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Lifting to the suspension of the real projective plane

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Abstract

The elements ε_3 and μ_3 of the homotopy groups of the 3-sphere are lifted to the suspension of the real projective plane.

1 Introduction

In this note all spaces, maps and homotopies are based. Let S^n be the n dimensional sphere. We denote by

$$\mathrm{P}^n(k) = S^{n-1} \cup_{k\iota_{n-1}} e^n$$

the Moore space of type $(Z_k, n-1)$, where ι_{n-1} is the homotopy class of the identity map of S^{n-1}. Note that $\mathrm{P}^3(2)$ is the suspension of the real projective plane. Let $\pi_n(X)$ be the n-th homotopy group of a space X. Let $p_n : \mathrm{P}^n(k) \to S^n$ be the collapsing map. Given an element $\alpha \in \pi_m(S^n)$ suppose that there exists an element $\beta \in \pi_m(\mathrm{P}^n(k))$ satisfying the relation $p_{n*}\beta = \alpha$, where $p_{n*} : \pi_m(\mathrm{P}^n(k)) \to \pi_m(S^n)$ is the homomorphism induced from p_n. In the other words, assume that the following diagram is commutative up to homotopy:

$$\begin{tikzcd} & \mathrm{P}^n(k) \arrow[d, "p_n"] \\ S^m \arrow[ur, dashed, "\beta"] \arrow[r, "\alpha"] & S^n. \end{tikzcd}$$

Then β is called a lift of α and we write

$$\beta = [\alpha].$$

The purpose of this note is to solve partially the problem proposed by Morisugi-Mukai [1]. Let $\varepsilon_3 \in \pi_{11}(S^3)$ and $\mu_3 \in \pi_{12}(S^3)$ be the elements given by Toda [4]. Then we show the following.

Theorem *ε_3 and μ_3 are lifted to* $\mathrm{P}^3(2)$.

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The key to proving the result is to use a relation in $\pi_7(\mathrm{P}^3(2))$ (Lemma 2.1) and Wu's result: $4\pi_8(\mathrm{P}^3(2))=0$ (Lemma 4.1). Moreover we need to reconstruct the Toda bracket defining μ_3 so that it is obtained from using $\mathrm{P}^n(4)$ instead of $\mathrm{P}^n(8)$ for some integers n. We use the notations and results of [4] freely, unless otherwise stated.

2 Lifting ε_3

We denote by EX a suspension of X. Let $\eta_n \in \pi_{n+1}(S^n)$ for $n \geq 2$, $\nu_n \in \pi_{n+3}(S^n)$ for $n \geq 4$ be the Hopf maps and set $\eta_n^2=\eta_n \circ \eta_{n+1}$. Let $i_n : S^{n-1} \to \mathrm{P}^n(k)$ be the inclusion map. As it is well known [2], $\pi_3(\mathrm{P}^3(2))=\mathrm{Z}_4\{i_3\eta_2\}$, $\pi_4(\mathrm{P}^3(2))=\mathrm{Z}_4\{[\eta_3]\}$ and $2[\eta_3]=i_3\eta_2^2$. Hereafter we write

$$\tilde{\eta}_2=[\eta_3].$$

Let ν' be a generator of the 2-primary component π_6^3 of $\pi_6(S^3)\cong\mathrm{Z}_{12}$. We denote by [,] the Whitehead product [5]. We recall the following [3].

Lemma 2.1 *The following relations hold in* $\pi_7(\mathrm{P}^3(2))$*:*

$$4(\tilde{\eta}_2\nu_4)=[i_3\eta_2,\ \tilde{\eta}_2]\eta_6$$

and

$$\tilde{\eta}_2E\nu'=\xi\eta_6,$$

where $\xi=[i_3\eta_2,\ \tilde{\eta}_2]+i_3\eta_2\nu'$.

Now we show the following.

Theorem ε_3 *is lifted to* $\mathrm{P}^3(2)$.

