JOUR. FAC. SCI., SHINSHU UNIV. Vol. 5, pp. 1–23, May. 1970

Connection of Topological Manifolds

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(Received May 16, 1970)

Introduction.

One of the most important notion in the differential geometry is connection. It has the origin in the notion of parallel displacement of Levi-Civita. Then, although connetion was defined only for manifolds at first, later it is defined for differentiable fibre bundles (cf. [9], [16]). In its analytical form, if a vector bundle ξ is given by its transition function $\{g_{UV}\}$, a connection $\{\theta_U\}$ of ξ is a collection of matrix valued *1* -forms θ_U such that

$$(d+\theta_U)g_{UV}=g_{UV}(d+\theta_V).$$

From the view point of differential geometry of fibre bundles, a connection of a (smooth) manifold is defined to be a connection of its tangent bundle.

Therefore, if we want to extend the notion of connection for topological manifolds, we must extend the notion of tangent bundle to topological manifolds in one hand, and on the other hand, to extend the notion of connection for topological fibre bundles.

The extension of the notion of tangent bundle has been done by Milnor ([17]). It is the tangent microbundle and defined as follows: Let $X=\{(U, h_U)\}$ be a topological manifold (h_U: $U\rightarrow \boldsymbol{R}^n$ is the homeomorphism by which the manifold structure of X is given), then the tangent microbundle τ of X is the sequence

$$X\xrightarrow{\Delta}X\times X\xrightarrow{p}X,\quad \Delta(x)=(x, x),\quad p((x, y))=x,$$

with commutative diagram

$$\begin{tikzcd}
& U\times U \arrow[dr, "p"] \arrow[dd, "\varphi_U"] & \\
U \arrow[ur, "\Delta"] \arrow[dr, "i"'] & & U \\
& U\times \boldsymbol{R}^n \arrow[ur, "p"'] &
\end{tikzcd}
\qquad \varphi_U(x, y)=(x, h_U(y)-h_U(x)).$$

Moreover, rescent development of differential geometry of higher order (cf. [18]) regards that the usual tangent bundle is the tangent bundle of order *1* and there constructed tangent bundles of higher orders by using higher jets, then from this point of view, the tangent microbundle is the tangent bundle constructed by germ ([5]). In fact, using the same notations as above, we may define the transition functions of τ to be the germs of

$$g_{UV}(x) = h_{U,x}h_{V,x}{}^{-1}, \quad h_{U,x}(y) = h_U(y) - h_U(x),$$

if X is paracompact ([5]).

On the other hand, the notion of connection has been extended for topological fibre bundles by the author ([2], [3]). If $\xi=\{g_{UV}\}$ is a G -vector bundle and G is contained in a topological ring $\mathfrak{R}$ as an open subset, then a (linear) connection $\{\theta_U\}$ of ξ is defined to be a collection of the germs at the diagonal (of $U\times U$) of $\mathfrak{R}$-valued functions $\theta_U(x^0, x_1)$ such that

$$\theta_U(x, x) = 0,$$

$$(\delta\times\theta_U)g_{UV} = g_{UV}(\delta + \theta_V),$$

where δ is the coboundary of Alexander-Spanier cohomology ([2]). Then since $s_U(x_0, x_1)=1+\theta_U(x_0, x_1)$ satisfies

$$s_U(x, x) = 1,$$

$$g_{UV}(x_0)s_V(x_0, x_1)g_{VU}(x_1) = s_U(x_0, x_1),$$

we define a connection $\{s_U\}$ of a G -bundle $\xi=\{g_{UV}\}$ to be a collection of germs at the diagonal (of $U\times U$) of G -valued functions $s_U(x_0, x_1)$ which satisfy the above two formulas. Then, it has been proved ([2], [3], [4]) the analogies of Ambrose -Singer -Nomizu's theorem ([1], [16]) and Chern's theorem ([8], [16]) are also true for topological connection (and topological curvature which is defined by

$$\Theta_U(x_0, x_1, x_2) = \delta\theta_U(x_0, x_1, x_2) + \theta_U(x_0, x_1)\theta_U(x_1, x_2),$$

$$\delta s_U(x_0, x_1, x_2) = s_U(x_1, x_2)s_U(x_0, x_2)^{-1}s_U(x_0, x_1)$$

for $\{\theta_U\}$ and $\{s_U\}$ respectively).

Then we must define a connection of a topological manifold to be a connection of the tangent microbundle. The purpose of this paper is to show the existence of a connection for any topological manifold if it is paracompact.

The outline of this paper is as follows: In § 1, we review the definition and main properties of the connection of topological fibre bundles which has been given in [2], [3], [4]. In § 2, we give an example of a fibre bundle which has no (topological) connection. The existence (or non existence) of such bundles remained as an open problem in our previous papers. In § 3, we review the definition of microbundle and state the representation of microbundles as the germ. Then we define the connection of micrbundles. But it is possible two kinds of definitions of connection for microbundles because a microbundle is considered to be a bundle constructed from the germ as was stated in the above (or [5]) in one hand, on the other hand it is considered to be an $H_0(n)$ -bundle ([12], [15]). Here $H_0(n)$ is the group of all homeomorphisms of $\boldsymbol{R}^n$ which fix the origin with compact topology. But by virtue of the annulus theorem ([13], [21] cf. [7]), we can prove the equivalence of the existence of connection in two definitions if the dimension of the microbundle is at least 5. In § 4, we prove the existence of a connection for the tangent microbundle of a paracompact topological manifold. Moreover, it is shown that the existence of a connection on a topological manifold X is equivalent to the existence of local parallel displacement on X. In fact, if $\{s_U(x_0, x_1)\}$ is a connection of X, then we can set

$$s_U(x_0, x_1)(a) = h_U x_0 q_U(x_0, x_1) h_U{}^{-1}(a + h_U(x_0) - h_U(x_1)),$$
$$a \in U(h_U(x_1)), \textit{ a neighborhood of } h_U(x_1) \textit{ in } \boldsymbol{R}^n,$$

where $q_U(x_0, x_1)$ is a homeomorphism from a neighborhood of x_0 to a neighborhood of x_0 such that

$$q_U(x_0, x_1)(x_0) = x_0, \quad q_U(x_0, x_0) \textit{ is the identity.}$$

Or, equivalently, setting

$$t(x_0, x_1)| V(\Delta(U)) = h_{U, x_0}{}^{-1} s_U(x_0, x^1) h_{U, x_1},$$

where $V(\Delta(U))$ is a neighborhood of the diagonal of $U \times U$, $t(x_0, x_1)$ is defined on some neighborhood of the diagonal of $X \times X$ and locally we have

$$\begin{aligned}&(t(x_0, x_1)| V(\Delta(U)))(y)\\ &= q_U(x_0, x_1)(h_U{}^{-1}(h_U(y) + h_U(x_0) - h_U(x))).\end{aligned}$$

Although we show the existence of a connection for the tangent microbundle, the general problem, that whether a microbundle always has a (topological) connection or not still remains open. The applications of connection of topological manifolds will be treated in forthcoming papers.

The outline of this paper is announced in Proc. Japan Acad., Vol. 46 (1970), 370–374, under the same title.

§ 1. Connection of topological fibre bundles.

1. Let X be a topological space, then we set

$$\Delta_s(x) = \{(x, \cdots\cdots, x) | x \in X\} \subset \overbrace{X \times \cdots\cdots \times X}^{s+1},$$

$$\Delta_0(x) = X.$$

Similarly, $\Delta_s(U)$, etc. mean same sets (cf. [3]).

Definition. *Let f_1 and f_2 are continuous maps from $V(\Delta_s(U))$, a neighborhood of Δ_s (U) in $U \times \cdots\cdots \times U$, into G, a topological group, then we call f_1 and f_2 are equivalent and denote $f_1 \sim f_2$ if and only if there is a neighborhood $W(\Delta_s(U))$ of $\Delta_s(U)$ in $V(\Delta_s(U))$ such that*

$$f_1 | W(\Delta_s(U)) = f_2 | W(\Delta_s(U)).$$

We call the equivalence class of f to be a germ of f (at $\Delta_s(U)$) and denote the equivalence class of f by $\overline{f}$ or simply by f. We set

(1) $$C^s(U, G) = \{\overline{f} \,|\, f(x_0, \cdots\cdots, x_s) = e, \text{ the unit of } G, \text{ if } x_i = x_{i+1} \text{ for some } i, \ 0 \leqq i \leqq s\}.$$

If $\{\xi = g_{UV}(x)\}$ be a (topological) G -bundle over X, then we set

(2) $$\mathscr{G}^r(\xi)\text{: the sheaf of germs of those maps } \overline{f}_U, \ \overline{f}_U \in C^r(U, G),$$
$$g_{UV}(x_0)^{-1} f_U(x_0, \cdots\cdots, x_r) g_{UV}(x_r) = f_V(x_0, \cdots\cdots, x_r).$$

Definition ([3]). *An element $\{\bar{s}_U\}$ of $H^0(X, \mathscr{G}^1(\xi))$ is called a connection of ξ.*

Note. We often denote $\{s_U\}$ instead of $\{\bar{s}_U\}$. Then we may consider $s_U \in \{s_U\}$ is a continuous map from $V(\Delta_1(U))$ into U such that

(1)′ $$s_U(x, x) = e,$$

(2)′ $$g_{UV}(x_0) s_V(x_0, x_1) g_{VU}(x_1) = s_U(x_0, x_1).$$

But in this definition, we must need following definition of equivalence.

