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An Alternative Method in the Foundation of Infinitesimal Analysis of Several Independent Variables

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§1 Introduction

As is seen in many references or textbooks on analysis, the part concerning functions of several independent variables has rather much rigour compared with that concerning functions of a single independent variable. In the present note the theorem on Jacobians and the theorem on the transformation of integration variables, both of which are fundamental in the analysis of several independent variables, are proved by a new method. Throughout this note the mean value theorem of integral form*) plays a principal role. So far the usual method is to utilize the uniform differentiability of the transformation and, in the present note, however, the fixed point theorem on a convex compact set is applied via the mean value theorem of integral form. Anyhow the idea of uniformity is necessarily indispensable to this theme, but our method seems to be preferable to the usual one for its easier argument and less ambiguity. Especially proofs of **Theorem 3. 3** and **Lemma (F)** are very simple.

In writing this note we received effective suggestions from some references. ([2], [5])

In this note there may be no original essentials but something instructive and pedagogical.

§2 Notations and Preliminaries

Throughout the following the continuous differentiability of the transformation will be assumed except in **Lemma (F)**.

In this section we introduce some notations.

R^n : the n-dimensional euclidean space

$\mathbf{x}$ or $\mathbf{y}$: n-dimensional vector

*) We owe this to prof. M. Nagumo. As to the papers published it is seen in [1] and [8].

$|\mathbf{x}|$: euclidean norm of $\boldsymbol{x}$ *i. e.* $(\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$

$\mathbf{f}$: the transformation (or vector valued function) defined on some domain in R^n into R^n.

$|A|$: euclidean norm of matrix A as a lineare operator or it is equal to $(\sum_{i,j=1}^{n} a_{ij}^2)^{\frac{1}{2}}$

B, D, or Q : the domain of independent variables

R, S, or T : the range of dependent variables

∂Q or ∂R : the boundary of Q or R

BA : the image of B by A

μ^* (or μ_*) : n-dimensional Jordan outer (or inner) measure. For the measurable set S its measure is denoted by $\mu(S)$.

Remark. The transform of $\mathbf{x}$ by A is denoted by $\mathbf{x}A$ and must not confused with $A(\mathbf{x})$.

Here we shall introduce the mean value theorem of integral form without proof.

$$\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) = (\mathbf{x} - \mathbf{a}) \int_0^1 J(\mathbf{a} + \theta(\mathbf{x} - \mathbf{a}))d\theta \qquad (1),$$

where J is the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$ and the integration represents the matrix of componentwise integration.

Assuming $\mathbf{a} = \mathbf{f}(\mathbf{a}) = \mathbf{0}$, we get $\mathbf{f}(\mathbf{x}) = \mathbf{x} \int_0^1 J(\theta\mathbf{x})\, d\theta$, the right side of which is denoted briefly by $\mathbf{x}A(\mathbf{x})$.

The following inequalities are evident, then proofs are omitted.

$$|\mathbf{x}A| \leqq |\mathbf{x}|\ |A|,$$
$$\left|\int_0^1 A(\theta)\ d\theta\right| \leqq \int_0^1 |A(\theta)|\, d\theta.$$

§ 3 Theorem on Jacobians

In this section the proof on the ontoness of the transformation (**Theorem 3.2.**) will be given and next the proof on the continuity of the inverse transformation (**Theorem 3.3**) will be given. Throughout the following let $\mathbf{f}$ be defined on some bounded open domain D into R^n.

Let us begin with the following theorem on the local schlichtness.

Theorem 3.1. Let $\mathbf{a} \in D$. If the matrix $J(\mathbf{a})$ is not singular *i. e.* det $J(\mathbf{a})$ does not vanish, then there exists a neighborhood U of $\mathbf{a}$ such that $\mathbf{f}$ is schlicht in U.

Proof. Without loss of generality, we assume that $J(\mathbf{a})$ is positive. Since the determinant is cotinuous as function of its n^2 components, there exists a positive number ε_0 such that if $\left|\eta_{ij} - \frac{\partial f_i}{\partial x_j}(\mathbf{a})\right| < \varepsilon_0$ for all (i, j), then

$$\det(\eta_{ij}) > 0 \tag{3. 1}$$

From the continuity of $\frac{\partial f_i}{\partial x_j}$ we can find a positive number δ such that

$$\left|\frac{\partial f_i}{\partial x_j}(\mathbf{x}) - \frac{\partial f_i}{\partial x_j}(\mathbf{a})\right| < \varepsilon_0 \tag{3. 2}$$

