

A Note on Spin (9)

By ICHIRO YOKOTA

Department of Mathematics, Faculty of Science

Shinshu University

(Received April 30, 1968)

1. Introduction

In the previous paper [1], we have used a lemma ([1], 6.1) on Spin(9) without its proof. That lemma was as follows.

Theorem. *For any given element $A \in \mathfrak{S}_{01}$ such that $(A, A) = 2$, choose any element $X_0 \in \mathfrak{S}_{01}$ such that $(A, X_0) = 0$, $(X_0, X_0) = 2$, choose any element $Y_0 \in \mathfrak{S}_{23}$ such that $2A \circ Y_0 = -Y_0$, $(Y_0, Y_0) = 2$ and put*

$$Z_0 = 2X_0 \circ Y_0.$$

Next choose any $X_1 \in \mathfrak{S}_{01}$ such that $(A, X_1) = (X_0, X_1) = 0$, $(X_1, X_1) = 2$,

choose any $X_2 \in \mathfrak{S}_{01}$ such that $(A, X_2) = (X_0, X_2) = (X_1, X_2) = 0$, $(X_2, X_2) = 2$

and put

$$Y_1 = -2Z_0 \circ X_1, \quad Z_2 = -2X_2 \circ Y_0, \quad X_3 = -2Y_1 \circ Z_2$$

Choose any $X_4 \in \mathfrak{S}_{01}$ such that $(A, X_4) = (X_0, X_4) = (X_1, X_4) = (X_2, X_4) = (X_3, X_4) = 0$, $(X_4, X_4) = 2$

and put

$$Z_4 = -2X_4 \circ Y_0, \quad Y_2 = -2Z_0 \circ X_2, \quad Y_3 = -2Z_0 \circ X_3,$$

$$X_5 = -2Y_1 \circ Z_4, \quad X_6 = 2Y_2 \circ Z_4, \quad X_7 = -2Y_3 \circ Z_4$$

and then put

$$Y_i = -2Z_0 \circ X_i \quad \text{for } i = 4, 5, 6, 7,$$

$$Z_i = -2X_i \circ Y_0 \quad \text{for } i = 1, 3, 5, 6, 7.$$

Now, let $f: \mathfrak{S} \rightarrow \mathfrak{S}$ be the \mathbf{R} -homomorphism satisfying

$$fE = E, \quad fE_1 = E_1, \quad f(E_2 - E_3) = A,$$

$$fF_1^i = X_i, \quad fF_2^i = Y_i, \quad fF_3^i = Z_i \quad \text{for } i = 0, 1, \dots, 7,$$

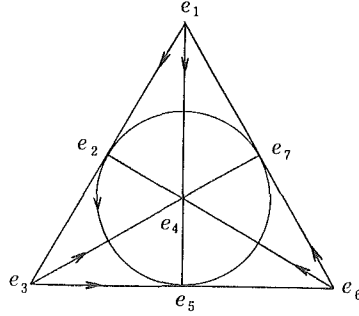
then we have $f \in F_4$ (a priori $f \in \text{Spin}(9)$).

The aim of the present paper is to give the proof of this theorem.

2. Notations

2.1 \mathfrak{C} is the division ring of Cayley numbers with the following multiplication figure among its \mathbf{R} -basis e_0, e_1, \dots, e_7 . (e_0 is the unit of \mathfrak{C}).

1) \mathbf{R} is the field of real numbers.



We describe some formulae in \mathfrak{G} used in later. For $a, x, y \in \mathfrak{G}$,

- (1) $(ax, y) = (x, \bar{a}y)$, $(xa, y) = (x, y\bar{a})$,
- (2) $x = \bar{\bar{x}}$, $\overline{x\bar{y}} = \bar{y}\bar{x}$,
- (3) $2(x, y) = x\bar{y} + y\bar{x} = \bar{x}y + \bar{y}x$, $\bar{x}x = x\bar{x} = |x|^2$,
- (4) $a(\bar{a}x) = (a\bar{a})x$, $a(x\bar{a}) = (ax)\bar{a}$, $x(a\bar{a}) = (xa)\bar{a}$,
 $a(ax) = (aa)x$, $a(xa) = (ax)a$, $x(aa) = (xa)a$,
- (5) $(ax)(ya) = a(xy)a$.