Definition. *If $\{s_U\}$ and $\{s'_{U'}\}$ are connections of ξ, then we call $\{s_U\}$ and $\{s'_{U'}\}$ are equivalent if and only if there exists common refinement $\{U''\}$ of $\{U\}$ and $\{U'\}$ such that*

$$s_U(x_0, x_1) | V(\Delta_1(U'')) = h_{U''}(x_0) s'_{U'}(x_0, x_1) h_{U''}(x_1) | V(\Delta_1(U'')),$$

where U'' is contained in $U \cap U'$ and the transition function $\{g_{UV}(x)\}$ and $(g'_{U'V'}(x)\}$ of ξ satisfy

$$g'_{U'V'}(x) | U'' \cap V'' = h_{U''}(x) g_{UV}(x) h_{V''}(x)^{-1} | U'' \cap V''.$$

We denote by X_ξ the associated principal bundle of ξ. Its projection to X is denoted by π, the homeomorphism from $\pi^{-1}(U)$ onto $U \times G$ is denoted by φ_U •

Denoting $\varphi_U^{-1}(x, a)$, by $\varphi_U^{-1}(x)(a)$, we set

$$p\alpha = \varphi_U^{-1}(x)(a\alpha), \quad p = \varphi_U^{-1}(x)(a) \in X_\xi, \quad \alpha \in G.$$

Then G operates on X_ξ. If $\{s_U\}$ is a connection of ξ, we set

$$s(p, q) = \alpha^{-1} s_U(x, y)\beta, \quad p = \varphi_U(x)^{-1}(\alpha), \quad q = \varphi_U(y)^{-1}(\beta).$$

Then $s(p, q)$ is a continuous map from some neighborhood of $\Delta_1(X_\xi)$ (in $X_\xi \times X_\xi$) into G and satisfies

(3) $$s(p, p) = e,$$

(4) $$s(p\alpha, q\beta) = \alpha^{-1} s(p, q)\beta.$$

Conversely, if there exists a continuous map s from a neighborhood of $\Delta_1(X_\xi)$ into G which satisfies (3), (4), then setting

$$s_U(x_0, x_1) = \alpha s(\varphi_U^{-1}(x_0)(\alpha), \ \varphi_U^{-1}(x_1)(\beta))\beta^{-1},$$

$\{s_U\}$ becomes connection of ξ. Therefore we have

Theorem 1. *ξ has a connection if and only if there exists a continuous map s from a neighborhood of $\Delta_1(X_\xi)$ in $X_\xi \times X_\xi$ into G which satisfies* (3), (4) *and there is a 1 to 1 correspondence between H^0 $(X, \mathscr{G}^1(\xi))$ and $C^1(X_\xi, G)_\xi$, where $C^1(X_\xi, G)_\xi$ is given by*

$$C^1(X_\xi, G)_\xi = \{\bar{s} \mid \bar{s} \in C^1(X_\xi, G), \ s \text{ satisfies } (3), (4) \ (s \in \bar{s})\}.$$

It $\{s_U\}$ and $\{s'_U\}$ are the connections of ξ, then setting $s'_U(x_0, x_1) = t_U(x_0, x_1) s_U(x_0, x_1)$, we have

(5) $$g_{UV}(x_0) t_V(x_0, x_1) g_{VU}(x_0) = t_U(x_0, x_1),$$

$$t_U(x, x) = e.$$

Conversely, if $\{t_U\}$ satisfies (5), $\{t_U s_U\}$ becomes a connection of ξ if $\{s_U\}$ is a connection of ξ. Similarly, setting

$$T^1(X_\xi, G) = \overline{\{t} \mid \bar{t} \in C^1(X_\xi, G), \ t \text{ satisfies}$$
$$t(x\alpha, \ y\beta) = \alpha^{-1} t(x, y)\alpha, \ \ t \in \overline{t\}},$$

we get

(5)′ $$C^1(X_\xi, G)_\xi = T^1(X_\xi, G)\bar{s}, \quad \bar{s} \in C^1(X_\xi, G)_\xi,$$

if $C^1(X_\xi, G)_\xi \neq \phi$.

2. If $\xi = \{g_{UV}(x)\}$ is a trivial bundle, then setting $g_{UV}(x) = h_U(x) h_V(x)^{-1}$ and

$$s_U(x_0, x_1) = h_U(x_0)h_U(x_1)^{-1},$$

$\{s_U\}$ is a connection of ξ. Hence a fibre bundle ξ always has a connection locally. Or, in other word, for a suitable open covering $\{U\}$ of X, there exists a continuous map s_U on $V(\Delta_1(\pi^{-1})U))$ which satisfies (3), (4). Then setting

$$\mathscr{T}(\xi): \textit{the sheaf of germs of elements of } T^1(\pi^{-1}(U), G),$$

$\{t_{UV}\}=\{s_{(U)}s_{(V)}^{-1}\}$ defines a cocycle on X with coefficients in $\mathscr{T}(\xi)$ and its class in $H^1(X, \mathscr{T}(\xi))$ does not depend on the choice of $\{s_{(U)}\}$.

Definition ([4], cf. [6]). *The class of* $\{t_{UV}\}=\{s_{(U)}s_{(V)}^{-1}\}$ *in* $H^1(X, \mathscr{T}(\xi))$ *is called the obstruction class (for the existence of connection) of* ξ *and denoted by* $o(\xi)$.

Theorem 2. ξ *has a connection if and only if* $o(\xi)$ *vanishes in* $H^1(X, \mathscr{T}(\xi))$.

Note. We may consider $\mathscr{T}(\xi)=\mathscr{T}(\xi)(G)$ to be the sheaf of germs of $\{t_U\}$, t_U satisfies (5).

As a consequence of this theorem, we show the existence of connection for the bundles whose structure groups are locally compact abelian groups as follows: We denote the sheaf of germs of the elements of $C^r(U, G)$ by $\mathscr{G}^r$. Then if G is an abelian group, we get $\mathscr{T}(\xi)= \mathscr{G}^1$. Hence we have

$$H^1(X, \mathscr{T}(\xi)) = H^1(X, \mathscr{G}^1).$$

If G is a locally compact, connected and locally connected abelian group, then we know ([19])

$$G \cong \boldsymbol{R}^\mu \times T^\nu$$

where $\boldsymbol{R}^\mu$ is the μ -direct product of $\boldsymbol{R}^1$, the additive group of real numbers, T^ν is the ν -direct product of $T^1=\boldsymbol{R}^1/\boldsymbol{Z}$. Then since we know

$$\mathscr{G}^r = (R^{\mu+\nu})^r, \quad r \geqq 1,$$

we get $H^1(X, \mathscr{G}^1)=\{0\}$, if X is normal paracompact. Hence we have

Theorem 3. *If G is a locally compact, connected and locally connected abelian group, then a G -bundle ξ over a normal paracompact space always has a connection.*

Corollary. *If G is a locally connected, connected and locally connected solvable group, then a G -bundle ξ over a normal paracompact space always has a connection.*

Note. If G is a locally compact, connected and locally connected abelian group, then we can define g^λ, $0 \leqq \lambda \leqq 1$, for any element g of G. Hence taking a partition of unity $\{e_W(x)\}$ for the locally finite open covering $\{W\}$ by which the given G -bundle $\xi=\{g_{UV}(x)\}$ is defined, we can construct a connection $\{s_U\}$ of ξ by

$$s_U(x_0, x_1) = \prod_{U\cap W\neq\phi} g_{UW}(x_0)^{e_W(x_0)}\, g_{WU}(x_1)^{e_W(x_1)}.$$

We note that this s_U satisfies $s_U(x_0, x_1)=s_U(x_1, x_0)$.