is satisfied for $|\mathbf{x} - \mathbf{a}| < \delta$.
$U : |x - a| < \delta$ is what we desire.
Indeed, if it took place that $\mathbf{f}(\mathbf{b}) = \mathbf{f}(\mathbf{c})$ for some pair $\mathbf{b}$, $\mathbf{c}$, $(\in U$ and $\mathbf{b} \neq \mathbf{c})$, then by virtue of the mean value theorem

$$\det\left(\int_0^1 \frac{\partial f_i}{\partial x_j}(\mathbf{b} + \theta(\mathbf{c} - \mathbf{b}))d\theta\right) = 0,$$

where integrand $\frac{\partial f_i}{\partial x_j}$ satisfies (3. 2) for all θ.

Therefore $\int_0^1 \frac{\partial f_i}{\partial x_j}(\mathbf{b} + \theta(\mathbf{c} - \mathbf{b}))\, d\theta$ also satisfies (3. 2), which contradicts (3. 1), setting

$\eta_{ij} = \int_0^1 \frac{\partial f_i}{\partial x_j}(\mathbf{b} + \theta(\mathbf{c} - \mathbf{b}))d\theta$.

Theorem 3. 2. U is homeomorphic to $\mathbf{f}(U)$, where U is the same above determined.

Before giving the proof we set the following

Lemma (A). Let a family of matrix $A(\mathbf{x})$ be given, where every components of $A(\mathbf{x})$ is continuous with respect to $\mathbf{x}$ in some closed ball $B(: |\mathbf{x}| \leqq \delta_0)$. And we assume that det $A(\mathbf{x})$ does not vanish.

Then there exists a solid sphere K with origin as its center such that

$$BA(\mathbf{x}) \supset K \quad \text{for every } \boldsymbol{x} \in B$$

Proof of Lemma (A).

Consider the image of the boundary ∂B By $A(\mathbf{x})$.

Let $y_i s$ be components of range space and $\alpha_{ij}(\mathbf{x})$ be component of inverse matrix of $A(\mathbf{x})$.
Then

$$\begin{aligned}\delta_0^2 &= \sum_{i=1}^n x_i^2 = \sum_{i=1}^n \Big(\sum_{j=1}^n \alpha_{ij}(\mathbf{x})y_j\Big)^2 = \sum_{i=1}^n \Big(\sum_{j,k=1}^n \alpha_{ij}(\mathbf{x})\ \alpha_{ik}(\mathbf{x})y_j y_k\Big) \\ &= \sum_{j,k=1}^n \Big(\sum_{i=1}^n \alpha_{ij}(\mathbf{x})\alpha_{ik}(\mathbf{x})\Big)y_j y_k = \sum_{j,k=1}^n \beta_{jk}(\mathbf{x})y_j y_k \end{aligned} \tag{3. 3}$$

The last side of the above equality is evidently a positive definite quadratic form, therefore

$$\sum_{j,k=1}^{n} \beta_{j,k}(\mathbf{x})y_j y_k \leq M(\mathbf{x})\sum_{j=1}^{n} y_j^2,$$

where $M(\mathbf{x})$ is the maximum value of the quadratic form on the unit sphere.

Since $M(\mathbf{x})$ is continuons and positive, we get

$$\frac{\delta_0^2}{M} \leq \sum_{j=1}^{n} y_j^2 \text{ setting } M = \max_{x \in B} M(\mathbf{x}).$$

From the above inequality we see that the image of the boundary ∂B by $A(\mathbf{x})$ lies outside of the sphere K with radius $\frac{\delta_0}{\sqrt{M}}$.

Since it is evident that the interior part of B is transformed into the interior part of the ellipsoid (3. 3), **Lemma (A)** is completely proved.

Proof of Theorem 3. 2 : We can assume $\mathbf{a} = \mathbf{f}(\mathbf{a}) = \mathbf{0}$ without loss of generality. To be easily seen, we shall have only to prove that for an arbitrary open set O which contains $\mathbf{0}$ and is contained in U, $\mathbf{f}(\mathbf{0})$ is inner point of $\mathbf{f}(\mathrm{O})$.

