J. FAC. SCI., SHINSHU UNIVESITY Vol. 20, No. 2, 1985

Curvature forms with singularities and Non-integral characteristic classes

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(Received, Oct. 10, 1985)

Introduction

This is the detailed exposition of [7]. The outline of the paper is as follows: In §1, we define the 2-dimensional non-abelian de Rham set $H^1(M, \mathscr{M}^1)$. Here M is a smooth manifold and $\mathscr{M}^1$ is the sheaf of germs of smooth matrix valued 1-forms θ such that $d\theta+\theta\wedge\theta=0$. A bijection between $H^1(M, \mathscr{M}^1)$ and a set of special classes of matrix valued 2-forms on M are also defined. In a sense, these, 2-forms can be regarded as curvature forms with singularities (singular gauge fields) (cf. [11], [17], [23], [24], [25]). Since we have obtained 2-dimensional non-abelian Poincaré lemma (local integration theorem of the eguation $d\theta+\theta\wedge\theta=\Theta$, [10]), the results of this section will be improved in future. In §2, we define the cohomology sets $H^2(M, G_t)$ and $H^2(M, G_d)$, $G=GL(n, C)$. Here G_t and G_d are sheaves of germs of constant and smmooth G-valued functions on M, respectively. Then we show the exactness of the following sequence

$$0\xrightarrow{i}H^0(M, G_t)\xrightarrow{\rho}H^0(M, G_d)\xrightarrow{\delta}H^0(M, \mathscr{M}^1)\xrightarrow{i^*}H^1(M, G_t)\longrightarrow$$
$$\longrightarrow H^1(M, G_d)\xrightarrow{\rho^*}H^1(M, \mathscr{M}^1)\xrightarrow{\delta}H^2(M, G_t)\xrightarrow{i^*}H^2(M, G_d).$$

This sequence is derived form the exact sequence of sheaves

$$0\longrightarrow G_t\xrightarrow{i}G_d\xrightarrow{\rho}\mathscr{M}^1\longrightarrow 0,$$

and the exactness of the first 6-terms of the above sequence are known and used by several authors ([3], [15], [21], [22], [26]). But the definitions and exactness of last 3-terms seem to be new. We note that our definition of $H^2(M, G_t)$ is different from former definitions ([13], [16], [19]). In fact, in our definition, $H^2(M, G_t)$ is not absolutely determined by the sheaf G_t. Its definition depends on the above sheaf exact sequence. This is the reason why we have the cohomology exact sequence (cf. [14]). In §3, we define the characteristic classes of the elements of $H^1(M, \mathscr{M}^1)$. These are non-integral generalisation of the Chern classes (cf. [11], [23], [24], [25]). The characteristic classes of the elements of $H^2(M, G_t)$ and H^2

(M, G_d) are also considered. Although there still remain many problems, we had some progress for the definition of the characteristic classes of the elements of $H^2(M, G_t)$ and $H^2(M, G_d)$ ([9]). The results of §§1–3 are constructed by the differential operator d and the group $G=GL(n, C)$ and G can be replaced by $U(n)$ or $SU(n)$. In §4, we give analogous theory replacing d by D, an arbitrary differential operator over M. It is a refinement of the appendix of [6] (cf. [1], [2], [5], [8]).

§1. Non-abelian de Rham theory of dimension 2

1. Let M be a connected paracompact smooth manifold. On M, we consider the following sheaves.

G_t : *the sheaf of germs of constant G-valued functions.*

G_d : *the sheaf of germs of smooth G-valued functions.*

$\mathfrak{g}^1$: *the sheaf of germs of complex (n, n)-matrix valued 1-forms.*

$\mathcal{M}^1$: *the subsheaf of $\mathfrak{g}^1$ consisting of forms θ such that $d\theta+\theta\wedge\theta=0$.*

$\mathcal{M}^2$: *the image sheaf of $\mathfrak{g}^1$ by the map d^e defined by $d^e\varphi=d\varphi+\varphi\wedge\varphi$.*

Using 2-dimensional non-abelian Poincaré lemma, we can give an intrinsic definition of $\mathcal{M}^2$. But in this paper, we use thisold definition.

Note 1. If G is a Lie group with the Lie algebra $\mathfrak{g}$ such that the exponential map is onto. Then we can define the similar sheaves for G-valued functions and $\mathfrak{g}$-valued forms.

Note 2. We denote G_ω, $\mathcal{M}^1_\omega$, *etc.*, the corresponding sheaves for holomorphic maps. The similar notations are used for other categories.

Defilnition 1. *We define the differential operators ρ and d^e respectively by*

$$\rho(g)=g^{-1}dg, \quad d^e\varphi = d\varphi + \varphi\wedge\varphi.$$

The induced maps of ρ and d^e on the sheaves G_d and $\mathfrak{g}^1$ are also denoted by ρ and d^e respectively.

Note. ρ is defined for Lie group valued functions, while d^e is defined for Lie algebra valued forms. This is the reason of the inhomogenity of notations. For Lie algebra valued functions, d^e is defined by $d^e f=\rho(e^f)=df+\sum_{n=1}^{\infty}((ad\ f)^n df)/(n+1)!$ ([8], [9]).

By definitions, we have the following exact sequence of sheaves.

$$(1)\qquad 0\longrightarrow G_t \xrightarrow{i} G_d \xrightarrow{\rho} \mathcal{M}^1 \longrightarrow 0,$$

$$(2)\qquad 0\longrightarrow \mathcal{M}^1 \xrightarrow{i} \mathfrak{g}^1 \xrightarrow{d^e} \mathcal{M}_2 \longrightarrow 0.$$

The 0-dimensional cohomology sets for these sheaves and 1-dimensional cohomology sets for G_t and G_d are known. But we need other 0-dimensional cohomology sets for $\mathscr{M}^1$, $\mathfrak{g}^1$ and $\mathscr{M}^2$ based on the action of gauge transformations. For these purposes, we take a locally finite open covering $\mathfrak{U}=\{U_i\}$ of M. We write c_i, c_{ij}, *etc.*, the sections on U_i, $U_i \cap U_j$, *etc.*,

Definition 2. *Let* $h=\{h_i\}$ be an element of $\mathrm{C}^0(\mathfrak{U}.\ \mathrm{G}_d)$. *Then we define the actions of* h *on* $\mathrm{C}^0(\mathfrak{U},\ \mathfrak{g}^1)$ *and on* $\mathrm{C}^0(\mathfrak{U},\ \mathscr{M}^2)$ *by*

$$h(\theta)=h_i(\theta_i-h_i^{-1}dh_i)h_i^{-1},\quad \theta=\{\theta_i\}\in \mathrm{C}^0(\mathfrak{U},\ \mathfrak{g}^1), \tag{3}$$

$$h(\Theta)=h_i(\Theta_i)h_i^{-1},\quad \Theta=\{\Theta\}_i\in \mathrm{C}^0(\mathfrak{U},\ \mathscr{M}^2). \tag{3$'$}$$

By definitions, we have

Lemma 1. *For the above actions, the followings hold*

$$\begin{aligned} &h_1(h_2(\theta))=(h_1h_2)(\theta),\quad h_1(h_2(\Theta))=(h_1h_2)(\Theta),\\ &\mathrm{e}(\theta)=\theta,\quad \mathrm{e}(\Theta)=\Theta,\quad \mathrm{e}=\mathrm{e}_i(x),\ \textit{the identity valued function.}\end{aligned} \tag{4}$$

$$d^e(h(\theta))=h(d^e\theta). \tag{5}$$

$$h(\mathrm{C}^0(\mathfrak{U},\ \mathscr{M}^1))=\mathrm{C}^0(\mathfrak{U},\ \mathscr{M}^1). \tag{6}$$

2. $h=\{h_i\}\in \mathrm{C}^0(\mathfrak{U},\ \mathrm{G}_d)$ also acts on $(\mathrm{C}^1(\mathfrak{U},\ \mathrm{G}_d)$ by the action

$$h(\xi)=h_ig_{ij}h_j^{-1},\quad \xi=\{g_{ij}\}. \tag{7}$$

By definition, we have

$$h_1(h_2(\xi))=(h_1h_2)(\xi),\quad \mathrm{e}(\xi)=\xi. \tag{4$'$}$$

Definition 3. *We set*

$$\mathrm{C}^1{}_a(\mathfrak{U},\ \mathrm{G}_d)=\{\{g_{ij}\}\in \mathrm{C}^1(\mathfrak{U},\ \mathrm{G}_d)\,|\,g_{ii}=\mathrm{e},\ g_{ij}=g_{ji}^{-1}\}.$$

By definition, we have

$$h(\mathrm{C}^1{}_a(\mathfrak{U},\ \mathrm{G}_d))=\mathrm{C}^1a(\mathfrak{U},\ G_d). \tag{6$'$}$$

Definition 4. *Let* $\xi=\{g_{ij}\}$ *be an element of* $\mathrm{C}^1(\mathfrak{U},\ \mathrm{G}_d)$ *and* $\mathscr{C}$ *be one of* $\mathscr{M}^1$, $\mathfrak{g}^1$ *or* $\mathscr{M}^2$. *Then we define the map* δ_ξ *on* $\mathrm{C}^0(\mathfrak{U},\ \mathscr{C})$ *by*

$$\delta_\xi(c)_{ij}=c_j-g_{ji}c_ig_{ij},\quad c=\{c_i\}. \tag{8}$$

By definition, if ξ is in $\mathrm{C}^1{}_a(\mathfrak{U},\ \mathrm{G}_d)$, *we get*

$$\delta_\xi(c)_{ij}=c_j-g_{ji}c_ig_{ij}^{-1}=c_j-g_{ij}(c_i). \tag{8$'$}$$

Note. The image of δ_ξ is not in $\mathrm{C}^1(\mathfrak{U},\ \mathscr{C})$ in general. For example, if $\xi\in \mathrm{C}^1{}_a(\mathfrak{U},\ \mathrm{G}_d)$, we have

$$d^e(\delta_\xi(\theta))_{ij}=\delta_\xi(d^e\theta)_{ij}+[\rho(g_{ij})-\delta_\xi(\theta)_{ij},\ g_{ij}^{-1}\theta_ig_{ij}]. \tag{9}$$

Here $[\varphi,\ \phi]$ means $\varphi\wedge\phi-(-1)^{pq}\phi\wedge\varphi$, $p=deg\varphi$, $q=deg\phi$.

Hereafter, we denote t_{ijk} or δg_{ijk} the map $g_{ij}g_{jk}g_{ik}^{-1}$.

Lemma 2. (i). *If* $\delta_\xi(c)=0$, *then* $\delta_\xi\rho(c)$ *is epual to* 0 *if and only if*

(10) $$h_{ij}c_jh_{ji} = c_j,\ \xi = \{g_{ij}\}\ \text{and}\ \xi' = \{h_{ij}g_{ij}\}.$$

(ii). *If* $\xi \in C^1{}_a(\mathfrak{U}, G_d)$ *and* $\delta_\xi(c)$ *is equal to* 0, *then*

(11) $$t_{ijk}{}^{-1}c_it_{ijk} = c_i.$$

(iii). *If* Θ *is in* $C^0(\mathfrak{U}, \mathscr{M}^2)$ *and* $\delta_\xi\Theta=0$, *then* $\delta_{h(\xi)}\ (h(\Theta))$ *is equal to* 0. *If* θ *is in* $C^0(\mathfrak{U}, g^1)$ *and* $\delta_\xi(\theta)=0$, *then* $\delta_{h(\xi)}\ (h(\theta))$ *is equal to* 0 *if and only if*

(12) $$\delta_\xi(\rho(h)) = 0.$$

(iv). *If* $\xi = \{g_{ij}\} \in C^1{}_a(\mathfrak{U}, G_d)$ *and* $\delta_\xi(\theta)=0$, *then* $\delta_\xi(d^e\theta)$ *is equal to* 0 *if and only if*

(13) $$[\rho(g_{ij}),\ \theta_j]=0.$$

Proof. (i) and (ii) follow from the definitions. Since

$$\delta_{h(\xi)}\ (h(\Theta))_{ij} = h_j(\delta_\xi(\Theta)_{ij})h_j{}^{-1},$$

we have the first assertion of (iii). The second assertion follows from $\delta_{h(\xi)}\ (h(\theta))_{ij}= h_j(\delta_\xi(\theta)_{ij} - \delta_\xi\rho(h)_{ij})h_j{}^{-1}$. Since $g_{ij}{}^{-1}\theta_ig_{ij}=\theta_j$ if $\delta_\xi(\theta)=0$, we have (iv) by (9).

Note. If h satisfies (12) for $\xi \in C^1{}_a(\mathfrak{U}, G_d)$, then we have

$$-\rho(h_i)\wedge\rho(h_j) = -[\rho(g_{ij}),\ g_{ij}{}^{-1}\rho(h_i)g_{ij}] - g_{ij}{}^{-1}\rho(h_i)g_{ij}\wedge g_{ij}{}^{-1}\rho(h_i)g_{ij}.$$

Hence we have

$$[\rho(g_{ij}),\ \rho(h_j)] = 0.$$

By this equality, if $\delta_\xi(d^e\theta) = \delta_{h(\xi)}\ (h(\theta)) = 0$, then we have $\delta_{h(\xi)}\ (d^e(h(\theta)) = 0$.