2.2 \mathfrak{Z} is the (non-associative) algebra of 3-hermitian matrices over \mathfrak{G} with the product $X \circ Y = \frac{1}{2}(XY + YX)$. The inner product in \mathfrak{Z} is defined by $(X, Y) = \text{tr}(X \circ Y)$. We know that $(A \circ X, Y) = (X, A \circ Y)$ for $A, X, Y \in \mathfrak{Z}$ ([1], 1.4).

2.3 F_4 is the automorphism group of \mathfrak{Z} , that is, $f \in F_4$ is an \mathbf{R} -isomorphism of \mathfrak{Z} satisfying $f(X \circ Y) = fX \circ fY$ for $X, Y \in \mathfrak{Z}$.

2.4

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E = E_1 + E_2 + E_3,$$

$$F_1^u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u \\ 0 & \bar{u} & 0 \end{pmatrix}, \quad F_2^u = \begin{pmatrix} 0 & 0 & \bar{u} \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix}, \quad F_3^u = \begin{pmatrix} 0 & u & 0 \\ \bar{u} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$F_k^{e_i}$ is denoted by F_k^i briefly.

2.5 $\text{Spin}(9)$ is the subgroup of F_4 consisting of f such that $fE_1 = E_1$.

2.6 \mathfrak{Z}_{01} is the subspace of \mathfrak{Z} consisting of X such that $E_1 \circ X = 0$ and $\text{tr}(X) = 0$. Such X is of the form $\xi(E_2 - E_3) + F_1^x$ for $\xi \in \mathbf{R}$, $x \in \mathfrak{G}$. And we have $(X, X') = 2(\xi\xi' + (x, x'))$, $2X \circ X' = (X, X')(E - E_1)$ for $X, X' \in \mathfrak{Z}_{01}$.

2.7 \mathfrak{Z}_{23} is the subspace of \mathfrak{Z} consisting of Y such that $2E_1 \circ Y = Y$. Such Y is of the form $F_2^y + F_3^{y'}$ for $y, y' \in \mathfrak{G}$. And we have $(Y_1, Y_2) = 2((y_1, y_2) + (y_1', y_2'))$.

3. Preliminary lemmas

3.1 Lemma. (1) For $Y, Z \in \mathfrak{S}_{23}$, if $(Y, Z) = 0$, then we have $Y \circ Z \in \mathfrak{S}_{01}$.

(2) For $X \in \mathfrak{S}_{01}$, $Y \in \mathfrak{S}_{23}$, we have $X \circ Y \in \mathfrak{S}_{23}$.

Proof. (1) Since $(Y, Z) = 0$ means $(y, z) + (y', z') = 0$ we have $Y \circ Z = (y, z)(E_1 + E_3) + (y', z')(E_1 + E_3) + \frac{1}{2}F_1^{\overline{yzI} + z\overline{yI}} = -(y, z)(E_2 - E_3) + \frac{1}{2}F_1^{\overline{yzI} + z\overline{yI}} \in \mathfrak{S}_{01}$. (2)

By the direct calculation, we have $X \circ Y = \frac{1}{2}(F_2^{-\xi y + \overline{yI}x} + F_3^{\xi y' + \overline{xI}y}) \in \mathfrak{S}_{23}$.

3.2 Remark. In the following propositions 3.3, 3.6, 3.8, we have the formulae of type $P \circ (P' \circ Q) + P' \circ (P \circ Q) = \frac{(P, P')}{4}Q$. As the consequence we have

(1) if $(P, P') = 0$, then $P \circ (P' \circ Q) = -P' \circ (P \circ Q)$.

(2) $P \circ (P \circ Q) = \frac{(P, P)}{8}Q$. Especially, if $(P, P) = 2$, then $P \circ (P \circ Q) = \frac{1}{4}Q$.