Let B be a closed ball such that $\mathbf{0} \in B \subset \mathrm{O}$. Then setting $A(\mathbf{x}) = \int_0^1 J(\theta\mathbf{x})d\theta$, we get a solid sphere K by the preceding lemma such that

$$BA(\mathbf{x}) \supset K \text{ for every } \mathbf{x} \in B.$$

Let $\mathbf{y}(\in K)$ be arbitrary but fixed. For any $\mathbf{x}\,(\in B)$ there exists uniquely $\mathbf{z}(\in B)$ such that

$$\mathbf{z}A(\mathbf{x}) = \mathbf{y}.$$

Then from the fixed point theorem for the continuous transformation $\mathbf{z}=\mathbf{y}A^{-1}(\mathbf{x})$, there exists point $\mathbf{x}$ such that $\mathbf{x} = \mathbf{y}A^{-1}(\mathbf{x})$ *i. e.* $\mathbf{x}A(\mathbf{x}) = \mathbf{y}$. Since $\mathbf{y}$ is arbitrary in K, we see that $\mathbf{f}(\mathrm{O})$ contains K.

Later on we shall necessitate the quantitative estimate from below on the measure of the range (**Lemma (F)**), where another application of the fixed point theorem will be introduced.

From the preceding two theorems the local existence of the inverse transformation $\mathbf{f}^{-1}$ on $\mathbf{f}(U)$ is established.

We can see further the following

Theorem 3. 3. The inverse transformation $\mathbf{f}^{-1}$ is continuously differentiable.

Before giving the proof we set the following

Lemma (B). Let $\mathbf{g}(\mathbf{y})$ be defined on a convex domain T.

If there exists a matrix $A(\mathbf{y}_1, \mathbf{y}_2)$ which is continuous in $T \times T$ and

$$\mathbf{g}(\mathbf{y}_2) - \mathbf{g}(\mathbf{y}_1) = (\mathbf{y}_2 - \mathbf{y}_1)A(\mathbf{y}_1, \mathbf{y}_2),$$

then $\mathbf{g}(\mathbf{y})$ is continuously differentiable.

Proof of Lemma (B). Let the (i, j) component of $A(\mathbf{y}_1, \mathbf{y}_2)$ be denoted by $a_{ij}(\mathbf{y}_1, \mathbf{y}_2)$. Setting $\mathbf{y}_2 = (y_1 + h, y_2, \cdots, y_n)$, $\mathbf{y}_1 = (y_1, y_2, \cdots, y_n)$, we get

$$\frac{\partial g_1}{\partial y_1} = \lim_{h \to 0} \frac{1}{h}\{g_1(\mathbf{y}_2) - g_1(\mathbf{y}_1)\} = \lim_{h \to 0} a_{11}(\mathbf{y}_1, \mathbf{y}_2) = a_{11}(\mathbf{y}_1, \mathbf{y}_1),$$

where $a_{11}(\mathbf{y}_1, \mathbf{y}_1)$ is cotinuous in T.

Proof of Theorem 3.3. To assert that $\mathbf{f}^{-1}$ is continuously differentiable, it is sufficient that we show in a small open (convex) sphere S.
Setting $\mathbf{g} = \mathbf{f}^{-1}$, we get

$$\begin{aligned} \mathbf{g}(\mathbf{y}_2) - \mathbf{g}(\mathbf{y}_1) &= (\mathbf{y}_2 - \mathbf{y}_1)\left[\int_0^1 J(\mathbf{x}_1 + \theta(\mathbf{x}_2 - \mathbf{x}_1))d\theta\right]^{-1} \\ &= (\mathbf{y}_2 - \mathbf{y}_1)\left[\int_0^1 J(\mathbf{g}(\mathbf{y}_1) + \theta(\mathbf{g}(\mathbf{y}_2) - \mathbf{g}(\mathbf{y}_1)))d\theta\right]^{-1}. \end{aligned}$$

In the right side of the above equality, the matrix $\left[\int_0^1 J(\mathbf{g}(\mathbf{y}_1) + \theta(\mathbf{g}(\mathbf{y}_2) - \mathbf{g}(\mathbf{y}_1))d\theta\right]^{-1}$ is evidently continuous in $S \times S$. Thus we see that $\mathbf{g} = \mathbf{f}^{-1}$ satisfies the assumption of **Lemma (B)**.

§4 A little geometry

In this section we shall prepare something concerning the elementary geometry in R^n.

Lemma (C). A set R of diameter l in a hyperplane is contained in some $n-1$ dimensional closed cube, the length of whose side is equal to $2l$.

Proof of Lemma (C).

In the following proof geometry only in the hyperplane is considered.

From the definition of diameter there exist two points A, B, in R such that

$$\overline{AB}^{*} > l - \varepsilon$$

Denoting the middle point between A and B by M, we shall consider the sphere $S_l(M)$ with radius l and with center M. If $P \in R$, then

$$\overline{PM}^2 = \frac{\overline{PA}^2 + \overline{PB}^2}{2} - \overline{AM}^2 < l^2$$

Therefore $R \subset S_l(M)$. What we desire is the smallest cube containing $S_l(M)$.