Definition 5. *We assume* $\xi=\{g_{ij}\} \in C^1{}_a(\mathfrak{U}, G_d)$. *Then we set*

$$Z^0{}_d(\mathfrak{U}, \mathscr{M}^1)=\{\theta\in C^0(\mathfrak{U}, \mathscr{M}^1)\,|\,\delta_\xi\theta=0\ \text{for some}\ \xi\ \text{such that}\ [\rho(g_{ij}),\ \theta_j]=0\}.$$

$$Z^0{}_d(\mathfrak{U}, \mathfrak{g}^1)=\{\theta\in C^0(\mathfrak{U}, \mathfrak{g}^1)\,|\,\delta_\xi\theta=0\ \text{for some}\ \xi\ \text{such that}\ [\rho(g_{ij}),\ \theta_j]=0\}.$$

$$Z^0{}_d(\mathfrak{U}, \mathscr{M}^2)=\{\Theta\in C^0(\mathfrak{U}, \mathscr{M}^2)\,|\,\delta_\xi\Theta=0\ \text{for some}\ \xi\ \text{such that}\ \rho(\xi)=\delta_\xi(\theta),\ \Theta=d^e\theta,\ \text{and}\ t_{ijk}{}^{-1}\theta_it_{ijk}=\theta_i\}.$$

Definition 6. *For* $\theta\in Z^0{}_d(\mathfrak{U}, \mathscr{M}^1)$ *or* $Z^0{}_d(\mathfrak{U}, \mathfrak{g}^1)$, *we call* $h\in C^0(\mathfrak{U}, G_d)$ *is an admissible action if*

$$\delta_\xi(\rho(h)) = 0,\ \ \delta_\xi\theta = 0.$$

Definition 7. *For* $\Theta\in Z^0{}_d(\mathfrak{U}, \mathscr{M}^2)$, *we call* h *is an admissible action if*

$$\delta_\xi\Theta=0,\ \ \rho(h_i) = t_{ijk}{}^{-1}\rho(h_i)t_{ijk}.$$

Since we have

$$\rho(h_1h_2) = \rho(h_2) + h_2^{-1}\rho(h_1)h_2,$$
$$\delta_\xi(\rho(h_1h_2))_{ij} = \delta_\xi\rho(h_2)_{ij} + h_{2,j}^{-1}(\delta_{h_2(\xi)}\rho(h_2))_{ij}h_{2,j},$$

these actions are well defined.

Definition 8. *The limit sets of* $Z^0{}_d(\mathfrak{U}, \mathscr{M}^1)$, $Z^0{}_d(\mathfrak{U}, \mathfrak{g}^1)$ *and* $Z^0{}_d(\mathfrak{U}, \mathscr{M}^2)$ *with respect to the refinement of the covering* $\mathfrak{U}$, *are denoted by* $Z^0{}_d(M, \mathscr{M}^1)$, $Z^0{}_d(M, \mathfrak{g}^1)$ *and* $Z^0{}_d(M, \mathscr{M}^2)$ *respectively.*

By definitions 6, 7, admissible actions are defined on $Z^0{}_d(M, \mathscr{M}^1)$, $Z^0{}_d(M, \mathfrak{g}^1)$ and $Z^0{}_d(M, \mathscr{M}^2)$.

Definition 9. *The quotient sets of* $Z^0{}_d(M, \mathscr{M}^1)$, $Z^0{}_d(M, \mathfrak{g}^1)$ *and* $Z^0{}_d(M, \mathscr{M}^2)$ *by the admissible actions are denoted* $H^0{}_d(M, \mathscr{M}^1)$, $H^0{}_d(M, \mathfrak{g}^1)$ *and* $H^0{}_d(M, \mathscr{M}^2)$. *They called the 0-dimensional cohomolgy sets.*

By the definitions and Lemma 2, we have the following exact sequence

$$0 \longrightarrow Z^0{}_d(M, \mathscr{M}^1) \xrightarrow{i} Z^0{}_d(M, \mathfrak{g}^1) \xrightarrow{d^e} Z^0{}_d(M, \mathscr{M}^2).$$

By (5) and Lemma 2, d^e also induces the map $d^e: H^0{}_d(M, \mathfrak{g}^1) \longrightarrow H^0{}_d(M, \mathscr{M}^2)$.

3. **Definition 10.** *Let* $\mathscr{C}$ *be* $\mathscr{M}^1$ *or* $\mathfrak{g}^1$ *an* $\xi \in C^1(\mathfrak{U}, G_d)$. *Then we define the map* δ_ξ *on* $C^1(\mathfrak{U}, \mathscr{C})$ *by*

$$\delta_\xi(\omega)_{ijk} = \omega_{jk} - \omega_{ik} + g_{kj}\omega_{ij}g_{jk}, \quad \xi = \{g_{ij}\}. \tag{14}$$

Note. In general, $\delta_\xi(C^1(\mathfrak{U}, \mathscr{M}^1))$ is not contained in $C^2(\mathfrak{U}, \mathscr{M}^1)$.

By definition, $\delta_\xi(\delta_\xi\theta)$ is equal to 0 if and only if $g_{ki}\theta_i g_{ik} = g_{kj}g_{ji}\theta_i g_{ij}g_{jk}$ for any i, j, k. If $\xi \in C^1{}_a(\mathfrak{U}, G_d)$, then

$$\delta_\xi(\omega)_{ijk} = \omega_{jk} - \omega_{ik} + g_{jk}^{-1}\omega_{ij}g_{jk}. \tag{14$'$}$$

Definition 11. *Let* ξ *be in* $C^1{}_a(\mathfrak{U}, G_d)$. *Then we set*

$$Z^1(\mathfrak{U}, \mathscr{M}^1) = \{\omega \in C^1(\mathfrak{U}, \mathscr{M}^1) \mid \omega = \rho(g_{ij}) \text{ and } \delta_\xi\omega = 0 \text{ for some } \xi = \{g_{ij}\}\},$$
$$Z^1(\mathfrak{U}, \mathfrak{g}^1) = \{\omega \in C^1(\mathfrak{U}, \mathfrak{g}^1) \mid \delta_\xi\omega = 0 \text{ for some } \xi\}.$$

Definition 12. *The elements* ω *and* ω' *of* $Z^1(\mathfrak{U}, \mathscr{M}^1)$ *are said to be cohomologous and denoted* $\omega \sim \omega'$ *if there exists* $h = h_1 \in C^0(\mathfrak{U}, G_d)$ *such that*

$$d(h_i t_{ijk} h_i^{-1}) = 0 \text{ for any } i, j, k, \ \omega_{ij} = \rho(g_{ij}), \tag{15}$$
$$\xi = \{g_{ij}\} \in C^1{}_a(\mathfrak{U}, G_d) \text{ and } \delta_\xi(\omega) = 0.$$

$$\omega_{ij}' = h_j(\omega_{ij} - \delta_\xi\rho(h)_{ij})h_j^{-1}. \tag{16}$$

Note. By (16), we have $\omega_{ij}' = \rho(h_i g_{ij} h_j^{-1})$. This ω' satisfies $\delta_{\xi'}(\omega') = 0$, where $\xi' = \{h_i g_{ij} h_j^{-1}\} \in C^1{}_a(\mathfrak{U}, G_d)$.

Definition 13. *The elements* ω *and* ω' *of* $Z^1(\mathfrak{U}, \mathfrak{g}^1)$ *are said to be cohomologous and denoted* $\omega \sim \omega'$ *if there exist* $h = \{h_i\} \in C^0(\mathfrak{U}, G_d)$ *and* $\theta = \{\theta_i\} \in C^0(\mathfrak{U}, \mathfrak{g}^1)$ *such that*

(15)′ $t_{ijk}{}^{-1}\theta_i t_{ijk}=\theta_i$, *for any* $i,\ j,\ k,\ \delta_\xi\omega=0,\ \xi=\{g_{ij}\}$,

(16)′ $\omega_{ij}{}'=h_j(\omega_{ij}-\delta_\xi(\theta)^i{}_j)h_j{}^{-1}$.

Definition 14. $\mathrm{H}^1(\mathfrak{U},\ \mathscr{M}^1)$ *and* $\mathrm{H}^1(\mathfrak{U},\ \mathfrak{g}^1)$ *are defined to be the quotient sets of* $\mathrm{Z}^1(\mathfrak{U},\ \mathscr{M}^1)$ *and* $\mathrm{Z}^1(\mathfrak{U},\ \mathfrak{g}^1)$ *by the cohomology relations.*

Note. The sets of coboundaries are given by $\mathrm{B}^1(\mathfrak{U},\ \mathscr{M}^1)=\{\omega\,|\,\omega=\delta_\xi(\theta),\ \theta\in\mathrm{C}^0(\mathfrak{U},\ \mathscr{M}^1),\ \omega=\rho(\xi)\}$ and $\mathrm{B}^1(\mathfrak{U},\ \mathfrak{g}^1)=\{\omega\,|\,\omega=\delta_\xi(\theta),\ t_{ijk}{}^{-1}\theta_i t_{ijk}=\theta_i\}$.

Note. If $\omega\in\mathrm{C}^1(\mathfrak{U},\ \mathscr{M}^1)$ satisfies $\omega_{jk}-\omega_{ik}+g_{jk}{}^{-1}\omega_{ij}g_{jk}=0,\ \omega_{ij}=\rho(g_{ij})$ for some $\{g_{ij}\}$, then we have

(i) $g_{jk}{}^{-1}\omega_{ij}g_{jk}$ *does not depend on the choice of* $\{g_{ij}\}$,

(ii) $\omega_{ii}=0$ *and* $\omega_{ij}=-g_{ji}\omega_{ji}g_{ji}{}^{-1}$.

By (ii), we may assume $\omega=\rho(\xi),\ \xi\in\mathrm{C}^1{}_a(\mathfrak{U},\ \mathrm{G}_d)$.

Proposition 1. (i) *Let* $\xi=\{g_{ij}\}$ *be in* $\mathrm{C}^1{}_a(\mathfrak{U},\ \mathrm{G}_d)$. *Then* $\rho(\xi)$ *belongs to* $\mathrm{Z}^1(\mathfrak{U},\ \mathscr{M}^1)$ *if and only if* $t_{ijk}=(\delta_\xi)_{ijk}$ *is a constant map for any* $i,\ j,\ k$.

(ii) $\mathrm{Z}^1(\mathfrak{U},\ \mathfrak{g}^1)$ *is equal to* $\mathrm{B}^1(\mathfrak{U},\ g^1)$.

Proof. Since we have

$$t_{ijk}{}^{-1}d(t_{ijk})=g_{ik}(g_{jk}{}^{-1}\omega_{ij}g_{jk}+\omega_{jk}-\omega_{ik})g_{ik}{}^{-1},$$

$d(t_{ijk})$ is equal to 0 if $\omega\in\mathrm{Z}^1(\mathfrak{U},\ \mathscr{M}^1)$. On the other hand, if $d(t_{ijk})$ is equal to 0, $\omega=\rho(\xi)$ satisfies $\delta_\xi\omega=0$. Hence we have (i).

Since we have $g_{jk}{}^{-1}g_{ij}{}^{-1}\omega_{ki}g_{ij}g_{jk}=g_{jk}{}^{-1}(\omega_{kj}-\omega_{ij})g_{jk}=-\omega_{jk}+(\omega_{ij}-\omega_{ik})=-\omega_{ik}=g_{ik}{}^{-1}\omega_{ki}g_{ik}$ if $\delta_\xi\omega=0$, we get $t_{ijk}{}^{-1}\omega_{ki}t_{ijk}=\omega_{ki}$ if $\delta_\xi\omega=0$. Hence to take a smooth partition of unity $\{e_i\}$ subordinate to U,

$$\theta_i=\sum_{U_k\cap U_i\neq\phi}e_k\omega_{ki}$$

satisfies (15)′. Then, since

$$\theta_j-g_{ij}{}^{-1}\theta_i g_{ij}=\sum_{U_k\cap U_i\cap U_j\neq\phi}e_k(\omega_{kj}-g_{ij}{}^{-1}\omega_{ki}g_{ij})=\omega_{ij},$$

ω is equal to $\delta_\xi(\theta)$. Therefore we obtain (ii).

Corollary 1. *For elements of* $\mathrm{Z}^1\ \mathfrak{U},\ \mathscr{M}^1)$, (15)′ *follows from* (15).

Proof. Since t_{ijk} is a constant, we get $h_i{}^{-1}(d(h_i t_{ijk}h_i{}^{-1}))h_i=\rho(h_i)t_{ijk}-t_{ijk}\rho(h_i)=0$. Hence wehave Corollary.

Corollary 2. $\mathrm{H}^1(\mathfrak{U},\ \mathfrak{g}^1)$ *is equal to* $\{0\}$.

Note. Corollary 2 does not hold in another category. For example, $\mathrm{H}^1(\mathfrak{U},\ \mathfrak{g}^1{}_\omega)$ may not be equal to $\{0\}$ (cf. [6], [7], [22]).

Definition 15. *We set* $\mathrm{C}^1{}_c(\mathfrak{U},\ \mathrm{G}_d)=\{\{g_{ij}\}\in\mathrm{C}^1{}_a(\mathfrak{U},\ \mathrm{G}_d)\,|\,\delta g_{ijk}$ *is a constant for any* $i,\ j,\ k\}$.

Let $\mathfrak{B}=\{V_j\,|\,j\in\mathbf{J}\}$ be a refinement of $\mathfrak{U}=\{U_i\,|\,i\in\mathbf{I}\}$ *and* $\tau,\ \tau'\colon\ \mathbf{J}>\mathbf{I}$ the maps such that $V_j \subset U_{\tau(i)} \cap U_{\tau'(i)}$. Then $\{h_i\} = \{g_{\tau(i)\tau'(i)}\}$ gives a chain homotopy between τ^* and τ'^*.

Definition 16. *We set* $\mathrm{H}^1(M, \mathscr{M}^1)$ *the limit set* $\lim \{\mathrm{H}^1(\mathfrak{U}, \mathscr{M}^1) | \tau^*\}$.

Note. $\mathrm{H}^1(M, \mathfrak{g}^1)$ is defined ay the same way. But by Corollary 2 of proposition 1, we have

$$\mathrm{H}^1(M, \mathfrak{g}^1) = \{0\}.$$

4. **Definition 17.** *Let* $\overline{\Theta}$ *and* $\overline{\Theta}'$ *be elements of* $\mathrm{H}^0{}_d(M, \mathscr{M}^2)$. $\overline{\Theta}$ *and* $\overline{\Theta}'$ *are cohomologous if they have representatives* $\Theta = \{\Theta_i\}$ *and* $\Theta' = \{\Theta_i'\}$ *such that* $\delta_\xi \Theta = 0$, δ_ξ, $\Theta' = 0$, $\Theta_i = d^e\theta_i$, $\Theta_i' = d^e\theta_i'$, $\rho(\xi) = \delta_\xi \theta$, $\rho(\xi') = \delta_\xi$, θ', $\xi = \{g_{ij}\}$ *and* $\xi' = \{c_{ij}g_{ij}\}$, *where* c_{ij} *is a constant map.*

Note. $\rho(\xi) = \delta_\xi \theta$ if and only if $\theta_j - g_{ij}{}^{-1}(\theta_i) = 0$.

