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# *Borel Transformation in Non-Analytic Category*

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## Introduction

For a power series $\varphi(z)=\sum_{i_1,\cdots,i_n} a_{i_1,\cdots,i_n} z_1^{i_1}\cdots z_n^{i_n}$, its Borel transformation $\mathscr{B}[\varphi]$ is given by $\sum_{i_1,\cdots i_n} a_{i_1,\cdots,i_n}/i_1!\cdots i_n!\zeta_1^{i_1}\cdots\zeta_n^{i_n}$ ([3], [5], [14], [15]). Borel transformation is linear and has following properties.

$$\mathscr{B}[\varphi\psi]=\mathscr{B}[\varphi]\,\sharp\,\mathscr{B}[\psi],\ \ f\,\sharp\,g(x)=\frac{\partial^n}{\partial x_1\cdots\partial x_n}\int_0^x f(x-t)g(t)dt,$$

$$\frac{\partial}{\partial\zeta_i}\mathscr{B}[\varphi]=\mathscr{B}[(z_i^{-1}\varphi)_+],\ \ \zeta_i\mathscr{B}[\varphi]=\mathscr{B}[z_i\varphi+2z_i\frac{\partial\varphi}{\partial z_i}],$$

where $\varphi_+$ is the holomorphic part of $\varphi$ ([1]). Therefore, since

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}(\log x)^{\sharp n}=\frac{e^{-\gamma t}}{\Gamma(1+t)}x^t,\ \ \gamma \text{ is Euler constant},$$

we may define

$$(a)\qquad \mathscr{B}[\log z](\zeta)=\log\zeta+\gamma,$$

and by $(a)$, we can define Borel transformation of many-valued analytic functions ([1]). This is used, for example, to give an explicit formula of the solution of constant coefficients linear partial differential equations with finite exponential type, meromorphic or many-valued analytic Cauchy data ([1], [1]').

We note that since the inverse of Borel transformation of a function $f$ is given by $\int_{R^{n,+}} e^{-t}f(zt)dt$ ([2], [2]) and $\int_0^\infty e^{-t}\log zt dt=\log z-\gamma$, which is the base of Volterra's theory of logarithm of the functions of composition ([18], [18]'). Hence, $(a)$ has been essentially used by Volterra.

The purpose of this paper is to extend Borel transformation for non-analytic functions (or distributions). Since

$$\mathscr{B}[e^{az}](\zeta)=J_0(\sqrt{-1}\sqrt{2a\zeta}),\ \ J_0(z) \text{ is 0-th Bessel function},$$

if $\varphi\in\mathscr{S}(\mathrm{R}^n)$, the space of rapidly decreasing $\mathrm{C}^\infty$-functions, and the Fourier transform $\mathscr{F}[\varphi]$ of $\varphi$ satisfies $|\mathscr{F}[\varphi](x)|=O(\mathrm{e}^{-\|x\|^{1/2+\delta}})$, $\|x\|\to\infty$, for some $\delta>0$, then we may define $\mathscr{B}[\varphi]$ by

$$\mathscr{B}[\varphi](x)=\int_{\mathrm{R}^n}J_0(\sqrt{-4\pi\sqrt{-1}\,x_1\zeta_1})\cdots J_0(\sqrt{-4\pi\sqrt{-1}\,x_n\xi_n})\mathscr{F}[\varphi](\xi)d\xi,$$

because $\varphi=\mathscr{F}^*[\mathscr{F}[\varphi]]$. Then, since to denote 0-th Hankel transformation of $f$ by $\mathrm{H}_0(f)$, $f$ is a function of 1-variable, we have

$$2\pi\sqrt{-1}\,\mathrm{H}_0(g\Big(\frac{x^2}{4\pi\sqrt{-1}}\Big)/x^2)(\sqrt{\xi})=\int_0^{\sqrt{-1}\infty}J_0(\sqrt{-4\pi\sqrt{-1}\,\zeta\xi})g(\zeta)d\zeta,$$

we may define Borel transformation of $T$, an element of the dual space of a suitable function space $\mathbf{F}$, by

$$(b)\qquad \mathscr{B}[T](f)=(2\pi\sqrt{-1})^n\mathscr{F}[T](\mathrm{H}_{(0)}(f\Big(\frac{x^2}{4\pi\sqrt{-1}}\Big)/x^2)(\sqrt{\xi})),$$

$$\mathrm{H}_{(0)}(g(x))(\xi)=\int_{\mathrm{R}^{n,+}}J_0(x_1\xi_1)\cdots J_0(x_n\xi_n)x_1\cdots x_n g(x)dx,$$

$$f\Big(\frac{x^2}{4\pi\sqrt{-1}}\Big)/x^2=f\Big(\frac{x_1^2}{4\pi\sqrt{-1}},\cdots,\frac{x_n^2}{4\pi\sqrt{-1}}\Big)/x_1^2\cdots x_n^2,$$

$$g(\sqrt{\xi})=g(\sqrt{\xi_1},\cdots,\sqrt{\xi_n}),\quad (\mathrm{H}_{(0)}(f\Big(\frac{x^2}{4\pi\sqrt{-1}}\Big)/x^2)(\sqrt{\xi})\in\mathbf{F}.$$

To give exact meaning of $(b)$, we define and treat function space $\mathscr{S}(\mathbf{R}^n,-1)$ and related spaces in §1. Here $\mathscr{S}(\mathbf{R}^n,-1)$ is the space of rapidly decreasing holomorphic functions on $\mathbf{R}^n$. In §2, we study Hankel transformations of these spaces and show that to set

$$\mathscr{B}_{\sqrt{-1}\,\mathrm{R}^{n,+}}=\{\mathrm{H}_{(0)}(\mathscr{S}^2(\mathbf{R}^n,-1))\cap\widehat{\mathscr{S}}_0(\mathbf{R}^{n,+}),$$

$$\mathscr{S}^2(\mathbf{R}^n,-1)=\{f(z_1^2,\cdots,z_n^2)\,|\,f\in\mathscr{S}(\mathbf{R}^n,-1)\},$$

$$\widehat{\mathscr{S}}_0(\mathbf{R}^{n,+})=f\,|\,f \textit{ is a rapidly decreasing even function and } f|_{x_i}=0,$$
$$i=1,\cdots,n\},$$

we have

$$\mathscr{S}(\mathbf{R}^n,-1)=\{\mathrm{H}_{(0)}(g\Big(\frac{x^2}{4\pi\sqrt{-1}}\Big)/x^2)(\sqrt{\xi})\,|\,g\in\mathscr{B}_{\sqrt{-1}\,\mathrm{R}^{n,+}}\}.$$

In §3, we define Borel transformation of $T\in(\mathscr{F}(\mathscr{S}(\mathbf{R}^n,-1)))^*$ as an element of $(\mathscr{B}_{\sqrt{-1}\,\mathrm{R}^{n,+}})^*$. Borel transformation of the element of other spaces (defined in §1) are also defined. The necessity of the use of other spaces is follows from the fact that $\mathscr{B}[T]$ is not always differentiable as the element of $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+})^*$. In § 4, first we define the product $fT$ of a many-valued analytic function $f$ and $T\in(\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^*$ and define its Borel transformation. Using these generalized Borel transformation, we show the explicit formula of the solution of Cauchy problem given in [1] is also applicable for non-analytic data in § 5.

We note that in our definition of Borel transformation, $\mathscr{B}[\delta^{(k)}]=0$ for all $k\geqq 0$, $\delta^{(0)}=\delta$. But if we use other type of function space, $\mathscr{B}[\delta^{(k)}]$ may not be equal to 0 in general. In fact, since we have by du Bois Raymond's formula ([4], [19])

$$\int_0^\infty udu\int_{\mathbf{R}^2}\varphi(x, y)g(u\sqrt{x^2+y^2})dxdy=2\pi\varphi(0, 0)\int_0^\infty\int_0^t g(s)ds\,\frac{dt}{t},$$

if $\int_{\mathbf{R}^2}|\varphi(x, y)|(x^2+y^2)^{-1/4}dxdy<\infty$, $f(r,\theta)$ is the function of bounded variation on $(0, \infty)$ for all $\theta$, $f(r, \theta)=\varphi(x, y)$, and its total variation tends to 0, $r\to 0$, uniformly in $\theta$, and $g(r)\sqrt{r}$ is bounded on $[0, \infty)$, $\int_0^\infty\int_0^t g(s)sdst^{-1}dt<\infty$ and especially have by Neumann's formula ([19])

$$\int_0^\infty udu\int_{\mathbf{R}^2}\varphi(x, y)J_0(u\sqrt{x^2+y^2})dxdy=2\pi\varphi(0, 0),$$

$$\int_0^\infty udu\int_0^\infty\int_0^{2\pi}\varphi(r, \theta)J_0(ur)rdrd\theta=2\pi\varphi(0, 0),$$

$\mathscr{B}[\delta^{(k)}](f)$, for suitable $f$, should be

$$\begin{aligned}
&\mathscr{B}[\delta^{(k)}](f)\\
&=(2\pi\sqrt{-1})^{1+k}(-1)^k\int_0^{\sqrt{-1}\infty}\int_{-\infty}^\infty \xi^kJ_0(\sqrt{-4\pi\sqrt{-1}\,\zeta\xi})d\xi f(\zeta)d\zeta\\
&=(-1)^{k+1}(\sqrt{-1})^k(2\pi)^k\int_0^\infty u^{2k+1}du\int_{-\infty}^\infty J_0(u\sqrt{\eta})f\Big(\frac{\sqrt{-1}}{4\pi}\eta\Big)d\eta \qquad \zeta=\frac{-1}{4\pi}\eta,\ \xi=u^2,\\
&=\frac{(-1)^{k+1}(-1)^k}{(k+1)^2}(2\pi)^k\int_0^\infty wdw\int_{-\infty}^\infty F_k(ws)f\Big(\frac{-1}{4\pi}{}^{k+1}\sqrt{s}\Big)s^{-k/(k+1)}ds,\quad u^{k+1}=w,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \eta^{k+1}=s,
\end{aligned}$$

because $\mathscr{F}[\delta^{(k)}]=(-2\pi\sqrt{-1}\,\xi)^k$. Here $F_k(x)=J_0({}^{k+1}\sqrt{x})$ and assume $f({}^{k+1}\sqrt{s})$ is 1-valued as an analytic function on $\mathbf{C}$. Then, to denote $K_R(r, \theta)$ the Poisson kernel on $\{s\,|\,|s|<R\}\subset\mathbf{C}$, we have

$$\mathscr{B}[\delta^{(k)}](f)$$
$$= \lim_{R\to\infty} \frac{(-1)^{k+1}(-1)^k}{(k+1)^2}(2\pi)^k \int_0^\infty w dw \int_0^{R-c} \int_0^{2\pi} K_R(r,\theta) F_k(wr) f\Big(\frac{{}^{k+1}\sqrt{r}}{4\pi}, \frac{\pi}{2}\Big) r^{-k/(k+1)} dr\, d\theta,$$

where $c>0$ is arbitrary. Hence by du Bois Raymond's formula, if $f$ satisfies suitable condition, then, it must be

$$(c) \qquad \mathscr{B}[\delta^{(k)}](f)$$
$$= \frac{(1)^{k+1}(\sqrt{-1})^k}{(k+1)^2}\Big)(2\pi)^k \lim_{r\to\infty} r^{-k/(k+1)} f\Big(\sqrt{-1}\frac{{}^{k+1}\sqrt{r}}{4\pi}\Big) \int_0^r \int_0^t F_k(s) s ds \frac{dt}{t}.$$

Especially, by Neumann's formula, it must be

$$(d) \qquad \mathscr{B}[\delta] = -\delta_{\sqrt{-1}\infty}, \quad \delta_{\sqrt{-1}\infty}(f) = \lim_{x\to+\infty} f(\sqrt{-1}x).$$

But, since our testing function $f$ used in this paper always satisfy

$$\lim_{X\to+\infty} \frac{d^k}{dx^k} f(\sqrt{-1}x) = 0, \; k = 0, 1, 2, \cdots,$$

$\mathscr{B}[\delta^{(k)}](f)$ is equal to 0 although we use $(c)$.

## § 0. *Review of Borel transformation in analytic category.*

**0. -0. Usual Borel transformation.** Let $\varphi(z) = \sum_{i_1,\cdots,i_n} a_{i_1,\cdots,i_n} z_1^{i_1}\cdots z_n^{i_n}$ be a germ of holomorphic function at the origin of $\mathbf{C}^n$, then its Borel transformation $\mathscr{B}[\varphi]$ is defined by

$$(1) \qquad \mathscr{B}[\varphi](\zeta) = \sum_{i_1,\cdots,i_n} \frac{a_{i_1,\cdots,i_n}}{i_1!\cdots i_n!} \zeta_1^{i_1}\cdots\zeta_n^{i_n}$$
$$= \frac{1}{(2\pi\sqrt{-1})^n} \int_{|z_1|=\varepsilon_1,\cdots,|z_n|=\varepsilon_n} \frac{\varphi(z)}{z_1\cdots z_n} \exp\Big(\frac{\zeta_1}{z_1}+\cdots+\frac{\zeta_n}{z_n}\Big) dz_1\cdots dz_n.$$

Here $\varphi$ is holomorphic on $\{z \mid |z_i| < \varepsilon_i,\ i=1,\cdots,n\}$ ([5], [15]). For example, we have (cf. [7])

$$(2) \qquad \mathscr{B}\Big[\frac{1}{1-az}\Big](\zeta) = e^{a\zeta},$$

$$(3) \qquad \mathscr{B}[e^{az}](\zeta) = J_0(\sqrt{-1}\sqrt{2a\zeta}), \quad J_0(z) = \sum_{n=0}^\infty \frac{(-1)^n}{(n!)^2}\Big(\frac{z}{2}\Big)^n \text{ is 0-th Bessel function,}$$

$$(4) \qquad \mathscr{B}[\log(z+\lambda)](\zeta) = \gamma + \log\zeta - \mathrm{Ei}\Big(-\frac{\zeta}{\lambda}\Big),$$

$$\gamma \text{ is Euler constant,} \quad \mathrm{Ei}(-\zeta) = \int_\zeta^\infty e^{-t} t^{-1} dt.$$

By definition, Borel transformation has following properties ([1]).

$$\text{(I)}' \quad \mathscr{B}[a\varphi + b\phi] = a\mathscr{B}[\varphi] + b\mathscr{B}[\phi],\ \mathscr{B}[\varphi\phi] = \mathscr{B}[\varphi] \sharp \mathscr{B}[\phi],\ \mathscr{B}[\varphi \otimes \phi]$$

$$= \mathscr{B}[\varphi] \otimes \mathscr{B}[\phi],\ \frac{\partial}{\partial \zeta_i}\mathscr{B}[\varphi] = \mathscr{B}[(z_i{}^{-1}\varphi)_+],\ \int_0^{\zeta_i} \mathscr{B}[\varphi] d\zeta_i = \mathscr{B}[z_i\varphi],$$

$$\zeta_i \mathscr{B}[\varphi] = \mathscr{B}\left[z_i\varphi + 2z_i\frac{\partial \varphi}{\partial z_i}\right].$$

Here, $f \sharp g(x) = \partial^n/\partial z_1 \cdots \partial z_n \int_0^x f(x-t)g(t)dt$, $(f \otimes g)(z_1, \cdots, z_{n+m}) = f(z_1, \cdots, z_n)g(z_{n+1}, \cdots, z_{n+m})$ and $(\varphi)_+$ is the holomorphic part of the Laurent expansion of $\varphi$.

It is also known that to denote $\mathscr{O}^n$ the ring of germs of holomorphic functions at the origin of $\mathbf{C}^n$ with the local ring topology, $\mathrm{Exp}(\mathbf{C}^n)$ the ring of finite exponential type functions on $\mathbf{C}^n$ with the $\sharp$-multiplication and the induced topology of the local ring topology of $\mathscr{O}^n$, $\mathscr{B}$ gives a topological ring isomorphism between $\mathscr{O}^n$ and $\mathrm{Exp}(\mathbf{C}^n)$ and we have the following commutative diagram.

$$\begin{tikzcd}
(\mathscr{E}_{\mathbf{R}^n})^* \arrow[r, "{}^t(-2\pi\sqrt{-1})"] \arrow[dr, "\mathscr{F}"'] & \mathscr{O}^n \arrow[d, "\cong"', "\mathscr{B}"] \\
& \mathrm{Exp}(\mathbf{C}^n).
\end{tikzcd}$$

Here $(\mathscr{E}_{\mathbf{R}^n})^*$ is the space of compact support distributione, $\iota_{(\alpha)}$ is given by

$$\iota_{(\alpha)}(T)(z) = \frac{1}{(2\pi\sqrt{-1})^n} T_\zeta\left[\frac{1}{(1-\alpha_1\zeta_1 z_1)\cdots(1-\alpha_n\zeta_n z_n)}\right],\ (\alpha) = (\alpha_1, \cdots, \alpha_n),$$

and $\mathscr{F}$ is the Fourier transformation.

**0. -1. Extension of Borel transformation.** We have the following formulas ([1]).

$$(5) \quad z^a \sharp z^b = \frac{(a+1)(b+1)}{(a+b+1)} z^{a+b},\ \ Re.\, a > -1,\ \ Re.\, b > -1,$$

$$(6) \quad \sum_{n=0}^{\infty} \frac{t^n}{n!}(\log x)^{\sharp n} = \frac{e^{-\gamma t}}{\Gamma(1+t)} x^t,\ \ \gamma \textit{ is Euler constant},\ x,\ t \textit{ are real positive.}$$

Hence we may dfine

$$(7) \quad \mathscr{B}[z^\alpha] = \frac{1}{\Gamma(1+\alpha)}\zeta^\alpha,\ \ \alpha \textit{ is not a negative integer},$$

$$(7)' \quad \mathscr{B}[z^{-m}] = 0, \quad m \geqq 1,$$

$$(8) \quad \mathscr{B}[\log z](\zeta) = \log \zeta + \gamma.$$

By (8), we get

(9) $$\mathscr{B}[(-1)^{m-1}\frac{1}{(m-1)!}z^{-m}\log z](\zeta)=\zeta^{-m},\quad m\geqq 1.$$

Since we know

$$Gal(\widetilde{k}^n/k^n)=\overbrace{\widehat{\mathbf{Z}}\oplus\cdots\oplus\widehat{\mathbf{Z}}}^{n},\quad \widehat{\mathbf{Z}}=\varprojlim_m[\mathbf{Z}/m\mathbf{Z}],$$

where $k^n$ is the qutient field of $\mathcal{O}^n$ and $\widetilde{k}^n$ is its algebraic closure ([12]), we may define Borel transformation on $\widetilde{k}^n$ by (7), (7)'. Moreover, to denote the completion of $\widetilde{k}^n[\log z_1,\cdots,\log z_n]$ by the topology

(*) $\lim f_m=f$ *if and only if for any* $\delta_1>0,\cdots,\delta_n>0$, *there exist* $\varepsilon_i,\ \varepsilon_i',\ i=1,\cdots,n$, *such that* $\delta_i\geqq\varepsilon_i>\varepsilon_i'\geqq 0$, *and* $\{\pi^*(f_m)\}$ *converges uniformly in wider sence on* $\widetilde{D}(\varepsilon_1,\varepsilon_1',\cdots,\varepsilon_n,\varepsilon_n')$ *to* $\pi^*(f)$, *where* $D(\varepsilon_1,\varepsilon_1',\cdots,\varepsilon_n,\varepsilon_n')=\{z\mid\varepsilon_i>|z_i|>\varepsilon_i',\ i=1,\cdots,n\}$, $\widetilde{D}$ *is the universal covering space of* $D$ *and* $\pi$ *is its projection,*

by $\mathfrak{k}^n$, Borel transformation is defined on $\mathfrak{k}^n$ and has following properties.

