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Generalized Derivatives and Their Integrations, II

By AKIRA ASADA

Department of Mathematics, Faculty of Science,
Shinshu University
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Introduction.

As was announced in the introduction of I of this paper, we treat the integration of $\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(t)$, t is a variable, in this paper. But, since $\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(a)$ is a probabilistic distribution on R^1, we treat the integration (in some sence) of $\mu(t)$, a probability distribution valued function on S, a measurable subset of R^1, in this paper.

In § 4, we first treat the integration of $\log\mathscr{F}[\mu](t)$, or in other word, the integration of $\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(t)$ (cf. § 3 of I). Then, since $\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(t)$ can be considered to be a function of 2-variables and $\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(t,\ 0)=0$, we can reduce the integration of $\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi$, to the integration of Alexander -Spanier 1-cochain $\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(t)(s-t)=\Phi_{d_r\varphi}(t,\ s)$ (cf. 15) and we have

$$\int_a^t \Phi_{d_r\varphi}=\varphi(t)-\varphi(a),$$

under the suitable condition about φ. We also change this to the integration of $\mathscr{F}[\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi](t)$ by using the product integral (cf. [17]).

Next, we consider direct integration of $\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(t)$, or $\mu(t)$. For this purpose, we define the $*$-product integral of $\mu(t)$ on [a, b], denoted by $*-\widehat{\int}_{[a,b]}\mu$, by

$$\left(*-\widehat{\int}_{[a,b]}\mu\right)(f)$$

$$=\lim_{|a_{i+1}-a_i|\to 0}\int_{R^1}\cdots\int_{R^1} f\left(\sum_{i=0}^{m}(a_{i+1}-a_i)t_i\right)d\mu(a_0')_{t_0}\cdots d\mu(a_m')_{t_m},$$

$$a=a_0<a_1<\cdots<a_m<a_{m+1}=b,\quad a_i\leqq a_i'<a_{i+1}.$$

Then we get

$$\left(*-\widehat{\int}_{[a,b]}\mathfrak{d}\,\mathscr{F}_{(R^1)}{}^{+}\varphi\right)(t)=\varphi(\mathrm{b})-\varphi(a),$$

if $\varphi(t)$ is (right) Borel derivable on $[a, b]$ and $B_r\varphi(t)$ is Riemannian integrable on $[a, b]$. In general, if $E(\mu(t))=\int_{R^1} s d\mu(t)_s$ exists for $t\in[a, b]$ and $E(\mu(t))$ is Riemannian integrable on $[a, b]$, then we have

$$*-\widehat{\int}_{[a,b]}\mu=\delta_{\int_b^a E(\mu(t))dt},\ \textit{the Dirac measure concentrated at}\ \int_a^b E(\mu(t))dt.$$

By this reason, we define the $*$-product integral of $\mu(t)$ on S, a measurable set of R^1, by

$$*-\widehat{\int}_S\mu=\delta_{\int_S E(\mu(t))dt},\ \textit{the Dirac measure concentrated at}\ \int_S E(\mu(t))dt,$$

if $E(\mu(t))$ exists almost everywhere on S and $E(\mu(t))$ is Lebesgue integrable on S (§5, n°15).

In §6, we also consider the $*$-product integral of those $\mu(t)$ that do not have $E(\mu(t))$ on S but for some α, $0<\alpha<1$, $\int_{R^1} s^\alpha d\mu(t)_s$ exists almost everywhere on S. But, since we can show that under this condition,

$$\lim_{h\to+0} h^{-\alpha}(\mathscr{F}[\mu(t)](h)-1)=c_{+\alpha},$$
$$\lim_{h\to-0} h^{-\alpha}(\mathscr{F}[\mu(t)](h)-1)=c_{-\alpha},$$

both exist, we consider the $*$-product integral of those $\mu(t)$ that the Fourier transformation $\mathscr{F}[\mu(t)]$ of $\mu(t)$ has the form

$$\begin{aligned}\mathscr{F}[\mu(t)](s)&=1-2\pi\sqrt{-1}e_{\alpha,+}(\mu(t))s^\alpha+o(s^\alpha),\quad s\geqq 0,\\ &=1-2\pi\sqrt{-1}e_{\alpha,-}(\mu(t))s^\alpha+o(|s|^\alpha),\quad s<0,\\ s^\alpha&=|s|^\alpha e^{\alpha\pi\sqrt{-1}},\ \textit{if}\ s<0,\end{aligned}$$

almost everywhere on S. Then we can show that, if $e_{\alpha,+}(\mu(t))$ and $e_{\alpha,-}(\mu(t))$ both absolutely Riemannian integrable on $[a, b]$ and f is the Fourier transformation of a holomorphic function φ such that

$$\int_0^\infty s^{\frac{1-\alpha}{\alpha}}(s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}dst^{\frac{\alpha}{\alpha-1}}\quad \textit{has the mean value in}\ \int_S e_{\alpha,+}(\mu(t))dt\textit{-direction},$$

$$\int_{\infty e^{\sqrt{-1}\alpha\pi}}^{0} s^{\frac{1-\alpha}{\alpha}}(s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}ds t^{\frac{\alpha}{\alpha-1}} \quad \textit{has the mean value in}$$

$$\int_S e_{\alpha,-}(\mu(t))dt\textit{-direction,}$$

we have

$$\lim_{s\to 0}.\frac{1}{s}\lim_{\varepsilon\to 0}.\int_{\varepsilon}^{s}\int_{R^1}\cdots\int_{R^1} f\Big(\sum_{i=0}^{n(h)} ht_i\Big)d\mu(a+\theta_0 h)_{t_0}\cdots$$

$$\cdots d\mu(a+(n(h)+\theta_{n(h)})h)_{t_{n(h)}}dh$$

$$=\frac{1}{\alpha-1}\Big[\int_a^b e_{\alpha,+}(\mu(t))dt\Big]^{\frac{1}{1-\alpha}}.$$

$$\bullet M_{arg.(\int_a^b e_{\alpha,+}(\mu(t))dt)}\Big[\int_0^{\infty} s^{\frac{1-\alpha}{\alpha}}(s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}ds\ t^{\frac{\alpha}{\alpha-1}}\Big]+$$

$$+\frac{1}{\alpha-1}\Big[\int_a^b e_{\alpha,-}(\mu(t))dt\Big]^{\frac{1}{1-\alpha}}.$$

$$\bullet M_{arg.(\int_a^b e_{\alpha,-}(\mu(t))dt)}\Big[\int_{-\infty e^{\sqrt{-1}\alpha\pi}}^{0} s^{\frac{1-\alpha}{\alpha}}(s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}}ds\ t^{\frac{\alpha}{\alpha-1}}\Big]$$

Here, $n(h)$ means $[(b-a)/h]$, the integer part of $(b-a)/h$, θ_i, $0\leqq i\leqq n(h)$ are real numbers such that $0\leqq\theta_i<1$, and $M_\theta[g]$ is the mean value of g in θ-direction, that is

$$M_\theta[g]=\lim_{T\to\infty}.\frac{1}{T\theta}\int_{a\theta}^{(a+T)\theta} g(t)dt.$$

Therefore, if $e_{\alpha,+}(\mu(t))$ and $e_{\alpha,-}(\mu(t))$ both exist almost everywhere on S, a measurable set of $\boldsymbol{R}^1$, and both Lebesgue integrable on S, and if f is the Fourier transformation of a holomorphic function φ (defined on suitable domain of $\boldsymbol{C}^1$) such that

$$\int_0^{\infty} s^{\frac{1-\alpha}{\alpha}}\varphi(s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}ds\ t^{\frac{\alpha}{\alpha-1}} \quad \textit{has the mean value in}$$

$$\int_S e_{\alpha,+}(\mu(t))\ dt\ \textit{-direction,}$$

$$\int_{-\infty e^{\sqrt{-1}\alpha\pi}}^{0} s^{\frac{1-\alpha}{\alpha}}\varphi(s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}ds\ t^{\frac{\alpha}{\alpha-1}} \quad \textit{has the mean value in}$$

$$\int_S e_{\alpha,-}(\mu(t))\ dt\ \textit{-direction,}$$

then we define the $*$ -product integral of $\mu(t)$ on S by

$$(* -\int_S \mu)(f)$$

$$=\frac{1}{\alpha-1}\Big[\int_S e_{\alpha,+}(\mu(t)(dt\Big]^{\frac{1}{1-\alpha}}.$$

$$\cdot M_{arg.(\int_S e_{\alpha,+}(\mu(t))dt)}\Big[\int_{\infty}^{0} s^{\frac{1-\alpha}{\alpha}}(s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}ds\ t^{\frac{1}{1-\alpha}}\Big]+$$

$$+\frac{1}{\alpha-1}\Big[\int_S e_{\alpha,+}(\mu(t))dt\Big]^{\frac{1}{1-\alpha}}.$$

$$\cdot M_{arg.(\int_S e_{\alpha,-}(\mu(t))dt)}\Big[\int_{-\infty e^{\sqrt{-1}\alpha\pi}}^{0} s^{\frac{1-\alpha}{\alpha}}(s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}ds\ t^{\frac{\alpha}{\alpha-1}}\Big].$$

We note that if $e_{\alpha,+}(\mu(t))e_{\alpha,-}(\mu(t))$ $(=e_\alpha(\mu(t)))$ almost everywhere on S, then we may write simbolically (cf. [16])

$$(* - \widehat{\int}_S)(f)$$

$$=\int_{R^1}\varphi(r)\exp.(-2\pi\sqrt{-1}\int_S e_\alpha(\mu(t))(dt)^\alpha r^\alpha)dr,$$

or, in other word, we may set

$$\int_{R^1}\varphi(r)\exp.(-2\pi\sqrt{-1}\int_S e_\alpha(\mu(t))(dt)^\alpha r^\alpha)dr$$

$$=\frac{1}{\alpha-1}\Big[\int_S e_\alpha(\mu(t))\,dt\Big]^{\frac{1}{1-\alpha}}.$$

$$\cdot M_{arg.(\int_S e_\alpha(\mu(t))\,dt)}\Big[\Big(\int_0^{\infty} s^{\frac{1-\alpha}{\alpha}}\varphi(s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}ds+$$

$$+\int_{-\infty e^{\sqrt{-1}\alpha\pi}}^{0} s^{\frac{1-\alpha}{\alpha}}\varphi(s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}ds\Big)t^{\frac{\alpha}{\alpha-1}}\Big].$$

Here, $f=\mathscr{F}[\varphi]$ and φ is assumed to have these mean values.