Proof. We recall that the Toda bracket $\{\eta_3, E\nu', \nu_7\}$ consists of the single element ε_3. Since $\eta_5\nu_6=0$, $\pi_8(S^3)=\mathrm{Z}_2\{\nu'\eta_6^2\}$ and $p_3i_3=0$, we have $\pi_8(S^3)\circ\nu_8=0$ and

$$p_3\xi=p_3\circ[i_3\eta_2,\ \tilde{\eta}_2]=[p_3i_3\eta_2,\ \eta_3]=0.$$

So, by the properties of Toda brackets and Lemma 2.1, we obtain

$$\begin{aligned}
\varepsilon_3 &= \{\eta_3, E\nu', \nu_7\} \\
&= \{p_3\tilde{\eta}_2, E\nu', \nu_7\} \\
&\subset \{p_3, \tilde{\eta}_2E\nu', \nu_7\} \\
&= \{p_3, \xi\eta_6, \nu_7\} \\
&\supset \{p_3, \xi, \eta_6\nu_7\} \\
&= \{p_3, \xi, 0\} \ni 0 \\
&\quad \bmod p_{3*}\pi_{11}(\mathrm{P}^3(2))+\pi_8(S^3)\circ\nu_8=p_{3*}\pi_{11}(\mathrm{P}^3(2)).
\end{aligned}$$

This means $\varepsilon_3 \in p_{3*}\pi_{11}(\mathrm{P}^3(2))$ and the proof is complete. □

3 Reconstructing μ_3

First we recall that $\pi_{n+2}(S^n)=\mathrm{Z}_2\{\eta_n^2\}$ for $n \geq 2$. By use of the properties of Toda brackets and by [4, Corollary 3.7], we get that

$$\{2a\iota_n, \eta_n, 2b\iota_{n+1}\} = ab\eta_n^2, \text{ where } a, b \text{ are integers and } n \geq 3. \tag{3.1}$$

Hereafter we assume that $k=2^m$ for $m \geq 2$. Let $\tilde{\nu}'$ be a coextension of ν', that is, a lift of $E\nu'$ and it is taken as a representative of the Toda bracket $\{i_4, k\iota_3, \nu'\}$. By the properties of Toda brackets,

$$\begin{aligned}\{k\iota_3, \nu', 4\iota_6\} &\subset \{\tfrac{k}{2}\iota_3, 2\iota_3 \circ \nu' \circ 2\iota_6, 2\iota_6\}\\ &= \{\tfrac{k}{2}\iota_3, 0, 2\iota_6\}\\ &\ni 0 \bmod 2\pi_7(S^3)=0.\end{aligned}$$

So we obtain

$$4\tilde{\nu}' \in \{i_4, k\iota_3, \nu'\} \circ 4\iota_7 = -(i_4 \circ \{k\iota_3, \nu', 4\iota_6\}) = 0$$

and hence, $\tilde{\nu}'$ is of order 4. Hereafter we assume $k=4$ or 8 and the element $\tilde{\nu}'$ is used for $k=4$, unless otherwise stated.

From the definition of μ_3 [4, p.55-6],

$$\mu_3 \in \{\eta_3, E\beta, E^2\gamma\}_1, \tag{3.2}$$

where β is an extension of ν' with respect to $8\iota_6$ and γ is a coextension of ν_5 with respect to $8\iota_5$. We consider the commutative diagram between cofiber sequences for $n \geq 2$:

$$\begin{tikzcd} S^{n-1} \arrow[r, "{[8]}"] \arrow[d, "{[2]}"] & S^{n-1} \arrow[r, "i_n"] \arrow[d, "="] & \mathrm{P}^n(8) \arrow[r, "p_n"] \arrow[d, "d_n"] & S^n \arrow[d, "{[2]}"] \\ S^{n-1} \arrow[r, "{[4]}"] & S^{n-1} \arrow[r, "i'_n"] & \mathrm{P}^n(4) \arrow[r, "p'_n"] & S^n, \end{tikzcd} \tag{3.3}$$