3.3 Proposition. For $X, X' \in \mathfrak{S}_{01}$, $Y \in \mathfrak{S}_{23}$, we have

$$X \circ (X' \circ Y) + X' \circ (X \circ Y) = \frac{(X, X')}{4}Y.$$

Proof. $8X \circ (X' \circ Y) = 4X \circ (F_2^{-\xi y' + \overline{y'I}x'} + F_3^{\xi y' + \overline{x'I}y'}) = 2F_2^{-\xi(-\xi y' + \overline{y'I}x')} + \frac{\xi(\xi y' + \overline{x'I}y')x}{(\xi y' + \overline{x'I}y')x} + 2F_3^{\xi y' + \overline{x'I}y'}$
 $= 2F_2^{(\xi^2 + |x|^2)y'} + 2F_3^{(\xi^2 + |x|^2)y'} = 2(\xi^2 + |x|^2)(F_2 y' + F_3 y') = (X, X')Y$.

By the polarization $X \rightarrow X + X'$, we have the desired formula.

3.4 Lemma. For $A \in \mathfrak{S}_{01}$, $Y, Z \in \mathfrak{S}_{23}$, if $2A \circ Y = -Y$, $2A \circ Z = Z$, then we have $(Y, Z) = 0$.

Proof. $2(Y, Z) = -8(A \circ Y, A \circ Z) \stackrel{2.2}{=} -8(A \circ (A \circ Y), Z) \stackrel{3.3}{=} -((A, A)Y, Z)$, hence $(2 + (A, A))(Y, Z) = 0$, that is $(Y, Z) = 0$.

3.5 Lemma. (1) For $A \in \mathfrak{S}_{01}$, $Y \in \mathfrak{S}_{23}$, we have

(i) $2A \circ Y = -Y$ is equivalent to $(1 + \alpha)y' + \overline{a}y = 0$, $\overline{y'a} + (1 - \alpha)y = 0$,

(ii) if $(A, A) = 2$, then $2A \circ Y = -Y$ is equivalent to either $(1 + \alpha)y' + \overline{a}y = 0$ for $\alpha \neq -1$ or $\overline{y'a} + (1 - \alpha)y = 0$ for $\alpha \neq 1$. And we have $|y|^2 = \frac{(Y, Y)}{4}(1 + \alpha)$, $|y'|^2 = \frac{(Y, Y)}{4}(1 - \alpha)$, $yy' = -\frac{(Y, Y)}{4}\overline{a}$.

(2) For $A \in \mathfrak{S}_{01}$, $Z \in \mathfrak{S}_{23}$, we have

(i) $2A \circ Z = Z$ is equivalent to $(1 - \alpha)z' - \overline{a}z = 0$, $\overline{z'a} - (1 + \alpha)z = 0$,

(ii) if $(A, A) = 2$, then $2A \circ Z = Z$ is equivalent to either $(1 - \alpha)z' - \overline{a}z = 0$ for $\alpha \neq 1$ or $\overline{z'a} - (1 + \alpha)z = 0$ for $\alpha \neq -1$. And we have $|z|^2 = \frac{(Z, Z)}{4}(1 - \alpha)$, $|z'|^2 = \frac{(Z, Z)}{4}(1 + \alpha)$, $zz' = \frac{(Z, Z)}{4}\overline{a}$.

Proof. (1)(i) $2A \circ Y = -Y$ means $F_2^{-\alpha y + \overline{yI}a} + F_3^{\alpha y' + \overline{aI}y} = -F_2 y - F_3 y'$, namely $-\alpha y + \overline{yI}a = -y$, $\alpha y' + \overline{aI}y = -y'$. These prove (1)(i). (1)(ii) If $(1 + \alpha)y' + \overline{a}y = 0$, $\alpha \neq -1$, then $\overline{y'a} + (1 - \alpha)y = -\frac{\overline{a}y\overline{a}}{1 + \alpha} + (1 - \alpha)y = \frac{-y}{1 + \alpha}(|a|^2 + \alpha^2 - 1) = \frac{-y}{1 + \alpha}$

$\left(\frac{(A, A)}{2} - 1\right) = 0$. Conversely $\overline{y'}a + (1 - \alpha)y = 0$, $\alpha \neq 1$ imply $(1 + \alpha)y' + \overline{a}y = 0$.