3. Defintion ([3]). *If* $\{s_U\}$ *is a connection of* ξ, *then*

(6) $$\{\delta s_U(x_0, x_1, x_2)\} = \{s_U(x_1, x_2)s_U(x_0, x_2)^{-1}s_U(x_0, x_1)\},$$

is called the curvature of $\{s_U\}$.

Theorem 4 ([2], [3] cf. [1], [16]). *If the value of the curvature of a connection of* ξ *is contained in* H, *a subgroup of* G, *then the connected component of the identity of the structure group of* ξ *is reduced to* H *as a* G *-bundle.*

Proof. We take a set of points $\{a_\alpha\}$ of X such that there exists a neighborhood $U(a_\alpha)$ of a_α for any a_α which satisfies

(i) $$\cup_\alpha U(a_\alpha) = X,$$

(ii) *some* s_U *is defined on* $U(a_\alpha) \times U(a_\alpha)$.

Since $s_U(x_1, x_2)s_U(x_0, x_2)^{-1}s_U(x_0, x_1)$ belongs in H, we have

$$s_U(x_0, x_2) = s_U(x_0, x_1)h_U(x_0, x_1, x_2)s_U(x_1, x_2),$$
$$h_U(x_0, x_1, x_2) \in H.$$

Then setting $x_1=x_2=x$ and $X_0=a_\alpha$, we get

$$s_U(a_\alpha, x)^{-1}g_{UV}(a_\alpha)s_V(a_\alpha, x)g_{VU}(x) = h_U(a_\alpha, x, x) \in H.$$

Hence we have the theorem.

Note. By definition, $\{\delta s_U\}$ satisfies

(7) $$g_{UV}(x_1)^{-1}\delta s_U(x_0, x_1, x_2)g_{UV}(x_1) = \delta s_V(x_0, x_1, x_2).$$

Similarly, if the connection is given as an element s of $C^1(X_\xi, G)_\xi$, then we define the curvature δs of s by

(6)′ $$\delta s(p_0, p_1, p_2) = s(p_1, p_2)s(p_0, p_2)^{-1}s(p_0, p_1).$$

This δs also satisfies

(7)′ $$\delta s(p_1\alpha, p_2\beta, p_3\gamma) = \beta^{-1}\delta s(p_1, p_2, p_3)\beta, \quad \alpha, \beta, \gamma \in G.$$

It G is an abelian group, then by (7)′, δs defines an Alexander -Spanier *2* -cocycle of X with coefficients in G, and its cohomology class does not depend on the choice of s.

Especially, if G is a locally compact, connected and locally connected abelian group, then δs defines an element of $H^2(X, R^{\mu+\nu})$.

Theorem 5 ([4]). *If G is a locally compact, connected and locally connected abelian group and X is paracompact normal, then the following diagram is commutative.*

$$\begin{tikzcd} H^1(X, G_c) \arrow[r, "\delta"] \arrow[dr, "\chi"'] & H^2(X, Z^\nu) \arrow[d, "\lambda"] \\ & H^2(X, R^{\mu+r}), \end{tikzcd}$$

where G_c is the sheaf of germs of continuous maps from X to G, δ is the coboundary homomorphism induced from the exact sequence

$$0 \to Z^\nu \xrightarrow{i} R^{\mu+\nu} \xrightarrow{j} G_c \to 0,$$

τ is the map induced from the inclusion and χ is the map defined by the curvature of the G -bundles (regarded as the elements of $H^1(X, G_c)$ (cf. [11])).

Corollary 1. *If $G=C^*$, the multiplicative group of complex numbers without 0, then for a C^* -bundle ξ, $\chi(\xi)$ is the 1 -st (complex) Chern class of ξ.*

Corollary 2. *Under the same assumptions about G and X as Theorem 5, the following sequence is exact.*

$$H^1(X, G) \xrightarrow{i} H^1(X, G_c) \xrightarrow{\chi} H^2(X, R^{\mu+\nu}),$$

where i is the map induced from the inclusion by regarding G to be the sheaf of germs of constant maps from X to G (which is a subsheaf of G_c).

Note. For $GL(n, C)$ -bundles, we can construct the analogy of Chern's theory of Characteristic classes ([4], cf. [8], [9]).

4. If ξ is a vector bundle with structure group G, fibre L, a topological vector space on which G acts as a linear transformation group, then we can give another definition of connection which is a natural generalization of usual definition of connection ([2]).

Definition. *If the collection $\{f_U\}$, $f_U \in C^s(U, L)$, satisfies*

$$g_{UV}(x_0) f_V(x_0, \cdots\cdots, x_s) = f_U(x_0, \cdots\cdots, x_s), \quad x_0 \in U \cap V, \tag{8}$$

then we call $\{f_U\}$ is an s -cross -section of $\xi=\{g_{UV}(x)\}$, and denote the space of all s -cross -sections of ξ by $C^s(X, \xi)$.

We assume that there is a topological ring $\mathfrak{R}$ of linear transformations of L

such that $G \subset \mathfrak{R}$ and some neighborhood of e in $\mathfrak{R}$ is contained in G.

Definition. *A collection* $\{\theta_U\}$, $\theta_U \in C^1(U, \mathfrak{R})$, *is called a (linear) connection of* ξ *if* $\{\theta_U\}$ *satisfies*

$$((\delta+\theta_U)(f_U)(x_0, \cdots\cdots, x_{s+1}))$$
$$= g_{UV}(x_0)((\delta+\theta_V)f_V)(x_0, \cdots\cdots, x_{s+1}), \quad \textit{for all } f_U \in C^s(X, \xi), \ s \geqq 0,$$

where $\delta f_U(x_0, \cdots\cdots, x_{s+1})$ *and* $\theta_U f_U(x_0, \cdots\cdots, x_{s+1})$ *are given by*

$$\delta f_U(x_0, \cdots\cdots, x_{s+1})$$
$$= \sum_{i=0}^{s+1}(-1)^i f_U(x_0, \cdots\cdots, x_{i-1}, x_{i+1}, \cdots\cdots, x_{s+1}),$$
$$\theta_U f_U(x_0, \cdots\cdots, x_{s+1}) = \theta_U(x_0, x_1) f_U(x_1, \cdots\cdots, x_{s+1}).$$

Definition. *If* $\{\theta_U\}$ *and* $\{\theta'_{U'}\}$ *are the connections of* ξ, *then we call* $\{\theta_U\}$ *and* $\{\theta'_{U'}\}$ *are equivalent if and only if there exists a common refinement* $\{U''\}$ *of* $\{U\}$ *and* $\{U'\}$ *such that*

$$\theta'_{U'} | V((\Delta_1(U'')) = h_{U''}(\theta_U - h_{U''}{}^{-1}\delta h_{U''}) h_{U''}{}^{-1} | V(\Delta_1(U'')), \quad U'' \subset U \cap U',$$
$$g'_{U'V'} | U'' \cap V'' = h_{U''} g_{UV} h_{V''}{}^{-1} | U'' \cap V'',$$

where $\{g_{UV}\}$ *and* $\{g'_{U'V'}\}$ *are the transition functions of* ξ *by which* $\{\theta_U\}$ *and* $\{\theta'_{U'}\}$ *are defined.*

Definition. *We set*

(9) $$\Theta_U(x_0, x_1, x_2) = \delta\theta_U(x_0, x_1, x_2) + \theta_U(x_0, x_1)\theta_U(x_1, x_2),$$

and call $\{\Theta_U\}$ *the curvature of* $\{\theta_U\}$.

Then we can prove ([2], [3]),

(i). *If* $\{\theta_U\}$ *is a (linear) connection of* ξ, *then setting* $s_U(x_0, x_1) = e + \theta_U(x_0, x_1)$, $\{s_U\}$ *is a connection of* ξ.

(i)'. *If* $\{s_U\}$ *is a connection of* ξ *and* $\mathfrak{R}$ *exists for* G, *then setting* $\theta_U = s_U - e$, $\{\theta_U\}$ *is a (linear) connection of* ξ.

(ii). *If* $\mathfrak{R}$ *exists for* G, *then a* G *-bundle always has a (linear) connection.*

(iii). *If the value of* $1+\Theta_U$ *is contained in* H, *a subgroup of* G, *then the connected component of the identity of the structure group of* ξ *is recuced to* H *as a* G*-bundle.*

§ 2. Bundles which have no connections.