Definition 18. $\mathrm{H}^2{}_{dR}(M, \mathfrak{g})$ *is defined as the quotient set of* $\mathrm{H}^0{}_d(M, \mathscr{M}^2)$ *by the cohomology relation.*

Theorem 1. *There is a bijection between* $\mathrm{H}^2{}_{dR}(M, \mathfrak{g})$ *and* $\mathrm{H}^1(M, \mathscr{M}^1)$.

Proof. We denote $\langle\Theta\rangle$ theclass of $\Theta \in \mathrm{Z}^0{}_d(\mathfrak{U}, \mathscr{M}^2)$ in $\mathrm{H}^2{}_{dR}(M, \mathfrak{g})$. Then we $\delta_\xi \Theta = 0$, $\rho(\xi) = \delta_\xi \theta$, $\xi \in \mathrm{C}^1{}_a(\mathfrak{U}, \mathrm{G}_d)$ and $\theta \in \mathrm{C}^0(\mathfrak{U}, \mathfrak{g}^1)$. Then, since $t_{ijk}{}^{-1}\theta_i t_{ijk} = \theta_i$, we obtain

$$\delta_\xi(\rho(\xi)) = \delta_\xi(\delta_\xi \theta) = 0.$$

Hence $\rho(\xi)$ belongs to $\mathrm{Z}^1(\mathfrak{U}, M^1)$. If Θ' is another representative of $\langle\Theta\rangle$ such that δ_ξ, $\Theta' = 0$, set $\xi = \{g_{ij}\}$ and $\xi' = \{g_{ij}'\}$, we have

$$g_{ij}' = h_i c_{ij} g_{ij} h_j{}^{-1},$$

where c_{ij} is a constant and $\{h_i\}$ is admissible. Then, since $\rho(c_{ij}g_{ij}) = \rho(g_{ij})$, we may assume $g_{ij}' = h_i g_{ij} h_j{}^{-1}$. Hence we have $\rho(\xi')_{ij} = h_j(\rho(\xi)_{ij} - \delta_\xi \rho(h)_{ij})h_j{}^{-1}$ and $d(h_i t_{ijk} h_i{}^{-1}) = 0$. Therefore $\rho(\xi) \sim \rho(\xi')$ and we can define the map $\iota: \mathrm{H}^2{}_{dR}(M, \mathfrak{g}) \to \mathrm{H}^1(M, \mathscr{M}^1)$ by

$$\iota(\langle\Theta\rangle) = \langle\rho(\xi)\rangle: \textit{ the class of } \rho(\xi) \textit{ in } \mathrm{H}^1(M, \mathscr{M}^1).$$

Since $\theta_j = \rho(g_{ij}) + g_{ij}{}^{-1}\theta_i g_{ij}$ if $\rho\ (\xi) = \delta_\xi(\theta)$, we have $\delta_\xi d^e\theta = 0$ if $\rho(\xi) = \delta_\xi(\theta)$. Hence we have

$$d\theta_j = -\rho(g_{ij}) \wedge \rho(g_{ij}) - \rho(g_{ij}),\ g_{ij}{}^{-1}\theta_i g_{ij} + g_{ij}{}^{-1} d\theta_i g_{ij}.$$

Therefore $d^e\theta_j = g_{ij}{}^{-1} d^e\theta_i g_{ij}$, that is $\delta_\xi d^e\theta = 0$. Let $\langle\omega\rangle$ be an element of $\mathrm{H}^1(M, \mathscr{M}^1)$. Then we set $\omega = \rho(\xi)$, $\xi \in \mathrm{C}^1{}_c(\mathfrak{U}, \mathrm{G}_d)$. By Proposition 1, (ii), we can set $\rho(\xi) = \delta_\xi(\theta)$, $\theta \in \mathrm{C}^0(\mathfrak{U}, \mathfrak{g}^1)$. Then by the above calculation, $d^e\theta$ belongs to $\mathrm{Z}^0{}_d(\mathfrak{U}, \mathscr{M}^2)$. If $\omega \sim \omega'$, we set $\omega' = \rho(h(\xi))$. Then $h \in \mathrm{C}^0(\mathfrak{U}, \mathrm{G}_d)$ defines an admissible action of $d^e\theta$. Hence we can define the map $\kappa: \mathrm{H}^1(M, \mathscr{M}^1) \to \mathrm{H}^2{}_{dR}(M, \mathfrak{g})$ by

$$\kappa(\langle\omega\rangle) = \langle d^e\theta\rangle,\quad \omega = \rho(\xi) = \delta_\xi \theta.$$

By the definitions of ι and κ, $\iota\kappa$ is the identity map of $H^2_{dR}(M, \mathfrak{g})$ and $\kappa\iota$ is the identity map of $H^1(M, \mathscr{M}^1)$. Hence we have Theorem.

Note 1. The kernel of the surjection $\delta: H^0_d(M, \mathscr{M}^2) \to H^1(M, \mathscr{M}^1)$ is given by $d^e(H^0_t(M, \mathfrak{g}^1))$, $H^0_t(M, \mathfrak{g}^1) = \{\langle\theta\rangle \in H^0_d(M, \mathfrak{g}^1) \mid \delta_\xi\theta = 0$ *for some* $\xi \in C^1_a(\mathfrak{U}, G_t)\}$. Here $C^1_a(\mathfrak{U}, G_t)$ means $C^1(\mathfrak{U}, G_t) \cap C^1_a(\mathfrak{U}, G_d)$.

Note 2. In holomorphic category, $H^1(M, \mathfrak{g}^1_\omega)$ may not be epual to $\{0\}$ and we have the following exact sequence

$$0 \longrightarrow H^0_t(M, \mathscr{M}^1_\omega) \xrightarrow{i^*} H^0_t(M, \mathfrak{g}^1_\omega) \xrightarrow{d^e} H^0_d(M, \mathscr{M}^2_\omega) \longrightarrow H^1(M, \mathscr{M}^1_\omega) \xrightarrow{i^*} H^1(M, \mathfrak{g}^1_\omega)$$

The image $i^*(\langle\omega\rangle)$ of $\langle\omega\rangle \in H^1(M, \mathscr{M}^1_\omega)$ is the obstruction class of $\langle\omega\rangle$ to be in δ image. Especially, if $\omega = \rho^*(\xi)$, $\xi \in H^1(M, G_\omega)$, $i^*\rho^*(\xi)$ is the obstruction class of ξ to have a holomorphic connection.

§2. The cohomology sets $H^2(M, G^t)$ and $H^2(M, G_d)$

5. **Definition 19.** *Let* $\xi = \{g_{jj}\}$ *be in* $C^1_a(\mathfrak{U}, G_d)$. *Then we define a map* δ_ξ *on* $C^2(\mathfrak{U}, G_t)$ *(resp. on* $C^2(\mathfrak{U}, G_d)$*) by*

$$(17) \qquad (\delta_\xi c)_{i_0i_1i_2i_3} = g_{i_0i_1}c_{i_1i_2i_3}g_{i_0i_1}{}^{-1}c_{i_0i_1i_3}c_{i_0i_2i_3}{}^{-1}c_{i_0i_1i_2}{}^{-1}.$$

By definition, we have

Lemma 3. (i) δ_ξ *maps* $C^2(\mathfrak{U}, G_d)$ *into* $C^3(\mathfrak{U}, G_d)$.

(*ii*) $\delta_\xi\ c = e$ *if and only if*

$$(18) \qquad c_{i_0i_1i_2}c_{i_0i_2i_3} = g_{i_0i_1}c_{i_1i_2i_3}g_{i_0i_1}{}^{-1}c_{i_0i_1i_3}.$$

(ii)′ *If* $c \in C^2(\mathfrak{U}, G_t)$ *satisfies* $\delta_\xi\ c = e$, $g_{i_0i_1}c_{i_1i_2i_3}g_{i_0i_1}{}^{-1}$ *is a constant map.*

(iii) *If* $\delta_\xi\ c = \delta_{\xi}$, $c = e$, *we get* $h_{i_0i_1}c_{i_1i_2i_3}h_{i_0i_1}{}^{-1} = c_{i_1i_2i_3}$, *where* $\xi = \{g_{ij}\}$ *and* $\xi' = \{h_{ij}g_{ij}\}$.

(iii)′ *If* $\delta_\xi\ c = \delta_\xi$, c, *we get* $h_{i_0i_1}g_{i_0i_1}c_{i_1i_2i_3}g_{i_0i_1}{}^{-1}h_{i_0i_1}{}^{-1} = g_{i_0i_1}c_{i_1i_2i_3}g_{i_0i_1}{}^{-1}$, *where* $\xi = \{g_{ij}\}$ *and* $\xi' = \{h_{ij}g_{ij}\}$.

(iv) *If* $c = \delta_\xi$, *that is,* $c_{i_0i_1i_2} = g_{i_0i_1}g_{i_1i_2}g_{i_0i_2}{}^{-1}$, *then* $\delta_\xi(\delta\xi) = e$.

(v) *Define the action of* $h = \{h_i\} \in C^0(\mathfrak{U}, G_d)$ *on* $C^2(\mathfrak{U}, G_t)$ *(on* $C^2(\mathfrak{U}, G_d)$*) by* $h(c) = h_{i_0}c_{i_0i_1i_2}h_{i_0}{}^{-1}$, *we get*

$$\delta_{h(\xi)}(h(c)) = h(\delta_\xi(c)), \quad h(a) = h_{i_0}a_{i_0i_1i_2i_3}h_{i_0}{}^{-1}.$$

Example. If $\delta_\xi c = e$ and $\xi = h_ih_j{}^{-1}$, then $c' = h^{-1}(c)$ satisfies

$$c'_{i_1i_2i_3}c'_{i_0i_1i_3}c'_{i_0i_2i_3}{}^{-1}c'_{i_0i_1i_2} = e.$$

Definition 20. *Let c be an element of* $C^2(\mathfrak{U}, G_t)$. *Then* $h \in C^0(\mathfrak{U}, G_d)$ *is called c-admissible if* $h(c)$ *belongs to* $C^2(\mathfrak{U}, G_t)$.

Lemma 4. *Let* $\delta_\xi c = e$, $\xi = \{g_{ij}\}$ *and assume* $\xi' = \{a_{ij}g_{ij}\}$ *is in* $C^1_a(\mathfrak{U}, G_d)$. *Then* *to set*

(19) $$c'_{i_0i_1i_2}=a_{i_0i_1}g_{i_0i_1}a_{i_1i_2}g_{i_0i_1}^{-1}c_{i_0i_1i_2}a_{i_0i_2}^{-1},$$

$_{\xi'}c'=\mathrm{e}$ *if and only if* $a=\{a_{ij}\}$ *satisfies*

(20) $$(g_{i_0i_1}^{-1}\delta c_{i_0i_1i_2})\,(g_{i_0i_2}a_{i_2i_3}g_{i_0i_2}^{-1})=(g_{i_1i_2}a_{i_2i_3}g_{i_1i_2}^{-1})\;g_{i_0i_1}^{-}c_{i_0i_1i_2}).$$

Proof. By (18), $\delta_{\xi'}\ c'=\mathrm{e}$ if and only if

(18)' $$c'_{i_0i_1i_2}c'_{i_0i_2i_3}=a_{i_0i_1}g_{i_0i_1}c'_{i_1i_2i_3}g_{i_0i_1}^{-1}a_{i_0i_1}^{-1}c'_{i_0i_1i_3}.$$

By (19), we get

$$c'_{i_0i_1i_2}c'_{i_0i_2i_3}=a_{i_0i_1}g_{i_0i_1}a_{i_1i_2}g_{i_0i_1}^{-1}c_{i_0i_1i_2}a_{i_0i_2}^{-1}a_{i_0i_2}g_{i_0i_2}\ a_{i_2i_3}g_{i_0i_2}^{-1}c_{i_0i_2i_3}a_{i_0i_3}^{-1},$$

$$a_{i_0i_1}g_{i_0i_1}c'_{i_1i_2i_3}g_{i_0i_1}^{-1}a_{i_0i_1}^{-1}c'_{i_0i_1i_3}=a_{i_0i_1}g_{i_0i_1}a_{i_1i_2}g_{i_1i_2}\cdot\ a_{i_2i_3}g_{i_1i_2}^{-1}c_{i_1i_2i_3}a_{i_1i_3}^{-1}g_{i_0i_1}^{-1}$$
$$a_{i_0i_1}^{-1}a_{i_0i_1}g_{i_0i_1}a_{i_1i_3}g_{i_0i_1}^{-1}\ c_{i_0i_1i_3}a_{i_0i_3}^{-1}.$$

Hence (18)' follows from

(18)'' $$g_{i_0i_1}^{-1}c_{i_0i_1i_2i_0i_2}a_{i_2i_3}g_{i_0i_2}^{-1}c_{i_0i_2i_3}=g_{i_1i_2}a_{i_2i_3}g_{i_1i_2}^{-1}c_{i_1i_2i_3}g_{i_0i_1}^{-1}c_{i_0i_1i}g_3.$$

But since $g_{i_0}^{-1}c_{i_0i_1i_3}=c_{i_1i_2i_3}^{-1}g_{i_0i_1}^{-1}c_{i_0i_1i_2}c_{i_0i_2i_3}$ by (18), (18)'' holds if and only if the equality (20) holds. Hence we have Lemma.