Since this paper is the continuation of *I*, the numbers of §, formulas and references etc. are continued from *I*. Theorems, formulas and references *etc.* in *I* are refered by their numbers in *I*.

§ 4. ***Integration.***

11. Let $F(t,\ s)$ be a (*continuous*) function of 2-variables such that

$$F(t,0)=0.$$

Then, to set

$$\Phi_F(t_0,\ t_1)=F(t_0, t_1-t_0), \tag{41}$$

Φ_F defines an Alexander -Spanier 1 -cochain of D_0, the intersection of D, the open set on which F is defined, and t_0 -axis.

***Lemma* 16.** *If F is differentiable in s, then*

$$\int_a^b \Phi_F=\int_a^b \frac{\partial F}{\partial s}(t,\ 0)dt, \tag{42}$$

if $[a,\ b]$ is contained in D_0. Here, the integral in the left hand side is the integral of the Alexander-Spanier 1-cochain Φ_F on the chain γ: $[a,\ b]\to[a,\ b]$, the identity map, and the right hand side is the Riemannian integral.

Proof. By assumption, we get

$$F(t,\ s)=\frac{\partial F}{\partial s}(t,\ 0)s+o(|s|),\quad s\downarrow 0.$$

Hence to set $a=a_0<a_1<\cdots<a_m=b$, we have by the definition of the integral (cf. [15]),

$$\begin{aligned}&\int_a^b \Phi_F\\ &=\lim_{|a_{i+1}-a_i|\to 0}\sum_{i=1}^{m}F(a_i,\ (a_{i+1}-a_i))\\ &=\lim_{|a_{i+1}-a_i|\to 0}\sum_{i=1}^{m}\left(\frac{\partial F}{\partial s}(a_i,\ 0)(a_{i+1}-a_i)+o(|a_{i+1}-a_i|)\right)\\ &=\lim_{|a_{i+1}-a_i|\to 0}\sum_{i=1}^{m}\frac{\partial F}{\partial s}(a_i,\ 0)(a_{i+1}-a_i)\\ &=\int_a^b\frac{\partial F}{\partial s}(t,\ 0)dt.\end{aligned}$$

Definition. *If $\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(t)$ exists for all t, $a\leqq t\leqq b$, then we set*

$$d_r\varphi(t,\ s)=(\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(t))(s).$$

We note that by the definition of $\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(t)$, $d_r\varphi(t,\ s)$ is defined on $[a,\ b]\times R^1$ and we have

$$d_r\varphi(t,\ 0)=0. \tag{43}$$

Example. If φ is differentiable on $[a,\ b]$, then

$$d_r\varphi(t, s)=\frac{d\varphi}{dt}(t)s.$$

Theorem 8. *If φ is continuous and (right) Borel derivable almost everywhere on $a \leqq t \leqq b$, and $B_r\varphi(t)$ is Riemannian integrable on $a \leqq t \leqq b$, then*

$$\int_a^t \Phi_{d_r\varphi}=\varphi(t)-\varphi(a), \quad t\leqq b. \tag{44}$$

Proof. By assumption and theorem 7, we have

$$d_r\varphi(t, s)=B_r\varphi(t)(s-t)+o(|s-t|),$$

almost everywhere on $[a, b]$. Hence, to set

$$[a, b]_{\varepsilon,\delta}=\{t|\ t\in[a, b],\ |d_r\varphi(t, s)-B_r\varphi(t)(s-t)|<\varepsilon,\ \text{if } |s-t|<\delta\},$$

we have

$$\int_a^b \chi_{\cup_{\delta>0}[a, b]_{\varepsilon,\delta}}(t)dt=b-a,$$

for any $\varepsilon>0$. Here χ_E is the characteristic function and the integral is the Riemannian integral. Therefore, by the same calculation as in lemma 16, we have

$$\int_a^t \Phi_{d_r\varphi}=\int_a^t B_r\varphi(u)du,$$

because $B_r\varphi$ is Riemannian integrable by assumption.

On the other hand, we know (cf. [9], § 46)

$$\int_a^t B_r\varphi(u)du=\varphi(t)-\varphi(a),$$

holds almost everywhere on $[a, b]$. But, since $\int_a^t B_r\varphi(u)du$ and $\varphi(t)-\varphi(a)$ are both continuous by assumption, we have the theorem.

12. If φ is an Alexander -Spanier 1 -cochain with the value in $\boldsymbol{R}^*$, the multiplicative group of non zero real numbers, (or in C^*), then we can define the product integral $\widehat{\int}_\gamma \varphi$ of φ on γ, a singular 1 -chain, by

$$\widehat{\int}_\gamma \varphi=\lim_{|a_{i+1}-a_i|\to 0} \prod_{i=1}^{m} \varphi(\gamma(a_i), \gamma(a_{i+1})).$$

By definition, if $\widehat{\int}_{\gamma}\varphi_i$, $i=1$, 2, or $\widehat{\int}_{\gamma_i}\varphi$, $i=1$, 2 are both exist, then we have

$$\widehat{\int}_{\gamma}(\varphi_1\varphi_2)=\left(\widehat{\int}_{\gamma}\varphi_1\right)\left(\widehat{\int}_{\gamma}\varphi_2\right),$$

$$\widehat{\int}_{\gamma_1+\gamma_2}\varphi=\left(\widehat{\int}_{\gamma_1}\varphi\right)\left(\widehat{\int}_{\gamma_2}\varphi\right).$$

***Lemma* 17.** *If φ is (an $\boldsymbol{R}^*$ -valued) 1 -cochain of $\boldsymbol{R}^1$, γ is the identity map and φ is smooth in s, then*

$$\widehat{\int}_{\gamma}\varphi=\widehat{\int}_{\gamma}\left(1+\frac{\partial\varphi}{\partial s}(t,\ t)dt\right), \tag{45}$$

where the integral of the right hand side is the usual product integral of $\varphi_s(t,\ t)$ ([17]).

Proof. By assumption, we have

$$\varphi(t,\ s)=1+\frac{\partial\varphi}{\partial s}(t,\ t)(s-t)+o(|s-t|).$$

Therefore, we get (45) by the definition of the product integral.

Corollary. *Under the same assumption*, we have

$$\widehat{\int}_{\gamma}\varphi=\exp.\left(\int_{\gamma}\frac{\partial\varphi}{\partial s}(t,\ t)dt\right). \tag{45$'$}$$

Proof. Since we know

$$\widehat{\int}_{\gamma}\left(1+\frac{\partial\varphi}{\partial s}(t,\ t)dt\right)=\exp.\left(\int_{\gamma}\frac{\partial\varphi}{\partial s}(t,\ t)dt\right),$$

because $\varphi_s(t,\ t)$ is a scalar valued function, we have the corollary.

If φ is $\mathscr{F}(\boldsymbol{R}^1)$ -derivable on $[a,\ b]$, then we set

$$\mathfrak{d}_{\gamma}\varphi(t,\ s)=(\mathscr{F}[\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^{+}\varphi(t)])(s). \tag{46}$$

By definition, if $d_{\gamma}\varphi(t,\ s)$ exists, then we have

$$d_{\gamma}\varphi(t,\ s)=\frac{-1}{2\pi\sqrt{-1}}\log\mathfrak{d}_{\gamma}\varphi(t,\ s). \tag{47}$$

Since $\mathfrak{d}_{\gamma}\varphi(t)$ is the characteristic function of some probability distribution on $\boldsymbol{R}^1$, we get

(43)' $\quad \mathfrak{d}_r\varphi(t,\ 0)=1.$

Therefore, to set

$$\Phi_{\mathfrak{d}_r\varphi}(t_0,\ t_1)=\mathfrak{d}_r\varphi(t_0,\ t_1-t_0),$$

as in n°1, $\Phi_{\mathfrak{d}_r\varphi}$ defines a C^* -valued Alexanded -Spanier 1 -cochain. Hence we can consider $\widehat{\int}_r \Phi_{\mathfrak{d}_r\varphi}$.

Theorem 8'. *If φ satisfies same assumptions as in theorem* 8, *then*

(48) $$\widehat{\int}_{[a,t]}\Phi_{\mathfrak{d}_r\varphi}=\exp.\left(\frac{-1}{2\pi\sqrt{-1}}(\varphi(t)-\varphi(a))\right),\quad t\leqq b.$$

Proof. By assumption and theorem 7, we have

$$\Phi_{\mathfrak{d}_r\varphi}(t,\ s)=1+\frac{-1}{2\pi\sqrt{-1}}B_r\varphi(t)(s-t)+o(|s-t|).$$

Hence we have

$$\widehat{\int}_{[a,\ t]}\Phi_{\mathfrak{d}_r\varphi}=\widehat{\int}_{[a,\ t]}\left(1+\frac{-1}{2\pi\sqrt{-1}}B_r\varphi(u)du\right),$$

by the same reason as in lemma 17 and theorem 8.

Then, since we know (*cf.* (45)')

$$\widehat{\int}_{[a,\ b]}\left(1+\frac{-1}{2\pi\sqrt{-1}}B_r\varphi(u)du\right)$$

$$=\exp.\left(\frac{-1}{2\pi\sqrt{-1}}\int_a^u B_r\varphi(u)du\right),$$

and $\int_a^t B_r\varphi(u)du=\varphi(t)-\varphi(a)$, we obtain the theorem.

13. If $\varphi(t)$ is $\mathscr{F}'(\boldsymbol{R}^1)$ -derivable on a, b, then $\mathfrak{d}_{\mathscr{F}'(\boldsymbol{R}^1)^+}\varphi(t)$ is a probability distribution valued function on $[a,\ b]$. To construct directly the (indefinite) integral of $\mathfrak{d}_{\mathscr{F}'(\boldsymbol{R}^1)^+}\rho(t)$, we define

Definition. *Let $\mu(t)$ be a probability distribution valued function on $[a,\ b]$. Then the $*$ -product integral of $\mu(t)$ at f, denoted by*

$(*-\widehat{\int}_{[a,\ b]}\mu(f)$, *is defined by*

(49) $$(*-\widehat{\int}_{[a,\ b]}\mu)(f)$$

$$=\Big(\lim_{|c_{i+1}-c_i|\to 0}\prod_{i=0}^{m}{}^{*}(\rho_{(c_{i+1}-c_i)}{}^{*}\mu(c_i'))(f),$$

$$\mathrm{a}=c_0<c_1<\cdots<C_m<c_{m+1}=b,\quad c_i\leqq c_i'<c_{i+1},$$

if the limit of the right hand side exists for any partition of $[a, b]$. *Here* $\prod_{i=0}^{m}{}^{*}\nu_i$ *means* $\nu_0 *\nu_1*\cdots*\nu_m$.