We can make **Lemma (C)** more precise, however it is sufficient in the applications.

Lemma (D). Let A be a singular matrix, Q a cube and l, the length of its

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* By $\overline{AB}$ we denote the euclidean metric between A and B.

side. Then QA is contained in some $n-1$ dimensional set of diameter kl, where the positive constant k depends only on A.

Proof. It is well known that QA is at most $n-1$ dimensional set. Setting $\mathbf{y}_s = \mathbf{x}_s A$ $(s = 1, 2)$, we get easily

$$|\mathbf{y}_1 - \mathbf{y}_2| \leqq na_0 |\mathbf{x}_1 - \mathbf{x}_2|,$$

where $a_0 = max\ |a_{ij}|$. na_0 may be viewed as k in **Lemma (D)**. Here we shall add further notations and a lemma. Let R be an n-dimensional (non-degenerate) parallelepiped.

R_ε : the smallest parallelepiped which contains the ε-neighborhood of R.

$R_{-\varepsilon}$: the largest parallelepiped such that $(R_{-\varepsilon})_\varepsilon = R$, if it exists.

$A-B$: the set of the elements belonging to A, but not belonging to B.

Lemma (E). Let R be a n-dimensional non-degenerate parallelepiped in R^n. Then the following inequalities hold

$$\mu(R_\varepsilon - R) \leqq \varepsilon\mu_{n-1}(\partial R) + K\varepsilon^n \tag{1}$$

$$\mu(R - R_{-\varepsilon}) \leqq \varepsilon\mu_{n-1}(\partial R), \tag{2}$$

where $\mu_{n-1}(\partial R)$ is $n-1$ dimensional Jordan measure of ∂R and K is a positive constant which depends only on the size and the form of R, and besides independent of ε.

Proof : For simplicity we shall prove in the case $n = 2$. Divide $R_\varepsilon - R$ into two parts; one is the part consisting of the parallerograms on each side of ∂R (we denote it by S), the other is the part consisting of the parallerograms situated at each corner of R_ε (we denote them by Q_1, Q_2, Q_3, and Q_4). It is evident that

$$\mu(S) = \varepsilon\ \mu_{n-1}(\partial R)$$

Let the length of two sides of Q_1 be denoted by q_1 and q_2. Then $q_1 = k_1\varepsilon$ and $q_2 = k_2\varepsilon$, where $k_i(i = 1, 2)$ depend only on the size and the form of R. Setting $K = k_1k_2$, we get

$$\mu(Q_1) \leqq K\varepsilon^2$$

It is easily seen that in the n-dimensional case we get the inequality (1).

As to the inequality (2), we omitt the proof, because it is evident.

Remark. Later on we shall use the inequality (2) in the following form

$$\mu(R_{-\varepsilon}) \geqq \mu(R) - \varepsilon\mu_{n-1}(\partial R),$$

where it should be noted that if $R_{-\varepsilon}$ does not exist, then the right side is negative.

§ 5 Sard's Theorem

Theorem (A. Sard) Let $\mathbf{f}$ be defined in a bounded domain D_1 into R^n. We shall consider it on a compact part, say $\overline{D}$ ($\overline{D} \subset D_1$). Let C be the set of critical points of $\mathbf{f}$ in $\overline{D}$. Then $\mu(\mathbf{f}(C)) = 0$.

Proof : By the compactness of $\overline{D}$, we can assume that C is contained in some closed cube Q, for C can be covered with a finite number of closed cubes which are contained in D_1.

To prove this theorem, we divide up Q into a large number of small cubes Q_j by a network of hyperplanes We consider a small Q_j which intersects with C and estimate the outer measure of $\mathbf{f}(Q_j)$.

For an arbitrary positive ε we can determine a network of hyperplanes such that in each Q_j

$$|\int_0^1 J(\mathbf{a}_j + \theta(\mathbf{x} - \mathbf{a}_j))d\theta - J(\mathbf{a}_j)| < \varepsilon$$

is satisfied, where $\mathbf{x} \in Q_j$ and $\mathbf{a}_j \in Q_j \cap C$. This fact means the uniform continuity of the derivatives of $\mathbf{f}$.

Let q and m be respectively the length of a side of Q and the division number of the network.