Finally, if $\alpha \neq -1$, then $(Y, Y) = 2(|y|^2 + |y'|^2) = 2\left(|y|^2 + \frac{|\overline{a}y|^2}{(1 + \alpha)^2}\right) = 2|y|^2\left(1 + \frac{|a|^2}{(1 + \alpha)^2}\right) = 2|y|^2\left(1 + \frac{1 - \alpha^2}{(1 + \alpha)^2}\right) = \frac{4|y|^2}{1 + \alpha}$. If $\alpha = -1$, then $y = 0$, therefore $|y|^2 = 0 = \frac{1 + \alpha}{4}(Y, Y)$. The proofs of the others are analogous to the above.

3.6 Proposition. For $A \in \mathfrak{F}_{01}$, $Y, Y', Z, Z' \in \mathfrak{F}_{23}$,

(1) if $(A, A) = 2$, $2A \circ Y = -Y$, $2A \circ Y' = -Y'$, $2A \circ Z = Z$, then we have

$$Y \circ (Y' \circ Z) + Y' \circ (Y \circ Z) = \frac{(Y, Y')}{4} Z,$$

(2) if $(A, A) = 2$, $2A \circ Z = Z$, $2A \circ Z' = Z'$, $2A \circ Y = -Y$, then we have

$$Z \circ (Z' \circ Y) + Z' \circ (Z \circ Y) = \frac{(Z, Z')}{4} Y.$$

Proof. Since $(Y, Z) = 0$ by 3.4 we have $2Y \circ Z = -2(y, z)(E_2 - E_3) + F_1^{\overline{yz} + \overline{zy}}$ as in the proof of 3.1, hence $4Y \circ (Y \circ Z) = F_2^{2(y, z)y + y'(\overline{yz'} + \overline{zy'})y'} + F_3^{-2(y, z)y' + (\overline{yz'} + \overline{zy'})y}$. The part of $F_2 = 2(y, z)y + (yz' + zy')\overline{y}' = 2(y, z)y + (yz')\overline{y}' + z|y'|^2$. In the case $\alpha \neq \pm 1$, since $(yz')\overline{y}' \stackrel{2.1}{=} 2(yz', y') - y'(\overline{y'z}) \stackrel{3.5}{=} 2(yz', y') + \frac{(\overline{y}a)(\overline{z'}\overline{y})}{1 + \alpha} \stackrel{2.1}{=} 2(\overline{y}'y, \overline{z}') + \frac{\overline{y}(\overline{a}\overline{z}')\overline{y}}{1 + \alpha} \stackrel{3.5}{=} 2\left(-\frac{ayy}{1 + \alpha}, \frac{az}{1 - \alpha}\right) + \overline{y}z\overline{y} \stackrel{2.1}{=} -2\frac{|a|^2}{1 - \alpha^2}(yy, z) + \overline{y}z\overline{y} \stackrel{2.6}{=} -2(y, \overline{y}z) + \overline{y}z\overline{y}$, the part of F_2 can be continued to $2(y, z)y - 2(y, \overline{y}z) + \overline{y}z\overline{y} + z|y'|^2 \stackrel{2.1}{=} (y\overline{z}y + z\overline{y}y) - (y\overline{z}y + \overline{y}z\overline{y}) + \overline{y}z\overline{y} + z|y'|^2 = (|y|^2 + |y'|^2)z = \frac{(Y, Y)}{2}z$. If $\alpha = -1$ (resp. $\alpha = 1$), then $a = 0$, whence $y = z' = 0$ (resp. $y' = z = 0$), therefore the part of $F_2 = z|y'|^2$ (resp. 0) $= \frac{(Y, Y)}{2}z$. Similarly the part of $F_3 = \frac{(Y, Y)}{2}z'$. Thus we have $4Y \circ (Y \circ Z) = \frac{(Y, Y)}{2}Z$. By the polarization $Y \rightarrow Y + Y'$, we have the desired result (1).

3.7 Lemma. (1) For $A, X \in \mathfrak{F}_{01}$, $Y \in \mathfrak{F}_{23}$ such that $(A, X) = 0$, $2A \circ Y = -Y$, if we set $Z = \pm 2X \circ Y$, then we have $2A \circ Z = Z$.

(2) For $A, X \in \mathfrak{F}_{01}$, $Z \in \mathfrak{F}_{23}$ such that $(A, X) = 0$, $2A \circ Z = Z$, if we set $Y = \pm 2Z \circ X$, then we have $2A \circ Y = -Y$.