where $[m]$ means a mapping of degree m, the maps i'_n, p'_n, d_n are the canonical maps such that $i'_n = Ei'_{n-1}$, $p'_n = Ep'_{n-1}$ and $d_n = Ed_{n-1}$ for $n \geq 3$, respectively. Let $\bar{\eta}_3 \in [\mathrm{P}^5(8), S^3]$ and $\bar{\eta}'_3 \in [\mathrm{P}^5(4), S^3]$ be extensions of η_3. By (3.3), $\bar{\eta}'_3 d_5 \circ i_5 = \bar{\eta}'_3 i'_5 = \eta_3$. So we take

$$\bar{\eta}_3 = \bar{\eta}'_3 d_5. \tag{3.4}$$

We set $\bar{\eta}_n = E^{n-3}\bar{\eta}_3$ and $\bar{\eta}'_n = E^{n-3}\bar{\eta}'_3$ for $n \geq 3$. We show the following.

Lemma 3.1 (i) *$\{\eta_4, 8\iota_5, \nu_5\} \ni 0 \bmod (E\nu')\eta_7^2$ and $\{\eta_3, E\nu', 4\iota_7\}_1 = \nu'\eta_6^2$.*

(ii) *The orders of $\bar{\eta}'_n$ and $\bar{\eta}_n$ are 2 for $n \geq 3$, respectively.*

(iii) $[\mathrm{P}^8(8), S^3] = Z_2\{\nu'\bar{\eta}_6\} \oplus Z_2\{\nu'\eta_6^2 p_8\}$, $[\mathrm{P}^9(8), S^3] = Z_2\{\nu'\eta_6\bar{\eta}_7\}$,

$[\mathrm{P}^8(4), S^3] = Z_2\{\nu'\bar{\eta}'_6\} \oplus Z_2\{\nu'\eta_6^2 p'_8\}$ *and* $[\mathrm{P}^9(4), S^3] = Z_2\{\nu'\eta_6\bar{\eta}'_7\}$.

(iv) $\bar{\eta}'_5 \circ E^3\tilde{\nu}' = 0$ *and* $\bar{\eta}_5 \circ E\gamma = 0$.

Proof. By the fact that $4\nu_5 = \eta_5^3$, $\{\eta_4, 2\iota_5, \eta_5\} = \pm E\nu'$, $\eta_3\nu_4 = \nu'\eta_6$ and $\pi_9(S^5) = Z_2\{\nu_5\eta_8\}$, we obtain

$$\begin{aligned}\{\eta_4, 8\iota_5, \nu_5\} &\supset \{\eta_4, 2\iota_5, 4\nu_5\} \\ &= \{\eta_4, 2\iota_5, \eta_5^3\} \\ &\supset \{\eta_4, 2\iota_5, \eta_5\} \circ \eta_7^2 \\ &\ni (E\nu')\eta_7^2 \bmod \eta_4 \circ \pi_9^5 + \pi_6^4 \circ \nu_6 = \{(E\nu')\eta_7^2\}.\end{aligned}$$

This leads to the first of (i).

By [4, Proposition 2.6], by the fact that $H(\nu') = \eta_5$ and $\Delta(\nu_5) = \pm\eta_2\nu'$,

$$H\{\eta_3, E\nu', 4\iota_7\}_1 = -\Delta^{-1}(\eta_2\nu') \circ 4\iota_8 = 4\nu_5.$$

So, by use of the EHP sequence and the fact that $E\pi_7(S^2) = 0$, we obtain the second of (i).

We have $2\bar{\eta}_3' = 2\iota_3 \circ \bar{\eta}_3'$ and $\bar{\eta}_3' \in \{\eta_3, 4\iota_4, p_4'\} \subset [\mathrm{P}^5(4), S^3]$. By (3.1),

$$2\bar{\eta}_3' \in 2\iota_3 \circ \{\eta_3, 4\iota_4, p_4'\} = -\{2\iota_3, \eta_3, 4\iota_4\} \circ p_5' = 0.$$

This implies $2\bar{\eta}_3' = 0$. By (3.4), $2\bar{\eta}_3 = 0$. This leads to (ii). The assertion (iii) is easily obtained by using the cofiber sequences starting with i_n and i_n' for $n = 8,9$.