5. For a normal paracompact space $\mathscr{E}$, we set

$Z(\mathscr{E})$: *the topological group of all continuous maps from* $\mathscr{E}$ *into* $\boldsymbol{Z}$, *the additive* *group of integers, with compact open topology.*

$\boldsymbol{C}(\mathscr{E})$: *the topological group of all continuous maps from* $\mathscr{E}$ *into* $\boldsymbol{C}$, *the complex number field, with compact open topology.*

$\boldsymbol{C}^*(\mathscr{E})$: *the topological group of all continuous maps from* $\mathscr{E}$ *into* $\boldsymbol{C}^*$, *the multiplicativegroup of complex numbers without 0, with compact open topology.*

We denote the sheaves of germs of continuous maps from $\mathscr{E}$ to $\boldsymbol{Z}$, $\boldsymbol{C}$ and $\boldsymbol{C}^*$ by $\boldsymbol{Z}_c$, $\boldsymbol{C}_c$, and $\boldsymbol{C}^*_c$. Then we have the exact sequence

$$0 \to \boldsymbol{Z}_c \xrightarrow{i} \boldsymbol{C}_c \xrightarrow{j} \boldsymbol{C}^*_c \to 0,$$

where i is the inclusion, j is given by $j(f) = \exp(2\pi\sqrt{-1}\, f)$. Hence for the above groups, we have the following exact sequence.

$$0 \to \boldsymbol{Z}(\mathscr{E}) \xrightarrow{i^*} \boldsymbol{C}(\mathscr{E}) \xrightarrow{j^*} \boldsymbol{C}^*(\mathscr{E}) \xrightarrow{\delta} H^1(\mathscr{E}, \boldsymbol{Z}) \to 0. \tag{10}$$

In the sequence (10), we set

$$j^*(\boldsymbol{C}(\mathscr{E})) = \boldsymbol{C}^*_0(\mathscr{E}). \tag{11}$$

Then by (10), we have the following exact sequences.

$$0 \to \boldsymbol{Z}(\mathscr{E}) \xrightarrow{i^*} \boldsymbol{C}(\mathscr{E}) \xrightarrow{j^*} \boldsymbol{C}^*_0(\mathscr{E}) \to 0, \tag{10'}$$

$$0 \to \boldsymbol{C}^*_0(\mathscr{E}) \xrightarrow{\delta} \boldsymbol{C}^*(\mathscr{E}) \xrightarrow{\tau} H^1(\mathscr{E}, \boldsymbol{Z}) \to 0, \tag{10''}$$

where τ is the inclusion.

On a normal paracompact space X, we denote the sheaves of germs of continuous maps from X to $\boldsymbol{Z}(\mathscr{E})$, $\boldsymbol{C}(\mathscr{E})$, $\boldsymbol{C}^*(\mathscr{E})$, $\boldsymbol{C}^*_0(\mathscr{E})$ and $H^1(\mathscr{E}, \boldsymbol{Z})$ by $\boldsymbol{Z}(\mathscr{E})_c$, $\boldsymbol{C}(\mathscr{E})_c$, $\boldsymbol{C}^*(\mathscr{E})_c$, $\boldsymbol{C}^*_0(\mathscr{E})_c$ and $H^1(\mathscr{E}, \boldsymbol{Z})_c$. Then by (10)′ and (10)″, we have the exact sequences of sheaves

$$0 \to \boldsymbol{Z}(\mathscr{E})_c \xrightarrow{\hat{\imath}} \boldsymbol{C}(\mathscr{E})_c \xrightarrow{\hat{\jmath}} \boldsymbol{C}^*_0(\mathscr{E})_c \to 0, \tag{12$_a$}$$

$$0 \to \boldsymbol{C}^*_0(\mathscr{E})_c \xrightarrow{\hat{\tau}} \boldsymbol{C}^*(\mathscr{E})_c \xrightarrow{\hat{\delta}} H^1(\mathscr{E}, \boldsymbol{Z})_c \to 0, \tag{12$_b$}$$

where $\hat{\imath}$, $\hat{\jmath}$, $\hat{\tau}$ and $\hat{\delta}$ are the maps induced from i^*, j^*, τ and δ. Moreover, we know

$$H^p(X, \boldsymbol{Z}(\mathscr{E})_c) = H^p(X, \boldsymbol{Z}(\mathscr{E})), \quad p \geqq 0, \tag{13$_a$}$$

$$H^p(X, H^1(\mathscr{E}, \boldsymbol{Z})_c) = H^p(X, H^1(\mathscr{E}, \boldsymbol{Z})), \quad p \geqq 0, \tag{13$_b$}$$

because the topologies of $\boldsymbol{Z}(\mathscr{E})_c$ and $H^1(\mathscr{E}, \boldsymbol{Z})_c$ are discrete.

By $(12)_a$ and $(13)_a$, we get

$$\delta:\ H^i(X,\ \boldsymbol{C}^*_0(\mathscr{E}))\cong H^{i+1}(X,\ \boldsymbol{Z}(\mathscr{E})),\ i\geqq 1. \tag{14}$$

On the other hand, by $(12)_b$ and $(13)_b$, we have the following exact sequence.

$$\cdots\to H^1(X,\ \boldsymbol{C}^*_0(\mathscr{E})_c)\xrightarrow{\hat{\delta}^{*\prime}} H^1(X,\ \boldsymbol{C}^*(\mathscr{E})_c)\xrightarrow{\tau^*} \tag{15}$$

$$\to H^1(X, H^1(\mathscr{E},\ \boldsymbol{Z}))\xrightarrow{\delta} H^2(X,\ \boldsymbol{C}^*_0(\mathscr{E})_c)\to\cdots.$$

By (14), (15) is rewritten as

$$\cdots\to H^2(X,\ \boldsymbol{Z}(\mathscr{E}))\xrightarrow{\hat{\iota}^{*\prime}} H^1(X,\ \boldsymbol{C}^*(\mathscr{E})_c)\xrightarrow{\hat{\ }^{*\prime}} \tag{15$'$}$$

$$\to H^1(X,\ H^1(\mathscr{E},\ \boldsymbol{Z}))\xrightarrow{\delta'} H^3(X,\ \boldsymbol{Z}(\mathscr{E}))\to\cdots,$$

where $\hat{\iota}^{*\prime}$ and δ' are the maps induced from (14) and $\hat{\iota}^*$ and δ. We note that $H^1(X, \boldsymbol{C}^*(\mathscr{E})_c)$ is the group of equivalence classes of all $C^*(\mathscr{E})$ -bundles over X ([11]).

6. Lemma 1. *If $\mathscr{E}$ is connected and H is a subgroup of $\boldsymbol{C}^*(\mathscr{E})$ such that whose connected component of 1 is $\boldsymbol{C}^*_0(\mathscr{E})$, then H is equal to $\boldsymbol{C}^*_0(\mathscr{E})$.*

Theorem 6. *If each connected component of $\mathscr{E}$ is compact then a $\boldsymbol{C}^*(\mathscr{E})$ -bundle over X, a normal paracompact space, has a connection if and only if it belongs in $\hat{\iota}^*$ -image.*

Proof. If a $\boldsymbol{C}^*(\mathscr{E})$ -bundle ξ over X has a connection, then its curvature gives an Alexander -Spanier 2-cocycle of X with coefficients in $\boldsymbol{C}^*(\mathscr{E})$. If $\mathscr{E}$ is connected, then by assumption, $\mathscr{E}$ is compact. Hence some neighborhood of 1 in $\boldsymbol{C}^*(\mathscr{E})$ is contained in $\boldsymbol{C}^*_0(\mathscr{E})$ by the definition of $\boldsymbol{C}^*_0(\mathscr{E})$. Hence we may consider the value of the curvature of ξ is contained in $\boldsymbol{C}^*_0(\mathscr{E})$, because the value of the curvature belongs in (arbitrary) neighborhood of 1. Therefore we have the nescessity by Theorem 4 and Lemma 1 if $\mathscr{E}$ is connected. If $\mathscr{E}$ is not connected, then setting $\mathscr{E}=\cup_\alpha \mathscr{E}_\alpha$, each $\mathscr{E}_\alpha$ is connected, a $\boldsymbol{C}^*(\mathscr{E})$ -bundle over X is a direct sum of $\boldsymbol{C}^*(\mathscr{E}_\alpha)$ -bundles over X. Hence we obtain the nescessity for the non -connected case.