For a fixed j, translating $\mathbf{a}_j$ to $\mathbf{0}$ and $\mathbf{f}(\mathbf{a}_j)$ to $\mathbf{0}$, we consider the following decomposition of $\mathbf{f}(\mathbf{x})$

$$\mathbf{f}(\mathbf{x}) = \mathbf{x}A(\mathbf{x}) = \mathbf{x}A(\mathbf{0}) + \mathbf{x}(A(\mathbf{x}) - A(\mathbf{0})),$$

where $A(\mathbf{x}) = \int_0^1 J(\theta\mathbf{x})d\theta$ and $|A(\mathbf{x}) - A(\mathbf{0})| < \varepsilon$.
By **Lemma (D)** $\mathbf{x}A(\mathbf{0})$ is an element of $n-1$ dimensional set $Q_jA(\mathbf{0})$ with diameter $k\frac{q}{m}$ and $|\mathbf{x}(A(\mathbf{x}) - A(\mathbf{0}))| \leqq \frac{q\varepsilon}{m}$.

Then, using **Lemma** (C), we get

$$\mu^*(\mathbf{f}(Q_j)) \leqq [2(k\frac{q}{m} + 2\frac{q\varepsilon}{m})]^{n-1}\ 2\frac{q\varepsilon}{m} = \frac{\varepsilon}{m^n}K,$$

where $K = 2^n(k + 2\varepsilon)^{n-1}\mu(Q)$.
Here it should be noted that since $A(\mathbf{x})$ is uniformly bounded in Q, k can be selected uniformly for all Q_j, and further K can be replaced by another positive K_1 ($K < K_1$) which is independent of small ε.

Therefore, $\mu^*(\mathbf{f}(Q_j)) \leqq \frac{\varepsilon}{m^n}K_1$

and

$$\mu^*((\mathbf{f}(C)) \leqq \sum_j \mu^*(f(Q_j)) \leqq m^n \frac{\varepsilon}{m^n}K_1 = \varepsilon K_1.$$

Since ε is arbitrary, we conclude the proof.

§ 6 Theorem on Transformation of Integration Variables

Our object in this section is **Theorem 6.3.** We shall begin from the following important theorem

Theorem 6.1 Let $\mathbf{f}$ be defined in a bounded domain D and Q be a closed cube in D. If $\mathbf{f}$ is assumed to be one-to-one and det $J(x)$ does not vanish in Q, then

$$\mu(\mathbf{f}(Q)) = \int_Q |\det J(\mathbf{x})| d\mathbf{x} \tag{6. 1},$$

where the right side is meant by n-dimensional Riemann integral.

Proof : $(I)\ \mu^*(\mathbf{f}(Q)) \leqq \int_Q |\det J(\mathbf{x})|\ d\mathbf{x}$

The method of proof of this inequality is analogous to that of Sard's Theorem. Dividing up Q into a large number of small cubes Q_j by a network of lines and using the decomposition of $\mathbf{f}(\mathbf{x})$ into two parts as the same that appears in the proof of Sard's Theorem, we can assert that $\mathbf{f}(\mathbf{x})$ is belonging to the $\frac{q\varepsilon}{m}$— neighborhood of $Q_j\ J(\mathbf{x}_j)(\mathbf{x}_j$ is arbitrary in Q_j).

From the well known lemma

$$\mu(Q_j J(\mathbf{x}_j)) = |\det J(\mathbf{x}_j)| \mu(Q_j)$$

and the inequality (1) of the **Lemma (E)**, we get

$$\mu^*(\mathbf{f}(Q_j)) \leqq |\det J(\mathbf{x}_j)| \mu(Q_j) + \frac{q\varepsilon}{m}\mu_{n-1}(\partial(Q_j J(\mathbf{x}_j))) + K(\frac{q\varepsilon}{m})^n.$$

Then, we get

$$\mu^*(\mathbf{f}(Q)) \leqq \sum_j \mu^*(\mathbf{f}(Q_j)) \leqq \sum_j |\det J(\mathbf{x}_j)| \mu(Q_j) + \frac{q\varepsilon}{m}\sum_j \mu_{n-1}\ (\partial(Q_j J(\mathbf{x}_j))) + K\mu(Q)\varepsilon^n.$$

In the second sum of the right side, each $\mu_{n-1}\ (\partial(Q_j J(\mathbf{x}_j)))$ is not greater than $L(\frac{q}{m})^{n-1}$, where L is a posivive constant that is determined from the uniform boundedness of $J(\mathbf{x})$ on Q.

Therefore the sum of the second term and the third term is not greater than $\varepsilon\mu(Q)(L+K)$ which can be arbitrarily small. Thus the inequality (I) is deduced.