Since we know

$$\Big(\sum_{i=0}^{m}{}^{*}(\rho_{(c_{i+1}-c_i)}{}^{*}\mu(c_i'))(f)$$

$$=\int_{R^1}\cdots\int_{R^1}f\Big(\sum_{i=0}^{m}(c_{i+1}-c_i)t_i\Big)d\mu(c_0')_{t_0}\cdots d\mu(c_m')_{t_m},$$

we have that, if $(*-\widehat{\int}_{[a,b]}\mu)(f)$ exists, then

$$(49)'\qquad (*-\widehat{\int}_{[a,b]}\mu(f)$$

$$=\lim_{|c_{i+1}-c_i|\to 0}\int_{R^1}\cdots\int_{R^1}f\Big(\sum_{i=0}^{m}(c_{i+1}-c_i)t_i\Big)d\mu(c_0')_{t_0}\cdots d\mu(c_m')_{t_m}.$$

***Lemma* 18.** $*$ *-product integral has the following properties.*

(i). *If* f *is a constant function* c, *then* $(*-\widehat{\int}_{[a,b]}\mu)(f)$ *exists for any* $\mu(t)$ *and*

$$(50)_{\mathrm{i}}\qquad (*-\widehat{\int}_{[a,b]}\mu)(c)=c.$$

(ii). *If* $\mu(t)$ *has the expectation* $E(\mu(t))$ *for any* t, *and* $E(\mu(t))$ is *Riemannian integrable on* $[a, b]$, *then* $(*-\widehat{\int}_{[a,b]}\mu)(t)$ *exists and*

$$(50)_{\mathrm{ii}}\qquad (*-\widehat{\int}_{[a,b]}\mu)(t)=\int_a^b E(\mu(t))dt.$$

(iii). *If* $\mu(t)$ *satisfies the assumptions of (ii) and* $(*-\widehat{\int}_{[a,b]}\mu)(e^{-2\pi\sqrt{-1}t})$ *exists, then using same notations as in (ii),*

$$(50)_{\mathrm{iii}}\qquad (*-\widehat{\int}_{[a,b]}\mu)(e^{-2\pi\sqrt{-1}t})=\widehat{\int}_{[a,b]}(1-2\pi\sqrt{-1}E(\mu(t))dt)$$

$$=\exp.\Big(\int_a^b-2\pi\sqrt{-1}E(\mu(t))dt\Big).$$

Proof. Since $\int_{R^1} d\mu(t)=1$ for any t, we have (i by (49)'. By (49)', we also obtain

$$\int_{R^1}\cdots\int_{R^1}\Big(\sum_{i=0}^{m}(c_{i+1}-c_i)t_i\Big)d\mu(c_0')_{t_0}\cdots d\mu(c_m')_{t_m}$$
$$=\sum_{i=1}^{m}(c_{i+1}-c_i)\int_{R^1} td\mu(c_i')\prod_{j\neq i}\int_{R^1} d\mu(c_j')$$
$$=\sum_{i=1}^{m}(c_{i+1}-c_i)E(\mu(c_i')).$$

Hence we have (ii). Similarly, since we get

$$\int_{R^1}\cdots\int_{R^1}\exp.(-2\pi\sqrt{-1}\Big(\sum_{i=0}^{m}(c_{i+1}-c_i)t_i\Big)d\mu(c_0')_{t_0}\cdots d\mu(c_m')_{t_m}$$
$$=\sum_{i=0}^{m}\int_{R^1} e^{-2\pi\sqrt{-1}(c_{i+1}-c_i)t}d\mu(c_i'),$$

we obtain (iii). Because by (the proof of) theorem 7, we have

$$\int_{R^1} e^{-2\pi\sqrt{-1}(c_{i+1}-c_i)t}d\mu(c_i')$$
$$=1-2\pi\sqrt{-1}(c_{i+1}-c_i)E\mu(c_i'))+o(|c_{i+1}-c_i|).$$

Theorem 8″. *Under the same assumptions about* φ *as in theorem* 8, *we have*

(i). $(*-\widehat{\int}_{[a,\ s]}\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(u))(t)$ *exists and we have*

(51)$_{\text{i}}$ $$(*-\widehat{\int}_{[a,\ s]}\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(u))(t)=\varphi(s)-\varphi(a).$$

(ii). $(*-\widehat{\int}_{[a,\ s]}\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(u))(e^{-2\pi\sqrt{-1}t})$ *exists and we have*

(51)$_{\text{ii}}$ $$(*-\widehat{\int}_{[a,\ s]}\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(u))(e^{-2\pi\sqrt{-1}t})$$
$$=\exp.(-2\pi\sqrt{-1}\int_a^s B_r\varphi(u)du.$$

Proof. By assumption, $\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(a)$ has the expectation, and we have

(52) $$E(\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(a))=B_r\varphi(a).$$

Hence we obtain (i) by (ii) of lemma 18 (and the proof of theorem 8).

Similarly, by (52) and (iii) of lemma 18, we have (ii).

14. ***Lemma*** 19. *For any integer $k\geqq 1$, we have*

$$m^k-k!\,{}_mC_k=O(m^{k-1}). \tag{53}$$

Proof. Since $k!\,{}_mC_k=m!/(m-k)!=m(m-1)\cdots(m-k+1)$, we get (53).

Lemma **20.** *Let $f(t)$ be (Riemannian) integrable on $[a, b]$ and set $F(s)=\int_a^s f(t)dt$, then*

$$j!\int_a^s\int_a^{t_j}\cdots\int_a^{t_2} f(t_j)\cdots f(t_1)dt_1\cdots dt_j=\{F(s)\}^j. \tag{54}$$

Proof. Since $\int_a^s f(t)\{F(t)\}^{k-1}dt=1/k\cdot\{F(t)\}^k$, we have (54) by induction.

Theorem **9.** *If $\mu(t)$ has the j-th moment $\alpha_j(\mu(t))=\int_{R^1}s^jd\mu(t)_s$ for $j\leqq k$ and $a\leqq t\leqq b$, and assume that, each $\alpha_j(\mu(t))$ is uniformly bounded on $[a,b]$, $1\leqq j\leqq k$, and $\alpha_1(\mu(t))=E(\mu(t))$ is Riemannian integrable on $[a, b]$, then $(*-\overset{\frown}{\int}_{[a,b]}\mu(t^j)$ exists for $j\leqq k$ and we have*

$$(*-\overset{\frown}{\int}_{[a,b]}\mu(t^j)=\{\mathscr{S}(\mu)(b)\}^j. \tag{55}$$

Here, $\mathscr{S}(\mu)$ is given by

$$\mathscr{S}(\mu)(s)=\int_a^s E(\mu(t))dt,\quad s\leqq b.$$

Proof. Since we know

$$\Big(\sum_i c_iu_i\Big)^j=j!\sum_{i_1\neq\cdots\neq i_j}c_{i_1}\cdots c_{i_j}u_{i_1}\cdots u_{i_j}+$$

$$+\sum_{i_\alpha=i_\beta,\text{ for some }\alpha,\beta}c_{i_1}\cdots c_{i_j}u_{i_1}\cdots u_{i_j},$$

we get

$$\int_{R^1}\cdots\int_{R^1}\Big(\sum_i(c_{i+1}-c_i)t_i\Big)^jd\mu(c_0)_{t_0}\cdots d\mu(c_m)_{t_m}$$

$$=j!\sum_{i_1\neq\cdots\neq i_j}(c_{i_1+1}-c_{i_1})\cdots(c_{i_j+1}-c_{i_j})\cdot$$

$$\cdot\int_{R^1}\cdots\int_{R^1}t_{i_1}\cdots t_{i_j}d\mu(c_0)_{t_0}\cdots d\mu(c_m)_{t_m}+$$

$$+\sum_{i_\alpha=i_\beta,\ for\ some\ \alpha,\beta}(c_{i_1+1}-c_{i_1})\cdots(c_{i_j+1}-c_{i_j})\cdot$$

$$\cdot\int_{R^1}\cdots\int_{R^1}t_{i_1}\cdots t_{i_j}d\mu(c_0)_{t_0}\cdots d\mu(c_m)_{t_m}.$$

Then, since each $\mu(c_k)$ is a probability distribution, for the first term of this right hand side, we get

$$\int_{R^1}\cdots\int_{R^1}(t_{i_1}\cdots t_{i_j})_{i_1\neq\cdots\neq i_j}d\mu(c_0)_{t_0}\cdots d\mu(c_m)_{t_m}$$

$$=E(\mu(c_{i_1}))\cdots E(\mu(c_{i_j})).$$

Hence we have

$$\lim_{|c_{i+1}-c_i|\to 0}.\sum_{i_1\neq\cdots\neq i_j}(c_{i_1+1}-c_{i_1})\cdots(c_{i_j+1}-c_{i_j})\cdot$$

$$\cdot\int_{R^1}\cdots\int_{R^1}t_{i_1}\cdots t_{i_j}d\mu(c_0)_{t_0}\cdots d\mu(c_m)_{t_m}$$

$$=\lim_{|c_{i+1}-c_i|\to 0}.\sum_{i_1\neq\cdots\neq i_j}(c_{i_1+1}-c_{i_1})\cdots(c_{i_j+1}-c_{i_j})\cdot$$

$$\cdot E(\mu(c_{i_1}))\cdots E(\mu(c_{i_j}))$$

$$=\int_a^b\int_a^{t_j}\cdots\int_a^{t_2}E(\mu(t_j))\cdots E(\mu(t_1))dt_1\cdots dt_j$$

$$=\frac{1}{j!}\{\mathscr{S}(\mu)(a)\}^j,$$

by lemma 20. On the other hand, for the second term we get

$$\left|\int_{R^1}\cdots\int_{R^1}t_{i_1}\cdots t_{i_j}d\mu(c_0)_{t_0}\cdots d\mu(c_m)_{t_m}\right|$$

$$=|\alpha_{\nu_1}(\mu(t_{\tau_1}))\cdots\alpha_{\nu_\kappa}(\mu(t_{\tau_\kappa}))|\leqq M,$$

$$\nu_1+\cdots+\nu_\kappa=j,\ \ (\tau_1,\cdots,\tau_\kappa)\subset(i_1,\cdots,i_j),$$

for some constant M, by assumption. Because, each $\mu(c_i)$ is a probabilistic distribution. Hence we get

$$\Big|\sum_{i_\alpha=i_\beta,\ for\ some\ \alpha,\beta}(c_{i_1+1}-c_{i_1})\cdots(c_{i_j+1}-c_{i_j})\cdot$$

$$\cdot\int_{R^1}\cdots\int_{R^1}t_{i_1}\cdots t_{i_j}d\mu(c_0)_{t_0}\cdots d\mu(c_m)_{t_m}\Big|$$

$$\leq \sum_{i_\alpha = i_\beta,\ for\ some\ \alpha,\beta} |c_{i_1+1}-c_{i_1}|\cdots|c_{i_j+1}-c_{i_j}|M.$$

Then by lemma 20, we have

$$\lim_{|c_{i+1}-c_i|\to 0}. \sum_{i_\alpha=i_\beta,\ for\ some\ \alpha,\beta} (c_{i_1+1}-c_{i_1})\cdots(c_{i_j+1}-c_{i_j})\cdot$$

$$\cdot\int_{R^1}\cdots\int_{R^1} t_{i_1}\cdots t_{i_j} d\mu(c_0)_{t_0}\cdots d\mu(c_m)t_m$$

$$=0.$$

Therefore, we have the theorem.