On the other hand, if $\{\xi=g_{UV}(x)\}$ is a $\boldsymbol{C}^*_0(\mathscr{E})$ -bundle, then we can define $g_{UV}(x)^\lambda$, $0\leqq\lambda\leqq 1$, and setting

$$s_U(x_0,\ x_1)=\prod_{U\cap W\neq\phi} g_{UW}(x_0)^{e_W(x_0)}\, g_{WU}(x_1)^{e_W(x_1)},$$

where $\{U\}$ is a locally finite covering and $\{e_U(x)\}$ is a partition of unity corresponding to $\{U\}$, $\{s_U\}$ gives a connection of ξ. This shows the sufficiency.

We take S^1 as X and $\mathscr{E}$. Then we get

$$H^1(S^1,\ C^{*0}(S^1)) \cong H^2(S^1,\ Z) = \{0\},$$

$$H^2(S^1,\ C^*{}_0(S^1)) \cong H^3(S^1,\ Z) = \{0\},$$

$$H^1(S^1,\ H^1(S^1,\ Z)) \cong H^1(S^1,\ Z) = Z.$$

Hence by the exact sequence $(15)'$, we obtain

$$(16) \qquad H^1(S^1,\ C^*(S^1)_c) \cong Z,$$

and τ^* -image in $H^1(S^1,\ C^*(S^1)_c)$ is equal to 0. Therefore there exists infinitely many non -trivial $C^*(S^1)$ -bundles over S^1 and they have no (topological) connection because S^1 is compact.

Note. We set $T(\mathscr{E}) = \{f \mid f \in C^*(\mathscr{E}),\ |f| = 1\}$, then by the map

$$C^*(\mathscr{E}) \times I \ni (f,\ t) \to \frac{f}{|f|^t} \in C^*(\mathscr{E}),$$

$T(\mathscr{E})$ is a deformation retract of $C^*(\mathscr{E})$. Hence a $C^*(\mathscr{E})$ -bundle over a normal paracompact space is considered to be a $T(\mathscr{E})$ -bundle ([14], [20]).

§ 3. Connection of microbundles.

7. A microbundle $\mathfrak{X}$ over X, a topological space, is a sequence $X \xrightarrow{i} E \xrightarrow{j} X$ with open coverings $\{U\}$ and $\{\mathfrak{U}\}$ of X and $i(X)$ in E such that the diagram

$$\begin{tikzcd} & \mathfrak{U} \arrow[dr, "j"] \arrow[dd, "\varphi_{\mathfrak{U}}"] & \\ U \arrow[ur, "i"] \arrow[dr, "\lambda"'] & & U \\ & U \times R^n \arrow[ur, "p"'] & \end{tikzcd} \qquad \tau(x) = x \times 0,\ \ p((x,\ a)) = x,\ \ x \in U,\ \ a \in R^n,$$

is commutative (17), where $\varphi_{\mathfrak{U}}$ is a homeomorphism.

It is known ([12], [15]), that a microbundle over X is regarded to be an $H_0(n)$ -bundle over X if X is normal paracompact. Here $H_0(n)$ is the group of all homeomorphisms of R^n which fix the origin with compact open topology. Therefore, a connection $\{s_U\}$ of $\mathfrak{X}$ should be considered to be a collection of continuous maps $s_U(x_0,\ x_1)$ from a neighborhood of $\Delta_1(U)$ to $H_0(n)$ such that

$$s_U(x,\ x) = e,$$

$$g_{UV}(x_0) s_V(x_0,\ x_1) g_{VU}(x_1) = s_U(x_0,\ x_1),$$

where $\{g_{UV}(x)\}$ is the transition function of $\mathfrak{X}$ regarded to be an $H_0(n)$ -bundle.

On the other hand, setting

$$\varphi_{\mathfrak{U}}\varphi_{\mathfrak{V}}^{-1}(x, a) = (x, \hat{\varphi}_{UV}(x)(a)), \quad x \in U \cap V, \quad a \in \boldsymbol{R}^n, \tag{17}$$

$\hat{\varphi}_{UV}(x)$ is a homeomorphism from a neighborhood of the origin in $\boldsymbol{R}^n$ to a neighborhood of the origin in $\boldsymbol{R}^n$ such that

$$\hat{\varphi}_{VU}(x)(0) = 0.$$

As in n^o *1*, we call two homeomorphisms φ, ψ from a neighborhood of the origin in $\boldsymbol{R}^n$ to a neighborhood of the origin of $\boldsymbol{R}^n$ such that

$$\varphi(0) = \psi(0) = 0,$$

are equivalent each other and denote $\varphi \sim \psi$ if and only if there exists a neighborhood W of the origin in $\boldsymbol{R}^n$ such that

$$\varphi \mid W = \psi \mid W.$$

Then the set of all those classes forms a group. We denote it by $H_*(n)$. Although $H_*(n)$ is not a topological group, we define a map from S, a topological space, to $H_*(n)$ to be continuous as follows.

Definition. *We call f: $S \to H_*(n)$ to be continuous if there exists an open covering $\{U\}$ of S and continuous maps $\hat{f}_U$: $U \to E_0(Q_U, \boldsymbol{R}^n)$ such that the class of $\hat{f}_U(x)$ is $f(x)$ for each $x \in U$. Here Q_U is a neighborhood of the origin in $\boldsymbol{R}^n$, $E_0(Q_U, \boldsymbol{R}^n)$ is the topological space of all homeomorphisms from Q_U into $\boldsymbol{R}^n$ which fix the origin with compact open topology.*

By this definition, $\hat{\varphi}_{UV}(x)$ given by (17) defines a continuous map φ_{UV} from $U \cap V$ into $H_*(n)$. Moreover, it satisfies

$$\varphi_{UV}(x)\varphi_{VW}(x)\varphi_{WU}(x) = e, \text{ the identity of } H_*(n).$$

Therefore, denoting the sheaf of germs of continuous maps from X to $H_*(n)$ by $H_*(n)_c$, we can construct a *1* -cocycle over X with coefficients in $H_*(n)_c$ and we can show ([5]) its cohomology class does not depend on the choice of $\varphi_{\mathfrak{U}}$.

Definition. *We call $\{\varphi_{UV}\}$ the transition function of $\mathfrak{X}$ as an $H_*(n)$ -bundle.*

On the other hand, if $\{U_\alpha\}$ is a locally finite open covering of X, $\{\varphi_{\alpha\beta}\} = \{\varphi_{U_\alpha U_\beta}\}$ belongs in $Z^1(X, H_*(n)_c)$, then we can construct a microbundle over X such that whose transition function as an $H_*(n)$ -bundle is $\{\varphi_{\alpha\beta}\}$ as follows: Take a representation $\hat{\varphi}_{\alpha\beta}$ of $\varphi_{\alpha\beta}$ for each α, β and assume

$$\hat{\varphi}_{\alpha\beta}(x)\hat{\varphi}_{\beta\gamma}(x)\hat{\varphi}_{\gamma\alpha}(x)(a) = a, \quad x \in U_\alpha \cap U_\beta \cap U_\gamma,$$

$$a \in Q_\alpha, \text{ a neighborhood of the origin of } \boldsymbol{R}^n.$$

Then in $U_\alpha \times \boldsymbol{R}^n \times \alpha$, we set

$$\mathfrak{U}_{\beta\alpha} = ((U_\alpha \cap U_\beta) \times Q_{\beta\alpha} \cap \hat{\varphi}_{\alpha\beta}((U_\alpha \cap U_\beta) \times Q_{\alpha\beta}))\,\alpha,$$

and identify $\mathfrak{U}_{\alpha\beta} \ni x \times a \times \beta$ and $x \times \hat{\varphi}_{\alpha\beta}(x)(a) \times \beta \in \mathfrak{U}_{\beta\alpha}$. Then setting the quotient space of $\cup_\alpha U_\alpha \times \boldsymbol{R}^n \times \alpha$ under this relation by E (the points which are not contained in $\mathfrak{U}_{\alpha\beta}$ are not identified each other), and set

$$i(x) = \text{the class of } x \times 0 \times \alpha \text{ in } E,$$

$$j(\{x \times a \times \alpha\}) = x, \quad \{x \times a \times \alpha\} \text{ is the class of } x \times a \times \alpha,$$

we obtain a microbundle $\mathfrak{X}$ and its transition function (as an $H_*(n)$ -bundle) is $\{\varphi_{\alpha\beta}\}$.

It is known (5) that:

Theorem 7. *There is a 1 to 1 correspondence between the set of all equivalence classes of n -dimensional microbundles over X, a normal paracompact space, and* $H^1(X, H_*(n)_c)$.