By the fact that $E^2\pi_8(S^3) = 0$, we have

$$\bar{\eta}_5' \circ E^3\tilde{\nu}' = E^2(\bar{\eta}_3' \circ E\tilde{\nu}') \in E^2\pi_8(S^3) = 0.$$

This leads to the first half of (iv). Similarly, by (i),

$$\bar{\eta}_5 \circ E\gamma \in E\{\eta_4, 8\iota_5, \nu_5\} \ni 0 \bmod E((E\nu')\eta_7^2) = 0.$$

This leads to the second half of (iv), completing the proof. □

Now we show the following.

Lemma 3.2 $\{\eta_3, E\bar{\nu}' + \nu_4\eta_7 p_8', E^4\tilde{\nu}'\} = \{\eta_3, E\beta, E^2\gamma\}_1$,
where $\bar{\nu}' \in [\mathrm{P}^7(4), S^3]$ *and* $\tilde{\nu}' \in \pi_7(\mathrm{P}^4(4))$ *are an extension and a coextension of* ν', *respectively.*

Proof. We obtain $\bar{\nu}' \circ E^3\tilde{\nu}' \in -\{\nu', 4\iota_6, E^3\nu'\} \subset \pi_{10}^3 = 0$ for any extension $\bar{\nu}'$ and coextension $\tilde{\nu}'$. And we have $\nu_4\eta_7 p_8' \circ E^4\tilde{\nu}' = \nu_4\eta_7 \circ 2\nu_8 = 0$. By [4, Proposition 1.9] and Lemma 3.1 (i),

$$\eta_3 \circ E\bar{\nu}' \in \{\eta_3, E\nu', 4\iota_7\}_1 \circ p_8' = \nu'\eta_6^2 p_8'. \tag{3.5}$$

Since $\eta_3\nu_4\eta_7 p_8' = \nu'\eta_6^2 p_8'$, the first Toda bracket is well-defined.

We know $\eta_3 \circ E\pi_{11}(S^3) = \eta_3 \circ \pi_{12}(S^4) = Z_2\{\eta_3\varepsilon_4\}$. By Lemma 3.1, $[\mathrm{P}^9(4), S^3] \circ E^5\tilde{\nu}' = 0$ and $[\mathrm{P}^9(8), S^3] \circ E^3\gamma = 0$. So indeterminacies of two Toda brackets coincide.

By the definition β and (3.3), we have

$$\begin{aligned}\beta &\in \{\nu', 8\iota_6, p_6\}\\ &\supset \{\nu', 4\iota_6, 2\iota_6\circ p_6\}\\ &= \{\nu', 4\iota_6, p_6'\circ d_6\}\\ &\supset \{\nu', 4\iota_6, p_6'\}\circ d_7\\ &\ni \bar{\nu}' d_7 \bmod \nu'\circ[\mathrm{P}^7(8), S^6]+\pi_7(S^3)\circ p_7.\end{aligned}$$

Since $[\mathrm{P}^7(8), S^6]=\{\eta_6 p_7\}\cong \mathrm{Z}_2$, we see that $\nu'\circ[\mathrm{P}^7(8), S^6]+\pi_7(S^3)\circ p_7=\{\nu'\eta_6 p_7\}$. So we obtain

$$\beta\equiv\bar{\nu}' d_7 \bmod \nu'\eta_6 p_7.$$

By (3.3), $\nu_4\eta_7 p'_8 d_8=\nu_4\eta_7\circ 2\iota_8\circ p_8=0$. Hence we get that