§5. Generalization of $*$-product integrals.

15. ***Theorem* 10.** If $\mu(t)=\delta_{\varphi(t)}$ *and* $\varphi(t)$ *is Riemannian integrable on* $[a, b]$, *then* $(*-\widehat{\int}_{[a,b]}\mu(f)$ *exists for any continuous* f *and we have*

$$(56)\qquad *-\widehat{\int}_{[a,b]}\delta_{\varphi(t)}=\delta_{\int_a^b \varphi(t)dt}.$$

Here, $*-\widehat{\int}_{[a,b]}\delta_{\varphi(t)}$ *means the element of* $C(R^1)^*$ *whose value at* f *is* $(*-\widehat{\int}_{[a,b]}\delta_{\varphi(t)}(f)$.

Proof. Since we know

$$\int_{R^1}\cdots\int_{R^1} f\left(\sum_i (c_{i+1}-c_i)t_i\right)\delta_{\varphi(c_0),t_0}\cdots\delta_{\varphi(c_m),t_m}$$

$$=f\left(\sum_i (c_{i+1}-c_i)\varphi(c_i)\right),$$

we have (56).

Corollary. *If* $\varphi(t)$ *is (right) approximately derivable at any point of* $[a, b]$ *and* $AD\ \varphi$ *is Riemannian integrable on* $[a, b]$, *then*

$$(57)\qquad *-\widehat{\int}_{[a,b]}\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(t)=\delta_{(\varphi(b)-\varphi(a))}.$$

To extend (57) for (right) Borel derivable φ, we show

***Lemma* 21.** *Let* $f(t)$ *be given by* $f(t)=\int_{R^1}g(s)\exp(-2\pi\sqrt{-1}st)ds$, $g\in L^1(R^1)$, *then* $(*-\int_{[a,b]}\mu(f)$ *exists if* $\mu(t)$ *has the expectation* $E(\mu(t))$ *for any* $t\in[a, b]$ *and* $E(\mu(t))$ *is Rimannian integrable on* $[a, b]$. *Moreover, we have*

$$(58)\qquad (* -\widehat{\int}_{[a,b]}\mu(f)=f\left(\int_a^b E(\mu(t))dt\right).$$

Proof. By assumption, we have

$$\int_{R^1}\cdots\int_{R^1} f\left(\sum_{i=0}^m (c_{i+1}-c_i)t_i\right)d\mu(c_0')_{t_0}\cdots d\mu(c_m')_{t_m}$$

$$=\int_{R^1}\cdots\int_{R^1}\int_{R^1} g(s)e^{-2\pi\sqrt{-1}(\sum_{i=0}^m (c_{i+1}-c_i)t_i)s}dsd\mu(c_0')_{t_0}\cdots du(c_m')_{t_m}$$

$$=\int_{R^1} g(s)\int_{R^1}\cdots\int_{R^1} e^{-2\pi\sqrt{-1}(\sum_{i=0}^m (c_{i+1}-c_i)t_i)s}du(c_0')_{t_0}\cdots d\mu(c_m')_{t_m}ds$$

$$=\int_{R^1} g(s)\prod_{i=0}^m\int_{R^1} e^{-2\pi\sqrt{-1}(c_{i+1}-c_i)ts}d\mu(c_i')_t ds.$$

Hence by lemma 18, we have

$$(* -\widehat{\int}_{[a,b]}\mu(f)=\int_{R^1} g(s)\exp(-2\pi\sqrt{-1}\,(\int_a^b E(\mu(t))dt)s)ds$$

$$=f(\int_a^b E(\mu(t))dt).$$

By lemma 21, we may consider

$$(58)'\qquad * -\widehat{\int}_{[a,b]}\mu=\delta_{\int_a^b E(\mu(t))dt},$$

if $\mu(t)$ satisfies the assumption of lemma 21.

By (58)$'$, we generalize the notion of $*$-product integral as follows.

Definition. *Let $\mu(t)$ be a probability distribution valued real variable function on S, a measurable set of $\boldsymbol{R}^1$, such that $E(\mu(t))$ exists almost everywhere on S and $E(\mu(t))$ is (Lebesgue) integrable on S. Then we define the $*$-product integral of μ on S, denoted by $* -\widehat{\int}_S \mu$, by*

$$(59)\qquad * -\widehat{\int}_S \mu=\delta_{\int_S E(\mu(t))dt}.$$

Here, $\int_S E(\mu(t))dt$ is the Lebesgue integral of $E(\mu(t))$ on S.

Example. If $\mu(t)$ is given by $\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(t)$, where $\varphi(t)$ is (right) Borel derivable almost everywhere on $[a, b]$ and $B_r(\varphi)(t)$ is measurable on $[a, b]$, then

$$(60)\qquad * -\widehat{\int}_S \mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(t)=\delta_{\int_S B_r(\varphi)(t)dt},$$

if $S\subset[a, b]$ and measurable. Especially, we have

$$(60)' \qquad * -\widehat{\int}_{[a,s]} \mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(t)=\delta_{(\varphi(s)-\varphi(a))},$$

almost everywhere on $[a, b]$, $s\leqq b$. We note that, by (60)′, we get

$$(51)_i' \qquad (* -\widehat{\int}_{[a,s]} \mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(u))(t)=\varphi(s)-\varphi(a).$$

16. ***Theorem* 11.** *$*$-product integral has the following properties.*

(i). *If μ_1 and μ_2 are both $*$-product integrable on S, then $\mu_1*\mu_2$, given by $(\mu_1*\mu_2)(t)=\mu_1(t)*\mu_2(t)$, is also $*$-product integrable on S and we have*

$$(61)_i \qquad * -\widehat{\int}_S (\mu_1*\mu_2)=(* -\widehat{\int}_S \mu_1)*(* -\widehat{\int}_S \mu_2).$$

(ii). *If μ is $*$-product integrable on S, then $\rho_c{}^*$, given by $(\rho_c{}^*\mu)(t)=\rho_c{}^*(\mu(t))$, is also $*$-product integrable on S and we have*

$$(61)_{ii} \qquad * -\widehat{\int}_S \rho_c{}^*\mu=\rho_c{}^*(* -\widehat{\int}_S \mu).$$

(iii). *If S_1 and S_2 are disjoint and μ is $*$-product integrable both on S_1 and S_2, then μ is $*$-product integrable on $S_1\cup S_2$ and we have*

$$(61)_{iii} \qquad * -\widehat{\int}_{S_1\cup S_2} \mu=(* -\widehat{\int}_{S_1} \mu)*(* -\widehat{\int}_{S_2} \mu).$$

(iv). *If $\{\mu_\nu\}$ is a series of probability distribution valued functions on S such that $\mu_\nu(t)$ converges to a probability distribution $\mu(t)$ almost everywhere on S in the $*$-weak topology as the elements of $C_{|x|}(R^1)^*$ and there exists a Lebesgue integrable function $g(t)$ on S such that $|E(\mu_\nu(t))|\leqq g(t)$ for all ν on s, then μ is $*$-product integrable on S and we have in the $*$-weak topology of $C_{|x|}(R^1)^*$,*

$$(61)_{iv} \qquad \lim_{\nu}. * -\widehat{\int}_S \mu_\nu=* -\widehat{\int}_S \mu.$$

Proof. Since we know $E(\mu_1*\mu_2)=E(\mu_1)+E(\mu_2)$, we have

$$\begin{aligned} * -\widehat{\int}_S (\mu_1*\mu_2) &=\delta_{(\int_S E(\mu_1(t))\,dt+\int_S E(\mu_2(t))dt)} \\ &=(\delta_{\int_S E(\mu_1(t))\,dt})*(\delta_{\int_S E(\mu_2(t))\,dt}) \\ &=(* -\widehat{\int}_S \mu_1)*(* -\widehat{\int}_S \mu_2), \end{aligned}$$

which shows (i). Similarly, since we know $E(\rho_c{}^*\mu)=cE(\mu)$ and $\delta_{ca}=\rho_c{}^*(\delta_a)$, we have (ii) and since $\delta_{(a+b)}=\delta_a*\delta_b$, we get (iii).

To show (iv), first we note that $E(\mu(t))$ exists almost everywhere on S and Lebesgue integrable by assumption. Then, by assumption, we have

$$lim._{\nu} \int_S E(\mu_\nu(t))dt=\int_S E(\mu(t))dt.$$

Then, since $lim._\nu f(a_\nu)=f(a)$, if $lim._\nu a_\nu=a$ *and* $f\in C_{|x|}(R^1)$, we obtain (iv).

Note. If $*-\widehat{\int}_S \mu$ is defined to satisfy (iii), (iv) of theorem 11 and

(a) $$*-\widehat{\int}_{[a,\ b]}\mu=\delta_{\int_{[a,\ b]}E(\mu(t))dt},$$ *if* $E(\mu(t))$ *is Riemannian integrable on* $[a,\ b]$.