Regarding $\mathfrak{X}$ to be an $H_*(n)$ -bundle with transition function $\{g_{UV}(x)\}$, we can define a connection $\{s_U\}$ of $\mathfrak{X}$ as an $H_*(n)$ -bundle as follows: $\{s_U\}$ is a collection of continuous maps $Su(x_0, x_1)$ from a neighborhood of $\Delta_1(U)$ to $H_*(n)$ such that

$$s_U(x, x) = e,$$

$$g_{UV}(x_0)s_V(x_0, x_1)g_{VU}(x_1) = s_U(x_0, x_1).$$

8. In $H_0(n)$, we set

$$H_e(n) = \{ f \mid f \in H_0(n),\ f(a) = a, \text{ if } a \text{ belongs in some neighborhood of the origin of } \boldsymbol{R}^n\}.$$

By definition, $H_e(n)$ is an (algebraic) normal subgroup of $H_0(n)$.

Lemma 2. *If* $n \geqq 5$, *then*

$$H_*(n) = H_0(n)/H_e(n). \tag{18}$$

Proof. Since an element of $H_0(n)/H_e(n)$ defines an element of $H_*(n)$, we need only to show if φ is an element of $H_*(n)$, then there is an element f of $H_0(n)$ such that whose class *mod.* $H_e(n)$ gives φ.

We set $B_\rho = \{a \mid a \in \boldsymbol{R}^n, |a| < \rho\}$ and take a representation $\hat{\varphi}$ of φ which is defined on $B_{r+\varepsilon}$, $r>0$, $\varepsilon>0$. We take a positive number R to satisfy

$$R>r,\ \ B_R \supset \hat{\varphi}(B_{r+\varepsilon}).$$

Then by annulus theorem ([13], [21]), there exists a homeomorphism h from $\overline{B_{2R}}-B_r$ onto $\overline{B_{2R}}-\hat{\varphi}(B_r)$ if $n\geqq 5$. Since $\hat{\varphi}^{-1}h$ is a homeomorphism of ∂B_r and it is extended to a homeomorphism $\overline{\hat{\varphi}^{-1}h}$ of $\overline{B_{2R}}-B_r$ by setting

$$\overline{\hat{\varphi}^{-1}h}\left(\frac{\rho}{r}b\right)=\frac{\rho}{r}\hat{\varphi}^{-1}h(b),\ \ b\in\partial B_r,\ \ r\leqq\rho\leqq 2R,$$

we may consider (if nescessary, taking $h(\overline{\hat{\varphi}^{-1}h})^{-1}$ instead of h), h is coincide to $\hat{\varphi}$ on ∂B_r.

Using this h, we define a homeomorphism $\bar{\varphi}$ from B_{2R} onto B_{2R} by

$$\bar{\varphi}(a)=\hat{\varphi}(a),\ \ |a|<r,$$

$$\bar{\varphi}(a)=h(a),\ \ r\leqq|a|<2R.$$

On the other hand, we define a homeomorphism T from B_{2R} onto $\boldsymbol{R}^n$ by

$$T(a)=a,\ \ |a|<R,$$

$$T(a)=\frac{R}{2R-|a|}a,\ \ R\leqq|a|<2R.$$

Then $T\bar{\varphi}T^{-1}$ is an element of $H_0(n)$ and by definition, we get

$$T\bar{\varphi}T^{-1}(a)=\hat{\varphi}(a),\ \ |a|<r.$$

Hence the class *mod.* $H_e(n)$ of $T\bar{\varphi}T^{-1}$ is φ. Therefore we have the lemma.

Note. By annulus theorem, $H_e(n)$ is dense in $SH_0(n)=H_0{}^+(n)$ (the connected component of the identity of $H_0(n)$, or equivalently, the subgroup of $H_0(n)$ consisted by all orientation preserving homeomorphisms (cf. [7])). Hence we can not give $H_*(n)$ any non -trivial topology.

Lemma 3. *$H_e(n)$ is a contractible group.*

9. **Lemma 4.** *Let $\mathfrak{X}$ be an n -dimensional microbundle over a normal paracompact space and $n\geqq 5$, then $\mathfrak{X}$ has a connection as an $H_0(n)$ -bundle if and only if $\mathfrak{X}$ has a connection as an $H_*(n)$ -bundle.*

Proof. If $\mathfrak{X}$ has a connection $\{s_U\}$ as an $H_0(n)$ -bundle, then denoting the class *mod.* $H_e(n)$ of $s_U(x_0, x_1)$ by $s_U{}^*(x_0, x_1)$, $\{s_U{}^*\}$ gives a connection of $\mathfrak{X}$ as an $H_*(n)$ -bundle. Hence we have the nescessity.

To show the sufficiency, we set the transition function of $\mathfrak{X}$ as an $H_0(n)$ -bundle by $\{g_{UV}(x)\}$. Then denoting the class *mod.* $H_e(n)$ of $g_{UV}(x)$ by $g_{UV}{}^*(x)$, $\{g_{UV}{}^*(x)\}$ is a transition function of $\mathfrak{X}$ as an $H_*(n)$ -bundle. We set

$\mathscr{T}_{\mathfrak{X}, H_0(n)}$: *the sheaf of germs of those* t_U, $\overline{t_U} \in C^1(U, H_0(n))$ and *(some)* $t_U \in \overline{t_U}$ *satisfies*
$g_{UV}(x_0) t_V(x_0, x_1) g_{VU}(x_0) = t_U(x_0, x_1)$.

$\mathscr{T}_{\mathfrak{X}, H_e(n)}$: *the sheaf of germs of those* t_U, $\overline{t_U} \in C^1(U, H_e(n))$ *and (some)* $t_U \in \overline{t_U}$ *satisfies*
$g_{UV}(x_0) t_V(x_0, x_1) g_{UV}(x_0) = t_U(x_0, x_1)$.

$\mathscr{T}_{\mathfrak{X}, H_*(n)}$: *the sheaf of germs of those* t_U, $\overline{t_U} \in C^1(U, H^*(n))$ *and (some)* $t_U \in \overline{t_U}$ *satisfies*
$g_{UV}{}^*(x_0) t_V(x_0, x_1) g_{VU}{}^*(x_0) = t_U(x_0, x_1)$.

Here $C^1(U, H_*(n))$ is defined similarly as $C^1(U, G)$. Then by Lemma 2, we have the exact sequence

$$0 \to \mathscr{T}_{\mathfrak{X}, H_e(n)} \xrightarrow{i} \mathscr{T}_{\mathfrak{X}, H_0(n)} \xrightarrow{j} \mathscr{T}_{\mathfrak{X}, H_*(n)} \to 0, \tag{19}$$

where i is the inclusion, j is the canonical map. By (19), we obtain the following exact sequence.

$$\cdots \to H^1(X, \mathscr{T}_{\mathfrak{X}, H_e(n)}) \xrightarrow{i^*} H^1(X, \mathscr{T}_{\mathfrak{X}, H_0(n)}) \xrightarrow{j^*} H^1(X, \mathscr{T}_{\mathfrak{X}, H_*(n)}). \tag{20}$$

In this sequence, by the definition of the obstruction class for the existence of connection, denoting the obstruction classes for the existence of connection in $H^1(X, \mathscr{T}_{\mathfrak{X}, H_0(n)})$ and in $H^1(X, \mathscr{T}_{\mathfrak{X}, H_*(n)})$ by $o(\mathfrak{X})$ and $o^*(\mathfrak{X})$, we get

$$j^*(o(\mathfrak{X})) = o^*(\mathfrak{X}). \tag{21}$$

On the other hand, taking $\overline{t_{UV}} \in Z^1(X, \mathscr{T}_{\mathfrak{X}, H_e(n)})$, we can construct a fibre bundle $\mathfrak{Y}$ over $Q(\Delta_1(X))$, a neighborhood of $\Delta_1(X)$ in $X \times X$, with fibre $H_e(n)$ as follows: Take the representations $t_{UV} \in \overline{t_{UV}}$ and assume that they satisfy

$$t_{UV}(x_0, x_1) g_{UV}(x_0) t_{VW}(x_0, x_1) g_{VU}(x_0) g_{UW}(x_0) t_{WU}(x_0, x_1) g_{WU}(x_0) = e,$$
$$(x_0, x_1) \in Q(\Delta_1(U \cap V \cap W)).$$

Then we define an equivalence relation $\sim$ between the elements of $Q(\Delta_1(U)) \times H_e(n)$ and the elements of $Q(\Delta_1(V)) \times H_e(n)$ by

$$Q(\Delta_1(U)) \times H_e(n) \ni ((x_0, x_1), \varphi)$$
$$\sim ((x_0, x_1), g_{UV}(x_0) t_{UV}(x_0, x_1) g_{VU}(x_0) \varphi) \in Q(\Delta_1(V)) \times H_e(n),$$

and construct the total space of $\mathfrak{Y}$ as the quotient space $\cup_U Q(\Delta_1(U)) \times H_e(n)/\sim$. Then, since $H_e(n)$ is contractible, $\mathfrak{Y}$ has a cross-section f ([14], [20]). Since we may consider f is a collection of $H_e(n)$ -valued function f_U on $Q(\Delta_1(U))$ such that

$$f_U(x_0, x_1) = g_{UV}(x_0) t_{UV}(x_0, x_1) g_{UV}(x_0) f_V(x_0, x_1),$$

$\{\bar{t}_{UV}\}$ belongs in $B^1(X, \mathscr{T}_{\mathfrak{X}, H_e(n)})$. Hence $H^1(X, \mathscr{T}_{\mathfrak{X}, H_e(n)})$ vanishes. Therefore by (20), j^* is *1* to *1*. Then by (21), if $o^*(\mathfrak{X})$ vanishes, then $o(\mathfrak{X})$ must vanish. This proves the lemma.