(I) $$\mathscr{B}[a\varphi+b\varphi]=a\mathscr{B}[\varphi]+b\mathscr{B}[\varphi],\ \mathscr{B}[\varphi\phi]=\mathscr{B}[\varphi]\,\sharp\,\mathscr{B}[\phi],\ \mathscr{B}[\varphi\otimes\phi]$$
$$=\mathscr{B}[\varphi]\otimes\mathscr{B}[\phi],\ \frac{\partial}{\partial z_i}\mathscr{B}[\varphi]=\mathscr{B}[z_i^{-1}\varphi],\ \zeta_i\mathscr{B}[\varphi]=\mathscr{B}\left[z_i\varphi+2z_i\frac{\partial\varphi}{\partial z_i}\right].$$

**Note 1.** As an operator, we have $\mathscr{B}[z^{-k}]=\delta^{(k)}$ (*cf.* [2]).

**Note 2.** To set

$$\mathscr{B}^{-1}[z^\alpha](\zeta)=\Gamma(1+\alpha)\zeta^\alpha,\ \alpha \text{ is not a negative integer},$$

$$\mathscr{B}^{-1}[z^{-m}](\zeta)=\frac{(-1)^{m-1}}{(m-1)!}\zeta^{-m}\log\zeta,\quad m=1,\ 2,\ \cdots,$$

$\mathscr{B}^{-1}$ is not continuous in $\alpha$. But, since we get

$$\mathscr{B}^{-1}[z^{-m+\varepsilon}]=\Gamma(1-m+\varepsilon)\zeta^{-m+\varepsilon}=\frac{\pi}{\Gamma(m-\varepsilon)\sin\pi(m-\varepsilon)}\zeta^{-m+\varepsilon}$$

$$=\frac{\pi}{\Gamma(m-\varepsilon)\sin\pi(m-\varepsilon)}\zeta^{-m}+\frac{\pi\varepsilon}{\Gamma(m-\varepsilon)\sin\pi(m-\varepsilon)}\zeta^{-m}\log\zeta+O(\varepsilon),$$

and $(\pi/\Gamma(m-\varepsilon)\sin\pi(m-\varepsilon))\zeta^{-m}\in ker\,\mathscr{B}$, $\mathscr{B}^{-1}$ is continuos as the map *mod. ker* $\mathscr{B}$.

## § 1. *Preliminaries on function spaces.*

**1.-1. The space $\mathscr{S}(\mathbf{R}^n, {}^{+}\!\sqrt{-1})$ and related spaces.** We set $\mathbf{R}^{n,+}=\{x\in\mathbf{R}^n\mid x_1\geqq 0,\ \cdots,\ x_n\geqq 0\}$ and define, $(\mathbf{R}^{n,+},\ {}^{q}\!\sqrt{1})\subset C^n$ by

(10) $$(\mathbf{R}^{n,+},\ {}^{p}\!\sqrt{1})=\mathbf{R}^{n,+}\cup e^{\frac{2\pi}{p}\sqrt{-1}}\mathbf{R}^{n,+}\cup\cdots\cup e^{\frac{p-1}{p}\sqrt{-1}}\mathbf{R}^{n,+}.$$

***Definition.*** *We set* $\mathscr{S}(\mathbf{R}^{n,+}, {}^p\sqrt{1})$ *the space of those holomorphic functions* $f$ *on* $(\mathbf{R}^{n,+}, {}^p\sqrt{1})$ *such that*

$$f(e^{-\frac{2k}{p}\sqrt{-1}}u)\,|\,\mathbf{R}^{n,+}\in\mathscr{S}(\mathbf{R}^{n,+}),\quad 0\leqq k\leqq p-1,$$

*where* $\mathscr{S}(\mathbf{R}^{n,+})$ *is the space of rapidly decreasing functions on* $\mathbf{R}^{n,+}$, *with the following topology*

(*) $\lim f_m = f$ *if and only if each* $f_m$ *is holomorphic on* $U((\mathbf{R}^{n,+}, {}^p\sqrt{1}))$ *(not depend on* $m$*) and* $\{f_m\}$ *converges uniformly to* $f$ *on* $U((\mathbf{R}^{n,+}, {}^p\sqrt{1}))$.

***Lemma 1.*** $\mathscr{S}(\mathbf{R}^{n,+}, {}^p\sqrt{1})$ *is complete.*

***Proof.*** If $\{f_m\}$ is a Cauchy series of $\mathscr{S}(\mathbf{R}^{n,+}, {}^p\sqrt{1})$, then for any $(i_1, \cdots, i_n)$, $\partial^{i_1+\cdots+i_n}f_m/\partial z_1^{i_1}\cdots\partial z_n^{i_n}$ converges uniformly to $\partial^{i_1+\cdots+i_m}f/\partial z_1^{i_1}\cdots\partial z_n^{i_n}$, where $f$ is holomorphic on $U((\mathbf{R}^{n,+}, {}^p\sqrt{1}))$. Hence
$\lim_{\|u\|\to\infty}(\partial^{i_1+\cdots+i_n}f/\partial z_1^{i_1}\cdots\partial z_n^{i_n})(e^{-(2k/p)\pi-1}u)=0$ for any $(i_1, \dots, i_n)$ and therefore $f\in\mathscr{S}(\mathbf{R}^{n,+}, {}^p\sqrt{1})$.

If $p=2q$, then we set $(\mathbf{R}^{n,+}, {}^p\sqrt{1})=(\mathbf{R}^n, {}^q\sqrt{-1})$. Especially, if $q=1$ *or* 2, then we denote $(\mathbf{R}^n, -1)$ and $(\mathbf{R}^n, \sqrt{-1})$ instead of $(\mathbf{R}^n, {}^1\sqrt{-1})$ and $(\mathbf{R}^n, {}^2\sqrt{-1})$. We set also

$$\mathscr{S}_{(k)}(\mathbf{R}^{n,+}, {}^p\sqrt{1})=\{f\,|\,f=z_1^{k_1}\cdots z_n^{k_n}g,\ g\in\mathscr{S}(\mathbf{R}^{n,+}, {}^p\sqrt{1})\}, (k)=(k_1, \cdots, k_n).$$

***Lemma 2.*** $\mathscr{S}(\mathbf{R}^{n,+}, {}^p\sqrt{1})$ *is dence in* $L^2(\mathbf{R}^{n,+}, w^{(p-1)}dw)$ *by the inclusion map* $\mathscr{S}(\mathbf{R}^{n,+}, {}^p\sqrt{1})\ni f\to f|\mathbf{R}^{n,+}$, *where* $L^2(\mathbf{R}^{n,+}, w^{(p-1)}dw)$ *means*
$\{f\,|\int_{\mathbf{R}^{n,+}}|f(w)|^2(w_1\cdots w_n)^{(p-1)}dw_1\cdots dw_n<\infty\}$.

***Proof.*** Since $P(z)e^{-z^p}\in\mathscr{S}(\mathbf{R}^{n,+}, {}^p\sqrt{1})$, if $P(z)$ is a polynomial, and Laguere functions form the O. N.-basis of $L^2(0, \infty)$ ([7], [20]), we have the lemma.

Similarly, since Hermite functions form the O. N.-basis of $L^2(-\infty, \infty)$ ([7], [20]), we have

***Lemma 2'.*** $\mathscr{S}(\mathbf{R}^n, {}^q\sqrt{-1})$ *is dence in* $L^2(\mathbf{R}^n, |w|^{(q-1)}dw)=$
$\{f\,|\int_{\mathbf{R}^n}|f(w)|^2|w_1\cdots w_n|^{(q-1)}dw_1\cdots dw_n<\infty\}$.

***Lemma 3.*** *We set*

$$\Delta_{k_1,\cdots,k_n}=\{z\,|\,e^{\frac{k_i}{p}2\pi\sqrt{-1}}<\arg z_i<e^{\frac{k_i+1}{p}2\pi\sqrt{-1}},\ i=1, n\},$$
$$0\leqq k_i\leqq p-1,$$

$\delta_{k_1,\cdots,k_n}=\{$*n-dimensional chain in* $\Delta_{k_1,\cdots,k_n}$ *which joins*
$(e^{\frac{k_1}{p}2\pi\sqrt{-1}}\infty, \cdots, e^{\frac{k_n}{p}2\pi\sqrt{-1}}\infty)$ *and* $(e^{\frac{k_1+1}{p}2\pi\sqrt{-1}}\infty, \cdots, e^{\frac{k_n+1}{p}2\pi\sqrt{-1}}\infty)\}$.

*Then for any* $T \in (\mathscr{S}(\mathbf{R}^{n,+}, {}^p\sqrt{1}))^*$, *there exists (not unique) a system of functions* $\{\tau_{k_1,\cdots,k_n}\}$, *such that each* $\tau_{k_1,\cdots,k_n}$ *is defined and holomorphic on* $\Delta_{k_1,\cdots,k_n}$ *and*

$$T(f) = \sum_{k_1,\cdots,k_n} \int_{\delta_{k_1,\cdots,k_n}} \tau_{k_1,\cdots,k_n}(z) f(z) dz, \tag{11}$$

*each* $\delta_{k_1,\cdots,k_n}$ *is takenin the domain on which* $f$ *is holomorphic.*

***Proof.*** By lemma 2, $L^2(\mathbf{R}^{n,+}, w^{p-1}dw)$ is dense in $(\mathscr{S}(\mathbf{R}^{n,+}, {}^p\sqrt{1}))^*$ and if $T \in (\mathscr{S}(\mathbf{R}^{n,+}, {}^p\sqrt{1}))^*$ can be regarded to be an element of $L^2(\mathbf{R}^{n,+}, w^{p-1}dw)$, then the lemma is true for such $T$.

Since we may rite for arbitrary $T \in (\mathscr{S}(\mathbf{R}^{n,+}, {}^p\sqrt{1}))^*$

$$T(f) = \lim_{m\to\infty} T_m(f), \quad T_m \in L^2(\mathbf{R}^{n,+}, w^{p-1}dw), \quad f \in \mathscr{S}(\mathbf{R}^{n,+}, {}^p\sqrt{1}),$$

and $\mathscr{S}(\mathbf{R}^{n,+}, {}^p\sqrt{1})$, is dense in $C(K)$, the space of continuous functions on $K$ with the uniform convergence topology, where $K$ is an arbitrary compact subset of $\{(e^{\sqrt{-1}\,\theta_1}x_1, \cdots, e^{\sqrt{-1}\,\theta_n}x_n) \mid x_1 > 0, \cdots, x_n > 0\}$, by the map $f \to f|K$ by lemma 2, there exists a system of type $(n, 0)$–currents $\{\sigma_{k_1,\cdots,k_n} dz_1 \cdots dz_n\}$ such that each $\sigma_{k_1,\cdots,k_n}$ is defined and measurable on $\Delta_{k_1,\cdots,k_n}$ and

$$T(f) = \sum_{k_1,\cdots,k_n} \int_{\delta_{k_1,\cdots,k_n}} \sigma_{k_1,\cdots,k_n}(x, y) f(z) dz, \quad z_i = x_i + \sqrt{-1}\, y_i,$$

by Riesz' theorem. But since $\int_\gamma \sigma_{k_1,\cdots,k_n}(x, y) f(z) dz = 0$ if $\gamma$ is an $n$–dimensional chain in $\Delta_{k_1,\cdots,k_n}$ such that $\partial\gamma = 0$ and $f$ is holomorphic, $\sigma_{k_1,\cdots,k_n}$ is a weak solution of the equation $\partial\sigma_{k_1,\cdots,k_n}/\partial\bar{z}_1 = \cdots = \partial\sigma_{k_1,\cdots,k_n}/\partial\bar{z}_n = 0$. Hence we have the lemma (cf. [13], [20]).

**1.-2. The space** $\mathscr{S}_{(\infty)}(\mathbf{R}^{*n,+}, {}^p\sqrt{1})$. We set $\mathbf{C}^* = \mathbf{C} - \{0\}$, $\mathbf{C}^{*n} = \overbrace{\mathbf{C}^* \times \cdots \times \mathbf{C}^{*n}}^{n}$ and also set

$$\begin{aligned}
&\mathbf{R}^{*n} = \mathbf{R}^n \cap \mathbf{C}^{*n}, \quad \mathbf{R}^{*n,+} = \mathbf{R}^{n,+} \cap \mathbf{C}^{*n}, \\
&\mathbf{C}^n - \mathbf{C}^{*n} = \mathbf{W}^n, \quad \mathbf{R}^n - \mathbf{R}^{*n} = \mathbf{X}^n, \quad \mathbf{R}^{n,+} - \mathbf{R}^{*n,+} = \mathbf{X}^{n,+}, \\
&(\mathbf{R}^{*n,+}, {}^p\sqrt{1}) = (\mathbf{R}^{n,+}, {}^p\sqrt{1}) \cap \mathbf{C}^{*n}, \quad (\mathbf{R}^{n,+}, {}^p\sqrt{1}) - (\mathbf{R}^{*n,+}, {}^p\sqrt{1}) \\
&= (\mathbf{X}^{n,+}, {}^p\sqrt{1}).
\end{aligned}$$

***Definition.*** *Let* $f$ *be a function such that*

(i). $f$ *is holomorphic on* $U - (\mathbf{X}^{n,+}, {}^p\sqrt{1})$, *where* $U$ *is a neighborhood of* $(\mathbf{R}^{n,+}, {}^p\sqrt{1})$ *in* $\mathbf{C}^n$.

(ii). $f(e^{-(2k/p)\pi-1}z) | \mathbf{R}^{*n,+} \in \mathscr{S}(\mathbf{R}^{*n,+})$, $0 \leqq k \leqq p-1$, *where* $\mathscr{S}(\mathbf{R}^{*n,+})$ *is the space of rapidly decreasing functions for* $x_i \to \infty$ *and* $x_i \to 0$, $i = 1, \cdots, n$.

*Then the set of all these functions with the topology*

(*) *lim* $f_m = f$ *if and only if there exists a neighborhood* $U$ *of* $(\mathbf{R}^{n,+}, \sqrt[p]{1})$ *in* $\mathbf{C}^n$ *and a neighborhood* $V$ *of* $(\mathbf{R}^{*n,+}, \sqrt[p]{1})$ *in* $\mathbf{C}^{*n}$ *such that both are independent with* $m$ *and* $U \supset V$, *and* $\{f_m\}$ *converges uniformly to* $f$ *on any compact subset of* $U - (\mathbf{X}^{n,+}, \sqrt[p]{1})$ *and converges uniformly to* $f$ *on* $V$,

*is denoted by* $\mathscr{S}_{(\infty)}(\mathbf{R}^{*n,+}, \sqrt[p]{1})$.

As in 1.-1, if $p = 2q$, then we denote $\mathscr{S}_{(\infty)}(\mathbf{R}^{*n}, \sqrt[q]{-1})$, *etc.*, instead of $\mathscr{S}_{(\infty)}(\mathbf{R}^{*n,+}, \sqrt[p]{1})$, *etc.*.

***Lemma 1'.*** $\mathscr{S}_{(\infty)}(\mathbf{R}^{*n,+}, \sqrt[p]{1})$ *is complete.*

***Lemma 2''.*** $\mathscr{S}_{(\infty)}(\mathbf{R}^{*n,+}, \sqrt[p]{1})$ *is dense in* $L^2(\overbrace{(c, \infty)\times\cdots\times(c, \infty)}^{n}, (w_1^{p-1}-c^{2p} w_1^{-p-1})\cdots(w_n^{p-1}-c^{2p}w_n^{-p-1})dw_1\cdots dw_n)$ *for any* $c > 0$.

***Proof.*** This follows from the fact that for any polynomial $P$, $P(z)e^{-z^p-cz^{-p}} \in \mathscr{S}_{(\infty)}(\mathbf{R}^{*,+}, \sqrt[p]{1})$, if $c > 0$.

By lemma 2'', lemma 3 is also true for $\mathscr{S}_{(\infty)}(\mathbf{R}^{*n,+}, \sqrt[p]{1})$.