Note. If $c=\delta_\xi$, (20) always holds. In fact, we get $(g_{i_0i_1}^{-1}c_{i_0i_1i_2})\,(g_{i_0i_2}a_{i_2i_3}g_{i_0i_2}^{-1})$ $=(g_{i_1i_2}a_{i_2i_3}g_{i_1i_2}^{-1})\ (g_{i_0i_1}^{-1}c_{i_0i_1i_2})=g_{i_1i_2}a_{i_2i_3}g_{i_0i_2}^{-1}$.

6. Definition 21. *We set*

$$\mathrm{Z}^2(\mathfrak{U},\ \mathrm{G}_t)=\{c\in\mathrm{C}^2(\mathfrak{U},\ \mathrm{G}_t)\,|\,\delta_\xi c=\mathrm{e}\ \textit{for some}\ \xi\in\mathrm{C}^1{}_c(\mathfrak{U},\ \mathrm{G}_d)\}.$$

$$\mathrm{Z}^2(\mathfrak{U},\ \mathrm{G}_d)=\{c\in\mathrm{C}^2(\mathfrak{U},\ \mathrm{G}_d)\,|\,\delta_\xi c=\mathrm{e}\ \textit{for some}\ \xi\in\mathrm{C}^1{}_c(\mathfrak{U},\ \mathrm{G}_d)\}.$$

Note 1. The definitions of $\mathrm{Z}^2(\mathfrak{U},\ \mathrm{G}_t)$ and $\mathrm{Z}^2(\mathfrak{U},\ \mathrm{G}_d)$ depend on the domain of ξ. From this point of view, the notations $\mathrm{Z}^2(\mathfrak{U},\ \mathrm{G}_t)_{\mathrm{G}_d}$ and $\mathrm{Z}^2(\mathfrak{U},\ \mathrm{G}_d)_{\mathrm{G}_d}$ are more exact. But we do not use these notations.

Note 2. In [8], we only assume the domain of ξ to be $\mathrm{C}^1{}_a(\mathfrak{U},\ \mathrm{G}_d)$. But the research on 3-dimensional theory suggests above definitions are more convenient.

Definition 22. $c,\ c'\in\mathrm{Z}^2(\mathfrak{U},\ \mathrm{G}_t)$ *(resp.* $\mathrm{Z}^2(\mathfrak{U},\ \mathrm{G}_d)$*) are said to be cohomologous and denoted in symbols* $c\sim c'$ *if there exists* $a\in\mathrm{C}^1(\mathfrak{U},\ \mathrm{G}_t)$ *(resp.* $\mathrm{C}^1(\mathfrak{U},\ \mathrm{G}_d)$*) such that* $\xi'=\{a_{ij}g_{ij}\}\in\mathrm{C}^1{}_c(\mathfrak{U},\ \mathrm{G}_d)$ *satisfies* (20) *for* c *and* c' *is expressed by* (19). *Here* $\delta_\xi\ c=\mathrm{e}$ *and* $\xi=\{g_{ij}\}$.

Lemma 5. (i) $c\sim c'$ *is an equivalence relation.*

(ii) *If* $c\sim c'$ *and* $h(a)\in\mathrm{C}^1(\mathfrak{U},\ \mathrm{G}_t)$, *then* $h(c)\sim h(c')$.

Proof. By Lemma 4, if $c\sim c'$ and $c\in\mathrm{Z}^2(\mathfrak{U},\ \mathrm{G}_t)$, c' belongs to $\mathrm{Z}^2(\mathfrak{U},\ \mathrm{G}_t)$. By definition, $c\sim c$. If $c\sim c'$, we get

$$c_{i_0i_1i_2}=g_{i_0i_1}a_{i_1i_2}^{-1}g_{i_0i_1}^{-1}a_{i_0i_1}^{-1}c'_{i_0i_1i_2}a_{i_0i_2}$$
$$=a_{i_0i_1}^{-1}(a_{i_0i_1}g_{i_0i_1})a_{i_1i_2}^{-1}(a_{i_0i_1}g_{i_0i_1})^{-}c'_{i_0i_1i_2}(a_{i_0i_2}^{-1})^{-1},$$

$$((a_{i_0i_1}g_{i_0i_1})^{-1}c'_{i_0i_1i_2})\ ((a_{i_0i_2}g_{i_0i_2})a_{i_2i_3}{}^{-1}(a_{i_0i_2}g_{i_0i_2})^{-1})$$
$$= a_{i_1i_2}(g_{i_0i_1}{}^{-1}c_{i_0i_1i_2}g_{i_0i_2})\ (a_{i_2i_3}{}^{-1}g_{i_0i_2}{}^{-1})a_{i_0i_2}{}^{-1},$$
$$((a_{i_1i_2}g_{i_1i_2})a_{i_2i_3}{}^{-1}(a_{i_1i_2}g_{i_1i_2})^{-1})\ ((a_{i_0i_1}g_{i_0i_1})^{-1}c'_{i_0i_1i_2})$$
$$=a_{i_1i_2}(g_{i_1i_2}a_{i_2i_3}{}^{-1}g_{i_1i_2}{}^{-1})\ (g_{i_0i_1}{}^{-1}c_{i_0i_1i_2})a_{i_0i_2})a_{i_0i_2}{}^{-1}.$$

Hence we have $c'\sim c$. If $c\sim c'$ and $c'\sim c''$, we set

$$c''_{i_0i_1i_2}=b_{i_0i_1}(a_{i_0i_1}g_{i_0i_1})b_{i_1i_2}(a_{i_0i_1}g_{i_0i_1})^{-1}c'_{i_0i_1i_2}b_{i_0i_2}{}^{-1}.$$

Then we get

$$c''_{i_0i_1i_2} = b_{i_0i_1}(a_{i_0i_1}g_{i_0i_1})b_{i_1i_2}(a_{i_0i_1}g_{i_0i_1})^{-1}a_{i_0i_1}g_{i_0i_1}{}^{-1}c_{i_0i_1i_2}\ a_{i_0i_2}{}^{-1}b_{i_0i_2}{}^{-1}$$
$$=(b_{i_0i_1}a_{i_0i_1})g_{i_0i_1}(b_{i_1i_2}a_{i_1i_2})g_{i_0i_1}{}^{-1}c_{i_0i_1i_2}\ (b_{i_0i_2}a_{i_0i_2})^{-1}.$$

On the other hand, since

$$(g'_{i_0i_1}{}^{-1}c'_{i_0i_1i_2})\ (g'_{i_0i_2}b_{i_2i_3}\ g'_{i_0i_2}{}^{-1})=(g'_{i_1i_2}b_{i_2i_3}g'_{i_1i_2}{}^{-1})\ (g'_{i_0i_1}c'_{i_0i_1i_2}),\quad g'_{ij}=a_{ij}g_{ij},$$

we get

$$a_{i_1i_2}g_{i_0i_1}{}^{-1}c_{i_0i_1i_2}g_{i_0i_2}b_{i_2i_3}g_{i_0i_2}{}^{-1}a_{i_0i_2}{}^{-1} = a_{i_1i_2}g_{i_1i_2}b_{i_2i_3}g_{i_1i_2}{}^{-1}g_{i_0i_1}{}^{-1}c_{i_0i_1i_2}a_{i_0i_2}{}^{-1}.$$

Hence we have

$$(g_{i_0i_1}{}^{-1}c_{i_0i_1i_2})\ (g_{i_0i_2}b_{i_2i_3}g_{i_0i_2}{}^{-1}) = (g_{i_1i_2}b_{i_2i_3}g_{i_1i_2}{}^{-1})\ (g_{i_0i_1}{}^{-1}c_{i_0i_1i_2}).$$

Therefore we obtain

$$(g_{i_0i_1}{}^{-1}c_{i_0i_1i_2})\ (g_{i_0i_2}b_{i_2i_3}a_{i_2i_3}g_{i_0i_2}{}^{-1})$$
$$=(g_{i_0i_1}{}^{-1}c_{i_0i_1i_2})\ (g_{i_0i_2}b_{i_2i_3}g_{i_0i_2}{}^{-1})\ (g_{i_0i_2}a_{i_2i_3}g_{i_0i_2}{}^{-1})$$
$$=(g_{i_1i_2}b_{i_2i_3}g_{i_1i_2}{}^{-1})(g_{i_0i_1}{}^{-1}c_{i_0i_1i_2})\ (g_{i_0i_1}a_{i_2i_3}g_{i_0i_2}{}^{-1})$$
$$=g_{i_1i_2}b_{i_2i_3}g_{i_1i_2}{}^{-1})\ (g_{i_1i_2}a_{i_2i_3}g_{i_1i_2}{}^{-1})\ (g_{i_0i_1}{}^{-1}c_{i_0i_1i_2})$$
$$=(g_{i_1i_2}b_{i_2i_3}g_{i_1i_2}{}^{-1})\ (g_{i_0i_1}c_{i_0i_1i_2}).$$

Hence $c''\sim c$, Therefore we have (i). (ii) follows from the following calculation:

$$(h_{i_0}a_{i_0i_1}h_{i_0}{}^{-1})\ (h_{i_0}g_{i_0i_1}h_{i_1}{}^{-1})\ (h_{i_1}a_{i_1i_2}h_{i_1}{}^{-1})\ (h_{i_0}c_{i_0i_1i_2}h_{i_0}{}^{-1}).\ (h_{i_0}a_{i_0i_2}{}^{-1}h_{i_0}{}^{-1})$$
$$=h_{i_0}(a_{i_0i_1}g_{i_0i_1}a_{i_1i_2}g_{i_0i_1}{}^{-1}c_{i_0i_1i_2}a_{i_0i_2}{}^{-1})h_{i_0}{}^{-1}$$
$$((h_{i_1}g_{i_1i_0}h_{i_0}{}^{-1})\ (h_{i_0}c_{i_0i_1i_2}h_{i_0}{}^{-1}))\ ((h_{i_0}g_{i_0i_2}h_{i_2}{}^{-1})\ (h_{i_2}a_{i_2i_3}h_{i_2}{}^{-1}).\ (h_{i_2}g_{i_2i_0}h_{i_0}{}^{-1}))$$
$$=h_{i_1}((g_{i_0i_1}{}^{-1}c_{i_0i_1i_2})\ (g_{i_0i_2}a_{i_2i_3}g_{i_0i_2}{}^{-1}))h_{i_0}{}^{-1},$$
$$((h_{i_1}g_{i_1i_2}h_{i_2}{}^{-1})\ (h_{i_2}a_{i_2i_3}h^{!}{}_{2}{}^{-1})\ (h_{i_2}g_{i_2i_1}h_{i_1}{}^{-1}))\ ((h_{i_1}g_{i_1i_0}h_{i_0}{}^{-1}).\ (h_{i_0}c_{i_0i_1i_2}h_{i_0}{}^{-1}))$$
$$=h_{i_1}((g_{i_1i_2}a_{i_2i_3}g_{i_1i_2}{}^{-1})\ (g_{i_0i_1}{}^{-1}c_{i_0i_1i_2}))h_{i_0}{}^{-1}.$$

Definition 23. *We denote* $\mathrm{H}^2(\mathfrak{U},\ \mathrm{G}_t)$ *(resp.* $\mathrm{H}^2(\mathfrak{U},\ \mathrm{G}_d)$*) the quotient set of* $\mathrm{Z}^2(\mathfrak{U},\ \mathrm{G}_t)$ *(resp.* $\mathrm{Z}^2(\mathfrak{U},\ \mathrm{G}_d)$*) by the relation of being cohomologous and admissible actions of the elements of* $\mathrm{C}^0(\mathfrak{U},\ \mathrm{G}_d)$.

Let $\mathfrak{V}=\{V_j\,|\,j\in \mathbf{J}\}$ be a refinement of $\mathfrak{U}=\{U_i\,|\,i\in \mathbf{I}\}$ and let $\tau,\ \tau':\mathbf{J}>\mathbf{I}$ the maps suchthat $V_j\subset U_{\tau(j)}\wedge U_{\tau'(j)}$. We denote $\tau(j)=i$ and $\tau'(j)=i'$. We set

$$a_{i_0i_1} = c_{i_0i_0'i_1'}c_{i_0i_1i_1'}^{-1}.$$

Then we get

$$a_{i_0i_1}g_{i_0i_1}a_{i_1i_2}g_{i_0i_1}^{-1}c_{i_0i_1i_2}a_{i_0i_2}^{-1}$$
$$= c_{i_0i_0'i_1'}c_{i_0i_1i_1'}^{-1}g_{i_0i_1}\ c_{i_1i_1'i_2'}c_{i_1i_2i_2'}^{-1}g_{i_0i_1}^{-1}c_{i_0i_1i_2}c_{i_0i_2i_2'}c_{i_0i_0'i_2'}^{-1}.$$

Since $c \in Z^2(\mathfrak{U},\ G_{(t)})$ or $\in Z^2(\mathfrak{U},\ G_d)$, we have

$$g_{i_0i_1}^{-1}c_{i_0i_1i_2} = c_{i_1i_2i_2'}g_{i_0i_1}^{-1}c_{i_0i_1i_2'}c_{i_0i_2i_2'},\quad c_{i_0i_1'i_2'} = c_{i_0i_1i_1'}^{-1}g_{i_0i_1}c_{i_1i_1'i_2'}g_{i_0i_1}^{-}c_{i_0{}^{!}{}_1i_2'}.$$

Hence we have

$$c_{i_0i_0'i_1'}c_{i_0i_1i_1'}^{-1}g_{i_0i_1}c_{i_1i_1'i_2'}c_{i_1i_2i_2'}^{-1}g_{i_0i_1}^{-}c_{i_0i_1i_2}c_{i_0i_2i_2'}\ c_{i_0i_0'i_2'}^{-1}$$
$$= c_{i_0i_0'i_1'}c_{i_0i_1i_1'}^{-1}g_{i_0i_1}c_{i_1i_1'i_2'}g_{i_0i_1}^{-1}c_{i_0i_1i_2'}\ c_{i_0i_0'i_2'}^{-1}$$
$$= c_{i_0i_0'i_1'}c_{i_0i_1'i_2'}c_{i_0i_0'i_2'}^{-1}.$$

There fore we have

$$a_{i_0i_1}g_{i_0i_1}a_{i_1i_2}g_{i_0i_1}^{-1}c_{i_0i_1i_2}a_{i_0i_2}^{-1} = g_{i_0i_0'}c_{i_0'i_1'i_2'}g_{i_0i_0'}^{-1}.$$

Since $g=\{g_{ii'}\}$ is c–admissible by Lemma 3 (ii)$'$, this shows $\tau^*(c)$ and $\tau'^*(c)$ give the same element in $H^2(M,\ G_t)$ (or $H^2(M,\ G_d)$).