Note. It is known that if $n = 1$ or 2, then $H_0(n)$ is contractible to $O(n)$. Then since $GL(n, \boldsymbol{R})$ is contained in $H_0(n)$ and a $GL(n, \boldsymbol{R})$ -bundle always has a connection by $n^o 4$, (ii), an n -dimensional microbundle over a normal paracompact space always has a connection as an $H_0(n)$ -bundle if $n \leqq 2$. Hence Lemma 4 is also true for $n \leqq 2$.

§ 4. Connection of tangent microbundles.

10. The tangent microbundle τ of a (paracompact) manifold is given by (17),

$$X \xrightarrow{\Delta} X \times X \xrightarrow{p} X, \quad \Delta(x) = (x, x), \quad p((x, y)) = x,$$

with the commntative diagram

$$\begin{tikzcd}
& U\times U \arrow[dr, "p"] \arrow[dd, "\varphi_U"] & \\
U \arrow[ur, "\Delta"] \arrow[dr, "i"'] & & U \\
& U\times \boldsymbol{R}^n \arrow[ur, "p"'] &
\end{tikzcd} \qquad \varphi_U(x, y) = (x, h_U(y) - h_U(x)),$$

where h_U is the homeomorphism from U onto $\boldsymbol{R}^n$ by which the manifold structure of X is given.

If we consider τ to be an $H_*(n)$ -bundle, then the transition function $\{g_{UV}(x)\}$ of τ is given by

(22) $$g_{UV}(x) = h_{U,x} h_{V,x}^{-1},$$

where $h_{U,x}$ is given by

(22)′ $$h_{U,x}(y) = h_U(y) - h_U(x).$$

Note. In (22), $g_{UV}(x)$ is a homeomorphism from $h_{V,x}(U \cap V)$ onto $h_{U,x}(U \cap V)$. We use $g_{UV}(x)$ to denote such homeomorphism in one hand, and on the other hand, by $g_{UV}(x)$ we mean the element of $H_*(n)$ defined by such homeomorphism.

Lemma 5. *τ has a connection as an $H_*(n)$ -bundle if and only if there exists a continuous function $t(x_0, x_1)$ on some neighborhood of $\Delta_1(X)$ in $X \times X$ such that*

(*). *$t(x_0, x_1)$ is a homeomorphism from a neighborhood of x_1 to a neighborhood of x_0 such that*

$$t(x_0, x_1)(x_1) = x_0, \quad t(x, x) \textit{ is the identity.}$$

Proof. If $\{s_U\}$ is a connection of τ as an $H_*(n)$ -bundle, then since

$$g_{UV}(x_0)\, s_V(x_0, x_1)\, g_{VU}(x_1) = s_U(x_0, x_1),$$

we obtain by (22),

$$h_{V,x_0}{}^{-1} s_V(x_0, x_1) h_{V,x_1} = h_{U,x_0}{}^{-1} s_U(x_0, x_1) h_{U,x_1}, \tag{23}$$

on some neighborhood of $\Delta_1(U \cap V)$.

By (23), setting

$$t(x_0, x_1) \mid V(\Delta_1(U)) = h_{U,x_0}{}^{-1} s_U(x_0, x_1) h_{U,x_1}, \tag{24}$$

$t(x_0, x_1)$ satisfies (*).

On the other hand, if t exists, then setting

$$s_U(x_0, x_1) = h_{U,x_0}\, t(x_0, x_1) h_{U,x_1}{}^{-1},$$

$s_U(x_0, x_1)$ is a homeomorphism from a neighborhood of the origin of $\boldsymbol{R}^n$ to a neighborhood of the origin of $\boldsymbol{R}^n$ such that $s_U(x_0, x_1)(0) = 0$ and satisfies

$$s_U(x, x) = e,$$

$$g_{UV}(x_0)\, s_V(x_0, x_1)\, g_{VU}(x_1) = s_U(x_0, x_1).$$

Hence $\{s_U\}$ gives a connection of τ.

11. On U, a coordinate neighborhood of X with homeomorphism h_U from U onto $\boldsymbol{R}^n$, we can construct t_U, which satisfies (*) on some neighborhood of $\Delta_1(U)$, by

$$t_U(x_0, x_1)(y) = h_U{}^{-1}(h_U(y) + h_U(x_0) - h_U(x_1)), \tag{25}$$
$$(x_0, x_1) \in W(\Delta_1(U)).$$

Then setting

$$r_{UV}(x, y) = t_U(x, y) t_V(x, y)^{-1},$$
$$(x, y) \in W(\Delta_1(U)) \cap W'(\Delta_1(V)),$$

$r_{UV}(x, y)$ is a homeomorphism from a neighborhood of x onto a neighborhood of x and satisfies

$$(26) \qquad r_{UV}(x,\ x) = \textit{the identity map},$$

$$r_{UV}(x,\ y)(x) = x.$$

For this r_{UV}, we set

$$(27) \qquad (r_{UV}(x)(b))(a)$$

$$= h_{U,\,x}(r_{UV}(x,\ h_{U,\,x}{}^{-1}(b))(h_{V,\,x}{}^{-1}(a)).$$

As usual, we call to continuous maps f_1 and f_2 from some neighborhood of x to $H_*(n)$ are equivalent if $f_1 \mid U(x) = f_2 \mid U(x)$ for some neighborhood $U(x)$ of x and call the equivalence class of f the germ f. Then we set

$$F_*(\boldsymbol{R}^n,\ H_*(n)) = \{\overline{f} \mid \overline{f} \textit{ is the germ at the origin of the continuous maps from } \boldsymbol{R}^n \textit{ into } H_*(n) \textit{ such that } f(0) = e,\ f \in \overline{f}\},$$

Then $F_*(R^n,\ H_*(n))$ is a group and $r_{UV}(x)$ given by (27) can be regard to be a continuous map from $U \cap V$ into $F_*(\boldsymbol{R}^n,\ H_*(n))$. Here the continuous map from a topological space to $F_*(\boldsymbol{R}^n,\ H_*(n))$ is defined similarly as the continuous map to $H_*(n)$ (cf. n°7).

If $\overline{f}$ is an element of $F_*(\boldsymbol{R}^n,\ H_*(n))$ and $\overline{\varphi}$ is an element of $H_*(n)$, then we define the operation $\overline{f}^{\overline{\varphi}}$ of $\overline{\varphi}$ to f by

$$\overline{f}^{\overline{\varphi}} \textit{ is the class of } f^{\varphi}, \textit{ where } f^{\varphi} \textit{ is given by}$$

$$f^{\varphi}(a) = f(\varphi(a)),\ \ a \in \boldsymbol{R}^n,\ \ f \in \overline{f},\ \ \varphi \in \overline{\varphi}.$$

Then by (27) and the definition of $r_{UV}(x,\ y)$, we get

$$(28) \qquad r_{UV}(x) r_{VW}(x)^{g_{UV}(x)} r_{WU}(x)^{g_{UW}(x)}$$

$$= e,\ \textit{ the identity of } F_*(\boldsymbol{R}^n,\ H_*(n)).$$

Here, for the simplicity, we denote the classes of $r_{VW}(x)^{g_{VU}(x)}$ etc. also by $r_{VW}(x)^{g_{VU}(x)}$ etc.