(II) $\mu_*(\mathbf{f}(Q)) \geqq \int_Q |\det J(\mathbf{x})|\ d\mathbf{x}$

The method of proof of this inequality is based on the fixed point theorem as was noted in the § **Introduction.**

Lemma (F) Let $\mathbf{f}$ be a continuous transformation in a closed cube Q into R^n with the form

$$\mathbf{f}(\mathbf{x}) = \mathbf{x}A + B(\mathbf{x}),$$

where A is a non-degenerate matrix and $|B(\mathbf{x})| < \varepsilon$ everywhere in Q. Then

$$\mathbf{f}(Q) \supset (QA)_{-\varepsilon} \qquad (6.2),$$

if the right side exists.

Proof of Lemma (F) :

Let $\mathbf{y}$ be chosen arbitrarily from $(QA)_{-\varepsilon}$, but fixed.
For any $\mathbf{x}(\in Q)$ there exists uniquely $\mathbf{z}(\in Q)$ such that

$$\mathbf{y} = \mathbf{z}A + B(\mathbf{x}).$$

This is possible since A is non-degenerate. Then from the fixed point theorem for $\mathbf{z} = (\mathbf{y} - B(\mathbf{x}))A^{-1}$ there exists a point $\mathbf{x}$ such that

$$\mathbf{x} = (\mathbf{y} - B(\mathbf{x}))A^{-1} \quad i.\ e.\ \ \mathbf{y} = \mathbf{x}A + B(\mathbf{x}).$$

Remark : If we assume that $\mathbf{f}$ is continuously differentiable, then we may consider both sides of (6.2) as open sets for small Q. This is evident by Theorem **3.2** and so both sides of the following (6.3) may be viewed as open sets.

In the proof of inequality (II), we divide up Q by a network and consider $\mathbf{x}(A(\mathbf{x}) - J(\mathbf{x}_j))$ as $B(\mathbf{x})$ in the **Lemma (F)** in each Q_j.

For an arbitrary positive number ε, we can determine a network over Q such that in each Q_j

$$|A(\mathbf{x}) - J(\mathbf{x}_j)| < \varepsilon \text{ for } \mathbf{x} \in Q_j \text{ and } \mathbf{x}_j \in Q_j.$$

After $\mathbf{x}_j(j=1, 2, \cdots, m^n)$ was chosen, we apply the preceding **Lemma (F)** to $\mathbf{f}(\mathbf{x}) = \mathbf{x}J(\mathbf{x}_j) + \mathbf{x}(A(\mathbf{x}) - J(\mathbf{x}_j))$ in each Q_j
Then, we get

$$\mathbf{f}(Q_j) \supset (Q_j J(\mathbf{x}_j))_{-\frac{q\varepsilon}{m}} \qquad (6.3)$$

if the right side exists.

Therefore, using the inequality (2) in the **lemma (E)**, we get

$$\mu_*(\mathbf{f}(Q_j)) \geqq \mu((Q_j J(\mathbf{x}_j))_{-\frac{q\varepsilon}{m}} \geqq \mu((Q_j J(\mathbf{x}_j))) - \frac{q\varepsilon}{m}\mu_{n-1}(\partial(Q_j J(\mathbf{x}_j)))$$

$$\geqq |\det J(\mathbf{x}_j)| \cdot \mu(Q_j) - \varepsilon L \mu(Q) \cdot \frac{1}{m^n} \qquad (6.4)$$

If the set $(Q_j J(\mathbf{x}_j))_{-\frac{q\varepsilon}{m}}$ dose not exist, the right side is negative (see the remark at the end of § 4) and so the above inequality is trivial.

Using the fact that for a finite of sets E_j which is mutually disjoint

$\mu_*(\cup E_j) \geqq \sum \mu_*(E_j)$, we get

$$\mu_*(\mathbf{f}(Q)) \geqq \sum_j \mu_*(\mathbf{f}(Q_j))^{\#} \geqq \sum_j |\det J(\mathbf{x}_j)| \cdot \mu(Q_j) - \varepsilon L \mu(Q),$$

from which the inequality (II) follows.

From both two inequalities (I) and (II), we get

$$\int_Q |\det J(\mathbf{x})| \, d\mathbf{x} \geqq \mu^*(\mathbf{f}(Q)) \geqq \mu_*(\mathbf{f}(Q)) \geqq \int_Q |\det J(\mathbf{x})| \, d\mathbf{x},$$

which deduces the Jordan measurability of $\mathbf{f}(Q)$ and the identity (6.1).

Comments. There are several methods of proof of **Theorem 6.1.** But as far is known from several references, every proof via the inequality (II) involves more or les ambiguity This is the reason why we necessitate (6.4) the estimate of $\mu_*(\mathbf{f}(Q_j))$ from below.