$$E\beta\equiv(E\bar{\nu}'+\nu_4\eta_7 p_8')d_8 \bmod (E\nu')\eta_7 p_8.$$

We have

$$\begin{aligned}\{\eta_3, (E\nu')\eta_7 p_8, E^2\gamma\} &\supset \{\eta_3, E\nu', \eta_7 p_8\circ E^2\gamma\}\\ &= \{\eta_3, E\nu', \eta_7\nu_8\}\\ &= \{\eta_3, E\nu', 0\}\ni 0\\ &\quad \bmod \eta_3\circ\pi_{12}(S^4)+[\mathrm{P}^9(8), S^3]\circ E^3\gamma=\{\eta_3\varepsilon_4\}.\end{aligned}$$

So we obtain

$$\{\eta_3, E\beta, E^2\gamma\}_1=\{\eta_3, (E\bar{\nu}'+\nu_4\eta_7 p_8')d_8, E^2\gamma\}.$$

On the other hand, by (3.3) and the fact that $\pi_6(\mathrm{P}^6(4))=\mathrm{Z}_2\{i_6'\eta_5\}$,

$$\begin{aligned}d_6\gamma &\in d_6\circ\{i_6, 8\iota_5, \nu_5\}\\ &\subset \{i_6', 8\iota_5, \nu_5\}\\ &\supset \{i_6', 4\iota_5, 2\nu_5\}\\ &\supset E^2\{i_4', 4\iota_3, \nu'\}\\ &\ni E^2\tilde{\nu}' \bmod i'_{6*}\pi_9(S^5)+\pi_6(\mathrm{P}^6(4))\circ\nu_6=\{i_6'\nu_5\eta_8\}.\end{aligned}$$

So, by the relation $\nu_6\eta_9=0$, we obtain $d_7\circ E\gamma=E^3\tilde{\nu}'$. Thus we conclude that

$$\begin{aligned}\{\eta_3, (E\bar{\nu}'+\nu_4\eta_7 p_8')d_8, E^2\gamma\} &\supset \{\eta_3, E\bar{\nu}'+\nu_4\eta_7 p_8', d_8\circ E^2\gamma\}\\ &= \{\eta_3, E\bar{\nu}'+\nu_4\eta_7 p_8', E^4\tilde{\nu}'\}\\ &\quad \bmod \eta_3\circ\pi_{12}(S^4)+[\mathrm{P}^9(8), S^3]\circ E^3\gamma\\ &\qquad =\{\eta_3\varepsilon_4\}.\end{aligned}$$

This completes the proof. □

4 Lifting μ_3

First we recall Wu's result [6].

Lemma 4.1 (Wu) $4\pi_8(\mathrm{P}^3(2))=0$.

We show the following.

Lemma 4.2 *There exists an element* $\delta\in\pi_8(\mathrm{P}^3(2))$ *satisfying* $\tilde{\eta}_2(E\bar{\nu}'+\nu_4\eta_7p_8')=\xi\bar{\eta}_6'+\delta p_8'$, *where* $p_3\delta=0$ *and* $2\delta=i_3\eta_2^2\nu_4\eta_7$.

Proof. By making use of the exact sequence induced from the cofiber map $i_8': S^7\to\mathrm{P}^8(4)$ and the relation $\tilde{\eta}_2E\nu'=\xi\eta_6$ in Lemma 2.1, there exists an element $\delta\in\pi_8(\mathrm{P}^3(2))$ satisfying

$$\tilde{\eta}_2(E\bar{\nu}'+\nu_4\eta_7p_8')-\xi\bar{\eta}_6'=\delta p_8'.$$

Since $\eta_3(E\bar{\nu}'+\nu_4\eta_7p_8')=p_3\xi=0$, we obtain the relation

$$p_3\delta p_8'=0.$$

So, by the fact that $p_3\delta\in\pi_8(S^3)$ and $\nu'\eta_6^2p_8'\neq 0$, we obtain $p_3\delta=0$.

