(FS)$.

Proposition 5.10. *Let $\chi, \varphi \in \text{Irr}(S)$. Then χ and φ are in the same block if and only if $\overline{\omega_\chi} = \overline{\omega_\varphi}$.*

Especially, if (X, S) is commutative, then $\chi = \omega_\chi$ for every $\chi \in \text{Irr}(S)$. So we have the following.

Proposition 5.11. *Suppose (X, S) is a commutative scheme. Let $\chi, \varphi \in \text{Irr}(S)$. Then χ and φ are in the same block if and only if $\overline{\chi} = \overline{\varphi}$.*

Note that the block decomposition of $\text{Irr}(S)$ depends on the choice of the p -modular system, though it is independent for group algebras. For $B \in \text{Bl}(S)$, put $\overline{\omega_B} = \overline{\omega_\chi}$ for $\chi \in \text{Irr}(B)$, since $\overline{\omega_\chi}$ is independent of the choice of $\chi \in \text{Irr}(B)$.

For the remainder of this subsection, we assume (X, S) to be commutative. Let ν_p denote the p -valuation on the rational number field \mathbb{Q} . For $B \in \text{Bl}(S)$, put

$$t(B) = \max\{\nu_p(n_s) \mid s \in S \text{ and } \overline{\omega_B}(\overline{\sigma_s}) \neq 0\}.$$

It is easy to see that this number is well-defined. For non-commutative schemes, the author does not know a good definition of $t(B)$ since, in general, $\sigma_s \notin Z(RS)$.

For a non-negative integer ℓ , put

$$I_\ell = \bigoplus_{p^\ell | n_s} F\overline{\sigma_s}.$$

Then I_ℓ is an ideal of FS . For $B \in \text{Bl}(S)$, put

$$t'(B) = \min\{\ell \mid \overline{e_B} \in I_\ell\}.$$

Then we have the following.

Theorem 5.12. *For every $B \in \text{Bl}(S)$, we have $t(B) = t'(B)$.*

We write

$$e_B = \sum_{s \in S} \beta_B(s) \sigma_s \quad (\beta_B(s) \in R).$$

Then we have $t(B) = \min\{\nu_p(n_s) \mid \overline{\beta_B(s)} \neq 0\}$. We put $\text{Bl}_\ell(S) = \{B \in \text{Bl}(S) \mid t(B) = \ell\}$ and $S_\ell = \{s \in S \mid \nu_p(n_s) = \ell\}$.

Proposition 5.13. *Let $B, B' \in \text{Bl}_\ell(S)$, $\chi \in \text{Irr}(B)$, and $\chi' \in \text{Irr}(B')$. Then $B = B'$ if and only if*

$$\overline{\chi(\sigma_s)} = \overline{\chi'(\sigma_s)},$$

for any $s \in S_\ell$. Moreover $\{(\overline{\omega_B} |_{S_\ell}) \mid B \in \text{Bl}_\ell(S)\}$ is linearly independent over F , so we have $|\text{Bl}_\ell(S)| \leq |S_\ell|$.

Theorem 5.12 and Proposition 5.13 impose severe constraints on character tables of commutative schemes. Let us give one more invariant for a block. For $B \in \text{Bl}(S)$, we have $\nu_p(\dim_F \overline{e_B} FX) \geq \nu_p(|X|)$. So

$$u(B) = \nu_p(\dim_F \overline{e_B} FX) - \nu_p(|X|)$$

is a non-negative integer. Having looked at many examples we come up with the following question.

Question 5.14. Is it true that $t(B) \leq u(B)$ for $B \in \text{Bl}(S)$? Is it true that $\nu_p(\beta_B(s)) \leq u(B) - \nu_p(n_s)$ for $B \in \text{Bl}(S)$ and $s \in S$?

We have $u(B) = \nu_p(\beta_B(1))$. So, if the second inequality holds in the above question, then the first inequality holds by putting $s = 1$.

Example 5.15. Let (X, S) be a group association scheme (conjugacy class scheme). Then the inequalities in Question 5.14 hold. In this case, $t(B) = u(B)$ and this number is closely related to the *defect* of the block of the group algebra.

5.5 Problems

In this final subsection, we state some problems and related facts.

5.5.1 Algebras as adjacency algebras

From the viewpoint of finite dimensional algebras, there is a natural problem.

Problem 5.16. Consider what kind of algebras are obtained as adjacency algebras or their blocks.

We do not know the answer to this question, even for group algebras.

Let K be an algebraically closed field. For $M \in \text{IRR}(KS)$, let $P(M)$ denote the projective cover of M . For $M, N \in \text{IRR}(KS)$, put $c_{M,N} = \dim_K \text{Hom}_{KS}(P(M), P(N))$ and call this number the *Cartan invariant*. Define the matrix $C = (c_{M,N})$, whose both rows and columns are indexed by $\text{IRR}(KS)$, and call this the *Cartan matrix* of KS . It is known that the algebra is a splitting symmetric algebra, then the Cartan matrix is a symmetric matrix [36, II. Theorem 8.21].