Definition 24. *We set lim* $\{H^2(\mathfrak{U},\ G_t)|\tau = H^2(M,\ G_t)\}$ *and lim* $\{H^2(\mathfrak{U},\ G_d)|\tau = H^2(M,\ G_d)\}$.

Note. These cohomology sets are not defined absolutely from the sheaves G_t and G_d. Therefore the notations $H^2(M,\ G_t)_{G_d}$ and $H^2(M,\ G_d)_{G_d}$ are more exact. But we do not use these notations.

7. **Theorem 2.** *There are maps* $\delta=\delta_0 : H^0(M,\ \mathscr{M}^1)\to H^1(M,\ G_t)$ *and* $\delta=\delta_1 : H^1(M,\ \mathscr{M}^1)\to H^2(M,\ G_t)$ *and the following sequence is exact.*

$$(21)\qquad 0\longrightarrow H^0(M,\ G_t)\xrightarrow{i} H^0(M,\ G_d)\xrightarrow{\rho^*} H^0(M,\ \mathscr{M}^1)\xrightarrow{\delta} H^1(M,\ G_t)\xrightarrow{i^*}$$
$$\longrightarrow H^1(M,\ G_d)\xrightarrow{\rho^*} H^1(M,\ \mathscr{M}^1)\xrightarrow{\delta} H^2(M,\ G_t)\xrightarrow{i^*} H^2(M,\ G_d).$$

Proof. The exactness of the first six terms of (21) has been known (cf. [3], [15], [21]). δ_0 is defined by

$$(22)_0\qquad \delta_0(\theta) = \{h_U h_V^{-1}\},\quad h_U^{-1}dh_U = \theta|U,\quad \theta\in H^0(M,\ \mathscr{M}^1).$$

We define δ_1 by

$$(22)_1\qquad \delta_1(\langle\omega_{ij}\rangle) = \{g_{ij}g_{jk}g_{ik}^{-1}\},\quad \omega_{ij} = g_{ij}^{-1}dg_{ij}.$$

By Proposition 1, (i) and Lemma 3, (iv), $c_{ijk} = g_{ij}g_{jk}g_{ik}^{-1}$ belongs to $Z^2(\mathfrak{U},\ G_t)$. If $\omega_{ij} = \rho(g_{ij}) = \rho(g_{ij}')$, set $g_{ij} = a_{ij}g_{ij}$, a_{ij} is a constant map. We set $c_{ijk}' = g_{ij}'g_{ki}'$. Then we have

$$c_{i_0i_1i_2}' = a_{i_0i_1}g_{i_0i_1}a_{i_1i_2}g_{i_0i_1}^{-1}c_{i_0i_1i_2}a_{i_0i_2}^{-1}.$$

Since (20) holds by the note at the end of Lemma 4, $\{c_{ijk}'\}$ is cohomologous to $\{c_{ijk}\}$. Hence δ_1 is well defined by Lemma 5, (ii). Then we have $ker\delta_1 = \{\omega_{ij} \mid \omega_{ij} = \rho(g_{ij}),\ g_{ij}g_{jk}g_{ki} = e\} = image\ \rho^*$. If $i^*(c)$ vanishes, we have

$$c_{ijk} = a_{ij}g_{ij}a_{ik}g_{ij}^{-1}a_{ik}^{-1} = b_{ij}g_{jk}g_{kj}g_{ji}g_{ki}^{-1}b_{ik}^{-1},\quad b_{ij} = a_{ij}g_{ij}.$$

Hence *ker* i^* is contained in *image* δ. Since *image* δ is contained in *ker* i^* by definition, we have Theorem 2.

Note. If M is a complex manifold, we can define the cohomology sets $H^2(M, G_t)_\omega = H^2(M, G_t)_{G_\omega}$ and $H^2(M, G_\omega) = H^2(M, G_\omega)_{G_\omega}$ by the same way. Then we get the exact sequence

$$0 \longrightarrow H^0(M, G_t) \longrightarrow H^0(M, G_\omega) \longrightarrow H^0(M, \mathscr{M}^1{}_\omega) \longrightarrow H^1(M, G_t) \longrightarrow H^1(M, G_\omega) \longrightarrow$$
$$\longrightarrow H^1(M, \mathscr{M}^1{}_\omega) \longrightarrow H^2(M, G_t)_\omega \longrightarrow H^2(M, G_\omega)_\omega.$$

Here, $H^2(M, G_t)_\omega$ may differ from $H^2(M, G_t)$.

If $n=1$, that is $G = C^*$, $\mathscr{M}^1$ becomes Φ^1, the sheaf of germs of closed 1-forms over M and we have $H^1(M, \mathscr{M}^1) = H^2(M, C)$, $H^1(M, G_d) = H^2(M, \mathbf{Z})$ and $H^2(M, G_d) = H^3(M, \mathbf{Z})$. The exact sequence (21) is rewritten as the following exact sequence

$$0 \longrightarrow H^0(M, C^*) \longrightarrow H^0(M, C^*_d) \longrightarrow H^0(M, \phi^1) \longrightarrow H^1(M, C^*) \longrightarrow H^2(M, \mathbf{Z}) \longrightarrow$$
$$\longrightarrow H^2(M, C) \xrightarrow{exp^*} H^2(M, C^*) \longrightarrow H^3(M, \mathbf{Z}).$$

This comes from the commutativity of the following diagram of sheaves

$$\begin{tikzcd}
 & 0 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C^*_t \arrow[r] \arrow[u] & C^*_d \arrow[r] & \mathscr{M}^1 = \phi^1 \arrow[r] & 0 \\
0 \arrow[r] & C_t \arrow[r] \arrow[u, "exp"] & C_d \arrow[r] \arrow[u, "exp"] \arrow[ur, phantom, "d"] & \theta^1 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathbf{Z} \arrow[r] \arrow[u] & \mathbf{Z} \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

The corresponding diagramfor $GL(n, C)$, $n \geqq 2$, takes the following form

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & G_t \arrow[r] \arrow[u] & G_d \arrow[r] \arrow[u] & \mathscr{M}^1 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathfrak{g}_{0,d} \arrow[r] \arrow[u, "exp"] & \mathfrak{g}_d \arrow[r] \arrow[u, "exp"] \arrow[ur, phantom, "d^e"] & \mathscr{M}^1 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & N_{,d} \arrow[r] \arrow[u] & N_{,d} \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Here $\mathfrak{g}_{0,d}$ and $N_{\ ,d}$ are defined as kernel sheaves of the maps d^e and exp. Detailed definitions of this diagram and related cohomology sets are given in [9].

8. We denote the natural map from $Z^0{}_d(M, \mathscr{M}^2)$ onto $H^2{}_{dR}(M, \mathfrak{g})$ by dR. Then $\delta_t\ dR$ maps $Z^0{}_d(M, \mathscr{M}^2)$ into $H^2(M, G_t)$. By Theorems 1, 2, we have

Proposition 2. *$\Theta=\{\Theta_i\}\in Z^0{}_d(M, \mathscr{M}^2)$ is realized as a curvature form of a* G*-bundle over M if and only if $\delta_t\ dR(\Theta)=0$.*

Corollary. *Let $\{M_G, \pi, M\}$ be a principal* G*-bundle over M and θ a matrix valued 1-form. Then $\Theta\in Z^0{}_d(M, \mathscr{M}^2)$ can be written as*

$$\pi^*(\Theta)=d\theta+\theta\wedge\theta, \tag{23}$$

if and only if $\delta_t\ dR(\Theta)=0$. Here M_G depends on Θ.

Proof. If (23) holds, Θ is realized as a curvature form. Hence we have the necessity. If Θ is a curvature form of $\xi=\{g_{ij}\}$ with the associated principal bundle $\{M_G, \pi, M\}$, we have

$$\pi^*(\Theta)\,|\,\pi^{-1}(U_i)=h_i{}^{-1}\theta_i h_i, \quad \pi^*(g_{ij})=h_i h_j{}^{-1}.$$

Since Θ is a curvature form, we can set $\Theta_i=d\theta_i'+\theta_i'\wedge\theta_i'$. Then we have θ to set

$$\theta\,|\,\pi^{-1}(U_i)=h_i{}^{-1}(\theta_i'+dh_i h_i{}^{-1})h_i.$$

Note 1. Usual de Rham groups are the obstructions of global solvabilities of the equations $d\varphi=\phi$, $deg\phi=1, 2, \ldots,$ From this point of view, $H^1(M, \mathscr{M}^1)$ is the obstruction of global solvability of the equation

$$d^e\theta=\Theta,\ \ \Theta=\{\Theta_i\},\ \ \theta=\{\theta_i\},\ \ \theta_j=g_{ji}(\theta_i),\ \ \Theta_j=g_{ji}(\Theta_i)$$

where $\{g_{ij}\}$ belongs to $C^1{}_c(U, G_d)$.

Note 2. If $d^e\theta=d^e\theta'$, set $\theta'=\theta+\eta$, η satisfies the equation

$$d\eta+\eta\wedge\eta+[\theta, \eta]=0. \tag{24}$$

Local solutions of this equation and the relation between gauge transformations are studied in [10] cf. [20].

§3. Characteristic classes for the elemens of $H^1(M, \mathscr{M}^1)$.

9. **Lemma 6.** *Let $\Theta=\{\Theta_i\}$ be an elemen of $Z^0d(M, \mathscr{M}^2)$. Then $tr(\overbrace{\Theta\wedge\cdots\wedge\Theta}^{p})$ is a closed $2p$-form on M and its cohomology class is determined by $dR(\Theta)$ for any p.*

Proof. Since $\Theta_j=g_{ij}{}^{-1}\Theta_i g_{ij}$, we get $tr(\Theta_j\wedge\cdots\wedge\Theta_j)=tr(g_{ij}{}^{-1}\Theta_i\wedge\cdots\wedge\Theta_i g_{ij})=tr(\Theta_i\wedge\cdots\wedge\Theta_i)$. Hence $tr(\Theta\wedge\cdots\wedge\Theta)$ defines a global form on M. Since $\Theta_i=d^e\theta_i$, we have the Bianchi identity $d\Theta=[\Theta, \theta]$. Then, since

$$tr(\overbrace{\Theta\wedge\cdots\Theta}^{r-1}\wedge[\theta, \Theta]\wedge\overbrace{\Theta\wedge\cdots\wedge\Theta}^{p-r})=0,$$

we get

$$d(tr(\Theta\wedge\overset{p}{\cdots}\wedge\Theta))=\sum_{r=1}^{p}tr(\Theta\wedge\overset{r-1}{\cdots}\Theta\wedge d\Theta\wedge\Theta\wedge\overset{p-r}{\cdots}\wedge\Theta)=0.$$

Hence $tr(\Theta\wedge\cdots\wedge\Theta)$ is closed. If $\rho(g_{ij})=\theta_j-g_{ij}^{-1}\theta_i g_{ij}=\theta_j'-g_{ij}^{-1}\theta_i' g_{ij}$, to set $\theta_i'=\theta_i+\eta_i$, we have

$$(25)\qquad \theta_j=g_{ij}^{-}\theta_i g_{ij},$$

$$(25)'\qquad d^e(\theta_i')=\Theta_i+d\theta_i+\eta_i\wedge\eta_i+[\theta_i,\ \eta_i].$$

Then, since $tr\ (\eta_i\wedge\eta_i)=tr\ [\theta_i,\ \eta_i]=0$, we get

$$tr(d^e(\theta_i')\wedge\cdots\wedge d^e(\theta_i'))=tr((\theta_i+d\eta_i)\wedge\cdots\wedge(\theta_i+d\eta_i)).$$

By (25), to set $\phi_{i,1}=\theta_i$ and $\phi_{i,-1}=d\eta_i$, we have

$$tr(\phi_j,\ \varepsilon_1\wedge\cdots\phi_{j,\varepsilon_p})=tr(\phi_{i,\varepsilon_1}\wedge\cdots\wedge\phi_{i,\varepsilon_p}),\quad \varepsilon_k=\pm 1.$$

Then, since

$$tr(\Theta_i\wedge\overset{r-1}{\cdots}\wedge\Theta_i\wedge d\eta_i\wedge\phi_{i,\varepsilon_{r+1}}\wedge\cdots\wedge\phi_{i,\varepsilon_p})$$
$$=d(tr(\Theta_i\wedge\overset{r-1}{\cdots}\wedge\Theta_i\wedge\phi_{i,\varepsilon_{r+1}}\wedge\cdots\wedge\phi_{i,\varepsilon_p})),\quad r\geqq 1,$$

$tr(d^e(\theta_i')\wedge\overset{p}{\cdots}\wedge d^e(\theta_i'))$ is cohomologous to $tr(\Theta\wedge\overset{p}{\cdots}\wedge\Theta)$. Hence we have Lemma.