(b) $$\lim_{m(s)\to 0} *-\widehat{\int}_S \mu=\delta,$$ $m(S)$ *means the Lebesgue measure of* S, then

$*-\int_S \mu$ should be equal to δ_c for some c. Because by (iii) and (b), $*-\widehat{\int}_S \mu$ is infinitely divisible and by (iv), (a) and (b), $|\mathscr{F}[*-\widehat{\int}_S \mu]|=1$ (cf. [7], [8].

To consider the (generalized) derivation of $*-\widehat{\int}_{[a,\ t]}\mu$, we first note that, since $\delta_{a+b}=\delta_a*\delta_b$, we get

$$\delta_a{}^{-1}|_*=\delta_{-a},$$ *where* $\delta_a{}^{-1}|_*$ *means the inverse of* δ_a *by the convolution product.*

Hence we obtain

$$\begin{aligned}&(\lim_{s\to 0}\frac{1}{s}\lim_{\varepsilon\to 0}\int_\varepsilon^s (\rho_t{}^*)^{-1}(\delta_{F(a+t)}*(\delta_{F(a)}{}^{-1}|_*)dt)(f)\\ &=\lim_{s\to 0}\frac{1}{s}\lim_{\varepsilon\to 0}\int_\varepsilon^s f(t^{-1}(F(a+t)-F(a)))dt\\ &=(\mathfrak{d}_{\mathscr{F}(R^1)}{}^+F(a))(f).\end{aligned}$$

Here the limit of the left hand side exists if and only if F is (right) $\mathscr{F}(R^1)$ -derivable at a and $f\in\mathscr{F}(R^1)$.

Especially, if F is given by the indefinit integral $F(t)=\int_a^t \varphi(u)du$, then since we know

$$\mathfrak{d}_{\mathscr{F}(R^1)}{}^+F(b)=\delta_{\varphi(b)},$$

almost everywhere, we get

$$(\lim_{s\to 0}.\frac{1}{s}\lim_{\varepsilon\to 0}.\int_{\varepsilon}^{s}(\rho_t{}^*)^{-1}((*-\widehat{\int}_{[a,\,b+t]}\mu)*((*-\widehat{\int}_{[a,\,b]}\mu)^{-1}|_*)))(f)$$
$$=f(E(\mu(b))),$$

almost everywhere. Hence, if $\mu(t)=\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi(t)$, then we have

$$(\lim_{s\to 0}.\frac{1}{s}\lim_{\varepsilon\to 0}.\int_{\varepsilon}^{s}(\rho_t{}^*)^{-1}((*-\widehat{\int}_{[a,\,b+t]}\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi)*$$
$$*((*-\widehat{\int}_{[a,\,b]}\mathfrak{d}_{\mathscr{F}(R^1)^+}\varphi^{-1}|_*)))(f)$$
$$=f(B_r(\varphi)(b)),$$

almost everywhere.

17. Since we get

$$(*-\widehat{\int}_{S}\mu)(e^{\sqrt{-1}t})=e^{\sqrt{-1}\int_S E(\mu(t))\,dt},$$

we define by $(50)_{\mathrm{iii}}$,

Definition. *If $E(\mu(t))$ is Lebesgue integrable on S, a measurable set in R^1, we define the product integral of $e\sqrt{-1}\,E(\mu(t))$ on S by*

$$\widehat{\int}_{S}(1+\sqrt{-1}E(\mu(t))dt)=e^{\sqrt{-1}\int_S E(\mu(t))dt}. \tag{62}$$

Similarly, if $\varphi(t_0,\ t_1)$ is (a representative of) an Alexander-Spanier 1 -cochain defined on some neighborhood of S, a measurable set of $\boldsymbol{R}^1$, then for sufficiently small s, we may consider a function $F_\varphi(t,\ s)$, $t\in S$, by

$$F_\varphi(t,\ s)=\varphi(t,\ t+s). \tag{63}$$

Moreover, by the definition of Alexander -Spanier cochain, we have

Lemma 22. *$(B_r)_sF_\varphi(t,\ s)|_{s=0}$ does not depend on the choice of (the representative of) φ.*

Definition. *We define $\int_S\varphi$ by*

$$\int_S\varphi=\int_S((B_r)_sF_\varphi(t,\ s)|_{s=0})dt, \tag{64}$$

where the right hand side is the Lebesgue integral on S.

Note. Although φ is an Alexander -Spanier 1 -cochain on a topological space M, if γ is a singular 1 -cochain on M given by $\gamma: I\rightarrow M$, then to set

$$\gamma^{*}\varphi(t_0, t_1)=\varphi(\gamma(t_0), \gamma(t_1)),$$

we have

$$\int_I\gamma^{*}\varphi=\int_\gamma\varphi, \tag{65}$$

by the definition of the integral. But since $\gamma^{*}\varphi$ is anAlexander -Spanier 1 -cochain on I, we can define $F_{\gamma*\varphi}$ and $(B_r)_s F_{\gamma*\varphi}(t, s)$ and we have

$$\int_I\gamma^{*}\varphi=\int_I((B_r)_s F_{\gamma*\varphi}(t, s)|_{s=0})dt,$$

if the left hand side and right hand side (and $(B_r)_s F_{\gamma*\varphi}(t, s)$) both exist. Hence, although $\int_\gamma\varphi$ does not defined in its original sence (cf. [15]), if $(B_r)_s F_{\gamma*\varphi}(t, s)|_{s=0}$ exists and Lebesgue integrable on I, then we define $\int_\gamma\varphi$ by

$$\int_\gamma\varphi=\int_I((B_r)_s F_{\gamma^{*}\varphi}(t, s)|_{s=0})dt. \tag{66}$$

Similarly, since we know to set

$$F_\varphi(t, s_1, \cdots, s_p)=\varphi(t, t+s_1, \cdots, t+s_p),$$
$$t\in I^p=\{(t_1, \cdots, t_p)|0\leqq t_i\leqq 1, 1\leqq i\leqq p\}, \quad s_i\in I^p, \quad 1\leqq i\leqq p,$$

we have

$$\int_{I^p}F_\varphi$$
$$=\int_{I^p}((B_r)_{s_{11}}\cdots(B_r)_{s_{pp}}F_\varphi(t, s_1, \cdots, s_p)|_{s_1=\cdots=s_p=0})dt_1\cdots dt_p,$$
$$s_i=(s_{i1}, \cdots, s_{ip}), \quad 1\leqq i\leqq p,$$

if the both sides exist, we can define $\int_\gamma\varphi$ by

$$\int_\gamma\varphi \tag{66$'$}$$
$$=\int_{I^p}((B_r)_{s_{11}}\cdots(B_r)_{s_{pp}}F_{\gamma*\varphi}(t, s_1, \cdots, s_p)|_{s_1=\cdots=s_p=0})dt_1\cdots dt_p,$$

$$\gamma*\varphi(t_0, t_1, \cdots, t_p)=\varphi(\gamma(t_0), \gamma(t_1), \cdots, \gamma(t_p)).$$

Moreover, if $S\subset I^p$ is Lebesgue measurable, then we can define $\int_{\gamma(s)}\varphi$ by

$$\int_{\gamma(s)}\varphi \tag{66}$$

$$=\int_S((B_r)_{s_{11}}\cdots(B_r)_{s_{pp}}F_{\gamma*\varphi}(t, s_1, \cdots, s_p)|_{s_1=\cdots=s_p=0})dt_1\cdots dt_p,$$

if $(B_r)_{s_{11}}\cdots(B_r)_{s_{pp}}F_{\gamma*\varphi}(t, s_1, \cdots, s_p)|_{s_1=\cdots=s_p=0}$ is Lebesgue integrable on S.

§ 6. Generalized * -product integrals.

18. In this §, we consider the problem to genealize the notion of * -product integrals for those $\mu(t)$ that does not have $E(\mu(t))$. For this purpose,. we first prove

***Lemma* 23.** *Let α be a real number such that $0<\alpha<1$. Then we have*

$$\mathscr{F}[\mu](t)=1+ct_\alpha+o(|t|^\alpha),\ t\downarrow 0, \tag{67}$$

if (and only if) $\int_{R^1}|t|^\alpha d\mu$ exists.

Proof. By assumption, to set

$$\chi_\alpha(\mu)(t)=\int_{-\infty}^t s^\alpha d\mu(s), \qquad s^\alpha=|s|^\alpha,\ s\geqq 0,\ s\geqq 0,\ s^\alpha=|s|^\alpha e^{\alpha\pi\sqrt{-1}},\ s<0,$$

$\chi_\alpha(\mu)(t)$ is continuous and bounded on R^1. Then, since

$$h^{n-\alpha}\int_{-\frac{1}{h}}^{\frac{1}{h}}t^n d\mu(t)$$

$$=h^{n-\alpha}\int_{-\frac{1}{h}}^{\frac{1}{h}}t^{n-\alpha}t^\alpha d\mu(t)$$

$$=h^{n-\alpha}[t^{n-\alpha}\chi_\alpha(\mu)(t)]_{-\frac{1}{h}}^{\frac{1}{h}}-(n-\alpha)h^{n-\alpha}\int_{-\frac{1}{h}}^{\frac{1}{h}}t^{n-\alpha-1}\chi_\alpha(\mu)(t)dt,$$

$\lim._{h\to 0}h^{n-\alpha}\int_{-(1/h)}^{(1/h)}t^n d\mu(t)=c_{n,\alpha}(\mu)$ exists and $|c_{n,\alpha}(\mu)|$ is uniformly bounded for all n, $n\geqq 1$, and the convergence is uniform in n. Because, since

$$\lim_{t\to-\infty}\chi_\alpha(\mu)(t)=0,$$

$$\lim_{t\to\infty}\chi_\alpha(\mu)(t)=E_\alpha(\mu)=\int_{-\infty}^{\infty}t^\alpha d\mu(t),$$

there exist positive numbers $A=A(\varepsilon)$ and $B=B(\varepsilon)$ for given $\varepsilon>0$, such that

$$|\chi_\alpha(\mu)(t)|<\varepsilon, \quad if\ t<B,$$
$$|E_\alpha(\mu)-\chi_\alpha(\mu)(t)|<\varepsilon, \quad if\ t>A,$$

and we have

$$|(n-\alpha)h_1^{n-\alpha}\int_{-\frac{1}{h_1}}^{\frac{1}{h_1}} t^{n-\alpha-1}\chi_\alpha(\mu)(t)dt-$$

$$-(n-\alpha)(h_2^{n-\alpha}\int_{-\frac{1}{h_2}}^{\frac{1}{h_2}} t^{n-\alpha-1}\chi_\alpha(\mu)(t)dt\,|$$

$$=|(n-\alpha)h_1^{n-\alpha}\left(\left(\int_{\frac{1}{h_2}}^{\frac{1}{h_1}}+\int_{-\frac{1}{h_1}}^{-\frac{1}{h_2}}\right)t^{n-\alpha-1}\chi_\alpha(\mu)(t)dt\right)+$$

$$+(n-\alpha)(h_1^{n-\alpha}-h_2^{n-\alpha})\int_{-\frac{1}{h_2}}^{\frac{1}{h_2}} t^{n-\alpha-1}\chi_\alpha(\mu)(t)dt\,|$$

$$\leqq 4\varepsilon,$$

if $1/h_1,\ 1/h_2 \geqq A$ and $-1/h_1,\ -1/h_2 \leqq B$ $(h_1\geqq h_2)$.