Next we shall attack the following generalized theorem on the basis of the preceding Theorem **6.1.**

Theorem 6.2 Let $\mathbf{f}$ be defined in a bounded domain D_1. We consider it in a compact part $\overline{D}$ ($\overline{D} \subset D_1$) and assume only that it is one-to-one in $\overline{D}$. If D has Jordan area $\mu(D)$, then also $f(D)$ has Jordan area and

$$\mu(\mathbf{f}(D)) = \int_D |\det J(\mathbf{x})| \, d\mathbf{x}$$

Proof : To prove this theorem, we start from the special case (i) and next attack the general case (ii) on the basis of (i).

(i) In the case that $\det J(\mathbf{x})$ does not vanish in $\overline{D}$. We set a network over $\overline{D}$ and denote the small cubes inside $\overline{D}$ by $Q_1, Q_2, \cdots, Q_l$. By the preceding Theorem **6.1**, we get

$$\mu(\mathbf{f}(Q_j)) = \int_{Q_j} |\det J(\mathbf{x})| \, d\mathbf{x}$$

Further from the finite additiveness of Jordan measure and Riemann integral, we get

$$\mu(\sum_j \mathbf{f}(Q_j)) = \int_{\cup Q_j} |\det J(\mathbf{x})| \, d\mathbf{x}.$$

Then,

Strictly speaking, $\mathbf{f}(Q_j)$ is not mutually disjoint, but by excluding ∂Q_j from Q_j we may consider Q_j as open such that $\mathbf{f}(Q_j)$ is mutually disjoint. Then the remark relating to **Lemma (F)** may be applied.

$$\mu_*(\mathbf{f}(D)) \geqq \mu(\mathbf{f}(\sum_j Q_j)) = \mu(\sum_j \mathbf{f}(Q_j))$$
$$\geqq \int_{\cup Qj} |\det J(\mathbf{x})| \ d\mathbf{x},$$

where $\int_{\cup Qj} |\det J(\mathbf{x})| d\mathbf{x} - \int_D |\det J(\mathbf{x})| \ d\mathbf{x}$ can be arbitrarily small.

Therefore, $\mu_*(\mathbf{f}(D)) \geqq \int_D |\det J(\mathbf{x})| \ d\mathbf{x}$.

Noting that $\overline{D}$ is compact in D_1, we may consider that for sufficiently large division number of network all Q_j which intersects D are contained in D_1. Then through the calculation analogous to the preceding Theorem **6.1** (I), we get

$$\mu^*(\mathbf{f}(D)) \leqq \int_D |\det J(\mathbf{x})| \ d\mathbf{x},$$

which deduces Jordan measurability of $(\mathbf{f}(D))$ and

$$\mu(\mathbf{f}(D)) = \int_D |\det J(\mathbf{x})| \ d\mathbf{x}.$$

(ii) In the general case

Let C be the set of critical points of $\mathbf{f}$ in $\overline{D}$. For an arbitrary positive number ε there exists an open neighborhood C_ε of C such that $|\det J(\mathrm{x})| < \varepsilon$ for $\mathrm{x} \in C_\varepsilon$. We may consider C_ε such as sum of a finite number of spheres and so C_ε has Jordan area. Therefore $D - C_\varepsilon$ has Jordan area and we get from the preceding special case (i)

$$\mu(\mathbf{f}(D-C_\varepsilon)) = \int_{D-C_\varepsilon} |\det J(\mathbf{x})| \ d\mathbf{x} = \int_D |\det J(\mathbf{x})| \ d\mathbf{x} - \int_{C_\varepsilon} |\det J(\mathbf{x})| \ d\mathbf{x}$$
$$\geqq \int_D |\det J(\mathbf{x})| \ d\mathbf{x} - K\varepsilon,$$

where K is a positive constant that is independent of ε. On the other hand, since $\mu_*(\mathbf{f}(D)) \geqq \mu(\mathbf{f}(D - C_\varepsilon))$, we see

$$\mu_*(\mathbf{f}(D)) \geqq \int_D |\det J(\mathbf{x})| \ d\mathbf{x}.$$

Before completion of our proof, we shall mention two propositions. One is the following.

For an arbitrary positive number ε there exists an open neighborhood O of C such that $\mu(\mathbf{f}(O)) < \varepsilon$.