We note that if X is paracompact, then taking a locally finite coverng $\{U_\alpha\}$ and denote $r_{\alpha\beta}(x) = r_{U_\alpha U_\beta}(x)$, etc., we may assume

$$(28)' \qquad r_{\alpha\beta}(x) r_{\beta\gamma}(x)^{g_{\beta\alpha}(x)} r_{\gamma\alpha}(x)^{g_{\gamma\alpha}(x)}(a)(b) = b,\ \textit{if}(a,\ b) \in Q_\alpha \times Q_\alpha,$$

where Q_α is an open ball of $\boldsymbol{R}^n$ with center the origin.

12. We set

$E_0(n)$: *the space of all homeomorphisms from* $\boldsymbol{R}^n$ *into* $\boldsymbol{R}^n$ *which fix the origin with* *compact open topology.*

$\overline{Q}_0(n)$: *the space of all homeomorphisms from* $\overline{Q^n}$ *into* R^n *which fix the origin with compact open topology.*

Here Q^n means an open ball of R^n with center the origin.

For these spaces, we set

$$F_e(R^n, E_0(n)) = \{f \mid f\colon R^n \to E_0(n),\ f(0) = e,\ \textit{the identity}\},$$
$$F_e(\overline{Q^n}, \overline{Q}_0(n)) = \{f \mid f\colon \overline{Q^n} \to \overline{Q}_0(n),\ f(0) = e,\ \textit{the identity}\},$$

where the topologies are given by the compact open topology. Then we get

Lemma 9. (i). $F_e(R^n, E_0(n))$ *and* $F_e(\overline{Q^n}, \overline{Q}_0(n))$ *are both contractible.*

(ii). *To define an equivalence relation* $\sim$ *of the elements of* $F_e(R^n, E_0(n))$ *by*

$$f_1 \sim f_2 \textit{ if and only if } f_1(a)(b) = f_2(a)(b),\ (a, b) \in Q \times Q,$$

we have

$$Fe(R^n, E_0(n))/\sim\ = F_e(\overline{Q^n}, \overline{Q}_0(n)).$$

Using these spaces and $\{r_{\alpha\beta}(x)\}$ given by (27), we construct an $F_*(R^n, H_*(n))$-bundle over X as follows: To define the equivalence between $U_\alpha \times F_e(R^n, E_0(n)) \times \alpha$ and $U_\beta \times F_e(R^n, E_0(n)) \times \beta$ by

(29) $(x, f, \alpha) \in U_\alpha \times F_e(R^n, E_0(n)) \times \alpha$ *and* $(x, g, \beta) \in U_\beta \times F_e(R^n, E_0(n)) \times \beta$ *are equivalant if and only if*

$$g(a)(b) = r_{\alpha\beta}(x) f(g_{\alpha\beta}(x)a)(b),\ (a, b) \in Q_\alpha \times Q_\alpha,$$

and set the quotient space of $\cup_\alpha U_\alpha \times F_e(R^n, E_0(n)) \times \alpha$ under this relation by $\mathfrak{F}$. Then $\mathfrak{F}$ is the total space of a fibre bundle over X and its fibre at x, $x \in U_\alpha$, is $F_e(\overline{Q^n}, Q_0(n))$ by (29) and Lemma 6. Then, since $F_e(\overline{Q^n}, E_0(n))$ is contractible by Lemma 6, $\mathfrak{F}$ has a cross-section $\rho\colon X \to \mathfrak{F}$ ([14], [17], 20). Then, using local coordinate, we get

$$r_{UV}(x) = \rho_U(x)(\rho_V(x)^{g_{UV}(x)})^{-1}. \tag{30$'$}$$

Here we denote the covering by $\{U\}$ instead of $\{U_\alpha\}$ and ρ_U means $\rho \mid U$.

By (30)$'$, setting

$$q_U(x, y) = h_{U,\,xU}(x, h_{U,\,x}{}^{-1}(y))h_{U,\,x}{}^{-1},$$

we get

$$r_{UV}(x, y) = q_U(x, y)q_V(x, y)^{-1}. \tag{30}$$

On the other hand, by the definition of $q_U(x, y)$, $q_U(x, y)$is a homeomorphism from a neighborhood of x such that

$$q_U(x, x) \text{ is the identity,}$$

$$q_U(x, y)(x) = x.$$

Hence setting

$$t(x, y) \mid W(\Delta_1(U)) = qU(x, y)^{-1} r_U(x, y), \tag{31}$$

$t(x, y)$ is defined on some neighborhood of $\Delta_1(X)$ and satisfies (*). Therefore we obtain

Theorem 8. *If X is a paracompact (topological) manifold, then the tangent microbundle τ of X has always a connection as an $H_*(n)$ -bundle.*

By Lemma 4, we also have

Theorem 8′. *If X is a paracompact (topological) manifold, and* dim. $X \geqq 5$, *then the tangent microbundle τ of X has a connection as an $H_0(n)$ bundle.*

Note. It is known that if dim. $X \leqq 3$, then we may consider X to be smooth. Since a smooth manifold always has a connection regarding its tangent microbundle to be an $H_0(n)$ -bundle, because the tangent microbundle of a smooth manifold is reduced to a $GL(n, \boldsymbol{R})$ -bundle. Hence we can rewrite Theorem 8′ as

Theorem 8″. *If X is a paracompact (topological) manifold, and* dim. $X \neq 4$, *then the tangent microbundle τ of X has a connection as an $H_0(n)$ -bundle.*

13. If $t_1(x, y)$ and $t_2(x, y)$ both satisfies (*), then setting

$$r(x, y) = t_1(x, y)t_2(x, y)^{-1}, \quad x, y \in V(\Delta_1(X)),$$

where $V(\Delta_1(X))$ is a neighborhood of $\Delta_1(X)$ in $X \times X$, we have

(**). *$r(x, y)$ is a homeomorphism from a neighborhood of x to a neighborhood of x such that*

$$r(x, y)(x) = x, \ r(x, x) \text{ is the identity.}$$

On the other hand, if a continuous function $r(x, y)$ on some neighborhood of $\Delta_1(X)$ satisfies (**), then $r(x, y)t(x, y)$ satisfies (*) if $t(x, y)$ satisfies (*). Hence we have

Theorem 9. *There is a 1 to 1 correspondence between the set of all connections of τ (as an $H_*(n)$ -bundle) and the set of germs of $r(x, y)$ at $\Delta_1(X)$ which satisfies* (**).

Note. If $r(x, y)$ satisfies (**), then we set

$$r_U(x, a)(b) = h_{U,x} r(x, h_{U,x}^{-1}(a)) h_{U,x}^{-1}(b),$$

$$a, b \in Q, \text{ a neighborhood of the origin in } \boldsymbol{R}^n.$$

By definition, $r_U(x, a)$ is a local homeomorphism of $\boldsymbol{R}^n$ near the origin which fix the origin and it satisfies

$$g_{UV}(x)r_V(x, g_{VU}(x)a)g_{VU}(x) = r_U(x, a). \tag{32}$$

Conversely, if there exists a collection $\{r_U(x, a)\}$ which satisfies (32), then we can construct $r(x, y)$ which satisfies (**).

By Theorem 9, if $t(x_0, x_1)$ is obtained from a connection of τ (as an $H_*(n)$-bundle), then we can write

$$\begin{aligned}(t(x_0, x_1) \mid W(\Delta_1(U)))(y)\\ = r_U(x_0, x_1)(h_U^{-1}(h_U(y) + h_U(x_0) - h_U(x_1))).\end{aligned} \tag{33}$$

Hence we may consider a connection of X (regarding τ to be an $H_*(n)$-bundle) to be a (local) parallel displacement of X.

We note that if r_U is given by (33) for given t, then we get

$$\begin{aligned}r_U(x_0, x_1)^{-1}r_V(x_0, x_1)(y)\\ = h_U^{-1}(h_U(h_V^{-1}(h_V(y) + h_V(x_0) - h_V(x_1)) + h_U(x_1) - h_U(x_0)).\end{aligned} \tag{34}$$

Hence $r_U^{-1}r_V$ does not depend on t.

Note. If M is a paracompact infinite dimensional manifold modeled on L, a topological linear space, then we can define the tangent microbundle of M and it is regarded to be an $H_*(L)$-bundle, where $H_*(L)$ is the group of germs of homeomorphisms of L which fix the origin. Then by the same method, we can show that the tangent microbundle of M has a connection if we consider it to be an $H_*(L)$-bundle. But by a recent result of Henderson ([10]), this is trivial at least L is an infinite dimensional separable Frechet space. Because of [10], such manifold is homeomorphic to an open subset of Hilbert space.

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