On the other hand, if $f(h,\ t)$ is bounded and continuous on $\boldsymbol{R}^1\times[0,\ \delta]$ for some $\delta>0$ and $lim._{h\to 0} f(h,\ t)=0$ (for any fixed t), then we have

$$\lim_{h\to\infty} h^{-\alpha}\left[\left(\int_{\frac{1}{h}}^{\infty}+\int_{-\infty}^{-\frac{1}{h}}\right) f(h,\ t)d\mu(t)\right]=0. \tag{68}$$

Because by assumption, to set

$$\chi_{\alpha,f}(\mu)(h,\ t)=\int_{-\infty}^{t} s^\alpha f(h,\ s)d\mu(s),$$

$\chi_{\alpha,f}(\mu)(h,\ t)$ exists and continuous and bounded on $\boldsymbol{R}^1\times[0,\ \delta]$ and we have (for any fixed t)

$$\lim_{h\to 0} \chi_{\alpha,f}(h,\ t)=0.$$

Then, since we have

$$h^{-\alpha}\int_{\frac{1}{h}}^{\infty} f(h,\ t)d\mu(t)$$

$$=h^{-\alpha}\int_{\frac{1}{h}}^{\infty} t^{-\alpha}t^\alpha f(h,\ t)d\mu(t)$$

$$=h^{-\alpha}[t^{-\alpha}\chi_{\alpha,f}(\mu)(h,\ t)]_{\frac{1}{h}}^{\infty}-h^{-\alpha}\int_{\frac{1}{h}}^{\infty}t^{-\alpha-1}\chi_{\alpha,f}(\mu)(h,\ t)dt,$$

we get $\lim._{h\to\infty}\int_{1/h}^{\infty}f(h,\ t)d\mu(t)=0$. Similarly we have $\lim._{h\to\infty}\int_{-\infty}^{-1/h}f(h,\ t)d\mu(t)=0$ and we obtain (68).

Then, since

$$\lim_{h\to 0}.\ h^{-\alpha}(\mathscr{F}[\mu](h)-1)$$

$$=\lim_{h\to 0}.\ h^{-\alpha}\left(\int_{-\infty}^{\infty}e^{-2\pi\sqrt{-1}sh}d\mu(s)-1\right)$$

$$=\lim_{h\to 0}.\ h^{-\alpha}\left(\int_{-\infty}^{\infty}(e^{-2\pi\sqrt{-1}sh}-1)d\mu(s)\right)$$

$$=\lim_{h\to 0}.\ h^{-\alpha}\left(\int_{-\frac{1}{h}}^{\frac{1}{h}}(e^{-2\pi\sqrt{-1}sh}-1)d\mu(s)+\left(\int_{\frac{1}{h}}^{\infty}+\int_{-\infty}^{-\frac{1}{h}}\right)(e^{-2\pi\sqrt{-1}sh}-1)d\mu(s)\right)$$

$$=\lim_{h\to 0}.\ h^{-\alpha}\left(\int_{-\frac{1}{h}}^{\frac{1}{h}}(e^{-2\pi\sqrt{-1}sh}-1)d\mu(s)\right)\qquad\text{by (68)}$$

$$=\lim_{h\to 0}.\ h^{-\alpha}\left(\sum_{n\geqq 1}\frac{(-2\pi\sqrt{-1})^{n}}{n!}h^{n}\int_{-\frac{1}{h}}^{\frac{1}{h}}s^{n}d\mu(s)\right)$$

$$=\lim_{h\to 0}.\ \sum_{n\geqq 1}\frac{(-2\pi\sqrt{-1})^{n}}{n!}h^{n-\alpha}\int_{-\frac{1}{h}}^{\frac{1}{h}}s^{n}d\mu(s)$$

$$=\sum_{n\geqq 1}\frac{(-2\pi\sqrt{-1})^{n}}{n!}c_{n,\alpha}(\mu),$$

we have the lemma.

Note. Similarly, under the same assumption, we can show the expansion

(67)′ $$\mathscr{F}[\mu](t)=1+c't^{\alpha}+o(|t|^{\alpha}),\quad t\uparrow 0.$$

In the rest, we set

(69) $$e_{\alpha,+}(\mu)=\frac{1}{2\pi\sqrt{-1}}\lim_{h\to+0}.\ h^{-\alpha}(\mathscr{F}[\mu](h)-1),$$

$$e_{\alpha,-}(\mu)=\frac{1}{2\pi\sqrt{-1}}\lim_{h\to-0}.\ h^{-\alpha}(\mathscr{F}[\mu](h)-1).$$

Note. if $h<0$, we set $h^{-\alpha}=|h|^{-\alpha}e^{\alpha\pi\sqrt{-1}}$. Hence $e_{\alpha,+}\neq e_{\alpha,-}$ in general.

By definition, we have

$$e_\beta(\mu)=0,\ \ \textit{if}\ e_\alpha(\mu)\ \textit{exists and}\ \beta<\alpha,\ \ 0<\alpha,\ \beta<1, \tag{70}$$

Here $e_\beta(\mu)$ means $e_{\beta,+}(\mu)$ and $e_{\beta,-}(\mu)$.

Hence we have

***Lemma* 24.** *$e_{\alpha,+}(\mu)$ and $e_{\alpha,-}(\mu)$ do not exist if $e_{\alpha,+}(\mu)\neq 0$* or *$e_{\alpha,-}(\mu)\neq 0$ and $\gamma>\alpha$, $0<\gamma<1$.*

Definition. *For a probability distribution μ on $\boldsymbol{R}^1$, we set*

$$\alpha(\mu)=\sup_\beta.\{\beta\,|\,|e_{\beta,+}(\mu)|=0\ \textit{and}\ |e_{\beta,-}(\mu)|=0,\ \ 0<\beta<1\}. \tag{71}$$

19. ***Lemma* 25.** *Let $a=c_0<c_1<\cdots<c_m<c_{m+1}=b$ be a partition of the interval $[a, b]$, α a real number such that $0<\alpha<1$, then*

$$\lim_{|c_{i+1}-c_i|\to 0}.\ \max_i.(|c_{i+1}-c_i|^{1-\alpha})\sum_{i=0}^{m}|c_{i+1}-c_i|^\alpha=b-a, \tag{72}$$

if and only if the partition satisfies

$$\lim_{|c_{i+1}-c_i|\to 0}.\ \frac{\max_i.\ (|c_{i+1}-c_i|^{1-\alpha})}{\min_i.\ (|c_{i+1}-c_i|^{1-\alpha})}=1. \tag{73}$$

Proof. Since we know

$$(\max_i.(|c_{i+1}-c_i|^{1-\alpha}))\sum_{i=0}^{m}|c_{i+1}-c_i|^\alpha$$

$$\geqq\sum_{i=0}^{m}|c_{i+1}-c_i|\ \ (=b-a)$$

$$\geqq(\min_i.\ (c_{i+1}-c_i|^{1-\alpha}))\sum_{i=0}^{m}|c_{i+1}-c_i|^\alpha,$$

we have the sufficiency.

Conversely, to set

$$c_{k,0}=a,\ \ c_{k,1}=a+(b-a)k^{-c},\ \ 0<c<1,$$

$$c_{k,i}=a+(b-a)k^{-c}+(i-1)\frac{(b-a)(1-k^{-c})}{k},\ \ 2\leqq i\leqq k+1,$$

we have

$$(\max_i.\ (|c_{k,i+1}-c_{k,i}|^{1-\alpha}))\sum_{i=0}^{k}|c_{k,i+1}-c_{k,i}|^\alpha$$

$$=(b-a)^{1-\alpha}k^{-c(1-\alpha)}(b-a)^{\alpha}(k^{-c\alpha}+(k-1)k^{-\alpha}(1-k^{-c})^{\alpha})$$
$$=(b-a)(k^{-c}+k^{(1-c)(1-\alpha)}+\frac{k^{1-\alpha}((1-k^{-c})^{\alpha}-1)-k^{-\alpha}(1-k^{-c})^{\alpha})}{k^{c(1-\alpha)}}.$$

Hence to get (72), it should be $lim._{k\to\infty}k^{(1-c)(1-\alpha)}=1$. But, since $min._i|c_{k,i+1}-c_{k,i}|=(b-a)(1-k^{-c})/k$, we get

$$\frac{\underset{i}{max.}(|c_{k,i+1}-c_{k,i}|^{1-\alpha})}{\underset{i}{min.}(|c_{k,i+1}-c_{k,i}|^{1-\alpha})}=k^{(1-c)(1-\alpha)}(1-k^{-c})^{\alpha-1},$$

(73) is necessary to get (72).

In the rest, we assume $\mu(t)$ is a probability distribution valued function defined on $[a, b]$ and satisfies the following conditions (i), (ii) and (iii).

(i). $\alpha(\mu(t))\neq 0$ *for any* t.

(ii). $e_{\alpha(\mu(t)),+}(\mu(t))$ *and* $e_{\alpha(\mu(t)),-}(\mu(t))$ *both exist for any* t *and the functions* $e_{\alpha(\mu(t)),+}(\mu(t))$ *and* $e_{\alpha(\mu(t)),-}(\mu(t))$ *are both absolutely Riemannian integrable on* $[a, b]$.