It is well known that every group algebra is symmetric algebra. Adjacency algebras are not necessary symmetric, but their Cartan matrices are symmetric matrices. This property restricts possibilities of algebras as adjacency algebras. For example, every non-semisimple hereditary algebra cannot be a block of an adjacency algebra.

Problem 5.17. Generalize known facts on Cartan matrices of group algebras to adjacency algebras.

5.5.2 Representation types

Let K be a field, A a finite dimensional K -algebra. We say that A is of *finite representation type* if there are only finitely many isomorphism classes

of indecomposable left A -modules. Otherwise A is said to be of *infinite representation type*.

Problem 5.18. Let K be a field, (X, S) a scheme. Determine when the adjacency algebra KS is of finite representation type.

Let G be a finite group, and let K be a field. Then KG is of finite representation type if and only if $p = 0$ or a Sylow p -subgroup of G is cyclic, where p is the characteristic of K (see [6, §64]). We want to generalize this fact.

Problem 5.19. For a scheme, define something like a Sylow p -subgroup of a finite group.

Closed Sylow subsets have been introduced for the so-called p -valenced schemes in [32]. (The definition follows in the next paragraph.) However, we wish to see a generalization to all schemes.

For a finite group G , if $\nu_p(|G|) = 1$, then a Sylow p -subgroup of G is cyclic and the group algebra is of finite representation type. So we consider the condition $\nu_p(|X|) = 1$. But the adjacency algebra of the group association scheme (conjugacy class scheme) (X, S) of the symmetric group of degree 3 over a field of characteristic 3 is of infinite representation type though $\nu_3(|X|) = 1$. So we strengthen our hypothesis and have the following problem. We say that a scheme (X, S) is *p' -valenced* if n_s is a p' -number for every $s \in S$.

Problem 5.20. Let (X, S) be a p' -valenced scheme, K a field of characteristic p . Suppose $\nu_p(|X|) = 1$. Is it true that, in this case, KS is of finite representation type?

The result on commutativity of schemes of prime order (Theorem 5.7) was obtained when we were considering this problem, since every scheme of prime order satisfies the assumptions.

For a finite group G with $\nu_p(|G|) = 1$, KG is not only of finite representation type, but also a direct sum of *Brauer tree algebras* (see [3, §4.18]). Also for adjacency algebras, all known examples of adjacency algebras of finite representation type are direct sums of Brauer tree algebras. A special case of this problem is considered in [22].

Algebras of infinite representation type are divided into those of *tame type* and those of *wild type*. Of course, one would like to know when an adjacency algebra of a scheme is tame. But this problem seems to be more difficult than Problem 5.18.

5.5.3 Standard modules

In [5] and [37], p -ranks of elements in an adjacency algebra were considered. Here the p -rank means the rank of an integer matrix modulo a prime number p . The p -rank was used in order to distinguish algebraically isomorphic schemes.

Let (X, S) and (X', S') be algebraically isomorphic schemes with $\psi : S \rightarrow S'$ such that $p_{st}^u = p_{\psi(s)\psi(t)}^{\psi(u)}$. For $\alpha = \sum_{s \in S} a_s \sigma_s \in \mathbb{Z}S$, define $\psi(\alpha) = \sum_{s \in S} a_s \sigma_{\psi(s)} \in \mathbb{Z}S'$. Then it is easy to see that the ranks of α and $\psi(\alpha)$ are equal. But, sometimes, the p -ranks of α and $\psi(\alpha)$ are different. This difference comes from the structures of standard modules.

Let F be a field of characteristic p . In this case, we have $FS \cong FS'$ as F -algebras. This isomorphism allows us to view the standard module FX' of (X', S') as an FS -module. Thus, the p -rank of α is equal to the dimension of $\bar{\alpha}FX$, where $\bar{\alpha}$ is the natural image of α to FS . So, if $FX \not\cong FX'$ as FS -modules, then the dimensions can be different. Some examples were given in [27].

In a certain sense, standard modules are similar to *permutation modules* of finite groups. Indecomposable direct summands of permutation modules are called *trivial source modules* ([36, IV, §8]). Trivial source modules have some special properties. So we have the following question.

Question 5.21. Does every indecomposable direct summand of a standard module have special properties ?

Especially, there is the unique direct summand M of the standard module with the property $\bar{\sigma}_S M \neq 0$, where $\sigma_S = \sum_{s \in S} \sigma_s$. We want to know what is M . If the adjacency algebra FS is self-injective, then M is isomorphic to the injective hull of the trivial module.

Finally, there is one more remark. Consider the elementary divisors $\{e_1, e_2, \dots, e_{|X|}\}$ of $\alpha = \sum_{s \in S} a_s \sigma_s \in \mathbb{Z}S$. Then the p -rank of α is equal to $\#\{i \mid e_i \not\equiv 0 \pmod{p}\}$. So the set of elementary divisors gives more information than the p -rank. But we do not know how to find good elements in $\mathbb{Z}S$ to see their elementary divisors.

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