Corollary. *If $\Theta\in Z^0{}_d(M,\ \mathscr{M}^2)$, the coefficients of det $(\mathrm{I}+(t/2\pi\sqrt{-1})\Theta)$ are closed anp their de Rham classes are determined by $\iota\ dr(\Theta)$.*

Definition 25. *Let $\langle\omega\rangle$ be an element of $H^1(M,\ \mathscr{M}^1)$, Θ an element of $Z^0{}_d(M,\ \mathscr{M}^2)$ such that $\iota\ dR(\Theta)=\langle\omega\rangle$. Then we define the p-th characteristic class $c^p(\langle\omega\rangle)$ of $\langle\omega\rangle$ by*

$$c^p(\langle\omega\rangle)=\langle\varphi_p\rangle,\ \textit{the de Rham class of}\ \varphi_p,$$

$$det\,(\mathrm{I}+\frac{t}{2\pi\sqrt{-1}}\Theta)=\mathrm{I}+\varphi_1 t+\cdots+\varphi_p t^p+\cdots+\varphi_n t^n.$$

Similarly, we define the p-th Chern character of $\langle\omega\rangle$ by

$$ch\ (\langle\omega\rangle)=\left(\frac{\sqrt{-1}}{2\pi}\right)^p\frac{1}{p!}rt\ (\Theta\wedge\overset{p}{\cdots}\wedge\Theta).$$

Proposition 3. (i) *$c^p(\langle\omega\rangle)$ and $ch^p(\langle\omega\rangle)$ aredetermined by $\langle\omega\rangle$.*

(ii) *Let ξ be a G-bundle with the p-th Chern class $c^p(\xi)$ and the p-th Chern character $ch^p(\xi)$, $i^*:H^{2p}M,\ \mathbf{Z})\to H^{2p}(M,\ \mathbf{C})$ the map induced from the in(clusion $i:\mathbf{Z}\to\mathbf{C}$. Then we have*

$$(26)\qquad c^p(\rho^*(\xi))=i^*(c^p(\xi)),$$

$$(26)' \qquad ch^p(\rho^*(\xi)) = ch^p(\xi).$$

(iii) *If* $\omega=\{\omega_{ij}\}$ *and each* ω_{ij} *is an antihermitian matrix valued* 1*-form,* $c^p(\langle\omega\rangle)$ *and* $ch^p(\langle\omega\rangle)$ *belong to* $\mathrm{H}^{2p}(M, \mathbf{R})$.

Proof. (i) follows from Lemma 6 and its Corollary. (ii) follows from the theorem of Chern ([12]) and the definition of the Chern character. If ω satisfies the assumption of (iii), there exists an antihermitian matrix valued $\Theta\in \mathrm{Z}^0{}_d(M, \mathscr{M}^2)$ such that $\iota dR(\Theta)=\langle\omega\rangle$. Then, since $\det(\mathbf{I}+(t/2\pi\sqrt{-1})\Theta)$ is a real form coefficients polynomial, we have (iii).

Corollary. $\Theta\in \mathrm{Z}^0{}_d(M, \mathscr{M}^2)$ *can not be realized as a curvature form of a* G*-bundle over* M *if* $c^p(\Theta)$ *is not an integral class for some* p.

Note. For an element θ of $\mathrm{H}^0(M, \mathscr{M}^1)$, we have defined its characteristic class $\beta^p(\theta)\in \mathrm{H}^{2p-1}(M, \mathrm{C})$ by

$$\beta^p(\theta) = \textit{the de Rham class of} \frac{(-1)^{p-1}}{(2\pi\sqrt{-1})^p}\, tr\, (\theta\wedge\overset{2p-1}{\cdots}\wedge\theta),$$

([6], cf. [27]). The definitions of $\beta^p(\theta)$ and $ch^p(\Theta)$ are parallel. Moreover, $\beta^p(\theta)$ comes from the p-th generator of $\mathrm{H}^*(\mathrm{G}, \mathbf{Z})$ and $c^p(\Theta)$ comes from the p-th generator of $\mathrm{H}^*(\mathrm{BG}, \mathbf{Z})$.

10. Example 1. If λ is a complex number, then we have

$$d^e(\lambda\theta) = \lambda d^e\theta + (\lambda^2-\lambda)\theta\wedge\theta.$$

Hence we have

$$(27)' \qquad \mathrm{tr}(d^e(\lambda\theta)\wedge\overset{p}{\cdots}\wedge d^e(\lambda\theta)) = \lambda^p tr(d^e\,\theta\wedge\overset{p}{\cdots}\wedge d^e\theta).$$

Therefore, if $det\,(\mathbf{I}+(t/2\pi\sqrt{-1})d^e\theta)=\mathbf{I}+\varphi_1 t+\cdots+\varphi_n t^n$, we have

$$(27) \qquad det\,\left(\mathbf{I}+\frac{t}{2\pi\sqrt{-1}}d^e\,(\lambda\theta)\right) = \mathbf{I}+\lambda\varphi_1 t+\cdots+\lambda^p\varphi_p t^p+\cdots+\lambda^n\varphi_n t^n.$$

On the other hand, if g is a smooth G-valued function such that

$$(28) \qquad (dg)g = g(dg),$$

and g^λ is defined, then

$$(29) \qquad \rho(g^\lambda) = \lambda\rho(g).$$

Hence if $\xi=\{g_{ij}\}\in \mathrm{H}^1(M, \mathrm{G}_d)$ satisfies (28) for any g_{ij} and for a fixed complex number λ, $g_{ij}{}^\lambda$ is defined for each i, j, we get

$$\rho(g_{ij}{}^\lambda) = (\lambda\theta_j) - g_{ij}{}^{-1}(\lambda\theta_i)g_{ij},\ \ \theta_j - g_{ij}{}^{-1}\theta_i g_{ij} = \rho(g_{ij}).$$

Therefore, if each θ_i satisfies $g_{ij}{}^{-\lambda}\theta_i g_{ij}\lambda = g_{ij}{}^{-1}\theta_i g_{ij}$, $\lambda\omega=\{\rho(g_{ij}\lambda)\}$ belongs to $\mathrm{Z}^1(M, \mathscr{M}^1)$. Here $\omega=\{\rho(g_{ij})\}$. Then by (27), we have

$$c^p(\langle\lambda\omega\rangle) = \lambda_p c^p(\langle\omega\rangle).$$

Example 2. Let M be the m-dimensional complex projective space and $\{\mu_1, \dots, \mu_m\}$ an arbitrary set of m complex numbers. We set

$$\omega_{ij} = \begin{pmatrix} \lambda_1\left(\frac{dz_j}{z_j} - \frac{dz_i}{z_i}\right) & & & \\ \cdots & & 0 & \\ & \cdots & & \\ 0 & & \cdots & \\ & & \lambda_m\left(\frac{dz_j}{z_j} - \frac{dz_i}{z_i}\right) \end{pmatrix},$$

$$\prod_{i=1}^{m} (1+\lambda_i t) = 1+\mu_1 t+\cdots+\mu_m t^m.$$

Then $\omega=\{\omega_{ij}\}$ defines an element $\langle\omega\rangle$ of $H^1(M, \mathscr{M}^1)$ such that

$$c^p(\langle\omega\rangle) = \lambda_p e^p, \ e^p \textit{ is the generator of } H^{2p}(M, \mathbf{Z}).$$

Note. Let $U=U(n)$ be the unitary group, $\mathfrak{h}$ its Lie algebra. Then the sequence $0\to U_t\to U_d\to \mathscr{M}_{\mathfrak{h}^1}\to 0$ is exact. Here $\mathscr{M}_{\mathfrak{h}^1}$ is the sheaf of germs of $\mathfrak{h}$-valued integrable connections. By this sequence, we get the following cohomology exact sequence

$$0\longrightarrow H^0(M, U_t)\longrightarrow H^0(M, U_d)\longrightarrow H^0(M, \mathscr{M}_{\mathfrak{h}^1})\longrightarrow H^1(M, U_t)\longrightarrow$$
$$\longrightarrow H^1(M, U_d)\longrightarrow H^1(M, \mathscr{M}_{\mathfrak{h}^1})\longrightarrow H^2(N, U_t)\longrightarrow H^2(M, U_d).$$

In this sequence, we know $H^1(M, G_d)=H^1(M, U_d)$. But example 2 and Proposition 3, (iii) show $H^1(M, \mathscr{M}^1)$ differes from $H^1(M, \mathscr{M}_{\mathfrak{h}^1})$ in general. We also know $H^1(M, G_t)$ differs from $H^1(M, U_t)$. For example, denote σ the generator of $\pi_1(C^*)$, the representation $\chi_\sigma=\begin{bmatrix} 1 & 2\pi\sqrt{-1} \\ 0 & 1 \end{bmatrix}$ defines an element of $H^1(C^*, GL(2, C))$ which is not in $H^1(C^*, U(2))$.

11. If $c=c_{ijk}$ belongs to $Z^2(\mathfrak{U}, G_t)$ resp. to $Z^2(\mathfrak{U}, G_d)$), $det\ c=\{det\ c_{ijk}\}$ belong to $Z^2(\mathfrak{U}, C^*)$ (resp. to $Z^2(\mathfrak{U}, C^*_d)$). Since the cohomology class of $det\ c$ is determined by the cohomology class of c, we define

$$d^1(\langle c\rangle) = \langle det\ c\rangle. \tag{30}$$

By definition, $d^1(\langle c\rangle)\in H^2(M, C^*)$ (resp. $H^2(M, C_{*d})$) and we have

$$exp^*(c^1(\langle\omega\rangle)) = d^1(\delta(\langle\omega\rangle)). \tag{31}$$

To treat higher dimensional chatacteristic classes for $\langle c\rangle$, we set $a_{i_0i_1i_2} = (1/2\pi\sqrt{-1})\log(c_{i_0i_1i_2})$. Then $a = \{a_{i_0i_1i_2}\}$ belongs to $C^2(\mathfrak{U}, \mathfrak{g})$. a is not uniquəly determined from c. But we can construct appropriate cohomology theory with coefficients in $\mathfrak{g}_{0,d}$ and $\mathfrak{g}_d$. To use this theory, if a satisfies

$$tr(\delta a\cup\cdots\cup\delta a)\in Z^{3p}(\mathfrak{U},\ \mathbb{Z}), \tag{30}$$

We can define higher order characteristic classes $d^p(\langle c\rangle)\in H^{3p-1}(M,\ \mathbf{C}^*)$ and $e^p(c)\in HH^{3p}(M,\ \mathbb{Z})$, $c\in H^2(M,\ G_d)$. For details, see [9].

Note. Researches on 3-dimensional theory suggest the possibility of the existence of a matrix valued 3-form (defined relative to gauge and potential) $\Phi=\Phi(c)$ such that

$$i^*(e^p(c))=\textit{the de Rham class of } tr(\overbrace{\phi\wedge\cdots\wedge\phi}^{p}).$$

If this is true, $e^{2p}(c)$ is a torsion class for any p, although it exists as an non zero class.

12. Let Y be a closed subset of M. Then we have the following commutative diagram with exact lines

$$\begin{array}{ccccc} H^1(M,\ G_d) & \xrightarrow{\rho^*} & H^1(M,\ \mathscr{M}^1) & \xrightarrow{\delta} & H^2(M,\ G_t) \\ \downarrow{\scriptstyle i_Y} & & \downarrow{\scriptstyle i_Y} & & \downarrow{\scriptstyle i_Y} \\ H^1(M-Y, G_d) & \xrightarrow{\rho^*} & H^1(M-Y,\ \mathscr{M}^1) & \xrightarrow{\delta} & H^2(M-Y,\ G^t). \end{array}$$

Hence we have

Lemma 7. *Let $\langle\omega\rangle$be an element of $H^1(M,\ \mathscr{M}^1)$. Then $i_Y(\langle\omega\rangle)$ is in ρ^*-image ifand only if $i_Y\delta(\langle\omega\rangle)$ is equal to 0.*

Corollary. *Let $\Theta\in Z^0{}_d(M,\ \mathscr{M}^2)$. On the tatal space of some G-bundle over $M-Y$, Θ is written*

$$\pi^*(\Theta)=d\theta+\theta\wedge\theta, \tag{23$'$}$$

if $i_Y\delta_t dR(\Theta)=0$.

We write r the codimension of Y and assume $r\geqq 2$. We also assume M and Y are cooriented and Y is a smooth submanifold (in the case M is a smooth manifold) or a real analytic subvariety in the case M is a real analytic manifold). Then we have the following commutative diagram with exact lines and columns

$$\begin{array}{ccccccc} & & H^p(M,\ \mathbb{Z}) & \xrightarrow{i_Y} & H^p(M-Y,\ \mathbb{Z}) & & \\ & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \\ \cdots\longrightarrow R^{p-r}(Y) & \xrightarrow{\delta_Y} & H^p(M,\ \mathbf{C}) & \xrightarrow{i_Y} & H^p(M-Y,\ \mathbf{C}) & \xrightarrow{res} & R^{p-r}{}_{+}{}^{1}(Y). \\ & & \downarrow{\scriptstyle exp^*} & & \downarrow{\scriptstyle exp^*} & & \\ & & H^p(M,\ \mathbf{C}^*) & \xrightarrow{i_Y} & H^p(M-Y,\ \mathbf{C}^*) & & \end{array}$$

Here $R^p(Y)=H^p(Y,\ \mathbf{C})$ if Y is topologically non-singular ([4]). By this diagram, we obtain

Lemma 8. *Let c be an element of $\mathrm{H}^p(M, \mathbf{C})$. Then $i_Y(c)$ is in i^*-image if and only if $i_Y exp^*(c) = 0$, and if c is in δ_Y-image, $i_Y exp^*(c) = 0$.*

By Lemma 8, we have

Proposition 4. *Let $\langle\omega\rangle$ be an element of $\mathrm{H}^1(M, \mathscr{M}^1)$ such that $i_Y(\langle\omega\rangle)$ is in ρ^*-image. Then we have*

$$i_Y exp^*(c^p(\langle\omega\rangle)) = 0, \ p \geqq 1. \tag{33}$$

(33) *holds if we have*

$$c^p(\langle\omega\rangle) = \delta_Y(\langle\alpha_p\rangle), \ \langle\alpha_p\rangle \in R^{2p-r}(Y). \tag{34}$$

Corollary. *If $\pi^*(\Theta) = d^e\theta$ on $(M-Y)_{\mathrm{G}'}$ the total space of some principal bundle over $M-Y$, then*

$$i_Y exp^*(c^p(\Theta)) = 0, \ p \geqq 1. \tag{33$'$}$$

(33)$'$ *holds if we have*

$$c^p(\Theta) = \delta_Y(\langle\alpha_p\rangle), \ \langle\alpha_p\rangle \in R^{2p-r}(Y). \tag{34$'$}$$

Note. we denote $\mathrm{G}_d\,[Y]$ and $\mathscr{M}^1\,[Y]$ the sheaves of germs of smooth G-valued functions and integrable connections over M with singularities on Y (cf. [4)]. Then we have the following exact sequences of sheaves

$$\begin{aligned} &0 \longrightarrow \mathrm{G}_t \xrightarrow{i_Y} \mathrm{G}_d\,[Y] \longrightarrow \rho(\mathrm{G}_d\,[Y]) \longrightarrow 0, \\ &0 \longrightarrow \rho(\mathrm{G}_d\,[Y]) \longrightarrow \mathscr{M}^1\,[Y] \longrightarrow Res_{\mathrm{G},Y} \longrightarrow 0. \end{aligned} \tag{35}$$

The stalk $Res_{\mathrm{G},Y,x}$ of $Res_{\mathrm{G},Y}$ at x given by

$$Res_{\mathrm{G},Y,x} = lim\ [\delta(\mathrm{H}^0(U(x) - Y, \ \mathscr{M}^1))].$$

Hence write $\pi_1(U-Y)_x$ the local fundamental group of $U-Y$ at x, $Res_{\mathrm{G},Y,x}$ is contained in Hom $(\pi_1(U-Y)_x, \mathrm{G})$ and therefore $Res_{\mathrm{G},Y} = 0$ if $r \geqq 3$.