**1.-3. Relations between $\mathscr{S}(\mathbf{R}^n, \sqrt[p]{1})$ and $\mathscr{S}(\mathbf{R}^n, \sqrt[p']{1})$.** By definition, if $p_1 = p_2 r$, then we can define the maps

$$i^{p_1}{}_{p_2} : \mathscr{S}(\mathbf{R}^{n,+}, \sqrt[p_1]{1}) \to \mathscr{S}(\mathbf{R}^{n,+}, \sqrt[p_2]{1}),\ i^{p_1}{}_{p_2}(f) = f,$$

$$j^{p_2}{}_{p_1} : \mathscr{S}(\mathbf{R}^{n,+}, \sqrt[p_2]{1}) \to \mathscr{S}(\mathbf{R}^{n,+}, \sqrt[p_1]{1}),\ (j^{p_2}{}_{p_1}(g))(z) = g(z^r),\ z^r = (z_1{}^r, \cdots, z_n{}^r).$$

By definition, we have

$$i^{p_1}{}_{p_2}(\mathscr{S}_{(k)}(\mathbf{R}^{n,+}, \sqrt[p_1]{1})) \subset \mathscr{S}_{(k)}(\mathbf{R}^{n,+}, \sqrt[p_2]{1}),$$
$$i^{p_1}{}_{p_2}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n,+}, \sqrt[p_1]{1})) \subset \mathscr{S}_{(\infty)}(\mathbf{R}^{*n,+}, \sqrt[p_2]{1}),$$
$$j^{p_2}{}_{p_1}(\mathscr{S}_{(k)}(\mathbf{R}^{n,+}, \sqrt[p_2]{1})) \subset \mathscr{S}_{(kr)}(\mathbf{R}^{n,+}, \sqrt[p_1]{1}),$$
$$j^{p_2}{}_{p_1}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n,+}, \sqrt[p_2]{1})) \subset \mathscr{S}_{(\infty)}(\mathbf{R}^{+n,+}, \sqrt[p_1]{1}).$$

In the rest, we set

$$\mathscr{S}^r(\mathbf{R}^{n,+}, \sqrt[p_2]{1}) = j^{p_2}{}_{p_1}(\mathscr{S}(\mathbf{R}^{n,+}, \sqrt[p_2]{1})), \tag{12}$$
$$\mathscr{S}^r{}_{(k)}(\mathbf{R}^{n,+}, \sqrt[p_2]{1}) = \mathscr{S}^r(\mathbf{R}^{n,+}, \sqrt[p_2]{1}) \cap \mathscr{S}_{(k)}(\mathbf{R}^{n,+}, \sqrt[p_1]{1}).$$

By the definitions $i^{p_1}{}_{p_2}$ and $j^{p_2}{}_{p_1}$, we may define the limit spaces $\lim[\mathscr{S}(\mathbf{R}^{n,+}, \sqrt[p]{1})\,;\, i^p{}_q]$ and $\lim[\mathscr{S}(\mathbf{R}^{n,+}, \sqrt[p]{1})\,;\, j^q{}_p]$. But these limits are both equal to $\{0\}$.

**Note.** We define $\pi_r : \mathbf{C}^n \to \mathbf{C}^n$ by $\pi_r(z) = (z)^r = (z_1{}^r, \cdots, z_n{}^r)$. Then $\pi_r{}^* = j^{p_2}{}_{p_1}$ as the map on $\mathscr{S}(\mathbf{R}^{n,+}, \sqrt[p_2]{1})$. Similarly, we define the map $\pi_{\infty,\alpha} : \mathbf{C}^n \to \mathbf{C}^{*n}$ by $\pi_{\infty,\alpha}((z_1, \dots, z_n)) = (e^{\alpha_1 z_1}, \cdots, e^{\alpha_n z_n})$ and set $\pi_\infty = \pi_{\infty,1}$. Then we have the following

$$\begin{tikzcd} \mathbf{C}^n \arrow[r, "\pi_{\infty,r}"] \arrow[d, "\pi_\infty"'] \arrow[dr, "\pi_{\infty,rs}" near start] & \mathbf{C}^{*n} \arrow[d, "\pi_s"] \\ \mathbf{C}^{*n} \arrow[r, "\pi_{rs}"'] \arrow[ur, "\pi_r"' near end] & \mathbf{C}^{*n}. \end{tikzcd}$$

commutative diagram

By definition, we have $\pi_\infty{}^{-1}(\mathbf{R}^{*n,+}) = \cup_{N\in\mathbf{Z}^n}(\mathbf{R}^n + 2\pi\sqrt{-1}N)$ and $\pi_\infty{}^{-1}(\mathbf{R}^{*n}) = \cup_{N\in\mathbf{Z}^n}(\mathbf{R}^n + \pi\sqrt{-1}N)$, $N=(N_1, \cdots, N_n)$. We set

$$\bigcup_{N\in\mathbf{Z}^n}(\mathbf{R}^n + 2\pi\sqrt{-1}N) = (\mathbf{R}^n,\ 2\pi\sqrt{-1}\mathbf{Z}^n),\quad \bigcup_{N\in\mathbf{Z}^n}(\mathbf{R}^n + \pi\sqrt{-1}N) = (\mathbf{R}^n,\ \pi\sqrt{-1}\mathbf{Z}^n).$$

***Definition.*** *Let $f$ be an entire function on $\mathbf{C}^n$ such that $f(z-2\pi\sqrt{-1}N)|\mathbf{R}^n \in \mathscr{S}(\mathbf{R}^n)$ for any $N\in\mathbf{Z}^n$. Then we denote the set of all those functions with the topology*

(*) $\lim_{m\to\infty} f_m = f$ *if and only if $\{f_m\}$ converges uniformly to $f$ on any compact subset of $\mathbf{C}^n$ and there exists a neighborhood $U$ of $(\mathbf{R}^n, 2\pi\sqrt{-1}\mathbf{Z}^n)$ such that $\{f_m\}$ converges uniformly to $f$ on $U$,*

*by $\mathscr{S}(\mathbf{R}^n, 2\pi\sqrt{-1}\mathbf{Z}^n)$.*

We also set $\mathscr{S}^\natural(\mathbf{R}^n, 2\pi\sqrt{-1}\mathbf{Z}^n) = \pi_\infty{}^*(\mathscr{S}(\mathbf{R}^{*n,+}, 1)\cap\mathscr{S}(\mathbf{R}^n, 2\pi\sqrt{-1}\mathbf{Z}^n)$. Then, since we have

$$\mathscr{S}^\natural(\mathbf{R}^n, 2\pi\sqrt{-1}\mathbf{Z}^n) = \pi_{\infty,p}{}^*(\mathscr{S}^p(\mathbf{R}^{*n,+}, 1)\cap\mathscr{S}(\mathbf{R}^n, 2\pi\sqrt{-1}\mathbf{Z}^n),$$

we may consider $\mathscr{S}^\natural(\mathbf{R}^n, 2\pi\sqrt{-1}\mathbf{Z}^n)$ to be a kind of limit space of the inverse system $[\mathscr{S}(\mathbf{R}^{n,+}, {}^p\sqrt{1})\,;\, j^q{}_p]$. Similarly, $\mathscr{S}(\mathbf{R}^n, 2\pi\sqrt{-1}\mathbf{Z}^n)$ can be considered to be a kind of limit space of the directed system $[\mathscr{S}(\mathbf{R}^{n,+}, {}^p\sqrt{1})\,;\, i^p{}_q]$.

## § 2. *Preliminaries on Hankel transformations*

**2.-0. Hankel transformations.** Let $J_\nu(z) = \sum_{n=0}^\infty (-1)^n(z/2)^{\nu+2m}/m!\Gamma(\nu+m+1)$ be $\nu$-th Bessel function and assume $\nu$ to be a real number and $\nu \geqq -1/2$. Then $(\nu)$-th Hankel transformation $\mathrm{H}_{(\nu)}(\varphi)(\xi)$ is defined by

$$\begin{aligned}\mathrm{H}_{(\nu)}(\varphi(x_1, \cdots, x_n))(\xi_1, \cdots, \xi_n) &= \int_{\mathbf{R}^{n,+}} J_{\nu_1}(x_1\xi_1)\cdots J_{\nu_n}(x_n\xi_n)\xi_1{}^{-\nu_1}x_1{}^{\nu_1+1}\cdots\xi_n{}^{-\nu_n}x_n{}^{\nu_n+1} \\ &\qquad \varphi(x_1,\cdots,x_n)dx_1\cdots dx_n,\quad (\nu) = (\nu_1,\cdots,\nu_n).\end{aligned} \tag{13}$$

By Hankel's formula ([10], [19]), for suitable $\varphi$, we have

$$\mathrm{H}_{(\nu)}(\mathrm{H}_{(\nu)}(\varphi(x))(\xi))(x) = \varphi(x). \tag{14}$$

Especially, we know ([8], [9], [17])

$$\mathrm{H}_{(\nu)}(\widehat{\mathscr{S}}(\mathbf{R}^{n,+}))=\widehat{\mathscr{S}}(\mathbf{R}^{n,+}), \tag{15}$$

$$\mathrm{H}_{(\nu)}(L^p(\mathbf{R}^{n,+}))=L^p(\mathbf{R}^{n,+}),\ 1\leqq p<\infty,\ \|\mathrm{H}_{(\nu)}\|=1, \tag{15$'$}$$

where $\widehat{\mathscr{S}}(\mathbf{R}^{n,+})$ is the space of rapidly decreasing even functions.

Since we know ([19], 3.2)

$$\frac{d}{dz}\{z^\nu J_\nu(z)\}=z^\nu J_{\nu-1}(z),\quad \frac{d}{dz}\{z^{-\nu}J_\nu(z)\}=-z^{-\nu}J_{\nu+1}(z),$$

we ge, for example, if $\varphi$ and $\varphi/x_i$ both belongs in $\widehat{\mathscr{S}}(\mathbf{R}^{n,+})$, *etc.*,

$$\frac{\partial}{\partial\xi_i}\mathrm{H}_{(\nu)}(\varphi)(\xi)=-\xi_i{}^2\mathrm{H}_{(\nu)+1_i}\Big(\frac{\varphi}{x_i}\Big)(\xi), \tag{16}$$

$$\mathrm{H}_{(\nu)}\Big(\frac{\partial\varphi}{\partial x_i}\Big)(\xi)=-\mathrm{H}_{(\nu)}\Big(\frac{\varphi}{x_i}\Big)(\xi)-\mathrm{H}_{(\nu)-1_i}(x_i\varphi)(\xi).$$

Here $(\nu)\pm 1_i$ means $(\nu_1,\cdots,\nu_{i-1},\nu_{i\pm 1},\nu_{i+1},\cdots,\nu_n)$.

By the asymptotic formula of Bessel functions ([19], 7.21, 22)

$$J_\nu(z)\sim\Big(\frac{2}{\pi z}\Big)^{1/2}\Big[\cos\Big(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\Big)\Big(1+O\Big(\frac{1}{z^2}\Big)\Big)-$$

$$-\frac{4\nu^2-1}{8}\frac{\sin(z-\nu\pi/2-\pi/4)}{z}\Big(1+O\Big(\frac{1}{z^2}\Big)\Big)\Big],\quad |\arg z|<\pi,\ |z|\to\infty,$$

$$J_\nu(z)\sim e^{(\nu+1/2)\pi\sqrt{+1}}\Big(\frac{2}{\pi z}\Big)^{1/2}\Big[\cos\Big(z+\frac{\nu\pi}{2}+\frac{\pi}{4}\Big)\Big(1+O\Big(\frac{1}{z^2}\Big)\Big)-$$

$$-\frac{4\nu^2-1}{8}\frac{\sin(z-\nu\pi/2-\pi/4)}{z}\Big(1+O\Big(\frac{1}{z^2}\Big)\Big)\Big],\quad 0<\arg z<2\pi,\ |z|\to\infty,$$

$\mathrm{H}_\nu(\varphi)$ is holomorphic on the domain $\{\zeta|\ |\mathrm{Re.}\,\zeta|<c\}$ if $\varphi(x)=O(\mathrm{e}^{-cx})$, $x\to\infty$ and if $\varphi(x)=O(x^{-\nu-3/2-\varepsilon}\mathrm{e}^{-cx})$, $\varepsilon>0$, $x\to\infty$, then $\mathrm{H}_\nu(\varphi)$ is continuous on $\{\zeta|\ |\mathrm{Re.}\ \zeta|=c\}$. Especially, if $\varphi(x)=O(\mathrm{e}^{-x^{1+\delta}})$, $\delta>0$, $x\to\infty$, then $\mathrm{H}_\nu(\varphi)$ is an entire function.

Since $\mathrm{H}_\nu(\varphi)(0)=1/2^\nu\Gamma(\nu+1)\int_0^\infty x^{\nu+1}\varphi(x)dx$, to set $2^{(\nu)}=2^{\nu_1+\cdots+\nu_n}$, $\Gamma((\nu)+1)=\Gamma(\nu_1+1)\cdots\Gamma(\nu_n+1)$, we have

$$\mathrm{H}_{(\nu)}(\varphi)(0)=\frac{1}{2^{(\nu)}\Gamma((\nu)+1)}\int_{\mathbf{R}^{n,+}}x^{(\nu)+1}\varphi(x)dx, \tag{17}$$

$$\mathrm{H}_{(\nu)}(\varphi)|_{\xi_i=0}=\frac{1}{2^\nu\Gamma(\nu+1)}\int_{\mathbf{R}^{n,+}}x_i{}^{\nu_i+1}\prod_{j\neq i}(J_{\nu_j}(x_j\xi_j)\xi_j{}^{-\nu_j}x_j{}^{\nu_j+1})\varphi(x)dx. \tag{17$'$}$$

In the rest, we set

$$(\mathrm{H}_{(\nu)_i}\varphi)(x_i)=\int_{\mathbf{R}^{n-1,+}}\prod_{j\neq i}(J_{\nu_j}(x_j\xi_j)\xi_j^{-\nu_j}x_j^{\nu_j+1})\varphi(x)dx_1\cdots dx_{i-1}dx_{i+1}\cdots dx_n.$$

We note that by the second formula of (16) and (17), to set $\mathscr{S}(\mathbf{R}^{n,+})=\{f\mid f$ *is written as* $g|\mathbf{R}^{n,+}$, *where* $g\in\mathscr{S}(\mathbf{R}^n)\}$, we also have

$$(15)'' \qquad \mathrm{H}_{(\nu)}(\mathscr{S}(\mathbf{R}^{n,+}))=\mathscr{S}(\mathbf{R}^{n,+}).$$

**2.–1. Hankel transformations of the spaces $\mathscr{S}(\mathbf{R}^n,{}^q\sqrt{-1})$ *etc.*.** Since we may consider $\mathscr{S}(\mathbf{R}^n,{}^q\sqrt{-1})\subset\mathscr{S}(\mathbf{R}^{n,+})$, we have by (15)″ and (17)′

***Lemma 4.*** *If* $f\in\mathscr{S}(\mathbf{R}^n,{}^q\sqrt{-1})$ *and* $\int_0^\infty x_i^{\nu_i+1}(\mathrm{H}_{(\nu)_i}f)(x_i)dx_i=0,\quad i=1,\cdots,n,$ *then* $\mathrm{H}_{(\nu)}[f(x)](re^{-(k/q)\pi\sqrt{-1}}(x^\mu))/x^{(\mu)}\in\mathscr{S}(\mathbf{R}^{n,+})$, *where* $(x^\mu)=(x_1{}^{\mu_1},\cdots,x_n{}^{\mu_n})$, $x^{(\mu)}=x^{\mu_1}\cdots x_n{}^{\mu_n}$ *and* $r$ *is a real number.*

***Lemma 4′.*** *If* $f\in\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},{}^q\sqrt{-1})$ *and* $\int_0^\infty x^{\nu_i+1}(\mathrm{H}_{(\nu)_i}f)(x_i)dx_i=0,\ i=1,\cdots,n,$ *then* $\mathrm{H}_{(\nu)}[f(x)](re^{-(k/q)\pi\sqrt{-1}}(x^\mu))/x^{(\mu)}\in\mathscr{S}(\mathbf{R}^{*n,+})$.

***Proof.*** By assumption, for any $(k_1,\cdots,k_n)$, $\varphi(x)/x_1{}^{k_1}\cdots x_n{}^{k_n}\in\mathscr{S}_{(\infty)}(\mathbf{R}^*{}_n,{}^q\sqrt{-1})$. Hence by (16), $\partial^{k_1+\cdots+k_n}/\partial x_1{}^{k_1}\cdots\partial x_n{}^{k_n}\mathrm{H}_{(\nu)}[f(x)](0)=0$, if $k_i\geqq 1$ for some $i$. On the other hand, $\mathrm{H}_{(\nu)}[f(x)](0)=0$ by assumption. Hence we have the lemma.

***Lemma 5.*** *If* $\varphi\in\mathscr{S}(\mathbf{R}^n,-1)$ *satisfies*

$$(18) \qquad \int_0^{\sqrt{-1}\infty}(\mathrm{H}_{(\nu)_i}\varphi)(x_i)dx_i=0,\quad i=1,\cdots,n,$$

*then to set* $g(x)=\mathrm{H}_{(0)}(\varphi(\xi^2))(x)$, *we have*

$$(19) \qquad \mathrm{H}_{(0)}\Big(\frac{g(x^2/4\pi\sqrt{-1})}{x^2}\Big)(\sqrt{\xi})\in\mathscr{S}(\mathbf{R}^n,-1),$$

$$\frac{g(x^2/4\pi\sqrt{-1})}{x^2}=\frac{g(x_1{}^2/4\pi\sqrt{-1},\cdots,x_n{}^2/4\pi\sqrt{-1})}{x_1{}^2\cdots x_n{}^2},\quad \sqrt{\xi}=(\sqrt{\xi_1},\cdots,\sqrt{\xi_n}).$$

***Proof.*** Since we have

$$\int_0^\infty x^{\nu+1}\frac{f(\alpha x^k)}{x^k}dx=\frac{1}{k}\alpha^{1-(\nu+2)/k}\int_0^\infty y^{\nu/k}f(y)dy,$$

we ge $\mathrm{H}_{(0)}(\varphi(\xi^2))|_{\xi_i=0}=0,\ i=1,\cdots,n$. On the other hand, since we may consider $j^p{}_{2p}(\mathscr{S}(\mathbf{R}^{n,+},{}^p\sqrt{1}))\subset\hat{\mathscr{S}}(\mathbf{R}^{n,+})$, we have

$$\mathrm{H}_{(\nu)}(j^p{}_{2p}(\mathscr{S}(\mathbf{R}^{n,+},{}^p\sqrt{1})))\subset\hat{\mathscr{S}}(\mathbf{R}^{n,+}),$$

we have the lemma by (15).