On the other hand, by use of Lemma 3.1 (ii) and (3.5),

$$\begin{aligned}2\delta p_8' &= \tilde{\eta}_2\circ 2(E\bar{\nu}'+\nu_4\eta_7p_8')\\ &= \tilde{\eta}_2\circ 2E\bar{\nu}'\\ &= 2\,\tilde{\eta}_2\circ E\bar{\nu}'\\ &= i_3\eta_2^2\nu_4\eta_7p_8'.\end{aligned}$$

Hence, by Lemma 4.1, we obtain $2\delta-i_3\eta_2^2\nu_4\eta_7\in 4\pi_8(\mathrm{P}^3(2))=0$. This completes the proof. □

Now we show the following.

Theorem μ_3 *is lifted to* $\mathrm{P}^3(2)$.

Proof. By Lemma 3.2, we have

$$\mu_3\in\{\eta_3,\ E\bar{\nu}'+\nu_4\eta_7p_8',\ E^4\tilde{\nu}'\}\subset\{p_3,\ \tilde{\eta}_2(E\bar{\nu}'+\nu_4\eta_7p_8'),\ E^4\tilde{\nu}'\}.$$

We have $(E\bar{\nu}'+\nu_4\eta_7p_8')\circ E^4\tilde{\nu}'\in\pi_{11}^4=0$ and $\bar{\eta}_6'\circ E^4\tilde{\nu}'=0$ by Lemma 3.1 (iv). So, by Lemma 4.2,

$$\begin{aligned}\{p_3,\ \tilde{\eta}_2(E\bar{\nu}'+\nu_4\eta_7p_8'),\ E^4\tilde{\nu}'\} &= \{p_3,\ \xi\bar{\eta}_6'+\delta p_8',\ E^4\tilde{\nu}'\}\\ &= \{p_3,\ \xi\bar{\eta}_6',\ E^4\tilde{\nu}'\}+\{p_3,\ \delta p_8',\ E^4\tilde{\nu}'\}.\end{aligned}$$

We see that

$$\begin{aligned}&\{p_3,\ \xi\bar{\eta}_6',\ E^4\tilde{\nu}'\}\supset\{p_3\xi,\ \bar{\eta}_6',\ E^4\tilde{\nu}'\}=\{0,\ \bar{\eta}_6',\ E^4\tilde{\nu}'\}\ni 0\\ &\bmod p_{3*}\pi_{12}(\mathrm{P}^3(2))+[\mathrm{P}^9(4),\ S^3]\circ E^5\tilde{\nu}'=p_{3*}\pi_{12}(\mathrm{P}^3(2)).\end{aligned}$$

That is,

$$\{p_3,\ \xi\bar{\eta}_6',\ E^4\tilde{\nu}'\}=p_{3*}\pi_{12}(\mathrm{P}^3(2)).$$

Finally, by Lemma 4.2 and the fact that $\pi_9(S^3)\cong Z_3$, we obtain

$$\begin{aligned}\{p_3, \delta p_8', E^4\tilde{\nu}'\} &\supset \{p_3, \delta, 2\nu_8\}\\ &\subset \{p_3, 2\delta, \nu_8\}\\ &= \{p_3, i_3\eta_2^2\nu_4\eta_7, \nu_8\}\\ &\supset \{p_3, i_3\eta_2^2\nu_4, 0\}\\ &= p_{3*}\pi_{12}(\mathrm{P}^3(2))\\ &\mathrm{mod}\ p_{3*}\pi_{12}(\mathrm{P}^3(2))+\pi_9(S^3)\circ\nu_9=p_{3*}\pi_{12}(\mathrm{P}^3(2)).\end{aligned}$$

This completes the proof. □

We recall the element $\bar{\mu}_3\in\{\mu_3, 2\iota_{12}, 8\sigma_{12}\}\subset\pi_{20}(S^3)$, where σ_n for $n\geq 9$ is a generator of $\pi_{n+7}^n\cong Z_{16}$. Finally, we show the following.