(iii). *To set* $\alpha_0(\mu)=inf._{t\in[a,b]}\alpha(\mu(t))$, $\alpha_0(\mu)\neq 0$, *and for any* t, *we have*

$$\mathscr{F}[\mu(t)](h)=1-2\pi\sqrt{-1}e_{\alpha(\mu(t)),+}(\mu(t))h^{\alpha(\mu(t))}+o(h^{\alpha_0(\mu)}),\quad h\uparrow 0,$$
$$=1-2\pi\sqrt{-1}e_{\alpha(\mu(t)),-}(\mu(t))h^{\alpha(\mu(t))}+o(|h|^{\alpha_0(\mu)}),\quad h\downarrow 0.$$

Under these conditions, if $\varphi(t)$ is given by the Fourier transformation of an entire function with suitable conditions about the degree of increase at ∞ (in some (suitable) direction), that is $\varphi(t)=\int_{R^1}f(r)e^{-2\pi\sqrt{-1}tr}dr$, where $f(r)$ is the entire function with the above condition. Then we have

$$\int_{R^1}\cdots\int_{R^1}\varphi\Big(\sum_{i=0}^{n(h)}ht_i\Big)d\mu(a)_{t_0}\cdots d\mu(a+n(h)h)_{t_{n(h)}}$$
$$=\int_{R^1}\cdots\int_{R^1}\int_{R^1}f(r)e^{-2\pi\sqrt{-1}r(\sum_{i=0}^{n(h)}ht_i)}drd\mu(a)_{t_0}\cdots d\mu(a+n(h)h)_{t_{n(h)}}$$
$$=\int_{R^1}f(r)\prod_{i=0}^{n(h)}\int_{R^1}e^{-2\pi\sqrt{-1}rht}d\mu(a+ih)(t)dr,$$

Where $n(h)=[b(-a)/h]$. In this last formula, we get by (iii),

$$\int_{R^1}e^{-2\pi\sqrt{-1}rht}d\mu(a+ih)(t)$$
$$=1-2\pi\sqrt{-1}e_{\alpha(\mu(a+ih)),\varepsilon}(\mu(a+ih))r^{\alpha(\mu(a+ih))}h^{\alpha(\mu(a+ih))}+$$
$$o(|r|^{\alpha_0(\mu)}|h|^{\alpha_0(\mu)})$$
$$=1-2\pi\sqrt{-1}e_{\alpha(\mu(a+ih)),\varepsilon}(\mu(a+ih))h^{\alpha(\mu(a+ih))-1}r^{\alpha(\mu(a+ih))}h+$$

$$+o(|h|^{\alpha_0(\mu)-1}|r|^{\alpha_0(\mu)}|h|),$$

where ε is either $+$ (if $h\geqq 0$) or $-$ (if $h<0$). Hence we have

$$\prod_{i=0}^{n(h)}\int_{R^1} e^{-2\pi\sqrt{-1}rht}d\mu(a+ih)(t)$$

$$=\widehat{\int}_{[a,b]}(1-2\pi\sqrt{-1}e_{\alpha(\mu(t)),\varepsilon}(\mu(t))h^{\alpha(\mu(t))-1}r^{\alpha(\mu(t))}dt\cdot$$

$$\cdot e^{o(|h|^{\alpha_0(\mu)-1}|r|^{\alpha_0(\mu)})}$$

$$=\exp.(-2\pi\sqrt{-1}\int_a^b e_{\alpha(\mu(t)),\varepsilon}\alpha(\mu(t))h^{\alpha(\mu(t))-1}r^{\alpha(\mu(t))}dt+$$

$$+o(|h|^{\alpha_0(\mu)-1}|r|^{\alpha_0(\mu)})).$$

Here, ε is same as above and $r^\alpha=|r|^\alpha e^{\alpha\pi\sqrt{-1}}$ if $r<0$.

By the above calculation, we get

$$\int_{R^1}\cdots\int_{R^1}\varphi\Big(\sum_{i=0}^{n(h)}ht_i\Big)d\mu(a)_{t_0}\cdots d\mu(a+n(h)h)_{t_{n(h)}}$$

$$=\int_0^\infty f(r)\exp.(-2\pi\sqrt{-1}\int_a^b e_{\alpha(\mu(t)),+}(\mu(t))h^{\alpha(\mu(t))-1}r^{\alpha(\mu(t))}dt+$$

$$+o(h^{\alpha_0(\mu)-1}|r|^{\alpha_0(\mu)})dr+$$

$$+\int_{-\infty}^0 f(r)\exp.(-2\pi\sqrt{-1}\int_a^b e_{\alpha(\mu(t)),-}(\mu(t))h^{\alpha(\mu(t))-1}r^{\alpha(\mu(t))}dt+$$

$$+o(|h|^{\alpha_0(\mu)-1}|r|^{\alpha_0(\mu)}))dr.$$

Here, $h\geqq 0$ in the first term of the right hand side and $h<0$ in the second term.

To calculate the limit of this formula, we assume that $\mu(t)$ satisfies (iv). $\alpha(\mu(t))$ *is a constant* α, $0<\alpha<1$.

Then, since we have

$$\int_0^\infty f(r)\exp.(-2\pi\sqrt{-1}h^{\alpha-1}\int_a^b e_{\alpha,+}(\mu(t))dtr^\alpha+o(h^{\alpha-1}))dr+$$

$$+\int_{-\infty}^0 f(r)\exp.(2\pi\sqrt{-1}h^{\alpha-1}\int_a^b e_{\alpha-}(\mu(t))dtr^\alpha+o(|h|^{\alpha-1}))dr$$

$$=\frac{1}{\alpha}\int_0^\infty q^{\frac{1}{\alpha}-1}f(q^{\frac{1}{\alpha}})\exp.(-2\pi\sqrt{-1}h^{\alpha-1}\int_a^b e_{\alpha,+}(\mu(t))dtq+o(h^{\alpha-1}))dq+$$

$$+\int_{\infty e^{\alpha\pi\sqrt{-1}}}^0 q^{\frac{1}{\alpha}-1}f(q^{\frac{1}{\alpha}})\exp.(-2\pi\sqrt{-1}\int_a^b e_{\alpha,-}(\mu(t))dtq+o(|h|^{\alpha-1}))dq$$

we obtain

$$\int_{R^1}\cdots\int_{R^1}\varphi\Big(\sum_{i=0}^{n(h)} ht_i\Big)d\mu(a)_{t_0}\cdots d\mu(a+n(h)h)_{t_{n(h)}}$$

$$=\frac{1}{\alpha}\int_0^\infty q^{\frac{1}{\alpha}-1}f(q^{\frac{1}{\alpha}})\exp.(2\pi\sqrt{-1}h^{\alpha-1}\int_a^b e_{\alpha,+}(\mu(t))dtq+o(h^{\alpha-1}))dq+$$

$$+\int_{\infty e^{\pi\alpha\sqrt{-1}}}^0 q^{\frac{1}{\alpha}-1}f(q^{\frac{1}{\alpha}})\exp.(-2\pi\sqrt{-1}h^{\alpha-1}\int_a^b e_{\alpha,-}(\mu(t))dtq+$$

$$+o(|h|^{\alpha-1}))dq.$$

Here $\int_{\infty e}^0 \alpha\pi\sqrt{-1}$ means the integration along the line $\{e^{\alpha\pi\sqrt{-1}}t\,|\,0\leqq t<\infty\}$ and $\int_0^\infty$ and $\int_{\infty e}^0 \alpha\pi\sqrt{-1}$ both mean $lim._{A\to\infty,\,\varepsilon\to 0}\int_\varepsilon^A$ and $lim._{A\to\infty,\,\varepsilon\to 0}\int_{Ae}^\varepsilon \alpha\pi\sqrt{-1}$.

But, since we know that to set $arg.\,c=\theta$, we have

$$\lim_{s\to 0}.\frac{1}{s}\lim_{\varepsilon\to 0}.\int_\varepsilon^s g(h^{\alpha-1}c)dh$$

$$=\lim_{S\to\infty}.S\theta\int_{S\theta}^{\infty\theta}\frac{c^{1/(1-\alpha)}}{\alpha-1}g(n)u^{1/(\alpha-1)-1}du$$

$$=\lim_{T\to\infty}.\frac{1}{T\theta}\int_{a\theta}^{(a+T)\theta}\frac{c^{1/(1-\alpha)}}{\alpha-1}g(u)u^{1/(\alpha-1)+1}du,$$

we may set

$$\lim_{h\to 0}.\int_{R^1}\cdots\int_{R^1}\varphi\Big(\sum_{i=0}^{n(h)} ht_i\Big)d\mu(a)_{t_0}\cdots d\mu(a+n(h)h)_{t_{n(h)}} \tag{74}$$

$$=\frac{1}{\alpha-1}\Big[\Big[\int_a^b e_{\alpha,+}(\mu(t))dt\Big]^{\frac{1}{1-\alpha}}M_{\theta+}\Big[\int_0^\infty s^{\frac{1}{\alpha}-1}\mathscr{F}^{-1}[\varphi](s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}dst^{\frac{1}{\alpha-1}}\Big]+$$

$$+\frac{1}{\alpha-1}\Big[\Big[\int_a^b e_{\alpha,-}(\mu(t))dt\Big]^{\frac{1}{1-\alpha}}M_{\theta-}\Big[\int_{\infty e^{\alpha\pi\sqrt{-1}}}^0 s^{\frac{1}{\alpha}-1}\mathscr{F}^{-1}[\varphi](s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}ds\cdot$$

$$\cdot t^{\frac{1}{\alpha-1}}\Big],$$

where $M_\theta[g(t)]$ means the mean value of $g(t)$ in θ-direction, that is

$$M_\theta[g(t)]=\lim_{T\to\infty}.\frac{1}{T\theta}\int_{a\theta}^{(a+T)\theta}g(t)dt,$$

and θ_+ and θ_- are given by

$$\theta_+=arg.\int_a^b e_{\alpha,+}(\mu(t))dt,\ \ \theta_-=arg.\int_b^a e_{\alpha,-}(\mu(t))dt.$$

20. ***Definition.*** *We denote by* $M_{\alpha,\theta+,\theta-}$ *the function space given by*

$$M_{\alpha,\theta+,\theta-}=\{f\,|\,f=\mathscr{F}[\varphi], \text{ where } \varphi \text{ satisfies the following conditions } (*)_+ \text{ and } (*)_-\},$$

$(*)_+$ $\displaystyle\int_0^\infty s^{\frac{1-\alpha}{\alpha}}\varphi(s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}dst^{\frac{\alpha}{\alpha-1}}$ *has the mean value in* θ_+ *-direction,*

$(*)_-$ $\displaystyle\int_{\infty e^{\alpha\pi\sqrt{-1}}}^0 s^{\frac{1-\alpha}{\alpha}}\varphi(s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}dst^{\frac{\alpha}{\alpha-1}}$ *has the mean value in* θ *-direction.*

Note. Since $0<\alpha<1$, $lim._{\varepsilon\to 0}\int_\varepsilon^\infty$ exists if φ is regular at 0. Here, $s^{1/\alpha}=|s|^{1/\alpha}$ if $s>0$ and in the second integral, $s^{1/\alpha}$ means $-|s|^{1/\alpha}$.