***Definition.*** *We set*

(20) $$\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}=\{\mathrm{H}_{(0)}(j^2{}_4(\varphi))\,|\,\varphi\in\mathscr{S}(\mathbf{R}^n,\ -1),\ \int_0^\infty(\mathrm{H}_{(0)i}\varphi)(x_i)dx_i=0,\ i=1,\cdots,n.$$

By definition, we have

$$\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}=\mathrm{H}_{(0)}(\mathscr{S}^2(\mathbf{R}^n,\ -1))\cap\widehat{\mathscr{S}}_0(\mathbf{R}^{n,+}),$$
$$\widehat{\mathscr{S}}_0(\mathbf{R}^{n,+})=\{f\,|\,f\in\widehat{\mathscr{S}}(\mathbf{R}^{n,+}),\ f|_{x_i=0}=0,\ i=1,\cdots,n.$$

***Lemma 6.*** *We have*

(21) $$\left\{\mathrm{H}_{(0)}\left(\frac{g(x^2/4\pi\sqrt{-1})}{x^2}\right)(\sqrt{\xi})\,|\,g\in\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}\right\}=\mathscr{S}(\mathbf{R}^n,\ -1).$$

***Proof.*** Since we get $\mathrm{H}_{(0)}(\mathrm{H}_{(0)}(\mathscr{S}^2(\mathbf{R}^n,\ -1)))=\mathscr{S}^2(\mathbf{R}^n,\ -1)$, and by definition, we have

$$\{f(\sqrt{\xi})\,|\,f\in\mathscr{S}^2(\mathbf{R}^n,\ -1)\}=\mathscr{S}(\mathbf{R}^n,\ -1),$$

we obtain the lemma.

Similarly, to set

$$\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+(\infty)}=\{\mathrm{H}_{(0)}(j^2{}_1(\varphi))\,|\,\varphi\in\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1)$$
$$\int_0^{\sqrt{-1}\infty}(\mathrm{H}_{(0)i}\varphi)(x_i)dx_i=0,\ \ i=1,\cdots,\mathrm{n}\},$$

we have by (16), (17)

(19)′ $$\mathrm{H}_{(0)}\left(\frac{g(x^2/4\pi\sqrt{-1})}{x^2}\right)(\sqrt{\xi})\in\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1)),\ g\in\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+(\infty)},$$

and since $z^2j^2{}_1(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1))\subset j^2{}_1(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1))$, we also obtain

***Lemma 6′.*** *We have*

(21)′ $$\left\{\mathrm{H}_{(0)}\left(\frac{g(x^2/4\pi\sqrt{-1})}{x^2}\right)(\sqrt{\xi})\,|\,g\in\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+(\infty)}\right\}=\mathscr{S}_{(\infty)}\ (\mathrm{R}^{*n},\ -1).$$

By (16), (17), lemma 6′ and (19)′, to set

$$\mathscr{S}_{(\infty)}(\mathbf{R}^n)=\{f\,|\,f\in\mathscr{S}(\mathbf{R}^n),\ f\ \textit{vanishes with order}\ \infty\ \textit{on}\ \mathbf{X}^n\},$$

we get

$$\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)} \subset \mathscr{S}_{(\infty)}(\mathbf{R}^n). \tag{22}$$

**Note.** Since $e^{-(x^p+1/x^p)} > 0$ on $\mathbf{R}^+$, to set $c_{(\nu)} = \mathrm{H}_{(\nu)}(\Pi_{i=1}^n e^{-(x_i^p+1/x_i^p)})(0)$, $c_{(\nu)}$ is positive and not equal to 0. Hence if $f \in \mathrm{H}_{(\nu)}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n,+}, {}^p\sqrt{1}))$, then we may set

$$f(\xi) = g(\xi) + \frac{f(0)}{c_{(\nu)}}\mathrm{H}_{(\nu)}\Big(\prod_{i=1}^{n} e^{-(x_i^p+1/x_i^p)}\Big)(\xi),\ \ g \in \mathscr{S}_{(\infty)}(\mathbf{R}^{*n,+},\ {}^p\sqrt{1}).$$

Here, $\mathrm{H}_{(\nu)}(\Pi_{i=1}^n e^{-(x_i^p+1/x_i^p)})(\xi)$ is an entire function, because $|e^{-(x^p+1/x^p)}| = O(e^{-x^p})$, $||x|| \to \infty$.

**2.-2. Fourier transformation on $\mathscr{S}_{(\infty)}(\mathbf{R})^n$.** Since we know

$$\mathscr{S}_{(\infty)}(\mathbf{R}^n) = \{\mathscr{F}[f] \mid f \in \mathscr{S}(\mathbf{R}^n),\ \int_{\mathbf{R}^n} x^{(m)} f(x)dx = 0,$$
$$(m) = (m_1, \cdots, m_n),\ m_1 \geqq 0, \cdots, m_n \geqq 0\},$$

we get

$$\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^n)) = \Big\{g \mid g \in \mathscr{S}(\mathbf{R}^n),\ \int_{\mathbf{R}^n} x^{(m)} f(x)dx = 0,$$
$$(m) = (m_1, \cdots, m_n),\ m_1 \geqq 0, \cdots, m_n \geqq 0\}.$$

Therefore, if $P(x)$ is a polynomial, then $P(x)$ is equal to 0 as an element of $(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^n))^*$. Hence to define indefinite integral operator $I_{(i_1, \cdots, i_n)}$ by

$$I_{(i_1, \cdots, i_n)}(h)(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \frac{(x_1-t_1)^{i_1-1} \cdots (x_n-t_n)^{i_n-1}}{(i_1-1)! \cdots (i_n-1)!} h(t)dt,$$

we get

$$I_{(i_1, \cdots, i_n)}(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^n)) = \mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^n)).$$

Hence $I_{(i_1, \cdots, i_n)}$ is defined also on $(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^n))^*$. Therefore we have

$$\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} : \mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^n)) = \mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^n)), \tag{23}$$

$$\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} : (\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^n))^* = (\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^n))^*.$$

On the other hand, although $\partial^{i_1+\cdots+i_n}/\partial x_1^{i_1} \cdots \partial x_n^{i_n}$ is 1 to 1 as the maps $\mathscr{S}(\mathbf{R}^n) \to \mathscr{S}(\mathbf{R}^n)$ and $\mathscr{S}_{(\infty)}(\mathbf{R}^n) \to \mathscr{S}_{(\infty)}(\mathbf{R}^n)$, but they are both not onto, and we have

(*) $T\in(\mathscr{S}(\mathbf{R}^n))^*$ *implies* $\xi_1{}^{-k_1}\cdots\xi_n{}^{-k_n}T\in\left(\mathscr{F}\left(\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1{}^{i_1}\cdots\partial x_n{}^{i_n}}\mathscr{S}(\mathbf{R}^n)\right)\right)^*$, $k_1\leqq i_1,\ \cdots,\ k_n\leqq i_n$,

$T\in(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^n))^*$ *implies* $\xi_1{}^{-k_1}\cdots\xi_n{}^{-k_n}T\in\left(\mathscr{F}\left(\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1{}^{i_1}\cdots\partial x_n{}^{i_n}}\mathscr{S}(\mathbf{R}^n)\right)\right)^*$, $k_1\leqq i_1,\ \cdots,\ k_n\leqq i_n$.

**Note.** By definition, if $T\in(\mathscr{S}_{(\infty)}(\mathbf{R}^n))^*$, then $\xi_1{}^{-k_1}\cdots\xi_n{}^{-k_n}T\in(\mathscr{S}_{(\infty)}(\mathbf{R}^n))^*$ for any $k_1\geqq 0,\ \cdots,\ k_n\geqq 0$ and we obtain

$$\mathscr{F}[I_{(i_1,\cdots,i_n)}]=(-2\pi\sqrt{-1})^{i_1+\cdots+i_n}\xi_1{}^{-i_1}\cdots\xi_n{}^{-i_n}.$$

**2.-3. The spaces $\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(k)}$.** ***Lemma 7.*** *We set*

$$\text{(24)}\qquad \mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(k)}=\{\mathrm{H}_{(0)}(j^2{}_4(\varphi))\,|\,\varphi\in\mathscr{S}_{(k)}(\mathbf{R}^n,-1),\ \int_0^{\sqrt{-1}\infty}(\mathrm{H}_{(0)i}\varphi)(x_i)dx_i=0,\quad i=1,\ \cdots,\ n\}.$$

*Then we have*

$$\text{(25)}\qquad \mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(k)}=\{f\,|\,f\in\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+},\ \frac{\partial^{i_1+\cdots+i_n}f}{\partial x_1{}^{i_1}\cdots\partial x_n{}^{i_n}}\in\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+},\ i_1\leqq k_1,\ \cdots,\ i_n\leqq k_n\}.$$

***Proof.*** If $g\in\mathscr{S}_{(1)}(\mathbf{R},-1)$, then

$$\begin{aligned}
&2\pi\sqrt{-1}\,\mathrm{H}_0\left(\frac{g'(x^2/4\pi\sqrt{-1})}{x^2}\right)(\sqrt{\xi})\\
&=\int_0^{-1\infty}J_0(\sqrt{-4\pi\sqrt{-1}\,\zeta\xi})g'(\zeta)d\zeta\\
&=[J_0(\sqrt{-4\pi\sqrt{-1}\,\zeta\xi})g(\zeta)]_{\zeta=0}^{\sqrt{-1}\infty}-\int_0^{\sqrt{-1}\infty}\left\{\frac{\partial}{\partial\zeta}J_0\left(\sqrt{-4\pi\sqrt{-1}\,\zeta\xi}\right)\right\}g(\zeta)d\zeta\\
&=-\int_0^{\sqrt{-1}\infty}\left\{\frac{\partial}{\partial\xi}J_0(\sqrt{-4\pi\sqrt{-1}\,\zeta\xi})\right\}g(\zeta)d\zeta\\
&=-\frac{d}{d\xi}\left(\int_0^{\sqrt{-1}\infty}J_0(\sqrt{-4\pi\sqrt{-1}\,\zeta\xi})\,g(\zeta)d\zeta\right)\\
&=-\frac{d}{d\xi}(2\pi\sqrt{-1}\,\mathrm{H}_0\left(\frac{g(x^2/4\pi\sqrt{-1})}{x^2}\right)(\sqrt{\xi}),
\end{aligned}$$

because we know

$$\text{(26)}\qquad 2\pi\sqrt{-1}\,\mathrm{H}_0\left(\frac{g(x^2/4\pi-1)}{x^2}\right)(\sqrt{\xi})=\int_0^{\sqrt{-1}\infty}J_0(\sqrt{-4\pi\sqrt{-1}\,\zeta\xi})\,g(\zeta)d\zeta,$$

since $2\pi\sqrt{-1}\,\mathrm{H}_0(g(x^2/4\pi\sqrt{-1})/x^2)(\sqrt{\xi}) = 2\pi\sqrt{-1}\int_0^\infty J_0(x\sqrt{\xi})\{g(x^2/4\pi\sqrt{-1})/x^2\}\,dx.$ Hence we have the lemma.

***Corollary.*** *If* $T\in(\mathscr{F}(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(k)}))^*$, *then* $\xi_1{}^{-i_1}\cdots\xi_n{}^{-i_n}T\in(\mathscr{F}(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(k+j)}))^*$, $i_1\leqq j_1, \cdots, i_n\leqq j_n$.

***Lemma 7'.*** *If* $f\in\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(\infty)}$, *then* $\partial^{i_1+\cdots+i_n}f/\partial x_1{}^{i_1}\cdots\partial x_n{}^{i_n}\in\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(\infty)}$ *for any* $i_1\geqq 0, \cdots, i_n\geqq 0$ *and if* $T\in(\mathscr{F}(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(\infty)})^*$, *then* $\xi_1{}^{-i_1}\cdots\xi_n{}^{-i_n}T\in(\mathscr{F}(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(\infty)}))^*$, $i_1\geqq 0, \cdots, i_n\geqq 0$.

***Lemma.*** *8. We have*

$$\{\mathrm{H}_{(0)}\Big(\frac{g(x^2/4\pi\sqrt{-1})}{x^2}\Big)(\sqrt{\xi})\,|\,g\in\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(k)}\} = \mathscr{S}_{(2k)}(\mathbf{R}^n, -1). \tag{27}$$

***Proof.*** This follows from (16) and Hankel's formula.

**Note.** By definition, we have

(*) *$P(z)$ vanishes as an element of $\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1))^*$, if $P(z)$ is a polynomial such that* $P(z)=\sum_{i_1\leqq 2k_1,\cdots,i_n\leqq 2k_n}c_{i_1,\cdots,i_n}z_1{}^{i_1}\cdots z_n{}^{i_n}$.

## § 3. *Borel transformation in non-analytic category*

**3.-1. Borel transformation of the elements of $(\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^*$.** ***Definition.*** *Let $T$ be an element of $(\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^*$, then we define its Borel transformation $\mathscr{B}[T]$ by*

$$\mathscr{B}[T](f) = (2\pi\sqrt{-1})^n\mathscr{F}[T](\mathrm{H}_{(0)}\Big(\frac{f(x^2/4\pi\sqrt{-1})}{x^2}\Big)(\sqrt{\xi})),\quad f\in\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}, \tag{28}$$

*where $\mathscr{F}[T]$ is the Fourier transform of $T$.*

By definition, $\mathscr{B}[T]$ is an element of $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+})^*$, and since $\mathscr{B}$ is linear, we have a homomorphism $\mathscr{B}:(\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^*\to(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+})^*$. For this map, we have

***Theorem 1.*** *$\mathscr{B}$ is an isomorphism. That is, we have*

$$\mathscr{B}:(\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^*\cong(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+})^*. \tag{29}$$

***Proof.*** By lemma 6, to set

$$\mathrm{H}_{(0),\,1/2}(f)(\xi)=(2\pi\sqrt{-1})^n\mathrm{H}_{(0)}\Big(\frac{f(x^2/4\pi\sqrt{-1})}{x^2}\Big)(\sqrt{\xi}),\quad f\ni\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}, \tag{30}$$

we have

$$\mathrm{H}_{(0),\,1/2}:\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}\cong\mathscr{S}(\mathbf{R}^n,\,-1).$$

On the other hand, we know that $\mathscr{F}:\mathscr{S}(\mathbf{R}^n,\,-1)\cong\mathscr{F}(\mathscr{S}(\mathbf{R}^n,\,-1))$. Therefore we have the theorem by the definition of $\mathscr{B}$.

**Note.** By (26), we may set

$$(30)'\qquad \mathrm{H}_{(0),\,1/2}(f)(\xi)$$
$$=\int_{\sqrt{-1}\mathbf{R}^n,+}J_0(\sqrt{-4\pi\sqrt{-1}\zeta_1\xi_1})\cdots J_0(\sqrt{-4\pi\sqrt{-1}\zeta_n\xi_n})f(\zeta)d\zeta.$$

**3. -2. Borel transformations of those T which satisfy $|\mathscr{F}[T](x)|=O(e^{-\|x\|^{1/2+\varepsilon}})$, $\varepsilon>0$, $\|x\|\to\infty$.** *Lemma 9. Let $\varphi\in\mathscr{S}(\mathbf{R}^n)$ be holomorphic on $(\mathbf{R}^n,\sqrt{-1})$ and satisfies*

$$(31)\qquad |\mathscr{F}[\varphi](x)|=O(e^{-\|x\|^{1/2+\varepsilon}}),\ \ \varepsilon>0,\ \ \|x\|\to\infty,$$

*then we get as an element of $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+})^*$,*

$$(32)\qquad \mathscr{B}[T_\varphi]=T_{\mathscr{B}[\varphi]},$$

*where $T_\varphi$ and $T_\phi$ are defined by*

$$(33)\qquad T_\varphi[f]=\int_{\mathbf{R}^n}\varphi(x)f(x)dx,\ \ f\in\mathscr{F}(\mathscr{S}(\mathbf{R}^n,\,-1)),$$

$$T_\phi[g]=\int_{\sqrt{-1}\mathbf{R}^n,+}\phi(x)g(x)dx,\ \ g\in\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}.$$

***Proof.*** Since we have by (3)

$$(3)'\qquad \mathscr{B}_x[e^{2\pi\sqrt{-1}x\xi}](\zeta)=J_0(\sqrt{-4\pi\sqrt{-1}\zeta_1\xi_1})\cdots J_0(\sqrt{-4\pi\sqrt{-1}\zeta_n\xi_n}),$$

and since $|J_0(\sqrt{z})|=O(e^{|z|^{1/2+\varepsilon}})$, for any $\varepsilon>0$, $|z|\to\infty$, we get by (26) and (32)

$$(2\pi\sqrt{-1})^n\int_{\mathbf{R}^n}\mathscr{F}[\phi](\xi)\mathrm{H}_{(0)}\left(\frac{f(x^2/4\pi\sqrt{-1})}{x^2}\right)(\sqrt{\xi})d\xi$$
$$=\int_{\sqrt{-1}\mathbf{R}^n,+}\left(\int_{\mathbf{R}^n}\mathscr{F}[\varphi](\xi)J_{(0)}(\sqrt{-4\pi\sqrt{-1}\zeta\xi})d\xi\right)f(\zeta)d\zeta$$
$$=\int_{\sqrt{-1}\mathbf{R}^n,+}\left(\int_{\mathbf{R}^n}\mathscr{F}[\phi](\xi)\mathscr{B}_\eta[e^{2\pi\sqrt{-1}\eta\xi}](\zeta)d\xi\right)f(\zeta)d\zeta$$
$$=\int_{\sqrt{-1}\mathbf{R}^n,+}\mathscr{B}_\eta[\mathscr{F}^*(\mathscr{F}[\varphi])](\zeta)f(\zeta)d\zeta$$
$$=\int_{\sqrt{-1}\mathbf{R}^n,+}\varphi(\zeta)f(\zeta)d\zeta.$$

This shows the lemma.

***Lemma 10.*** (i). If $\mathscr{F}[T]$ *is a function such that* $\mathscr{F}(T)$ *is holomorphic on* $(\mathbf{R}^n, \sqrt{-1})$ *and satisfies* (31), *then* $\mathscr{B}[T]$ *is an entire function.*

(ii). *If* $\mathscr{F}[T]$ *is a function and satisfies*

$$(34) \qquad |\mathscr{F}[T](\xi)| = O(e^{-\langle c, \sqrt{|\xi|}\rangle}), \ \ ||\xi|| \to \infty, \ \ \langle c, \sqrt{|\xi|}\rangle = \sum_{i=1}^{n} c_i\sqrt{|\xi_i|},$$

*then* $\mathscr{B}[T]$ *is holomorphic on the domain* $D$ *given by*

$$D = \{\xi \,|\, (\mathrm{Re.}\,\zeta_i)^2 < \frac{c_i}{4} + c_i|\mathrm{Im.}\,\zeta_i|, \ i = 1, \cdots, n\}.$$

*Here, a function is regarded to be an element of* $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+})^*$ *by* (33).