Proposition 4.3 *$\bar{\mu}_3$ is lifted to* $\mathrm{P}^3(2)$ *if* $8\pi_{12}(\mathrm{P}^3(2))=0$.

Proof. By the above theorem, there exists a lift $[\mu_3]\in\pi_{12}(\mathrm{P}^3(2))$ of μ_3. If $8\pi_{12}(\mathrm{P}^3(2))=0$, then we have

$$\begin{aligned}\bar{\mu}_3 &\in \{\mu_3, 2\iota_{12}, 8\sigma_{12}\}\\ &= \{p_{3*}[\mu_3], 2\iota_{12}, 8\sigma_{12}\}\\ &\subset \{p_3, 8[\mu_3], 2\sigma_{12}\}\\ &= \{p_3, 0, 2\sigma_{12}\}\\ &\ni 0\ \mathrm{mod}\ p_{3*}\pi_{20}(\mathrm{P}^3(2))+\pi_{13}(S^3)\circ 2\sigma_{13}.\end{aligned}$$

By the fact that $\pi_{13}^3=Z_4\{\varepsilon'\}\oplus Z_2\{\eta_3\mu_4\}$, $2\varepsilon'=\eta_3^2\varepsilon_5$ and $\varepsilon_3\sigma_{11}=0$, we obtain that $\pi_{13}(S^3)\circ 2\sigma_{13}=\pi_{13}^3\circ 2\sigma_{13}=\{\eta_3^2\varepsilon_5\sigma_{13}, 2(\eta_3\mu_4)\circ\sigma_{13}\}=0$. This implies that $\bar{\mu}_3\in p_{3*}\pi_{20}(\mathrm{P}^3(2))$ and the proof is complete. □

5 A remark on a result of the previous work

We recall a result of the paper (J. Mukai and T. Shinpo, Some homotopy groups of the mod 4 Moore space, J. Fac. Sci. Shinshu Univ. **13** (1978), 103-120). In the proof of Lemma 4.2 of this paper, the second author asserts that $\pi_8(\mathrm{P}^5(4), S^4)\cong Z\oplus Z_4$. This is not true and should be corrected as follows:

$$\pi_8(\mathrm{P}^5(4), S^4)\cong Z\oplus Z_{24}.$$

However, the group structure of $\pi_8(\mathrm{P}^5(4))$ in that paper is not wrong. Hereafter we determine the group structure of $\pi_8(\mathrm{P}^5(k))$ for $k=2^m$ with $m\geq 2$, by use of the James exact sequence (I. M. James, On the homotopy groups of certain pairs and triads, Quart. J. Math. Oxford, (2) **5** (1954), 260-270). Let X be a connected finite CW-complex and $X^*=X\cup_\theta e^n$ for $\theta: S^{n-1}\to X$ a complex formed by attaching an n-cell. We denote by

$$\gamma_n^{(X^*,X)}\in\pi_n(X^*, X)$$

the characteristic map of the n-cell e^n of X^*. Let CY be a cone of a space Y. For an element $\alpha \in \pi_m(Y)$, we denote by $\tilde{\alpha}' \in \pi_{m+1}(CY, Y)$ an element satisfying $\partial'\tilde{\alpha}'=\alpha$, where $\partial' : \pi_{m+1}(CY, Y) \to \pi_m(Y)$ is the connecting isomorphism. For $\alpha \in \pi_m(S^{n-1})$, we set

$$\hat{\alpha}=\gamma_n^{(X^*,X)} \circ \tilde{\alpha}' \in \pi_{m+1}(X^*, X).$$

We use the relation

$$[\iota_4, \iota_4]=2\nu_4-E\omega, \text{ where } \omega \text{ is a generator of } \pi_6(S^3)\cong Z_{12}.$$

By abuse of notation, we use the same letter ν_n for $n \geq 5$ to denote the generator of $\pi_{n+3}^n \cong Z_8$ or $\pi_{n+3}(S^n)\cong Z_{24}$.