Proof. We shall show this for only $Z = 2X \circ Y$. $2A \circ Z = 4A \circ (X \circ Y) \stackrel{3.3}{=} -4X \circ (A \circ Y) = 2X \circ Y = Z$.

3.8 Proposition. For $A \in \mathfrak{F}_{01}$, $Y, Y', Z, Z' \in \mathfrak{F}_{23}$,

(1) if $(A, A) = 2$, $(A, X) = 0$, $2A \circ Y = -Y$, $2A \circ Y' = -Y'$, then we have

$$Y \circ (Y' \circ X) + Y' \circ (Y \circ X) = \frac{(Y, Y')}{4} X,$$

(2) if $(A, A) = 2$, $(A, X) = 0$, $2A \circ Z = Z$, $2A \circ Z' = Z'$, then we have

$$Z \circ (Z' \circ X) + Z' \circ (Z \circ X) = \frac{(Z, Z')}{4} X.$$

Proof. We need to show that $4Y \circ (Y \circ X) = \frac{(Y, Y)}{2}X$. For this, using 3.1, 3.7, then we see first that $(Y, Y \circ X) = 0$ by 3.4. Hence as in the proof of 3.1, $4Y \circ (Y \circ X) = -2(y, -\xi y + \overline{y'x})(E_2 - E_3) + F_1 \frac{y(\xi y' + x\overline{y}) + (-\xi y + \overline{y'x})y'}{y(\xi y' + x\overline{y}) + (-\xi y + \overline{y'x})y'}$. If $\alpha \neq -1$, then the part of $E_2 - E_3$ is $2\xi|y|^2 - 2(\overline{yy'}, x) \stackrel{3.5}{=} 2\xi|y|^2 + 2\left(\frac{ay\overline{y}}{1+\alpha}, x\right) = \frac{2|y|^2}{1+\alpha}(\xi(1+\alpha) + (a, x))$. Since $(A, X) = 0$, namely $\alpha\xi + (a, x) = 0$, this can be continued to $\frac{2|y|^2\xi}{1+\alpha} \stackrel{3.5}{=} \frac{(Y, Y)}{2}$. And the part of F_1 is $(|y|^2 + |y'|^2)x = \frac{(Y, Y)}{2}x$. Thus we have $4Y \circ (Y \circ X) = \frac{(Y, Y)}{2}X$ (if $\alpha = -1$, then this result is easily seen because $y = 0$). By the polarization $Y \rightarrow Y + Y'$, we have the required formula (1).

3.9 Proposition. For $A \in \mathfrak{S}_{01}$, $Y, Y' \in \mathfrak{S}_{23}$, $Z, Z' \in \mathfrak{S}_{23}$,

(1) if $(A, A) = 2$, $2A \circ Y = -Y$, $2A \circ Y' = -Y'$, then we have

$$Y \circ Y' = \frac{(Y, Y')}{4}(E + E_1 - A),$$

(2) if $(A, A) = 2$, $2A \circ Z = Z$, $2A \circ Z' = Z'$, then we have

$$Z \circ Z' = \frac{(Z, Z')}{4}(E + E_1 + A).$$

Proof. $Y \circ Y = |y|^2(E_1 + E_3) + |y'|^2(E_1 + E_2) + F_1 \frac{\overline{yy'}}{y\overline{y'}} \stackrel{3.5}{=} \frac{(Y, Y)}{4}((1+\alpha)(E_1 + E_3) + (1-\alpha)(E_1 + E_2) + F_1^{-\alpha}) = \frac{(Y, Y)}{4}(2E_1 + E_2 + E_3 - \alpha(E_2 - E_3)F_1^{\alpha}) = \frac{(Y, Y)}{4}(E + E_1 - A)$. By the polarization $Y \rightarrow Y + Y'$, we have the required result (1).

3.10 Lemma. (1) For $A \in \mathfrak{S}_{01}$, $Y, Z \in \mathfrak{S}_{23}$ such that $(A, A) = (Y, Y) = (Z, Z) = 2$, $2A \circ Y = -Y$, $2A \circ Z = Z$, if we set $X = \pm 2Y \circ Z$, then we have $X \in \mathfrak{S}_{01}$ and $(X, X) = 2$.