***Proof.*** Since we know for $x \to \infty$, $x$ is real,

$$|J_0(x)| = \sqrt{\frac{2}{\pi|x|}}\left(|\cos\left(x - sgn(x)\frac{\pi}{4}\right)| + O\left(\frac{1}{x}\right)\right),$$

$$|J_0(\sqrt{-1}x)| = \frac{2}{\pi|x|}\left(\sqrt{e^{2x} + e^{-2x}} + O\left(\frac{1}{x}\right)\right),$$

we have the lemma by lemma 9.

**Note.** For these class of $T$, we may define its Borel transformation by

$$(28)' \qquad \mathscr{B}[T](\zeta) = \int_{\mathbf{R}^n} \mathscr{F}[T](\xi) J_{(0)}(\sqrt{-4\pi\sqrt{-1}\zeta\xi})d\xi.$$

**3.-3. Borel transformation of the elements of** $(\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}_n, -1)))^*$ **and** $(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n}, -1)))^*$. ***Definition.*** *Let* $T$ *be an element of* $(\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1)))^*$ *or* $(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n}, -1)))^*$, *then we define its Borel transformation* $\mathscr{B}[T]$ *by*

$$(28)' \qquad \mathscr{B}[T](f) = (2\pi\sqrt{-1})^n[T](\mathrm{H}_{(0)}\left(\frac{f(x^2/4\pi\sqrt{-1})}{x^2}\right)(\sqrt{\xi})),$$

$$f \in \mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}(k) \ \textit{if} \ T \in (\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1)))^*,$$

$$f \in \mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}(\infty) \ \textit{if} \ T \in (\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n}, -1)))^*.$$

***Theorem 1'.*** *We have*

$$(29)_k \qquad \mathscr{B} : (\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1)))^* \cong (\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}(k))^*,$$

$$\mathscr{B} : (\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n}, -1)))^* \cong (\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}(\infty)).$$

***Proof.*** Since we get by lemma 6' and lemma 8

$$\mathrm{H}_{(0),1/2}:\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(k)}\cong\mathscr{S}_{(2k)}(\mathbf{R}^{n},\,-1),$$

$$\mathrm{H}_{(0),1/2}:\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)}\cong\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\,-1),$$

we have the theorem because $\mathscr{F}:\mathscr{S}_{(2k)}(\mathbf{R}^{n},\,-1)\cong\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^{n},\,-1))$, $\mathscr{F}:\mathscr{S}_{(\infty)}(\mathbf{R}^{n},\,-1)\cong\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\,-1))$.

Since $\mathscr{S}_{(2k)}(\mathbf{R}^{n},\,-1)\subset\mathscr{S}_{(2j)}(\mathbf{R}^{n},\,-1)\subset\mathscr{S}(\mathbf{R}^{n},\,-1)$, $(j)<(k)$, to denote $\tau^{(k)*}$, $\tau^{(j)}{}_{(k)}{}^{*}$, $\mathscr{B}[\tau^{(k)}]^{*}$ and $\mathscr{B}[\tau^{(j)}{}_{(k)}]^{*}$ the maps induced from the inclusions, we have the following commutative diagram

$$\begin{tikzcd}
(\mathscr{F}(\mathscr{S}(\mathbf{R}^{n},\,-1)))^{*} \arrow[r, "\tau^{(j)*}"] \arrow[d, phantom, "\cong\ \mathscr{B}"] \arrow[rr, bend right, "\tau^{(k)*}"'] & (\mathscr{F}(\mathscr{S}_{(2j)}(\mathbf{R}^{n},\,-1)))^{*} \arrow[r, "\tau^{(j)}{}_{(k)}{}^{*}"] \arrow[d, "\cong\ \mathscr{B}"] & (\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^{n},\,-1)))^{*} \arrow[d, "\cong\ \mathscr{B}"] \\
(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+})^{*} \arrow[r, "\mathscr{B}[\tau^{(j)}]^{*}"] \arrow[rr, bend right, "\mathscr{B}[\tau^{(k)}]^{*}"'] & (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(j)})^{*} \arrow[r, "\mathscr{B}[\tau^{(j)}{}_{(k)}]^{*}"] & (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(k)})^{*},
\end{tikzcd}$$

but by definition, we have, for example

(35) $\quad ker.\ \mathscr{B}[\tau^{(k)}]^{*}$

$$=\left\{\sum_{i=1}^{n}\sum_{j_i\leqq k}\xi_i{}^{j_i}\otimes T_{(\xi_1,\,\cdots,\,\xi_{i-1},\,\xi_{i+1},\,\cdots,\,\xi_n)},\ i,\ j_i,\ \ T_{i,\,j_i}\in(\mathscr{F}(\mathscr{S}(\mathbf{R}^{n},\,-1)))^{*}\right\}.$$

On the other hand, we can not define $\tau^{(\infty)*}$ and $\mathscr{B}[\tau^{(\infty)}]^{*}$ because we get

$$\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}\cap\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)}=\{0\},\ \ \mathscr{S}(\mathbf{R}^{n},\,-1)\cap\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\,-1)=\{0\}.$$

***Lemma 9'.*** *Under the same assumptions as lemma 9, (32) is true regarding* $\mathscr{B}$ *to be the map* $(\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^{n},\,-1)))^{*}\rightarrow(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(k)})^{*}$ *or* $(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\,-1)))^{*}\rightarrow(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)})^{*}$.

We denote by $\mathfrak{A}^{n}$ the space of real analytic functions on $\mathbf{R}^{n}$ and let $f$ be an element of $\mathfrak{A}^{n}$. Then we define the maps $i$, $i_{(k)}$ and $i_{(\infty)}$ by

$$i(f)(\varphi)=\int_{\mathbf{R}^{n}}f(x)\varphi(x)dx,\ \ \varphi\in\mathscr{S}(\mathbf{R}^{n},\,-1),$$

$$i_{(k)}(f)(\varphi)=\int_{\mathbf{R}^{n}}f(x)\varphi(x)dx,\ \ \varphi\in\mathscr{S}_{(2k)}(\mathbf{R}^{n},\,-1),$$

$$i_{(\infty)}(f)(\varphi)=\int_{\mathbf{R}^{n}}f(x)\varphi(x)dx,\ \ \varphi\in\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\,-1).$$

We denote the domains of $i$, $i_{(k)}$ and $i_{(\infty)}$ by $\mathfrak{A}^{n}{}_{(0)}$, $\mathfrak{A}^{n}{}_{(k)}$ and $\mathfrak{A}^{n}{}_{(\infty)}$. By definition, $\mathfrak{A}^{n}{}_{(0)}\subset\mathfrak{A}^{n}{}_{(j)}\subset\mathfrak{A}^{n}{}_{(k)}$ if $j<k$.

***Theorem 2.*** *The following diagrams are commutative*

$$\begin{tikzcd} (\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^* \arrow[r, "\mathscr{B}"] & (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}})^* \\ \mathfrak{A}^n{}_{(0)} \arrow[u, "i"] \arrow[r, "\mathscr{B}"] & \mathscr{B}[\mathfrak{A}^n]_{(0)}], \arrow[u, "i'"'] \end{tikzcd} \qquad \begin{tikzcd} (\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1)))^* \arrow[r, "\mathscr{B}"] & (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}}{}^{(k)})^* \\ \mathfrak{A}^n{}_{(k)} \arrow[u, "i_{(k)}"] \arrow[r, "\mathscr{B}"] & \mathscr{B}[\mathfrak{A}^n{}_{(k)}], \arrow[u, "i_{(k)}'"'] \end{tikzcd}$$

$$\begin{tikzcd} (\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n}, -1)))^* \arrow[r, "\mathscr{B}"] & (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}}{}^{(\infty)})^* \\ \mathfrak{A}^n{}_{(\infty)} \arrow[u, "i_{(\infty)}"] \arrow[r, "\mathscr{B}"] & \mathscr{B}[\mathfrak{A}^n{}_{(\infty)}], \arrow[u, "i_{(\infty)}'"'] \end{tikzcd}$$

*Here,* $\mathscr{B}$ *in the under lines are the usual Borel transformation and* $i'$, $i_{(k)}'$ *and* $i_{(\infty)}'$ *are given by*

$$i'(g)(\phi) = \int_{\sqrt{-1}\mathbf{R}^{n,+}} g(x)\phi(x)dx, \quad \phi \in \mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}}$$

$$i_{(k)}'(g)(\phi) = \int_{\sqrt{-1}\mathbf{R}^{n,+}} g(x)\phi(x)dx, \quad \phi \in \mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}}{}^{(k)},$$

$$i_{(\infty)}'(g)(\phi) = \int_{\sqrt{-1}\mathbf{R}^{n,+}} g(x)\phi(x)dx, \quad \phi \in \mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}}{}^{(\infty)}.$$

***Proof.*** If $f$ belongs either of $\mathfrak{A}^n{}_{(0)}$, $\mathfrak{A}^n{}_{(k)}$ or $\mathfrak{A}^n{}_{(\infty)}$, then to set

$$f_c = (\mathscr{F}^*[e^{-c(x_1{}^4 + \cdots + x_n{}^4)}])^* f, \quad c > 0,$$

$f_c$ satisfies the assumptions of lemma 9 or lemma 9′. Therefore we get

$$\mathscr{B}[i(f_c)] = i'\mathscr{B}[f_c], \ f \in \mathfrak{A}^n{}_{(0)}, \ \mathscr{B}[i_{(k)}(f_c)] = i_{(k)}'\mathscr{B}[f_c], \ f_c \in \mathfrak{A}^n{}_{(k)},$$
$$\mathscr{B}[i_{(\infty)}(f_c)] = i_{(\infty)}'\mathscr{B}[f_c], \ f \subset \mathfrak{A}^n{}_{(\infty)},$$

by lemma 9 and lemma 9′. Hence we obtain the theorem because $\lim_{c \to 0} f_c = f$.

**Note.** $i$ is a monomorphism. But *ker.* $(i_{(k)})$ and *ker.* $(i_{(\infty)})$ are both not equal to 0.

**3.–4. Properties of Borel transformation.** We have

(36) $$\mathscr{B}[aS + bT] = a\mathscr{B}[S] + b\mathscr{B}[T], \quad \mathscr{B}[S \otimes T] = \mathscr{B}[S] \otimes \mathscr{B}[T],$$

where $a$, $b$ are constants, for the Borel transformations of $(\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^*$, $(\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1)))^*$ *or* $(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n}, -1)))^*$, because $\mathrm{H}_{(0)}(g \otimes h) = \mathrm{H}_{(0)}(g) \otimes \mathrm{H}_{(0)}(h)$.

***Theorem 3.*** *Let* $T$ *be an element of* $(\mathscr{F}(\mathscr{B}(\mathbf{R}^n, -1)))^*$, *then*

$\partial^{i_1+\cdots+i_n}/\partial x_1^{i_1}\cdots\partial x_n^{i_n}\mathscr{B}[T]$ *is defined to be an element of* $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(k)})^*$, $i_1\leqq k_1$, $\cdots i_n\leqq k_n$ *and we have*

$$\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}\mathscr{B}[T]=\mathscr{B}\Big[\frac{T}{x_1^{i_1}\cdots x_n^{i_n}}\Big], \tag{37}$$

*where* $\partial^{i_1+\cdots+i_n}/\partial x_1^{i_1}\cdots\partial x_n^{i_n}\,[T]$ *is defined by*

$$\Big(\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}\mathscr{B}[T]\Big)(f)=(-1)^{i_1+\cdots+i_n}\mathscr{B}[T]\Big(\frac{\partial^{i_1+\cdots+i_n}f}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}\Big).$$

***Proof.*** *For $n=1$ and $f\in\mathscr{B}_{\sqrt{-1}\mathbf{R}+}{}^{(1)}$, we get*

$$\begin{aligned}\mathscr{B}[T]\Big(\frac{df}{dx}\Big)&=(2\pi\sqrt{-1})\,\mathscr{B}[T](\mathrm{H}_0\Big(\frac{f'(x^2/4\pi\sqrt{-1})}{x^2}\Big)(\sqrt{\xi}))\\&=-2\pi\sqrt{-1}[T]\Big(\frac{d}{d\xi}(\mathrm{H}_0\Big(\frac{f(x^2/4\pi\sqrt{-1})}{x^2}\Big)(\sqrt{\xi}))\\&=-\Big[\frac{T}{x}\Big](\mathrm{H}_0\Big(\frac{f(x^2/4\pi\sqrt{-1})}{x^2}\Big)(\sqrt{\xi})),\end{aligned}$$

Hence we obtain the theorem by lemma 7.

***Corollary.*** *As the element of $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(k)})^*$, $\mathscr{B}[x_1^{i_1}\cdots x_n^{i_n}]=0$ if $i_1\leqq k_1,\cdots,$ $i_n\leqq k_n$, and for some $j$, $i_j\neq k_j$.*

***Proof.*** This follows from the note of 2.-2 and the definition of $\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(k)}$.

***Theorem 3′.*** *If $T$ is an element of $(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},-1)))^*$, then $\mathscr{B}[T]$ is infinitely differentiable as an element of $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)})^*$ and* (37) *is hold.*

***Proof.*** This follows from the proof of theorem 3 and lemma 7′.

***Corollary.*** *As the element of $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)})^*$, $[P(x)]=0$, $P\in\mathbf{C}(x_1,\cdots,x_n)$.*

***Theorem 4.*** *Let $\varphi$ be a (holomorphic) function and $T$ an element either of $(\mathscr{F}(\mathscr{S}(\mathbf{R}^n,-1)))^*$, $(\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n,-1)))^*$ or $(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},-1)))^*$ such that $\varphi T$ is defined to be an element either of $(\mathscr{F}(\mathscr{S}(\mathbf{R}^n,-1)))^*$, $(\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n,-1)))^*$ or $(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},-1)))^*$. Then we have*

$$\mathscr{B}[\varphi T]=\mathscr{B}[\varphi]\,\sharp\,\mathscr{B}[T]. \tag{38}$$

*Here $S\sharp T$ means $\partial^n/\partial x_1\cdots\partial x_n(S^*T)$, where $S^*T$ is taken as an element either of $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+})^*$, $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(k)})^*$ or $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)})^*$.*

***Proof.*** By definition, we have

$$\mathscr{B}[\varphi T](f)=(2\pi\sqrt{-1})^n\mathscr{F}[\varphi T](\mathrm{H}_{(0)}\Big(\frac{f(x^2/4\pi\sqrt{-1})}{x^2}\Big)(\sqrt{\xi}))$$
$$=(2\pi\sqrt{-1})^n(\mathscr{F}[\varphi]\otimes\mathscr{F}[T])(\mathrm{H}_{(0)}\Big(\frac{f(x^2/4\pi\sqrt{-1})}{x^2}\Big)(\sqrt{\xi+\eta}),$$

where $\sqrt{\xi+\eta}=(\sqrt{\xi_1+\eta_1},\ \cdots,\ \sqrt{\xi_n+\eta_n})$.

On the other hand, since we know $\mathscr{B}[fg]=\mathscr{B}[f]\,\sharp\,\mathscr{B}[g]$ for usual Borel transformation, we obtain

$$J_0(\sqrt{-1}\sqrt{2(a+b)\zeta})=\mathscr{B}[e^{(a+b)\eta}](\zeta)=\mathscr{B}[e^{a\eta}](\zeta)\,\sharp\,\mathscr{B}[e^{b\eta}](\zeta)$$
$$=J_0(\sqrt{-1}\sqrt{2a\zeta})\,\sharp\,J_0(\sqrt{-1}\sqrt{2b\zeta}),$$

that is

$$J_0(\sqrt{c\zeta(\xi_1+\xi_2)})=J_0(\sqrt{c\zeta\xi_1})\,\sharp\,J_0(\sqrt{c\zeta\xi_2}). \tag{39}$$

Hence, if $g$ belongs either of $\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}$, $\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(k)}$ or $\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)}$, we get

$$2\pi\sqrt{-1}\,\mathrm{H}_0\Big(\frac{g(x^2/4\pi\sqrt{-1})}{x^2}\Big)(\sqrt{\xi_1+\xi_2})$$
$$=\int_0^{\sqrt{-1}\infty}J_0(\sqrt{-4\pi\sqrt{-1}\,\zeta(\xi_1+\xi_2)})g(\zeta)d\zeta$$
$$=\int_0^{\sqrt{-1}\infty}(J_0(\sqrt{-4\pi\sqrt{-1}\,\zeta\xi_1})\,\sharp\,J_0(\sqrt{-4\pi\sqrt{-1}\,\zeta\xi_2}))g(\zeta)d\zeta$$
$$=-\int_0^{\sqrt{-1}\infty}\int_0^{\zeta}J_0(\sqrt{-4\pi\sqrt{-1}(\zeta-\tau)\xi_1})\ J_0(\sqrt{-4\pi\sqrt{-1}\,\tau\xi_2})d\tau\frac{dg(\zeta)}{d\zeta}d\zeta.$$

Then, since

$$\int_0^{\sqrt{-1}\infty}\int_0^{\sqrt{-1}\infty}\varphi(\xi_1)\phi(\xi_2)\int_0^{\sqrt{-1}\infty}\int_0^{\zeta}J_0(\sqrt{-4\pi\sqrt{-1}\,(\zeta-\tau)\xi_1})J_0(\sqrt{-4\pi\sqrt{-1}\,\tau\xi_2})d\tau$$
$$\frac{dg(\zeta)}{d\zeta}d\zeta d\xi_1 d_2$$
$$=\int_0^{\sqrt{-1}\infty}\int_0^{\zeta}(\int_0^{\sqrt{-1}\infty}J_0(\sqrt{-4\pi\sqrt{-1}\,(\zeta-\tau)\xi_1})\varphi(\xi_1)d\xi_1)(\int_0^{\sqrt{-1}\infty}J_0(\sqrt{-4\pi\sqrt{-1}\,\tau\xi_2})$$
$$\phi(\xi_2)d\xi_2)d\tau\frac{dg(\zeta)}{d\zeta}d\zeta,$$

if $\varphi$ and $\phi$ both rapidly decreasing on $\sqrt{-1}\,\mathbf{R}^+$, we obtain the theorem.