By (35), we have the following commutative diagram with exact lines

$$\begin{tikzcd}
\mathrm{H}^1(M, \mathrm{G}_d) \arrow[rr, "\rho^*"] \arrow[dd, "i_Y"'] \arrow[dr, "\iota_Y"] & & \mathrm{H}^1(M, \mathscr{M}^1) \arrow[rr, "\delta"] \arrow[dd, "i_Y"'] \arrow[dr, "\iota_Y"] & & \mathrm{H}^2(M, \mathrm{G}_t) \arrow[dd, "\iota_Y"'] \arrow[dr, "i_Y"] & \\
& \mathrm{H}^1(M, \mathrm{G}_d[Y]) \arrow[rr, "\rho^*"] & & \mathrm{H}^1(M, \rho(\mathrm{G}_d\,[Y])) \arrow[rr] & & \mathrm{H}^2(M, \mathrm{G}_t). \arrow[dl] \\
\mathrm{H}^1(M-Y, \mathrm{G}_d) \arrow[rr, "\rho^*"] \arrow[ur] & & \mathrm{H}^1(M-Y, \mathscr{M}^1) \arrow[rr, "\delta"] & & \mathrm{H}^2(M-Y, \mathrm{G}_t) &
\end{tikzcd}$$

If Y is smooth and r is even, $\mathrm{H}^1(M, \mathrm{G}_d\,[Y]) = \mathrm{H}^1(M-Y, \mathrm{G}_d)$. In this case, $\iota_Y(\langle\omega\rangle)$ is not in ρ^*-image if $\langle\omega\rangle$ is not in ρ^*-image although $i_Y(\langle\omega\rangle)$ is in ρ^*-image. Therefore, if the equivalence of $i_Y(\langle\omega\rangle)$ and an element in ρ^*-image is given by $\{h_i\}$, $\{h_i\}$ is not defined using open covering of $M-Y$ obtained to restrict an open covering of M to $M-Y$. It also suggests $\mathrm{H}^1(M, \mathscr{M}^1\,[Y])$ may differ from $\mathrm{H}^1(M-$

$Y, \mathscr{M}^1)$.

§4. The general case

13. We fix a differential operator $D : C^\infty(M, E) \to C^\infty(M, F)$, where E, F are complex vector bundles over M. We take a G-vector space H. Then $D_i \otimes 1_H : C^\infty(U_i, E \otimes H) \to C^\infty(U_i, F \otimes H)$ is a differential operator on U_i. Here $D = \{D_i\}$ and E and F are both trivial on each U_i. By assumption, a smooth G-valued function g on U_i acts on $C^\infty(U_i, E \otimes H)$ and on $C^\infty(U_i, F \otimes H)$. We set

$$\rho_D(g) = g^{-1}(D_i \otimes 1_H)g - D_i \otimes 1_H.$$

If $\rho_D(g) = 0$, g is called a $c(D)$-class function ([6], [8]). The sheaf of germs of $c(D)$-class G-valued functions is denoted by $G_{c(D)}$. $\rho_D(G_d)$ is denoted by $\mathscr{M}^1{}_D$ (in [6], this sheaf was denoted by $L_{G,D}$). By definitions, we have the following exact sequence

$$(1)_D \qquad 0 \longrightarrow G_{c(D)} \xrightarrow{\ i\ } G_d \xrightarrow{\ D\ } M^1{}_D \longrightarrow 0.$$

By the definition of $\rho_{D'}$ we have

$$\rho_D(gh) = h^{-1}\rho_D(g)h + \rho_D(g), \ \ \rho_D(e) = 0, \ \ e = 1_H.$$

Hence we have

$$\rho_D(g_{ij}g_{jk}g_{ik}{}^{-1}) = g_{ik}(\rho_D(g_{jk}) - \rho_D(g_{ik}) + g_{jk}{}^{-1}\rho_D(g_{ij})g_{jk})g_{ik}{}^{-1}.$$

By this formula, we can define the cohomology sets $H^1(M, \mathscr{M}^1{}_D)$ and $H^2(M, G_{c(D)})$ as follows;

Definitions 11_D and 21_D. *We set*

$$Z^1(\mathfrak{U}, \mathscr{M}^1{}_D) = \{\{\omega_{ij}\} \in C^1(\mathfrak{U}, \mathscr{M}^1{}_D) \mid \delta_\xi \omega = 0,\ \omega = \rho_D(\xi),\ \xi \in C^1{}_a(\mathfrak{U}, G_d)\},$$
$$Z^2(\mathfrak{U}, G_{c(D)}) = \{c \in C^2(\mathfrak{U}, G_{c(D)}) \mid \delta_\xi c = 0, \ \textit{for some}\ \xi \in C^1{}_{c(D)}(\mathfrak{U}, G_d)\}.$$

Here $C^1{}_{c(D)}(\mathfrak{U}, G_d)$ is $\{\xi \in C^1{}_a(\mathfrak{U}, G_d) \mid (\delta\xi)_{ijk}$ *is a $c(D)$-class fuction for anyi, j, k*$\}$.

Definitions 12_D and 22_D. *The cohomologous relations on $Z^1(\mathfrak{U}, \mathscr{M}^1{}_D)$ and on $Z^2(\mathfrak{U}, G_{c(D)})$ are defined as follows*:

$$(15)_D \qquad \{\omega_{ij}'\} \sim \{\omega_{ij}\}\ \ if\ \ \omega_{ij}' = h_j(\omega_{ij} - \rho_D(h_j) + g_{ij}{}^{-1}\rho_D(h_i)g_{ij})h_j{}^{-1},$$
$$\{h_i\} \in C^0(\mathfrak{U}, G_d), \ \ t_{ijk}\rho_D(h_i)t_{ijk} = \rho_D(h_i),$$

and

$$(19)_D \qquad \{c_{ijk}\} \sim \{h_i c_{ijk} h_i{}^{-1}\}, \ \ \{h_i\} \in C^0(\mathfrak{U}, G_d),$$
$$\{c_{ijk}\} \sim \{a_{ij}g_{ij}a_{jk}g_{ij}{}^{-1}c_{ijk}a_{ik}{}^{-1}\}, \ \ \{a_{ij}\} \in C^1(\mathfrak{U}, G_{c(D)}).$$

Here $\{a_{ij}g_{ij}\}$ belongs to $C^1{}_{c(D)}(\mathfrak{U}, G_d)$ and satisfies the following condition

$$(g_{i_0i_1}{}^{-1}c_{i_0i_1i_2})\ (g_{i_0i_2}a_{i_2i_3}g_{i_0i_2}{}^{-1}) = (g_{i_1i_2}a_{i_2i_3}g_{i_1i_2}{}^{-1})\ (g_{i_0i_1}{}^{-1}\ c_{i_0i_1i_2}).$$

$H^1(M, \mathscr{M}^1{}_D)$ and $H^2\ M, G_{c(D)})$ are defined by these relations. Then we obtain

Theorem 2_D. *The following sequence is exact*

$$(21)_D \qquad 0 \longrightarrow H^0(M, G_{c(D)}) \xrightarrow{i} H^0(M, G_d) \xrightarrow{\rho_D} H^0(M, \mathscr{M}^1{}_D) \longrightarrow H^1(M, G_{c(D)}) \xrightarrow{i^*}$$
$$H^1(M, G_d) \xrightarrow{\rho_D{}^*} H^1(M, \mathscr{M}^1{}_D) \longrightarrow H^2(M, G_{c(D)}) \xrightarrow{i^*} H^2(M, G_d).$$

Example. If $D=\bar\partial$, $G_{c(D_1)}$ is the sheaf of germs of holomorphic G-valued fuctions on M and $\mathscr{M}_D$ is the sheaf of germs of matrix valued (0, 1)-forms θ such that $\bar\partial\theta + \theta_\wedge\theta = 0$ ([18]). If $G=C^*$, $(21)_D$ reduces to

$$0 \longrightarrow H^0(M, C^*{}_\omega) \longrightarrow H^0(M, C^*{}_d) \longrightarrow H^0(M, \phi^{0,1}) \longrightarrow H^1(M, C^*{}_\omega) \longrightarrow$$
$$\longrightarrow H^1(M, C^*{}_d) \longrightarrow H^1(M, \phi^{0,1}) \longrightarrow H^2(M, C^*{}_\omega) \longrightarrow H^2(M, C^*{}_d).$$

Here $\Phi^{0,1}$ is the sheaf of germs of $\bar\partial$-closed (0, 1)-forms. If M is a compact Kaehler manifold, this sequence is rewritten to

$$0 \longrightarrow H^0(M, C^*{}_\omega) \longrightarrow H^0(M, C^*{}_d) \longrightarrow H^0(M, \phi^{0,1}) \longrightarrow H^1(M, C^*{}_\omega) \longrightarrow$$
$$\longrightarrow H^2(M, \mathbf{Z}) \longrightarrow H^{0,2}(M, C) \xrightarrow{exp^*} H^2(M, C^*{}_\omega) \longrightarrow H^3(M, \mathbf{Z}).$$

On the other hand, if M is a Stein manifold, $i^*: H^1(M, G_\omega) \to H^1(M, G_d)$ is a dijection. Hence the sequence $0 \to H^1(M, \mathscr{M}^1{}_{\bar\partial}) \to H^2(M, G_\omega) \to H^2(M, G_d)$ is exact.

14. In this n°, we assume $E=F$, and the principal symbol $\sigma(D)$ of D does not vanish on any open set of M. We consider a smooth G-valued function g on U to be a linear operator acting on $C^\infty(U, E\otimes H)$. For differential operators L, L_1, $L_2: C^\infty(U, E\otimes H) \to C^\infty(U, E\otimes H)$, we set

$$L^g = g^{-1}Lg, \quad [L_1, L_2] = L_1L_2 + L_2L_1.$$

Definition 26. *Let U be an open set of M and $L: C^\infty(U, E\otimes H) \to C^\infty(U, E\otimes H)$ a differential operator of order $k-1$, $k=ord\ D$, on U. Then we define a differential operator $D^e(L): C^\infty(U, E\otimes H) \to C^\infty(U, E\otimes H)$ by*

$$(36) \qquad D^e(L) = (D\otimes 1_H + L)^2 - D^2 \otimes 1_H = L^2 + [D\otimes 1_H, L].$$

$D^e(L)$ was denoted by $\rho_D(L)$ in [6]. The following Lemma is also given in [6].

Lemma 9. *D^e has following properties*

$$(37, \text{ i}) \qquad D^e(cL) = cD^e(L) + (c^2-c)L^2, \ c \text{ is a constant},$$

$$(37, \text{ ii}) \qquad D^e(L_1+L_2) = D^e(L_1) + D^e(L_2) + [L_1, L_2],$$

$$(37, \text{ ii})' \qquad D^e(L_1-L_2) = D^e(L_1) - D^e(L_2) - [L_1-L_2, L_2],$$

$$(37, \text{ iii}) \qquad D^e(L^g) = D^e(L)^g - [\rho_D(g), L^g],$$

$$(37, \text{ iv}) \qquad D^e(\rho_D g) = \rho_{D^2}(g).$$

Proof. (37, i) and (37, ii) follow from Definition. (37, ii)$'$ follows from (37, i) and (37, ii). Since we have

$$D^e(L^g)=(D\otimes 1_H)L^g+L^g(D\otimes 1_H)+(L^2)^g$$
$$=((D\otimes 1_H)L^g+L^g(D\otimes 1_H)+L^2)^g+[D\otimes 1_H-(D\otimes 1_H)^g,\ L^g],$$

we get (37, iii). (37, iv) foollows from Definition.

Corollary 1. *We have*

$$D^e(L^g+\rho_D(g))=D^e(L)^g+\rho_{D^2}(g). \tag{38}$$

Proof. By (37, ii) and (37, iv), we get

$$D^e(L^g+\rho_D(g))=D^e(L^g)+\rho_{D^2}(g)+[\rho_D(g),\ L^g].$$

Hence we have (38) by (37, iii).

Corollary 2. *If* $\rho_D(g)$ *is* $L_2-L_1{}^g$, $D^e(L_2)$ *is equal to* $D^e(L_1)^g+\rho_{D^2}(g)$. *Especially, if* $D^2=0$, $D^e(L_2)$ *is equal to* $D^e(L_1)^g$.

Proof. Set $L_2=L_1{}^g+\rho_D(g)$ and apply (38), we have Corollary.

Note. The reason of notational inhomogenity of ρ_D and D^e was exposed in n°1. For functions, $D^e(f)$ is defined by $\rho_D(e^f)$ (cf. [8]). We also note if f is a g-valued function, we say say f to be a $c(D)$-class function if $f(D\otimes 1_H)=(D\otimes 1_H)f$ (cf [6], [8]).