Definition *Let* $\mu(t)$ *be a probability distribution valued function defined on* S, *a measurable set on* $\boldsymbol{R}^1$, *and satisfies the following conditions* (i) and (ii).

(i). $\alpha(\mu(t))=\alpha$, *a constant not equal to* 0 *or* 1 *almost everywhere on* S.

(ii). $e_{\alpha,+}(\mu(t))$ *and* $e_{\alpha,-}(\mu(t))$ *both exist almost everywhere on* S *and* (*as the functions on* S) *they are both* (*Lebesgue*) *integrable on* S.

Then we define the generalized $*$ *-product integral of* $\mu(t)$ *on* S, *also denoted by* $*-\widehat{\int}_S\mu$, *to be the element of the dual space of*

$M_{\alpha,\,arg.\int_S e_{\alpha,+}(\mu(t))dt,\,arg.\int_S e_{\alpha,-}(\mu(t))dt}$ *given by*

$$(75)\qquad (*-\widehat{\int}_S\mu)(f)$$

$$=\frac{1}{\alpha-1}\Big[\int_S e_{\alpha,+}(\mu(t))dt\Big]^{\frac{1}{1-\alpha}}M_{arg.\int_S e_{\alpha,+}(\mu(t))dt}\Big[\int_0^\infty s^{\frac{1-\alpha}{\alpha}}\mathscr{F}^{-1}[f](s^{\frac{1}{\alpha}})\cdot$$

$$\cdot e^{-2\pi\sqrt{-1}st}dst^{\frac{\alpha}{\alpha-1}}\Big]+$$

$$\frac{1}{\alpha-1}\Big[\int_S e_{\alpha,-}(\mu(t))dt\Big]^{\frac{1}{1-\alpha}}M_{arg.\int_S e_{\alpha,-}(\mu(t))dt}\Big[\int_{\infty e^{\alpha\pi\sqrt{-1}}}^0 s^{\frac{1-\alpha}{\alpha}}\cdot$$

$$\cdot\mathscr{F}^{-1}[f](s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}dst^{\frac{\alpha}{\alpha-1}}\Big].$$

By definition, $*-\widehat{\int}_S$ is linear but since *arg.* $(a+b)$ may not be equal to *arg.* $a+$*arg.* b and although we get

$$\alpha(\mu_1*\mu_2)=\alpha, \quad if\ \alpha(\mu_1)=\alpha(\mu_2)=\alpha,$$
$$e_{\alpha,+}(\mu_1*\mu_2)=e_{\alpha,+}(\mu_1)+e_{\alpha,+}(\mu_2),$$
$$e_{\alpha,-}(\mu_1*\mu_2=e_{\alpha,-}(\mu_1)+e_{\alpha,-}(\mu_2),$$

$(a+b)^{1/(1-\alpha)} \neq a^{1/(1-\alpha)}+b^{1/(1-\alpha)}$, we can not obtain corresponding results of theorem 11 for generalized $*$ -product integrals.

If $\theta_+=\theta_-$ in the definition of $M_{\alpha,\theta+,\theta-}$, then to set $\theta=\theta_+$, we denote $M_{\alpha,\theta}$ instead of $M_{\alpha,\theta+,\theta-}$, that is $M_{\alpha,\theta}$ is given by

$$M_{\alpha,\theta}=[f \mid f=\mathscr{F}[\varphi], \textit{ where } \varphi \textit{ satisfies the following condition}(*)\},$$

$$(*) \qquad \left(\int_0^\infty+\int_{\infty e^{\alpha\pi\sqrt{-1}}}^0\right) s^{\frac{1-\alpha}{\alpha}}\varphi(s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}\,dst^{\frac{\alpha}{\alpha-1}} \quad \textit{has the mean value in } \theta\textit{-direction.}$$

We assume $\mu(t)$ satisfies the above (i), (ii) and the following condition (iii),

(iii). $e_{\alpha,+}(\mu(t))=e_{\alpha,-}(\mu(t))$ *almost everywhere on* S.

Then to set $e_\alpha(\mu(t))=e_{\alpha,+}(\mu(t))$, we can define $*-\widehat{\int}_S\mu$ to be an element of the dual space of $M_{\alpha,\theta}$ as follows:

$$(75)' \qquad (*-\widehat{\int}_S)(f)$$

$$=\frac{1}{\alpha-1}\left[\int_S e_\alpha(\mu(t))dt\right]^{\frac{1}{1-\alpha}} M_{arg.\int_S e_\alpha(\mu(t))dt}\left[\left(\int_0^\infty+\int_{\infty e^{\alpha\pi\sqrt{-1}}}^0\right)s^{\frac{1-\alpha}{\alpha}}\cdot\right.$$

$$\left.\cdot\mathscr{F}^{-1}[f](s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}\,dst^{\frac{\alpha}{\alpha-1}}\right],$$

$$f\in M_{\alpha,\theta}.$$

Note. Since we may write simbolically,

$$(*-\widehat{\int}_{[a,b]}\mu)(\varphi)$$

$$=\lim_{h\to 0}\left[\int_{R^1}\cdots\int_{R^1}\varphi\left(\sum_{i=0}^{n(h)}ht_i\right)d\mu(a)_{t_0}\cdots d\mu(a+n(h)h)_{t_{n(h)}}\right.$$

$$=\int_0^\infty f(r)\exp.(-2\pi\sqrt{-1}\int_a^b e_{\alpha(\mu(t)),+}(\mu(t))(dt)^{\alpha(\mu(t))}r^{\alpha(\mu(t))})dr+$$

$$+\int_{-\infty}^0 f(r)\exp.(-2\pi\sqrt{-1}\int_a^b e_{\alpha(\mu(t)),-}(\mu(t))(dt)^{\alpha(\mu(t))}r^{\alpha(\mu(t))})dr,$$

$$\varphi=\mathscr{F}[f],$$

if $\mu(t)$ satisfies the conditions (i), (ii), (iii) of n°19, we may write
if $\mu(t)$ satisfies the conditions (i), (ii) of n° 20,

$$(76) \qquad \int_0^\infty f(r)\exp.(-2\pi\sqrt{-1}\int_S e_{\alpha,+}(\mu(t))(dt)^\alpha r^\alpha)dr+$$

$$+\int_{-\infty}^{0} f(r)\exp.(-2\pi\sqrt{-1}\int_S e_{\alpha,-}(\mu(t))(dt)^\alpha r^\alpha)dr$$

$$=\frac{1}{\alpha-1}\Big[\int_S e_{\alpha,+}(\mu(t))dt\Big]^{\frac{1}{1-\alpha}} M_{arg.\int_S e_{\alpha,+}(\mu(t))dt}\Big[\int_\infty^0 s^{\frac{1-\alpha}{\alpha}}\mathscr{F}^{-1}[f](s^{\frac{1}{\alpha}})\cdot$$

$$\cdot e^{-2\pi\sqrt{-1}st}\,dst^{\frac{\alpha}{\alpha-1}}\Big]+$$

$$+\frac{1}{\alpha-1}\Big[\int_S e_{\alpha,-}(\mu(t))dt\Big]^{\frac{1}{1-\alpha}} M_{arg.\int_S e_{\alpha,-}(\mu(t))dt}\Big[\int_{\infty e^{\alpha\pi\sqrt{-1}}}^0 s^{\frac{1-\alpha}{\alpha}}\cdot$$

$$\cdot\mathscr{F}^{-1}[f](s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}\,dst^{\frac{\alpha}{\alpha-1}}\Big],$$

$$f\in M_{\alpha,\, arg.\int_S e_{\alpha,+}(\mu(t))dt,\ arg.\int_S e_{\alpha,-}(\mu(t))dt}.$$

Especially, if $\mu(t)$ satisfies conditions (i), (ii), (iii) of n°20, we may write

(76)′
$$\int_{-\infty}^{\infty} f(r)\exp.(-2\pi\sqrt{-1}\int_S e_\alpha(\mu(t))(dt)^\alpha r^\alpha)dr$$

$$=\frac{1}{\alpha-1}\Big[\int_S e_\alpha(\mu(t))dt\Big]^{\frac{1}{1-\alpha}} M_{arg.\int_S e_\alpha(\mu(t))dt}\Big[\Big(\int_0^\infty+\int_{\infty e^{\alpha\pi\sqrt{-1}}}^0\Big)s^{\frac{1-\alpha}{\alpha}}\cdot$$

$$\cdot\mathscr{F}^{-1}[f](s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}\,dst^{\frac{\alpha}{\alpha-1}}\Big],$$

$$f\in M_{\alpha,\, arg.\int_S e_\alpha(\mu(t))dt}.$$

By (76)′, we may also set

(77)
$$\mathscr{F}[g]\Big(\int_S f(t)(dt)^\alpha\Big)$$

$$=\int_{-\infty}^{\infty} g(r)\exp.-2\pi\sqrt{-1}\int_S f(t)(dt)^\alpha r)dr$$

$$=\frac{\alpha}{1-\alpha}\Big[\int_S f(t)dt\Big]^{\frac{1}{1-\alpha}} M_{arg.\int_S f(t)dt}\Big[\Big(\int_0^\infty+\int_{\infty e^{\alpha\pi\sqrt{-1}}}^0\Big)s^{\frac{1-\alpha}{\alpha}}\cdot$$

$$\mathscr{F}^{-1}[r^{\frac{\alpha 2}{1-\alpha}}g((sgn.\,r)|r|^\alpha)](s^{\frac{1}{\alpha}})e^{-2\pi\sqrt{-1}st}\,dst^{\frac{\alpha}{\alpha-1}}\Big],$$

Where g is the class of functions such that the mean value in the last formula exists and f is (Lebesgue) integrable on S.

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