**Note.** By lemma 3 and lemma 3′, if $T$ belongs either of of $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+})^*$, $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(k)})^*$ or $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)})$, then to set

$$\varDelta_{\varepsilon_1,\cdots,\varepsilon_n}=\{z\,|\,\mathrm{Im}.\,z_i>0,\ sgn\,\mathrm{Re}.\,z_i=\varepsilon_i,\ \ \varepsilon_i=\pm 1\},$$

$\delta_{\varepsilon_1,\cdots,\varepsilon_n;c}$ : *the n–chain in* $\varDelta_{\varepsilon_1,\cdots,\varepsilon_n}$ *which joins* $(\varepsilon_1 c_1,\dots,\varepsilon_n c_n)$ *and* $(\varepsilon_1\infty,\ \cdots,\ \varepsilon_n\infty)$, $c=(c_1,\ \cdots,\ c_n)\in\mathbf{R}^{*n,+}$,

there exists a system of functions $\{\tau_{\varepsilon_1,\cdots,\varepsilon_n}\}$ such that each $\tau_{\varepsilon_1,\cdots,\varepsilon_n}$ is holomorphic on $\varDelta_{\varepsilon_1,\cdots,\varepsilon_n}$ and

$$T(f)=\lim_{c\to 0}\sum_{\varepsilon_1,\cdots,\varepsilon_n}\int_{\delta_{\varepsilon_1,\cdots,\varepsilon_n;c}}\tau_{\varepsilon_1,\cdots,\varepsilon_n}(z)f(z)dz.$$

In this case, to define $\varphi\,\sharp\,\tau_{\varepsilon_1,\cdots,\varepsilon_n;c}$ by

$$\varphi\,\sharp\,\tau_{\varepsilon_1,\cdots,\varepsilon_n;c}(z)=\frac{\partial^n}{\partial z_1\cdots\partial z_n}\int_{(\varepsilon_1 c_1,\cdots,\varepsilon_n c_n)}^{z}\varphi(z-\zeta)\tau_{\varepsilon_1,\cdots,\varepsilon_n}(\zeta)d\zeta,$$

$z\in\varDelta_{\varepsilon_1,\cdots,\varepsilon_n}$, we have

$$(\varphi\,\sharp\,T)(f)=\lim_{c\to 0}\sum_{\varepsilon_1,\cdots,\varepsilon_n}\int_{\delta_{\varepsilon_1,\cdots,\varepsilon_n;c}}\varphi\,\sharp\,\tau_{\varepsilon_1,\cdots,\varepsilon_n}(z)f(z)dz.$$

Hence we may define $\varphi\,\sharp\,T$ by

$$\varphi\,\sharp\,T=\lim_{c\to 0}\{\varphi\,\sharp\,\tau_{\varepsilon_1,\cdots,\varepsilon_n;c}\}\ \textit{if } T \textit{ is defined by } \{\tau_{\varepsilon_1,\cdots,\varepsilon_n}\}.$$

## §4. *Product by the elements of* $\mathfrak{k}^n$.

**4.-1. Borel transformation of the elements of** $\mathfrak{k}^n+(\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,\ -1)))^*$. Let $\mathfrak{k}^n\oplus(\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,\ -1)))^*$ be the direct sum of $\mathfrak{k}^n$ and $(\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,\ -1)))^*$, then we set in $\mathfrak{k}^n\oplus(\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,\ -1)))^*$

(40) $\quad \mathfrak{k}^n\cap(\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,\ -1)))^*=\{f\oplus(-T_f)\,|\,f\in\mathfrak{k}^n,\ T_f\in(\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,\ -1))^*\}$,

where $T_f$ is given by $T_f[\varphi]=\int_{\mathbf{R}^n}f(x)\varphi(x)dx$ and assume $T_f$ is defined as an element of $(\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,\ -1)))^*$.

***Definition.*** *We set*

(41) $\quad \mathfrak{k}^n+(\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,\ -1)))^*=(\mathfrak{k}^n\oplus(\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,\ -1)))^*)/(\mathfrak{k}^n\cap(\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,\ -1)))^*)$.

Similarly, in $\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+})^*$, we set

(40)′ $\quad \mathscr{B}[\mathfrak{k}^n]\cap(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+})^*=\{\varphi\oplus(-T_\varphi)\,|\,\varphi\in\mathscr{B}[\mathfrak{k}^n],\ T_\varphi\in(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+})^*$, where $T\varphi(g) = \int_{\sqrt{-1}\mathbf{R}^{n,+}} \varphi(x)\, g(x)dx$, and set

$$(41)' \quad \mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}})^* = (\mathscr{B}[\mathfrak{k}^n] \oplus (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}})^*)/(\mathscr{B}[\mathfrak{k}^n] \cap (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}})^*).$$

By definition, we may consider $\mathfrak{k}^n \cap (\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^*$ to be a submodule either of $\mathfrak{k}^n$ or $(\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^*$ and by theorem 2 and corollaries of theorem 3 and theorem 3′, to define $\widehat{\mathscr{B}} : \mathfrak{k}^n \oplus (\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^* \to \mathscr{B}[\mathfrak{k}^n] \oplus (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}})^*$ by

$$(42)' \qquad \widehat{\mathscr{B}}[\varphi \oplus T] = \mathscr{B}[\varphi] \oplus \mathscr{B}[T],$$

where $\mathscr{B}[\varphi]$ and $\mathscr{B}[T]$ are the Borel transformations in $\mathfrak{k}^n$ and $(\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^*$, we have

$$\widehat{\mathscr{B}}[\mathfrak{k}^n \cap (\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^*] = \mathscr{B}[\mathfrak{k}^n] \cap (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}})^*.$$

Hence to denote the class of $f \oplus T$ in $\mathfrak{k}^n + (\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^*$ by $f + T$ and the class of $\varphi \oplus S$ in $\mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}})^*$ by $\varphi + S$, we may define

***Definition.*** *We define Borel transformation $\mathscr{B} : \mathfrak{k}^n + (\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^* \to \mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}})^*$ by*

$$(42) \qquad \mathscr{B}[f + T] = \mathscr{B}[f] + \mathscr{B}[T].$$

Similarly, we define $\mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1)))^*$, $\mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n}, -1)))^*$, $\mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}}{}^{(k)})^*$ and $\mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}}{}^{(\infty)})^*$ and the maps $\mathscr{B} : \mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1)))^* \to \mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}}{}^{(k)})^*$ and $\mathscr{B} : \mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n}, -1)))^* \to \mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}}{}^{(\infty)})^*$. By definition, the maps $\tau^{(k)*}$ and $\tau^{(j)}{}_{(k)}{}^*$ are extended to be the maps $\hat{\tau}^{(k)*} : \mathfrak{k}^n + (\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^* \to \mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1)))^*$ and $\hat{\tau}^{(j)}{}_{(k)}{}^* : \mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(2j)}(\mathbf{R}^n, -1)))^* \to \mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1)))^*$, $j < k$, and we have the following commutative diagram.

$$\begin{tikzcd}
\mathfrak{k}^n + (\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^* \arrow[r, "\cong"'] \arrow[d, "\hat{\tau}^{(j)*}"] \arrow[dd, bend right=60, "\hat{\tau}^{(k)*}"'] & \mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}})^* \arrow[d, "\mathscr{B}[\dot{\tau}^{(j)}]^*"'] \arrow[dd, bend left=60, "\mathscr{B}[\hat{\tau}^{(k)}]^*"] \\
\mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(2j)}(\mathbf{R}^n, -1)))^* \arrow[r, "\cong"'] \arrow[d, "\hat{\tau}^{(j)}{}_{(k)}{}^*"] & \mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}}{}^{(j)})^* \arrow[d, "\mathscr{B}[\hat{\tau}^{(j)}{}_{(k)}{}^*]"'] \\
\mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1)))^* \arrow[r, "\cong"'] & \mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}}{}^{(k)}).
\end{tikzcd}$$

By theorem 2, theorem 3 and theorem 4, these generalized Borel transformations also satisfy

$$(\mathrm{II}) \qquad \mathscr{B}[a\alpha + b\beta] = a\mathscr{B}[\alpha] + b\mathscr{B}[\beta], \ a, \ b \textit{ are constants},$$

$$\mathscr{B}\alpha \otimes \beta] = \mathscr{B}[\alpha] \otimes \mathscr{B}[\beta], \quad \mathscr{B}[\alpha\beta] = \mathscr{B}[\alpha] \,\sharp\, [\beta], \textit{ if } \alpha\beta \textit{ is defined},$$

$$\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}\mathscr{B}[\alpha]=\mathscr{B}\left[\frac{\alpha}{x_1^{i_1}\cdots x_n^{i_n}}\right],$$

$$\alpha\in\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1)))^* \ \textit{or}\ \alpha\in\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}(\mathbf{R}^n,\ -1)))^*,$$

$$\mathscr{B}\left[\frac{\alpha}{x_1^{i_1}\cdots x_n^{i_n}}\right]\in(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}^{(k)})^*,\quad i_1\leqq k_1,\ \cdots,\ i_n\leqq k_n.$$

**4.-2. Replenishment of the vanishing part of $\mathfrak{k}^n$.** By definitions, there are maps $\hat{\mathfrak{i}}:\mathfrak{k}^n\to\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}(\mathbf{R}^n,\ -1)))^*$, $\hat{\mathfrak{i}}_{(k)}:\mathfrak{k}^n\to\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n,\ -1)))^*$, $\hat{\mathfrak{i}}_{(\infty)}:\mathfrak{k}^n\to\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1)))^*$, $\hat{\mathfrak{i}}':\mathscr{B}[\mathfrak{k}^n]\to\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+})^*$, $\hat{\mathfrak{i}}_{(k)}':\mathscr{B}[\mathfrak{k}^n]\to\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}^{(k)})^*$ and $\hat{\mathfrak{i}}_{(\infty)}':\mathscr{B}[\mathfrak{k}^n]\to\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}^{(\infty)})^*$ and the diagrams

$$\begin{tikzcd}
& (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+})^* \arrow[rr] & & \mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+})^* \\
(\mathscr{F}(\mathscr{S}(\mathbf{R}^n,\ -1)))^* \arrow[rr] \arrow[ur, "\mathscr{B}"] & & \mathfrak{k}^n+(\mathscr{F}(\mathscr{S}(\mathbf{R}^n,\ -1)))^* \arrow[ur, "\mathscr{B}"] & \\
& \mathscr{B}[\mathfrak{A}^n_{(0)}] \arrow[rr] \arrow[uu, "\mathfrak{i}'"] & & \mathscr{B}[\mathfrak{k}^n] \arrow[uu, "\hat{\mathfrak{i}}'"'] \\
\mathfrak{A}^n_{(0)} \arrow[rr] \arrow[uu, "\mathfrak{i}"] \arrow[ur, "\mathscr{B}"] & & \mathfrak{k}^n \arrow[uu, "\hat{\mathfrak{i}}"] \arrow[ur, "\mathscr{B}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}^{(k)})^* \arrow[rr] & & \mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}^{(k)})^* \\
(\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n,\ -1)))^* \arrow[rr, dash] \arrow[ur, "\mathscr{B}"] & & \mathfrak{k}^n+(\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n,\ -1)))^* \arrow[ur, "\mathscr{B}"] & \\
& \mathscr{B}[\mathfrak{A}^n_{(k)}] \arrow[rr] \arrow[uu, "\mathfrak{i}_{(k)}'"] & & \mathscr{B}[\mathfrak{k}^n] \arrow[uu, "\hat{\mathfrak{i}}_{(k)}'"'] \\
\mathfrak{A}^n_{(k)} \arrow[rr, "\mathscr{B}"] \arrow[uu, "\mathfrak{i}_{(k)}"] \arrow[ur, "\mathscr{B}"] & & \mathfrak{k}^n \arrow[uu, "\hat{\mathfrak{i}}_{(k)}"] \arrow[ur, "\mathscr{B}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}^{(\infty)})^* \arrow[rr] & & \mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}^{(\infty)})^* \\
(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1)))^* \arrow[rr] \arrow[ur, "\mathscr{B}"] & & \mathfrak{k}^n+(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1))))^* \arrow[ur, "\mathscr{B}"] & \\
& \mathscr{B}[\mathfrak{A}^n_{(\infty)}] \arrow[rr] \arrow[uu, "\mathfrak{i}_{(\infty)}'"] & & \mathscr{B}[\mathfrak{k}^n] \arrow[uu, "\hat{\mathfrak{i}}_{(\infty)}'"'] \\
\mathfrak{A}^n_{(\infty)} \arrow[rr] \arrow[uu, "\mathfrak{i}_{(\infty)}"] \arrow[ur, "\mathscr{B}"] & & \mathfrak{k}^n \arrow[uu, "\hat{\mathfrak{i}}_{(\infty)}"] \arrow[ur, "\mathscr{B}"] &
\end{tikzcd}$$

are commutative. But, although $\hat{\mathfrak{i}}$ and $\hat{\mathfrak{i}}$ are monomorphisms, $\hat{\mathfrak{i}}_{(k)}$, $\hat{\mathfrak{i}}_{(k)}'$, $\hat{\mathfrak{i}}_{(\infty)}$ and $\hat{\mathfrak{i}}_{(\infty)}'$ are not monomorphisms. In these cases, to set (direct sums are taken as $\mathbf{C}$-vector spaces)

$$\mathfrak{k}^n=\mathfrak{k}^n_{(k)}+\mathit{ker}.\,\hat{\mathfrak{i}}_{(k)},\quad \mathfrak{k}^n=\mathfrak{k}^n_{(\infty)}+\mathit{ker}.\,\hat{\mathfrak{i}}_{(\infty)},$$

we have (although $\mathfrak{k}^n{}_{(k)}$ or $\mathfrak{k}^n{}_{(\infty)}$ are not determined uniquely by $ker.\,\hat{\mathfrak{i}}_{(k)}$ or $ker.\,\hat{\mathfrak{i}}_{(\infty)}$),

$$\mathscr{B}[\mathfrak{k}^n] = \mathscr{B}[\mathfrak{k}^n{}_{(k)}] \oplus ker.\,\hat{\mathfrak{i}}_{(k)}{}', \quad \mathscr{B}[\mathfrak{k}^n] = \mathscr{B}[\mathfrak{k}^n{}_{(\infty)}] \oplus ker.\,\hat{i}_{(\infty)}{}'. \tag{43}$$

Hence to define the maps $\mathscr{B} : ker.\,\hat{\mathfrak{i}}_{(k)} \oplus (k^n + (\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1)))^*) \to ker.\,\hat{\mathfrak{i}}_{(k)}{}' \oplus (\mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(k)})^*)$ or $\mathscr{B} : ker.\,\hat{\mathfrak{i}}_{(\infty)} \oplus (\mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n}, -1)))^*) \to ker.\,\hat{\mathfrak{i}}_{(\infty)}{}' \oplus (\mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(\infty)})^*)$ by

$$\mathscr{B}[f \oplus (g+T)] = \mathscr{B}[f] \oplus \mathscr{B}[g+T], \quad f \in ker.\,\hat{\mathfrak{i}}_{(k)} \ (or\ ker.\,i_{(\infty)}),$$
$$g + T \in \mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1)))^* \ (or\ \mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n}, -1)))^*),$$

the following diagrams are commutative

$$\begin{tikzcd}
\mathfrak{k}^n{}_{(k)} \arrow[r, "\mathscr{B}"] \arrow[d, "\hat{\mathfrak{i}}_{(k)}"] \arrow[dd, bend right=60, "\hat{\mathfrak{i}}"'] & \mathscr{B}[\mathfrak{k}^n{}_{(k)}] \arrow[d, "\hat{\mathfrak{i}}_{(k)}{}'"'] \arrow[dd, bend left=60, "\hat{\mathfrak{i}}'"] \\
\mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1)))^* \arrow[r, "\mathscr{B}"] \arrow[d, "\mathfrak{i}"] & \mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(k)})^* \arrow[d, "i'"'] \\
ker.\,\hat{\mathfrak{i}}_{(k)} \oplus (\mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1)))^*) \arrow[r, "\mathscr{B}"] & ker.\,\hat{\mathfrak{i}}'_{(k)} \oplus (\mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(k)})^*),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{k}^n{}_{(\infty)} \arrow[r, "\mathscr{B}"] \arrow[d, "\hat{\mathfrak{i}}_{(\infty)}"] \arrow[dd, bend right=60, "\hat{\mathfrak{i}}"'] & \mathscr{B}[\mathfrak{k}^n{}_{(\infty)}] \arrow[d, "\hat{\mathfrak{i}}_{(\infty)}{}'"'] \arrow[dd, bend left=60, "\hat{\mathfrak{i}}'"] \\
\mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n}, -1)))^*) \arrow[r, "\mathscr{B}"] \arrow[d, "\mathfrak{i}"] & \mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(\infty)})^* \arrow[d, "\mathfrak{i}'"'] \\
ker.\,\hat{\mathfrak{i}}_{(\infty)} \oplus (\mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n}, -1)))^*) \arrow[r, "\mathscr{B}"] & ker.\,\hat{\mathfrak{i}}_{(\infty)}{}' \oplus (\mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(\infty)})^*).
\end{tikzcd}$$

Here, $\mathfrak{i}$ and $\mathfrak{i}'$ are defined as natural inclusions and $\hat{\mathfrak{i}}$ and $\hat{\mathfrak{i}}'$ are the maps to the second factors. On the other hand, we also obtain following commutative diagrams with exact lines

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{k}^n \arrow[r, "j_{(k)}"] \arrow[d, "\mathscr{B}"] & ker.\,\hat{\mathfrak{i}}_{(k)} \oplus (\mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1)))^*) \arrow[d, "\mathscr{B}"] \\
0 \arrow[r] & \mathscr{B}[\mathfrak{k}^n] \arrow[r, "j_{(k)}{}'"] & ker.\,\hat{\mathfrak{i}}_{(k)}{}' \oplus (\mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(k)})^*),
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{k}^n \arrow[r, "j_{(\infty)}"] \arrow[d, "\mathscr{B}"] & ker.\,\hat{\mathfrak{i}}_{(\infty)} \oplus (\mathfrak{k}^n + (\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n}, -1)))^*) \arrow[d, "\mathscr{B}"] \\
0 \arrow[r] & [\mathfrak{k}^n] \arrow[r, "j_{(\infty)}{}'"] & ker.\,\hat{\mathfrak{i}}_{(\infty)}{}' \oplus (\mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(\infty)})^*).
\end{tikzcd}$$