The next Lemma generalizes gauge transformation

Lemma 10. *Let g be a smooth* G*-valued function on U and* $L:\mathrm{C}^\infty(U,\ E\otimes H)\to\mathrm{C}^\infty(U,\ E\otimes H)$ *a differential operator. We set*

$$g_D(L)=L^{g^{-1}}+\rho_D(g^{-1})=g(L-\rho_D(g))g^{-1}. \tag{39}$$

Then g_D is a G*-action and we have*

$$D^e(g_D(L))=g_{D^2}(D^e(L)). \tag{40}$$

Proof. Since $e_D(L)=L$, to show g_D to be a G-action, we need only to show $g_D(h_D(L))=(gh)_D(L)$. But this follows from $g(hLh^{-1}-h\rho_D(h)h^{-1})g^{-1}-g\rho_D(g)g^{-1}=(gh)L(gh)^{-1}-gh\rho_D(h)\,(gh)^{-1}-g\rho_D(g)g^{-1}=(gh)L(gh)^{-1}-gh\rho_D(gh)\,(gh)^{-1}$. Since g_D is a G-action, (40) follows from (38).

We set $\mathscr{D}^{k-1}_{E\otimes H}$ the sheaf of germs of differential operators $L:\mathrm{C}^\infty(U,\ E\otimes H)\to\mathrm{C}^\infty(U,\ E\otimes H)$ with the order at most $k-1$. By definition, $\mathscr{M}^1{}_D$ is a subsheaf of $\mathscr{D}^{k-1}_{E\otimes H}$. D^e induces a sheaf map (also denoted by D^e) on $\mathscr{D}^{k-1}_{E\otimes H}$. Then $\mathscr{M}^1{}_{D^2}$ is a subsheaf of $D^e\,(\mathscr{D}^{k-1}_{E\otimes H})$. We set $\mathscr{M}^1{}_D{}^\#=(D^e)^{-1}(\mathscr{M}^1{}_{D^2})$. By (37, iv), $\mathscr{M}^1{}_D$ is a subsheaf of $\mathscr{M}^1{}_D{}^\#$. If $D^2=0$, $\mathscr{M}^1{}_D{}^\#$ is the kernel sheaf oe D^e. By the actions g_D and $g_{D^2}{}'$ $(\mathscr{D}^{k-1}_{E\otimes H})$. and D^e $\mathscr{D}^{k-1}_{E\otimes H})$ are the G_d-sheaves and D^e is an equivariant map by Lemma 10. Since $\mathscr{M}^1{}_{D^2}$ is a G_d-subsheaf of $D^e\,(\mathscr{D}^{k-1}_{E\otimes H})$, $\mathscr{M}^1{}_D{}^\#$ is a G_d-subsheaf of

$\mathscr{D}^{k-1}_{E\otimes H}$ and $\mathscr{M}^1{}_D$ is a G_d-subsheaf of $\mathscr{M}^1{}_D{}^{\#}$.

Definition 27. *The quotient sheaves of* $\mathscr{M}^1{}_D{}^{\#}$, $\mathscr{D}^{k-1}_{E\otimes H}$ *and* D^e $(\mathscr{D}^{k-1}_{E\otimes H})$ *by* G_d-*actions are denoted by* $\mathscr{M}^1{}_D{}^{\#}$, $\mathscr{D}^{k-1}_{E\otimes H}$, *and* $\mathscr{M}^2{}_D$.

By Definitions, D^e induces the sheaf map $D^e: \mathscr{D}^{k-1}_{E\otimes H} \to \mathscr{M}^2{}_D$ and the following sequences are exact

$$(2)_D \qquad 0 \longrightarrow \mathscr{M}^1{}_D{}^{\#} \longrightarrow \mathscr{D}^{k-1}_{E\otimes H} \xrightarrow{D^e} \mathscr{M}^2{}_D \longrightarrow 0,$$

$$(41)_1 \qquad 0 \longrightarrow \mathscr{M}^1{}_D \longrightarrow \mathscr{M}^1{}_D{}^{\#} \longrightarrow \mathscr{M}^1{}_D{}^{\#} \longrightarrow 0,$$

$$(41)_2 \qquad 0 \longrightarrow \mathscr{M}^1{}_{D^2} \longrightarrow D^e(\mathscr{D}^{k-1}_{E\otimes H}) \longrightarrow \mathscr{M}^2{}_D \longrightarrow 0.$$

Note. If $D^2=0$, $\mathscr{M}^1{}_{D^2}$ is the 0- sheaf. But $\mathscr{M}^2{}_D$ may be different from D^e $(\mathscr{D}^{k-1}_{E\otimes H})$ since the G_d-action on D^e $(\mathscr{D}^{k-1}_{E\otimes H})$ may not be trivial unless $n=1$. But the following sequence is exact if $D^2=0$.

$$(2)_D' \qquad 0 \longrightarrow \mathscr{M}^1{}_D{}^{\#} \longrightarrow \mathscr{D}^{k-1}_{E\otimes H} \xrightarrow{D^e} D^e(\mathscr{D}^{k-1}_{E\otimes H}) \longrightarrow 0.$$

15. If $L \in H^0(M, (\mathscr{D}^{k-1}_{E\otimes H}), \bar{L}$ has a resentative $\{L_i\} \in C^0\mathfrak{U}$, $(\mathscr{D}^{k-1}_{E\otimes H})$ such that $L_j=g_{ji,D}(L_i)$, $\{g_{ji}\}\in C^1(\mathfrak{U}, G_d)$. For $\xi=\{g_{ij}\}$, we set $\delta_{\xi,D}(L)_{ij}=L_j-g_{ji,D}(L_i)$.

Definition 28. *Let* $\xi=\{g_{ij}\}$ *be in* $C^1{}_a(\mathfrak{U}, G_d)$. *Then we set*

$$H^0{}_dM, (\mathscr{D}^{k-1}_{E\otimes H})=\{L\in H^0(M, \mathscr{D}^{k-1}_{E\otimes H})\,|\,L=L_i,\ \delta_{\xi,D}L=0 \text{ and } L_i{}^{t_{ijk}}=L_i \text{ for some } \xi\}$$

D^e induces a map from $H^0(M, \mathscr{D}^{k-1}_{E\otimes H})$ into $H^0(M, \mathscr{M}^2{}_D)$. We set

$$(42) \qquad H^0{}_D(M, \mathscr{M}^2{}_D)=D^e(H^0{}_d(M, \mathscr{D}^{k-1}_{E\otimes H})).$$

Theorem 1_D. *There is a surjection* $dR_D: H^0{}_D(M, \mathscr{M}^2{}_D)\to H^1(M, \mathscr{M}^1{}_D)$.

Proof. If $\Theta\in H^0{}_D(M, \mathscr{M}^2{}_D)$, Θ is represented by $\{D^e(L_i\}$, $\{L_i\}\in H^0{}_d(M, \mathscr{D}^{k-1}_{E\otimes H})$ Hence there exists $\{g_{ij}\}\in C^1{}_a(\mathfrak{U}, G_d)$ such that to set $\omega_{ij}=L_j-L_i{}^{g_{ji}}=\rho_D g_{ij}$, we get $\omega_{jk}=\omega_{ik}+\omega_{ij}{}^{g_{jk}}=0$. That is, $\{\omega_{ij}\}$ defines an element of $H^1(M, \mathscr{M}^1{}_D)$. If $\{D^e(L_i')\}$ is another representative of Θ, we have $L_i'=h_{i,D}(L_i)$. Hence $\{\omega_{ij}'\}$ giues the same element of $H^1(M, \mathscr{M}^1{}_D)$ Therefore we can define the map $dR_D: H^0{}_D(M, \mathscr{M}^2{}_D)\to H^1$ $(M, \mathscr{M}^1{}_D)$. If $\{g_{ij}\}\in C^1{}_a(\mathfrak{U}, G_d)$ and $\{\omega_{ij}\}=\{\rho_D(g_{ij})\}$ is in $Z^1(\mathfrak{U}, \mathscr{M}^1{}_D)$, by the same calculation as in the proof of Proposition 1, we get $\omega_{ki}{}^{g_{ij}g_{jk}g_{ki}}=\omega_{ki}$. Hence we have

$$\omega_{ij}=L_j-L_i{}^{g_{ij}},\ L_i=\sum_{U_k\cap U_i\neq\phi} e_k\omega_{ki}.$$

Therefore $\{L_i\}$ represents an element of $H^0{}_d(M, \mathscr{D}^{k-1}_{E\otimes H})$ and we have $dR(\{L_i\})=\langle\omega\rangle$. Hence we have Theorem.

Note 1. If $\xi=\{g_{ij}\}$ is a G–bundle, then above decomposition of ω_{ij} gives a connection of D with respect to ξ(cf. [1], [2], [5], [6]).

Note 2. If $D=\bar{\partial}$, denote $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ the sheaves of germs of smooth matrix valued (1, 0)–and (0, 1)–type forms, we have the following commutative diagram with exact lines and columns

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathscr{M}^1{}_{\bar{\partial}} \arrow[u] \arrow[r] & \mathfrak{g}^{0,1} \arrow[u] \arrow[r, "\bar{\partial}^e"] & \mathscr{M}^2{}_{\bar{\partial}} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{M}^1 \arrow[u, "\pi^{0,1}"] \arrow[r] & \mathfrak{g}^1 \arrow[u, "\pi^{0,1}"] \arrow[r, "d^e"] & \mathscr{M}^2 \arrow[u, "0{,}2"] \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{M}^{1,0}=\mathscr{M}^1{}_\omega \arrow[u, "i"] \arrow[r] & \mathfrak{g}^{1,0} \arrow[u] \arrow[r, "d^e"] & \mathscr{M}^2 \arrow[u, "i"] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Hence $\langle\omega\rangle\in \mathrm{H}^1(M,\ \mathscr{M}^1)$ is in i^*–image if and only if there exists $\Theta\in \mathrm{Z}^0{}_d(M,\ \mathscr{M}^2)$ such that $\iota\ dR(\Theta)=\langle\omega\rangle$ and $\pi^{0,2}\Theta=0$ (cf. [18]).

16. We assume $\mathrm{C}_{c(D)}$ admits the following resolution (cf. [6])

$$0\longrightarrow \mathrm{C}_{c(D)}\xrightarrow{\ i\ }\mathrm{C}_d\xrightarrow{d^D}\mathrm{C}^{1,D}\xrightarrow{d^D}\mathrm{C}^{2,D}\longrightarrow\cdots\cdots.$$

We define the map $j:\mathscr{M}^1{}_D\to\mathscr{M}^1{}_{dD}$ by $j(\rho_D(g))=\rho_{dD}(g)$. Then j is abijection and therefore induces a bijection $j^*:\mathrm{H}^1(M,\ \mathscr{M}^1{}_D)\to \mathrm{H}^1(M,\ \mathscr{M}^1{}_{dD})$. By the definition of $\mathscr{M}^1{}_{dD}$, $\Theta=\{\Theta_i\}\in \mathrm{Z}^0{}_d(\mathfrak{U},\ \mathscr{M}^2{}_{dD})$ is regarded to be $\Theta_i\in \mathrm{H}^0(U_i,\ \mathrm{C}^{2,D}\otimes\mathfrak{g})$. Hence we may consider $tr(\Theta\wedge\overset{p}{\cdots}\wedge\Theta)$ to be a d^D–closed $2p$–form on M. The d^D–cohomology class of $tr(\Theta\wedge\overset{p}{\cdots}\wedge\Theta)$ is determined by $\iota dR_{dD}(\Theta)\in \mathrm{H}^1(M,\ \mathscr{M}^1{}_{dD})$. Since dR_D is onto, we can define the p–th D–Chern class $c^p{}_D(\langle\omega\rangle)$ and the p–th D–Chern character $ch^p{}_D(\langle\omega\rangle)$ by

$$c^p{}_D(\langle\omega\rangle)=\textit{the class of } c^p{}_D\in \mathrm{H}^{2p}(M,\ \mathrm{C}_{c(D)}),$$

$$ch^p{}_D(\langle\omega\rangle)=\textit{the class of }\left(\frac{\sqrt{-1}}{2\pi}\right)^p\frac{1}{p!}\ tr\ (\Theta\wedge\overset{p}{\cdots}\wedge\Theta)\in \mathrm{H}^{2p}(M,\ \mathrm{C}_{c(D)}).$$

Here, we set $det\ (\mathrm{I}+(2\pi/\sqrt{-1}))=\mathrm{I}+c^1{}_Dt+\cdots+c^p{}_Dt^p+\cdots+c^n{}_Dt^n$. Then, denoting $i_D:\mathbf{Z}\to \mathrm{C}_{c(D)}$ the inclusion, we have

$$(26)_D \qquad c^p{}_D(\rho_D{}^*(\xi))=i_D{}^*(c^p(\xi)),$$

$$(26)'_D \qquad ch^p{}_D(\rho_D{}^*(\xi))=i_D{}^*(ch^p(\xi)).$$

In [6], the right hand side of $(26)_D$ was called the p–th $c(D)$–characteristic class of $\xi\in \mathrm{H}^1(M,\ \mathrm{G}_d)$.

Example. If M is a compact Kaehler manifold and $D=\bar{\partial}$, $c^p{}_{\bar{\partial}}(\langle\omega\rangle)$ is a (0, $2p$) -type class. If $\langle\omega\rangle=\rho_D{}^*(\xi)$, $c^p{}_{\bar{\partial}}(\langle\omega\rangle)$ is the (0, $2p$)-type part of $c^p(\xi)$, the p-th Chern class of ξ.

By definition and $(26)_{D'}$ we have

Proposition 3_D. *Let $\langle\omega\rangle$ be an element of $\mathrm{H}^1(M, \mathscr{M}^1{}_D)$. Then $\langle\omega\rangle$ can not be in $\rho_D{}^*$-image if $c^p{}_D(\langle\omega\rangle)$ is not an integral class for some p.*

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