Proposition 5.1 (i) $\pi_8(P^5(4))=Z_4\{E\tilde{\nu}'\} \oplus Z_2\{i_5\nu_4\eta_7\} \oplus Z_2\{i_5(E\nu')\eta_7\}$.

(ii) *Let* $k=2^m$ *for* $m \geq 3$. *Then there exists a lift of* ν_5 *and*

$\pi_8(P^5(k))=Z_8\{[\nu_5]\} \oplus Z_2\{i_5\nu_4\eta_7\} \oplus Z_2\{i_5(E\nu')\eta_7\}$, *where*

$2[\nu_5]\equiv E\tilde{\nu}' \bmod i_5\nu_4\eta_7,\ i_5(E\nu')\eta_7$.

Proof. As it is well-known, the order of the identity class of $P^5(4)$ is 4. So, by the fact that $p_{5*}(E\tilde{\nu}')=E^2\nu'=2\nu_5$, the order of $E\tilde{\nu}'$ is 4. We consider the exact sequence

$$\pi_9(P^5(k), S^4) \xrightarrow{\partial} \pi_8(S^4) \xrightarrow{i_*} \pi_8(P^5(k)) \xrightarrow{j_*} \pi_8(P^5(k), S^4) \xrightarrow{\partial} \pi_7(S^4).$$

We set $\gamma_n=\gamma_n^{(P^n(k),S^{n-1})}$. By Lemma 3.4 of the paper (K. Morisugi and J. Mukai, Whitehead square of a lift of the Hopf map to a mod 2 Moore space, J. Math. Kyoto Univ. **42** (2002), 331–336), we obtain

$$[\gamma_5, k\iota_4]=-2\hat{\nu}_4+E'\hat{\omega},$$

where $E' : \pi_4(P^4(k), S^3) \to \pi_5(P^5(k), S^4)$ is the relative suspension homomorphism. So we have

$$2(\hat{\nu}_4+\frac{k}{2}[\gamma_5, \iota_4])=E'\hat{\omega}. \tag{5.1}$$

By (5.1) and the James exact sequence, we obtain the following:

$$\pi_8(P^5(k), S^4)=Z\{[\gamma_5, \iota_4]\} \oplus Z_{24}\{\hat{\nu}_4+\frac{k}{2}[\gamma_5, \iota_4]\}.$$

And

$$\pi_9(P^5(k), S^4)=Z_2\{\hat{\nu}_4\tilde{\eta}_7'\} \oplus Z_2\{[\gamma_5, \eta_4]\}.$$

By the properties of Whitehead products,

$$\partial\hat{\nu}_4=k\iota_4 \circ \nu_4=k^2\nu_4-\frac{k(k-1)}{2}E\omega$$

and

$$\partial[\gamma_5, \iota_4]=-[k\iota_4, \iota_4]=-2k\nu_4+kE\omega.$$

So we obtain

$$\partial(\hat{\nu}_4\tilde{\eta}_7')=k\iota_4\circ\nu_4\eta_7=(k^2\nu_4+\frac{k}{2}E\nu')\eta_7=0,$$

$$\partial[\gamma_5,\ \eta_4]=(-2k\nu_4+kE\nu')\eta_7=0$$

and

$$\partial(\hat{\nu}_4+\frac{k}{2}[\gamma_5,\ \iota_4])=\frac{k}{2}E\omega.$$

We note that $j_*(E\tilde{\nu}')=E'\hat{\nu}'$. Hence, by (5.1), we get (i). Since $\partial j_*=0$, the lift $[\nu_5]$ of $\nu_5 \in \pi_8^5$ has the property that

$$j_*[\nu_5]=3(\hat{\nu}_4+\frac{k}{2}[\gamma_5,\ \iota_4]) \text{ for } m\geq 3.$$

So, by (5.1), we obtain the group structure of (ii) and the relation $2[\nu_5]-E\tilde{\nu}'\in i_{5*}\pi_8(S^4)$. This leads to (ii), completing the proof. □

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