Here the maps $j_{(k)}$, $j_{(k)}{}'$, $j_{(\infty)}$ and $j_{(\infty)}{}'$ are defined by

$$j_{(k)}(f+g)=f+\hat{\mathfrak{i}}_{(k)}(g),\quad f\in ker.\ \hat{\mathfrak{i}}_{(k)},\quad g\in \mathfrak{k}^n{}_{(k)},$$

$$j_{(\infty)}(f+g)=f+\hat{\mathfrak{i}}_{(\infty)}(g),\quad f\in ker.\ \hat{\mathfrak{i}}_{(\infty)},\quad g\in \mathfrak{k}^n{}_{(\infty)},$$

$$j_{(k)}'(\varphi\oplus\psi)=\varphi\oplus\hat{\mathfrak{i}}_{(k)}'(\psi),\quad \varphi\in ker.\ \mathfrak{i}_{(k)}',\quad \psi\in\mathscr{B}[\mathfrak{k}^n{}_{(k)}],$$

$$j_{(\infty)}'(\varphi\oplus\psi)=\varphi\oplus\hat{\mathfrak{i}}_{(\infty)}'(\psi),\quad \varphi\in ker.\ \hat{\mathfrak{i}}_{(\infty)}',\quad \psi\in\mathscr{B}[\mathfrak{k}^n{}_{(\infty)}].$$

**4.-3. Borel transformation of the elements of $\mathfrak{k}^n(\mathfrak{k}^n+(\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,-1)))^*)$.** We set

(44) $$\mathfrak{k}^n(\mathfrak{k}^n+(\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,-1)))^*)$$
$$=\mathfrak{k}^n\otimes_{(\mathfrak{k}^n\cap(\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,-1)))^*)}(\mathfrak{k}^n+(\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,-1)))^*).$$

Since we may set

$$T(f)=\lim_{c=0}\sum_{\varepsilon_1,\dots,\varepsilon_n}\int_{\delta_{\varepsilon_1,\dots,\varepsilon_n;c}}\tau_{\varepsilon_1,\dots,\varepsilon_n}(z)\bar{f}(z)dz,\quad f\in\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,-1)),$$

for any $T\in(\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,-1)))^*$, where $\tau_{\varepsilon_1,\dots,\varepsilon_n}(z)$ is defined and holomorphic on $\varDelta_{\varepsilon_1,\dots,\varepsilon_n}=\{z\,|\,sgn(\mathrm{Im.}\,z_i)=i,\ i=\pm1\}$, $\delta_{\varepsilon_1,\dots,\varepsilon_n;c}=\mathbf{R}^n+\sqrt{-1}(\varepsilon_1c_1,\dots,\varepsilon_nc_n)$, $c\in\mathbf{R}^{*n,+}$, and $\bar{f}(z)$ is given by $f(\mathrm{Re.}\,z)$, we may define $\varphi T$ by

(45) $$\varphi T=\{\varphi\tau_{\varepsilon_1,\dots,\varepsilon_n}\,|\,\varepsilon_i=\pm1\},$$

where $T$ corresponds to $\{\tau_{\varepsilon_1,\dots,\varepsilon_n}\,|\,\varepsilon_i=\pm1\}$, although $\varphi T\notin(\mathscr{F}^{-}(\mathscr{S}(\mathbf{R}^n,-1)))^*$. Then, since $T$ corresponds to $\{\tau_{\psi;\varepsilon_1,\dots,\varepsilon_n}\}$, where $\tau_{\psi;\varepsilon_1,\dots,\varepsilon_n}$ are given by

$$\tau_{\psi;1,\dots,1}=\psi|\varDelta_{1,\dots,1},\quad \tau_{\psi;\varepsilon_1,\dots,\varepsilon_n}=0,\quad (\varepsilon_1,\cdots,\varepsilon_n)\neq(1,\cdots,1),$$

we heve

(46) $$\varphi T_\psi=T_{\varphi\psi},$$

if $T_{\varphi\psi}$ is defined. Therefore, we may identify $\varphi\otimes T$ and $\varphi T$.

Similarly, we set

(44)′ $$\mathscr{B}[\mathfrak{k}^n]\,\sharp\,(\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}})^*)$$
$$=\mathscr{B}[\mathfrak{k}^n]\otimes_{(\mathscr{B}[\mathfrak{k}^n]\cap(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}})^*)}(\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}})^*).$$

Then, using the correspondence $S\to\{\sigma_{\varepsilon_1,\dots,\varepsilon_n}$, $S\in(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n,+}})^*$, $\sigma_{\varepsilon_1,\dots,\varepsilon_n}$ is defined and holomorphic on $\{z\,|\,\mathrm{Im.}\,z_i<0,\ sgn(\mathrm{Re.}\,z_i)=\varepsilon_i,\ \varepsilon_i=\pm1\}$, we define $f\,\sharp\,S$ by

(45)′ $$f\,\sharp\,S=\{f\,\sharp\,\sigma_{\varepsilon_1,\dots,\varepsilon_n}\,|\,\varepsilon_i=\pm1\},$$

and we have, by the note of **3.-4**,

$$f\,\sharp\,T_g = T_{f\,\sharp\,g}, \tag{46)'}$$

if $T_{f\sharp g}$ is defined. Therefore, we may identify $f\otimes S$ and $f\,\sharp\,S$ in this case.

We define $\mathscr{B}:\mathfrak{k}^n(\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}(\mathbf{R}^n,\ -1)))^*)\to\mathscr{B}[\mathfrak{k}^n]\,\sharp\,(\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+})^*)$ by

$$\mathscr{B}[\varphi T]=\mathscr{B}[\varphi]\,\sharp\,\mathscr{B}[T].$$

Then, to define $i:\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}(\mathbf{R}^n,\ -1)))^*\to\mathfrak{k}^n(\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}(\mathbf{R}^n,\ -1)))^*)$ and $i':\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+})^*\to\mathscr{B}[\mathfrak{k}^n]\,\sharp\,(\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+})^*)$ by $i(f+T)=1(f+T)$ and $i'(\varphi+S)=1\,\sharp\,(\varphi+S)$, where 1 is considered to be an element of $\mathfrak{k}^n$ or $\mathscr{B}[\mathfrak{k}^n]$, we get the following commutative diagram

$$\begin{array}{ccc}
\mathfrak{k}^n(\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}(\mathbf{R}^n,\ -1)))^*) & \xrightarrow{\mathscr{B}} & \mathscr{B}[\mathfrak{k}^n]\,\sharp\,(\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+})^*)\\
i\uparrow & & \uparrow i'\\
\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}(\mathbf{R}^n,\ -1)))^* & \xrightarrow{\mathscr{B}} & \mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+})^*.
\end{array}$$

Similarly, we may define $\mathfrak{k}^n(\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n,\ -1)))^*)$, $\mathfrak{k}^n(\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1)))^*)$, $\mathscr{B}[\mathfrak{k}^n]\,\sharp\,(\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(k)})^*)$ and $\mathscr{B}[\mathfrak{k}^n]\,\sharp\,(\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(\infty)})^*)$ and the Borel transformations $\mathscr{B}:\mathfrak{k}^n(\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n,\ -1)))^*)\to\mathscr{B}[\mathfrak{k}^n]\,\sharp\,(\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(k)})^*)$ and $\mathscr{B}:\mathfrak{k}^n(\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1)))^*)\to\mathscr{B}[\mathfrak{k}^n]\,\sharp\,(\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(\infty)})^*)$.

By definition, we have

***Theorem 5.*** *The spaces* $\mathfrak{k}^n(\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}(\mathbf{R}^n,\ -1)))^*)$, $\mathfrak{k}^n(\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n,\ -1)))^*)$ *and* $\mathfrak{k}^n(\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1)))^*)$ *are* $\mathfrak{k}^n$*-vector spaces and* $\mathscr{B}[\mathfrak{k}^n]\,\sharp\,(\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+})^*)$, $\mathscr{B}[\mathfrak{k}^n]\,\sharp\,(\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(k)})^*)$ *and* $\mathscr{B}[\mathfrak{k}^n]\,\sharp\,(\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(\infty)})^*)$ *are* $\mathscr{B}[\mathfrak{k}^n]$*-modules and therefore* $\mathbf{C}(z_1,\ \cdots,\ z_n)$*-vector spaces, and the Borel transformations between these spaces are satisfy* (II) *of* **4.-1.**

**4.-4. The space $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}{}^{(\infty)})^*_b$ and related Borel transformation.** Since *ker.* $\hat{\mathfrak{i}}_{(k)}$ is not a $\mathfrak{k}^n$-vector space for any $(k)$, we can not extend Borel transformations of *ker.* $\hat{\mathfrak{i}}_{(k)}\oplus(\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n,\ -1)))^*)$ to the Borel transformation of some $\mathfrak{k}^n$-vector space. But, for the space *ker.* $\hat{\mathfrak{i}}_{(\infty)}\oplus(\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1)))^*)$, we can construct a $\mathbf{C}(z_1,\ \cdots,\ z_n)$-vector space which can be considered as a kind of extension of *ker.* $\hat{\mathfrak{i}}_{(\infty)}\oplus(\mathfrak{k}^n+(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1)))^*)$ by the following manner.

Since $\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1))\subset\mathscr{S}(\mathbf{R}^n)$, we get $\mathbf{C}[z_1,\ \cdots,\ z_n]\cap\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1))=\{0\}$. Hence we have

$$\mathbf{C}[z_1,\ \cdots,\ z_n]+\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1))=\mathbf{C}[z_1,\ \cdots,\ z_n]\oplus\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},\ -1)).$$

***Definition.*** *We set*

$$(47)\qquad (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)})^{*}{}_{b}=\mathscr{B}[(\mathbf{C}[z_1,\cdots,z_n]\oplus\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},-1)))^{*}.$$

We define $\mathfrak{k}^n+(\mathbf{C}[z_1,\cdots,z_n]\oplus\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},-1)))^*$ and $\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)})^*{}_b$ similarly as in 4.-1. The inclusions from $\mathfrak{k}^n$ and $\mathscr{B}[\mathfrak{k}^n]$ into $\mathfrak{k}^n+(\mathbf{C}[z_1,\cdots,z_n]\oplus\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{n},-1)))^*$ and $\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)})^*{}_b$ are denoted by $\hat{\mathfrak{i}}_{(\infty),b}$ and $\hat{\mathfrak{i}}_{(\infty),b'}$. Then by definition, we obtain

***Lemma 11.*** *$\hat{\mathfrak{i}}_{(\infty)}(\mathfrak{k}^n)$ is a $\mathbf{C}[z_1,\cdots,z_n]$-modul. That is, if $\varphi\in\mathfrak{k}^n$ and $T_\varphi\in(\mathbf{C}[z_1,\cdots,z_n]\oplus\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},-1)))^*$ is defined, then for any polynomial $P(z_1,\cdots,z_n)$ (or more general, for any algebraic function $a(z_1,\cdots,z_n)$ which has no poles and branching points on $\mathbf{R}^n$), $PT_\varphi=T_{P\varphi}(aT_\varphi=T_{a\varphi})$ is defined to be an element of $(\mathbf{C}[z_1,\cdots,z_n]\oplus\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},-1)))^*$.*

***Corollary.*** *$ker.\,\hat{\mathfrak{i}}_{(\infty),b}$ is a $\widetilde{\mathbf{C}(z_1,\cdots,z_n)}$-vector space.*

By this lemma, we decompose $\mathfrak{k}^n$ as follows

$$(48)\qquad \mathfrak{k}^n=\mathfrak{k}^n{}_{(\infty),b}+ker.\,\hat{\mathfrak{i}}_{(\infty),b},$$

$$\mathfrak{k}^n{}_{(\infty),b}\ \textit{and}\ ker.\,\hat{\mathfrak{i}}_{(\infty),b}\ \textit{are both}\ \widetilde{\mathbf{C}(z_1,\cdots,z_n)}\textit{-vector spaces.}$$

Then, as in 4.-2, we define Borel transformation $\mathscr{B}: ker.\,\hat{\mathfrak{i}}_{(\infty),b}\oplus(\mathfrak{k}^n+(\mathbf{C}[z_1,\cdots,z_n]\oplus\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},-1)))^*)\to ker.\,\hat{\mathfrak{i}}_{(\infty),b'}+(\mathscr{B}[\mathfrak{k}^n]\oplus(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)})^*{}_b)$ and by (48), this Borel transformation is extended as the map $\mathscr{B}:\widetilde{\mathbf{C}(z_1,\cdots,z_n)}\,(ker.\,\hat{\mathfrak{i}}_{(\infty),b'}\oplus(\mathbf{C}[z_1,\cdots,z_n]\oplus\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},-1)))^*)\to\mathscr{B}[\widetilde{\mathbf{C}(z_1,\cdots,z_n)}]\,\sharp\,(ker.\,\hat{\mathfrak{i}}_{(\infty),b'}\oplus(\mathscr{B}[\mathfrak{k}^n]\oplus(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)})^*{}_b)$. Then, since the inclusion maps $j_{(\infty),b}:\hat{\mathfrak{k}}^n\to ker.\,\mathfrak{i}_{(\infty),b}\oplus(\mathfrak{k}^n+(\mathbf{C}[z_1,\cdots,z_n]\oplus\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},-1)))^*)$ and $j_{(\infty),b'}:\mathscr{B}[\mathfrak{k}^n]\to ker.\,\hat{\mathfrak{i}}_{(\infty),b'}\oplus(\mathscr{B}[\mathfrak{k}^n]+(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)})^*{}_b)$ are both monomorphisms, we have the following commutative diagram with exact lines

$$\begin{array}{ccl}
0\to\mathfrak{k}^n & \xrightarrow{j_{(\infty),b}} & \widetilde{\mathbf{C}(z_1,\cdots,z_n)}(ker.\,\hat{\mathfrak{i}}_{(\infty),b}\oplus(\mathfrak{k}^n+(\mathbf{C}[z_1,\cdots,z_n]\oplus(\mathscr{F})\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},-1)))^*)\\
\downarrow\mathscr{B} & & \downarrow\mathscr{B}\\
0\to\mathscr{B}[\mathfrak{k}^n] & \xrightarrow{j_{(\infty),b}} & \mathscr{B}[\widetilde{\mathbf{C}(z_1,\cdots,z_n)}]\,\sharp\,(ker.\,\hat{\mathfrak{i}}_{(\infty),b'}\oplus(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)})^*{}_b).
\end{array}$$

***Theorem 5'.*** *$\mathscr{B}:\widetilde{\mathbf{C}(z_1,\cdots,z_n)}](ker.\,\hat{\mathfrak{i}}_{(\infty),b}\oplus(\mathfrak{k}^n+(\mathbf{C}[z_1,\cdots,z_n]\oplus\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},-1)))^*)\to\mathscr{B}[\widetilde{\mathbf{C}(z_1,\cdots,z_n)}]\,\sharp\,(ker.\,\hat{\mathfrak{i}}_{(\infty),b'}\oplus(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)})^*{}_b)$ satisfies* (II) *of* 4.-1 *and $\partial^{i_1+\cdots+i_n}T/\partial z_1{}^{i_1}\cdots\partial z_n{}^{i_n}$ always exists as an element of $\widetilde{\mathbf{C}(z_1,\cdots,z_n)}(ker.\,\hat{\mathfrak{i}}_{(\infty),b}\oplus(\mathfrak{k}^n+(\mathbf{C}[z_1,\cdots,z_n]\oplus\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^*{}_n,-1)))^*)$ for any $i_1\geqq 0,\cdots,i_n\geqq 0$ if $T\in\widetilde{\mathbf{C}(z_1,\cdots,z_n)}$ $(ker.\,\hat{\mathfrak{i}}_{(\infty),b}\oplus(\mathfrak{k}^n+(\mathbf{C}[z_1,\cdots,z_n]\oplus\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n},-1)))^*)$.*

## § 5. *Borel transformation and inverse Borel transformation of non-analytic functions.*

**5.-1. Non-analytic functions as the elements of $(\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^*$ and $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+})^*$.** ***Definition.*** *Let $f$ be a function on $\mathbf{R}^n$, then we define the elements $\alpha(f)$ and $\beta(f)$ of $(\mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)))^*$ and $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+})^*$ by*

$$\alpha(f)[g] = \int_{\mathbf{R}^n} f(x)g(x)dx, \quad g \in \mathscr{F}(\mathscr{S}(\mathbf{R}^n, -1)), \tag{49}$$

$$\beta(f)[\varphi] = \int_{\sqrt{-1}\mathbf{R}^{n},+} \varphi(z)\left\{\frac{1}{(2\pi\sqrt{-1})^n}\int_{\mathbf{R}^n} \frac{f(x)}{(x_1 - z_1)\cdots(x_n - z_n)}dx\right\}dz, \quad \varphi \in \mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}, \tag{49$'$}$$

*if the integrals in the right hand sides always exist.*

By definition, we obtain

***Lemma 12.*** *If $f$ is measurable and for some $k > 0$, $|(1 + \|x\|)^{-k}f(x)|$ is bounded on $\mathbf{R}^n$, then $\alpha(f)$ is defined. $\beta(f)$ is defined if $f \in L^1(\mathbf{R}^n)$, or for any $\varepsilon_1 > 0, \cdots, \varepsilon_n > 0$, $f(x)/(x_1+\varepsilon_1\sqrt{-1}) \cdots (x_n + \varepsilon_n\sqrt{-1}) \in L^1(\mathbf{R}^n)$.*

**Note.** We may also consider $\alpha(f)$ or $\beta(f)$ to be an element of $(\mathscr{F}(\mathscr{S}_{(2k)}(\mathbf{R}^n, -1)))^*$, $(\mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n}, -1)))^*$, $(\mathbf{C}[z_1, \cdots, z_n] \oplus \mathscr{F}(\mathscr{S}_{(\infty)}(\mathbf{R}^{*n}, -1)))^*$, $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(k)})^*$, $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)})^*$ and $(\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n},+}{}^{(\infty)})^*{}_b$. If there are nescessity to specify these, we denote $\alpha_{(k)}(f)$, $\alpha_{(\infty)}(f)$, $\alpha_{(\infty),b}(f)$, $\beta_{(k)}(f)$, $\beta_{(\infty)}(f)$ and $\beta_{(\infty),b}(f)$.