By Sard's Theorem we can find a finite number of spheres R_i such that

$$\mathbf{f}(C) \subset \cup R_i \text{ and } \mu(\cup R_i) < \varepsilon.$$

If we put $O = \mathbf{f}^{-1}(\cup R_i)$, then O is the desired one. Another proposition is that for an arbitrary open neighborhood O of C, there exists Jordan measurable set O_1

such that $C \subset O_1 \subset O$. For instance we may set O_1 as sum of a finite number of spheres. This is possible since C is compact. Using the above given O and O_1, we get

$$\begin{aligned}\mu^*(\mathbf{f}(D)) &\leqq \mu^*(\mathbf{f}(D-O)) + \mu(\mathbf{f}(O)) \\ &\leqq \mu(\mathbf{f}(D-O_1)) + \mu(\mathbf{f}(O)) \\ &\leqq \int_{D-O_1} |\det J(\mathbf{x})|\ d\mathbf{x} + \mu(\mathbf{f}(O)) \\ &\leqq \int_D |\det J(\mathbf{x})|\ d\mathbf{x} + \varepsilon,\end{aligned}$$

which shows

$$\mu^*(\mathbf{f}(D)) \leqq \int_D |\det J(\mathbf{x})|\ d\mathbf{x}.$$

Thus the proof of Theorem 6.2 is completed.

Theorem 6.3 Let $\mathbf{f}$ be defined in a bounded domain D_1, $\overline{D}$ be its compact part and $g(\mathbf{y})$ be a real valued continuous function defined in $\mathbf{f}(\overline{D})$. If we assume that D has Jordan area and $\mathbf{f}$ is one-to-one in $\overline{D}$, then

$$\int_{\mathbf{f}(D)} g(\mathbf{y})\ d\mathbf{y} = \int_D g(\mathbf{f}(\mathbf{x}))|\det J(\mathbf{x})|\ d\mathbf{x}$$

Proof. By the preceding theorem, $\mathbf{f}(D)$ has Jordan area and $\mu(\mathbf{f}(D)) = \int_D |\det J(\mathbf{x})| d\mathbf{x}$. The right side of this equality can be approximated by $\int_{\cup Q_j} |\det J(\mathbf{x})|\ d\mathbf{x}$, where Q_j is small cubes inside D. Then, we see that $\mu(\mathbf{f}(D))$ can be approximated by $\mu(\cup_j \mathbf{f}(Q_j))$. Therefore, we see that for an arbitrary positive number ε there exists mutually disjoint cubes $Q_j s$ inside D such that

$$\Big|\int_{\mathbf{f}(D)} g(\mathbf{y})\, d\mathbf{y} - \int_{\cup \mathbf{f}(Q_j)} g(\mathbf{y})\ d\mathbf{y}\Big| < \varepsilon \tag{1}$$

together with

$$\Big|\int_D g(\mathbf{f}(x))|\det J(\mathbf{x})|\ dx - \int_{\cup Q_j} g(\mathbf{f}(\mathbf{x}))|\det J(\mathbf{x})| d\mathbf{x}\Big| < \varepsilon \tag{2}$$

If we set $M_j = \sup_{\mathbf{y} \in \mathbf{f}(Q_j)} g(\mathbf{y}) = \sup_{\mathbf{x} \in Q_j} g(\mathbf{f}(\mathbf{x}))$ and $m_j = \inf_{\mathbf{y} \in \mathbf{f}(Q_j)} g(\mathbf{y}) = \inf_{\mathbf{x} \in Q_j} g(\mathbf{f}(\mathbf{x}))$,

then we get

$$\sum_j m_j \mu(\mathbf{f}(Q_j)) \leqq \int_{\cup Q_j} g(\mathbf{f}(\mathbf{x}))|\det J(\mathbf{x})|\ d\mathbf{x} \leqq \sum_j M_j \mu(\mathbf{f}(Q_j)) \tag{3}$$

and

$$\sum m_j \mu(\mathbf{f}(Q_j)) \leqq \int_{\cup \mathbf{f}(Q_j)} g(\mathbf{y})\ d\mathbf{y} \leqq \sum_j M_j \mu(\mathbf{f}(Q_j)) \tag{4}$$

From the above four inequalities (1), (2), (3) and (4), we get

$$\left| \int_{\mathbf{f}(D)} g(\mathbf{y})\ d\mathbf{y} - \int_D g(\mathbf{f}(\mathbf{x})) |\det J(\mathbf{x})|\ dx \right| < 2\varepsilon + \sum_j (M_j - m_j) \mu(\mathbf{f}(Q_j))$$

Since $\sum_j (M_j - m_j)\ \mu(\mathbf{f}(Q_j))$ can be arbitrarily small, the equality in **Theorem 6.3** follows.

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