Since we know for finite exponential type $f$, $\mathscr{B}^{-1}[f]$ is given by

$$\mathscr{B}^{-1}[f](z) = \int_{\mathbf{R}^{n},+} e^{-t}f(zt)dt$$

([2], [9], [11]), and since

$$\int_0^\infty e^{-t}\int_{-\infty}^\infty \frac{f(\xi)}{\xi - zt}d\xi dt = \frac{1}{z}\int_0^\infty e^{-s/z}\int_{-\infty}^\infty \frac{f(\xi)}{\xi - s}d\xi ds, \quad -\frac{\pi}{2} < arg.\, z < \frac{\pi}{2},$$

$$= -\frac{1}{z}\int_0^{-\infty} e^{-s/z}\int_{-\infty}^\infty \frac{f(\xi)}{\xi + s}d\xi ds, \quad \frac{\pi}{2} < arg.\, z < \frac{3\pi}{2},$$

for $f \in L^1(-\infty, \infty)$, we define $\mathscr{B}^{-1}[\beta(f)]$ by

$$\mathscr{B}^{-1}[\beta(f)](x_1, \cdots, x_n) = \frac{1}{|x_1 \cdots x_n|}\int_{\mathbf{R}^{n},+} e^{-\Sigma s/|x_i|}\int_{\mathbf{R}^{n},+} \frac{f(\xi)}{(\xi_1 - (sgn.\, x_1)s_1)\cdots(\xi_n - sgn.\, x_n)s_n)}d\xi ds. \tag{50}$$

Then, since $\mathscr{B}_z[1/(\xi \pm zt)](\zeta) = 1/\xi e^{\pm t\zeta/\xi}$, and for compact support $f$,

$$\int_0^\infty e^{-t}\int_{-\infty}^\infty f(\xi)\frac{1}{\xi}e^{t\zeta/\xi}d\xi dt=\int_{-\infty}^\infty f(\xi)\int_0^\infty \xi e^{t(\zeta/\xi-1)}dtd\xi$$

$$=\int_{-\infty}^\infty \frac{f(\xi)}{\xi-\zeta}d\xi,\ if\ \mathrm{Re.}\left(\frac{\zeta}{\xi}-1\right)<0,$$

$$\int_0^\infty e^{-t}\int_{-\infty}^\infty f(\xi)\frac{1}{\xi}e^{-t\zeta/\xi}d\xi dt=-\int_{-\infty}^\infty \frac{f(\xi)}{\xi-\zeta}d\xi,\ if\ \mathrm{Re.}\left(\frac{\zeta}{\xi}+1\right)>0,$$

we have

$$\mathscr{B}[\mathscr{B}^{-1}[\beta(f)]]=\beta(f), \tag{51}$$

or, in other word, the following diagram is commutative

$$\begin{tikzcd}
(\mathscr{F}(\mathscr{S}(\mathbf{R}^n,-1)))^* \arrow[r, "\mathscr{B}"] & (\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+})^* \\
 & L^1(\mathbf{R}^n). \arrow[u, "\beta"'] \arrow[ul, "\mathscr{B}^{-1}"]
\end{tikzcd}$$

On the other hand, by the definition of $\alpha$, to set $\mathrm{M}(\mathbf{R}^n)=\{f\,|\,f$ *is measurable on* $\mathbf{R}^n$, $|(1+||x||)^{-k}f(x)|$ *is bounded on* $\mathbf{R}^n$ *for some* $k>0\}$, we have the following commutative diagram

$$\begin{tikzcd}
(\mathscr{F}(\mathscr{S}(\mathbf{R}^n,-1)))^* \arrow[r, "\mathscr{B}"] & (\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+})^* & \mathscr{B}[\mathrm{M}(\mathbf{R}^n)\cap\mathscr{O}^n] \arrow[l, "i'"'] \\
\mathrm{M}(\mathbf{R}^n) \arrow[u, "\alpha"] & \mathrm{M}(\mathbf{R}^n)\cap\mathscr{O}^n \arrow[l] \arrow[ur, "\mathscr{B}"']
\end{tikzcd}$$

Here, $\mathscr{B}:\mathrm{M}(\mathbf{R}^n)\cap\mathscr{O}^n\to\mathscr{B}[\mathrm{M}(\mathbf{R}^n)\cap\mathscr{O}^n]$ is the usual Borel transformation.

**Note 1.** Since $\mathscr{B}[\mathrm{M}(\mathbf{R}^n)\cap\mathscr{O}^n]\subset\mathrm{Exp}(\mathrm{C}^n)$, $\beta$ can not defined if $0\neq f\in\mathscr{B}[\mathrm{M}(\mathbf{R}^n)\cap\mathscr{O}^n]$.

**Note 2.** We have same commutative diagrams for the maps $\alpha_{(k)}$, $\beta_{(k)}$, *etc.*.

**5.-2. An application.** It is shown in [1], that if $P(\partial/\partial z)$ is a constant coefficients linear partial differential operator of the form

$$P\left(\frac{\partial}{\partial z}\right)=\frac{\partial^m}{\partial z_1{}^m}+P_1\left(\frac{\partial}{\partial z_2},\cdots,\frac{\partial}{\partial z_n}\right)\frac{\partial^{m-1}}{\partial z_1{}^{m-1}}+\cdots+P_m\left(\frac{\partial}{\partial z_2},\cdots,\frac{\partial}{\partial z_n}\right), \tag{52}$$

then to set $P(z)=\Pi_{i=1}^s(z_1-\sigma_i(z_2,\cdots,z_n))^{r_i}$, $\sigma_i\in\widetilde{\mathrm{C}(z_1,\cdots,z_n)}$, define vector $((1-z_1\sigma(z_2{}^{-1},\cdots,z_n{}^{-1}))^{-1})$ and matrix $T\begin{pmatrix}r_1,\cdots,r_s\\ \sigma_1,\cdots,\sigma_s\end{pmatrix}$ by

$$((1 - z_1\sigma(z_2^{-1}, \cdots, z_n^{-1}))^{-1})$$
$$= ((1 - z_1\sigma_1(z_2^{-1}, \cdots, z_n^{-1}))^{-1}, \cdots, (1 - z_1\sigma_1(z_2^{-1}, \cdots, z_n^{-1}))^{-r_1},$$
$$(1 - z_1\sigma_2(z_2^{-1}, \cdots, z_n^{-1}))^{-1}, \cdots, (1 - z_1\sigma_s(z_2^{-1}, \cdots, z_n^{-1}))^{-r_s}),$$

$$T\begin{pmatrix} r_1, \cdots, r_s \\ \sigma_1, \cdots, \sigma_s \end{pmatrix} = \begin{pmatrix} 1, \cdots\cdots, 1, & 1, \cdots\cdots, 1 \\ 1, \cdots\cdots, c_{1,r_1}\sigma_1, & \sigma_2, \cdots\cdots, c_{1,r_s}\sigma_s \\ 1^2, \cdots\cdots, c_{2,r_1}\sigma_1^2, & \sigma_2^2, \cdots\cdots, c_{2,r_s}\sigma_s \\ \cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots & \\ 1^{m-1}, \cdots\cdots, c_{m,r_1}\sigma_1^{m-1}, & \sigma_2^{m-1}, \cdots\cdots, c_{m,r_s}\sigma_s^{m-1} \end{pmatrix},$$

$$c_{k,\rho_i} = (-1)^{\rho_i - 1}\frac{(k+\rho_i)(k+\rho_i-1)\cdots(k+1)}{(\rho_i-1)!},$$

then the solution of Cauchy problem

$$P\left(\frac{\partial}{\partial z}\right)u = 0,\ u(0, z_2, \cdots, z_n) = g_1(z_2, \cdots, z_n),\ \frac{\partial u}{\partial z_1}(0, z_2, \cdots z_n) \tag{53}$$
$$= g_2(z_2, \cdots, z_n), \cdots, \frac{\partial^{m-1}u}{\partial z_1^{m-1}}(0, z_2, \cdots, z_n) = g_m(z_2, \cdots, z_n),$$

is given by

$$u(z) = \mathscr{B}[\langle((1-\sigma_1(\zeta_2^{-1}, \cdots, \zeta_n^{-1}))^{-1}),\ \mathscr{B}^{-1}[T\begin{pmatrix} r_1, \cdots, r_s \\ \sigma_1, \cdots, \sigma_s \end{pmatrix}g](\zeta)\rangle(z), \tag{54}$$

where $g = (g_1, \cdots, g_m)$ if $\{g_i\}$ satisfies suitable condition. Hence by the commutativity of the diagrams in **4.-3** and **4.-4**, we get

***Theorem 6.*** *If $g_1, \cdots, g_m$ satisfy either of the conditions*

$$T\begin{pmatrix} r_1, \cdots, r_s \\ \sigma_1, \cdots, \sigma_s \end{pmatrix}\beta(g) \in (\mathscr{B}[\mathfrak{k}^{n-1}] \sharp (\mathscr{B}[\mathfrak{k}^{n-1}] + (\mathscr{B}_{\sqrt{-1}R^{n-1},+})^*)^m, \tag{55}$$

$$T\begin{pmatrix} r_1, \cdots, r_s \\ \sigma_1, \cdots, \sigma_s \end{pmatrix}\beta_{(\infty),b}(g) \in (\mathscr{B}[\widetilde{C(z_2, \cdots, z_n)}] \sharp (ker.\ \hat{\mathfrak{i}}_{(\infty),b'} \oplus (\mathscr{B}_{\sqrt{-1}R^{n-1},+^{(\infty)}})^*{}_b)^m, \tag{55$_{(\infty)}$}$$

where $\beta(g) = (\beta(g_1), \cdots, \beta(g_m))$, $\beta_{(\infty),b}(g) = (\beta_{(\infty),b}(g_1), \cdots, \beta_{(\infty),b}(g_m))$ and $(R)^m$ means $m$-direct sum of $R$, then the solution $u(x)$ of the equation (52) with Cauchy data (53) is given by (54) as the element of $\mathscr{B}[\mathfrak{k}^n] \sharp (\mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}R^n,+})^*)$ *or* $\mathscr{B}[\widetilde{C(z_1, \cdots, z_n)}] \sharp (ker.\ \hat{\mathfrak{i}}_{(\infty),b'} \oplus (\mathscr{B}_{\sqrt{-1}R^n,+^{(\infty)}})^*{}_b)$.

**Note.** If $P$ is a system of constant coefficients linear partial differential differential operators, then by the normalization theorem ([16]), by the change of variables, $P$ is equivalent to the system of operators

$$(52)' \qquad P_i\left(\frac{\partial}{\partial z}\right) = \frac{\partial^{m_i}}{\partial z_i^{m_i}} + P_{i,1}\left(\frac{\partial}{\partial z_{h+1}}, \cdots, \frac{\partial}{\partial z_m}\right)\frac{\partial^{m_i-1}}{\partial z_i^{m_i-1}} + \cdots$$

$$+ P_{i,m_i}\left(\frac{\partial}{\partial z_{k+1}}, \cdots, \frac{\partial}{\partial z_m}\right), \quad 1 \leqq i \leqq h,$$

and the solution of (52)′ with Cauchy data

$$(53)' \qquad \frac{\partial^{k_1+\cdots+k_h}u}{\partial z_1^{k_1}\cdots\partial z_h^{k_h}}(0, \cdots, 0, z_{h+1}, \cdots, z_n)$$

$$= g_{k_1+1,\cdots,k_h+1}(z_{h+1}, \cdots, z_n), \quad 0 \leqq k_i \leqq m_i - 1, \quad 1 \leqq i \leqq h,$$

is given by

$$(54)' \qquad u(z) = \mathscr{B}[<(1-\zeta_1\sigma_1(\zeta_{h+1}^{-1}, \cdots, \zeta_n^{-1}))^{-1}, \ (1-\zeta_2\sigma_2(\zeta_{h+1}^{-1}, \cdots, \zeta_n^{-1}))^{-1},$$

$$\cdots, (1-\zeta_h\sigma_h(\zeta_{h+1}^{-1}, \cdots, \zeta_n^{-1}))^{-1}), \ \mathscr{B}^{-1}[T\begin{pmatrix} r_{1,1}, & \cdots, & r_{1,s_1} \\ \sigma_{1,1}, & \cdots, & \sigma_{1,s_1}\end{pmatrix} \otimes$$

$$\cdots \otimes T\begin{pmatrix} r_{h,1}, & \cdots, & r_{h,s_h} \\ \sigma_{h,1}, & \cdots, & \sigma_{h,s_h}\end{pmatrix}(g_1, \cdots, g_h)>],$$

$$P_i(z) = \prod_{j=1}^{s_i}(z_i - \sigma_{i,j}(z_{h+1}, \cdots, z_n))^{r_{i,j}}, \quad 1 \leqq i \leqq h,$$

$$((1 - z_i\sigma_i(z_{h+1}^{-1}, \cdots, z_n^{-1}))^{-1})$$

$$= ((1 - z_i\sigma_{i,1}(z_{h+1}^{-1}, \cdots, z_n^{-1}))^{-1}, \cdots, (1 - z_i\sigma_{i,1}(z_{h+1}^{-1}, \cdots, z_n^{-1}))^{-r_{i,1}},$$

$$(1 - z_i\sigma_{i,2}(z_{h+1}^{-1}, \cdots, z_n^{-1}))^{-1}, \cdots, (1 - z_i\sigma_{i,s_i}(z_{h+1}^{-1}, \cdots, z_n^{-1}))^{-r_{i,s_i}}),$$

$$g_i = (g_{i,1}, \cdots, g_{i,m_i}), \quad 1 \leqq i \leqq h,$$

if $\{g_{i,j}\}$ satisfies suitable condition. Hence we get

***Theorem 6′.*** *The solution of* (52)′ *with data* (53)′ *is given by* (54)′ *as the element of* $\mathscr{B}[\mathfrak{z}^n] \# \mathscr{B}[\mathfrak{k}^n] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+})^*)$ *or* $\mathscr{B}[\widetilde{\mathbf{C}(z_1, \cdots, z_n)} \# (ker.\, \hat{\mathfrak{i}}_{(\infty),b'} \oplus (\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+}^{(\infty)})^{*b})$ *if* $\{g_{i,j}\}$ *satisfies either of the conditions*

$$(55)' \qquad T\begin{pmatrix} r_{1,1}, & \cdots, & r_{1,s_1} \\ \sigma_{1,1}, & \cdots, & \sigma_{1,s_1}\end{pmatrix} \otimes \cdots \otimes T\begin{pmatrix} r_{h,1}, & \cdots, & r_{h,s_h} \\ \sigma_{h,1}, & \cdots, & \sigma_{h,s_h}\end{pmatrix}(\beta(g_1), \cdots, \beta(g_h))$$

$$\in (\mathscr{B}[\mathfrak{k}^{n-h}] \# (\mathscr{B}[\mathfrak{k}^{n-h}] + (\mathscr{B}_{\sqrt{-1}\mathbf{R}^n,+})^*)^{\sum_{i=1}^h m_i},$$

$$(55)_{(\infty)}' \quad T\begin{pmatrix} r_{1,1}, & \cdots, & r_{1,s_1} \\ \sigma_{1,1}, & \cdots, & \sigma_{1,s_1} \end{pmatrix} \otimes \cdots \otimes T\begin{pmatrix} r_{h,1}, & \cdots, & r_{h,s_h} \\ \sigma_{h,1}, & \cdots, & \sigma_{h,s_h} \end{pmatrix} (\beta_{(\infty),b}(g_1), \cdots, \beta_{(\infty),b}(g_h))$$

$$\in (\mathscr{B}[\widetilde{\mathbf{C}(z_{h+1}, \cdots, z_n)}] \sharp (ker.\, \hat{\imath}_{(\infty),b'} \oplus (\mathscr{B}_{\sqrt{-1}\mathbf{R}^{n-h},+}(\infty))^*{}_b))^{\sum_{i=1}^{h} m_i}.$$

Here, in $(54)'$, $g_i$ means $\beta(g_i)$ or $\beta_{(\infty),b}\,(g_i)$.

## *Bibliography*


[1] ASADA, A. : Some extensions of Borel transformation, J. Fac. Sci. Shinshu Univ., 9 (1974), 71–89.

[1]′ ASADA, A. : An explicit formula of the solution of constant coefficients partial differential equation with the meromorphic Cauchy data, J. Fac. Sci. Shinshu Univ., 9 (1974), 91–92.

[2] ASADA, A. : Remarks on inverse Borel transformation, J. Fac. Sci. Shinshu Univ., 10 (1975), 61–70.

[3] AVANISSIAN, V.-GAY, R. : Sur une transformation des fonctionelles analytiques et ses applications aux fonctions entièrs de plusieurs variables, Bull. Soc. math. France, 103 (1975), 341–384.

[4] BOIS RAYMOND, P. DU : Die Theorie der Fourier' schen Integrale und Formeln, Math. Ann., 4 (1871), 362–390.

[5] BOREL, E. : *Leçons sur les série divergents*, Paris, 1928.

[6] DOETSCH, G. : *Theorie und Anwendung der Laplace-Transformation*, Berlin, 1937.

[7] FRDÈLYI, A *et al.* : *Higher Transcendental Functions*, I∼III, New York, 1953.

[8] GATESOUPE, M. : Sur les transformes de Fourier radialles, Bull. Soc. math. France, Mémoire 28 (1971), 1–133.

[9] GUY, D. : Hankel multiplier transformations and weighted p-norms, Trans. Amer. Math. Soc. 97 (1960), 137–189.

[10] HANKEL, H. : Die Fourierschen Reihen und Integralen für Cylinderfunktionen, Math. Ann., 8 (1875), 471–494.

[11] HARDY, G. H. : *Divergent Series*, Oxford, 1948.

[12] LEFSCHETZ, S. : *Algebraic Geometry*, Princeton, 1953.

[13] MARTINEAU, A. : Les hyperfunctions de M. Sato, Seminaire Bourbaki, n° 214, 1960.

[14] MARTINEAU, A. : Sur les fonctionelles analytiques et la transformation de Fourier-Borel, J. Anal. Math., 11 (1963), 1–164.

[15] POLYA, G. : Unterschungen über Lücken und Singularitäten von Potenzreihen, Math. Zeit., 29 (1929), 549–640.

[16] SAMUEL, P.-Zariski, 0. : *Commutative Algebra* II, Princeton, 1960.

[17] TRICOMI, F. : Sulla transformazione e il teoreme di reciprocitè di Hankel, Rend. Accad. Acci Lincei (6), 22 (1935), 564–571.

[18] VOLTERRA, V. : Teoria della potenze dei logaritmi e delle funzioni di composizione, Mem. Acci Lincei (5), 11 (1916), 167–250.

[18]′ VOLTERRA, V. : Functions of composition, The Rice Inst. Pamphlet, 7 (1920), 181–251.
[19] WATSON, G. N. : *A Treaties on the Theory of Bessel Functions*, Cambridge, 1941.
[20] YOSIDA, K. : *Functional Analysis*, Berlin, 1966.