(2) For $X \in \mathfrak{S}_{01}$, $Y \in \mathfrak{S}_{23}$ such that $(X, X) = (Y, Y) = 2$, if we set $Z = \pm 2X \circ Y$, then we have $Z \in \mathfrak{S}_{23}$ and $(Z, Z) = 2$.

Proof. (1) Since $(Y, Z) = 0$ by 3.4 we have $X \in \mathfrak{S}_{01}$ by 3.1. $(X, X) = 4(Y \circ Z, Y \circ Z) \stackrel{2.2}{=} 4(Y \circ (Y, Z), Z) \stackrel{3.6}{=} \frac{1}{2}(Y, Y)(Z, Z) = 2$. (2) $Z \in \mathfrak{S}_{23}$ is already shown in 3.1. $(Z, Z) = 4(X \circ Y, X \circ Y) \stackrel{2.2}{=} 4(X \circ (X \circ Y), Y) \stackrel{3.3}{=} \frac{1}{2}(X, X)(Y, Y) = 2$.

4. Elements A, X_i, Y_i, Z_i

We construct the elements A, X_i, Y_i, Z_i , for $i=0, 1, \dots, 7$ as in the theorem of the introduction.

4.1 Lemma. The set of A, X_0, X_1, \dots, X_7 is an orthonormal basis in \mathfrak{S}_{01} .

The proof of this lemma will be completed after the following I-VIII.

I. $(A, X_0) = (A, X_1) = (A, X_2) = (A, X_4) = 0$ by definition.

$$4.2 \quad Z_0 \in \mathfrak{S}_{23}, \quad 2A \circ Z_0 = Z_0.$$

In fact, $Z_0 \equiv 2X_0 \circ Y_0 \in \mathfrak{S}_{23}$ by 3.1. $2A \circ Z_0 \equiv 4A \circ (X_0 \circ Y_0) \stackrel{3.3}{=} -4X_0 \circ (A \circ Y_0) \equiv 2X_0 \circ Y_0 \equiv Z_0$. Next using $(A, X_1) = 0$, $Z_0 \in \mathfrak{S}_{23}$, $2A \circ Z_0 \stackrel{4.2}{=} Z_0$, $Y_1 \equiv -2Z_0 \circ X_1$, then we have

by 3.1, 3.7

$$4.3 \quad Y_1 \in \mathfrak{S}_{23}, \quad 2A \circ Y_1 = -Y_1.$$

Similarly, using $(A, X_2) = 0$, $2A \circ Y_0 \equiv -Y_0$, $Z_2 \equiv -2X_2 \circ Y_0$, then we have by 3.1, 3.7

$$4.4 \quad Z_2 \in \mathfrak{S}_{23}, \quad 2A \circ Z_2 = Z_2.$$

As a result of 4.3, 4.4, we have by 3.4

$$4.5 \quad (Y_1, Z_2) = 0.$$

And hence, using $X_3 \equiv -2Y_1 \circ Z_2$, we have by 3.1

$$4.6 \quad X_3 \in \mathfrak{S}_{01}.$$

$$(A, X_3) \equiv -2(A, Y_1 \circ Z_2) \stackrel{2.2}{=} -2(A \circ Y_1, Z_2) \stackrel{4.3}{=} (Y_1, Z_2) \stackrel{4.6}{=} 0.$$

By the analogous methods as 4.3–4.6, we have 4.7–4.9.

$$4.7 \quad Y_2, Y_3, Z_4 \in \mathfrak{S}_{23}, \quad 2A \circ Y_2 = -Y_2, \quad 2A \circ Y_3 = -Y_3, \quad 2A \circ Z_4 = Z_4,$$

$$4.8 \quad (Y_1, Z_4) = (Y_2, Z_4) = (Y_3, Z_4) = 0,$$

$$4.9 \quad X_5, X_6, X_7 \in \mathfrak{S}_{01}.$$

$$(A, X_5) \equiv -2(A, Y_1 \circ Z_4) \stackrel{2.2}{=} -2(A \circ Y_1, Z_4) \stackrel{4.3}{=} (Y_1, Z_4) \stackrel{4.8}{=} 0.$$

$$(A, X_6) \equiv 2(A, Y_2 \circ Z_4) \stackrel{2.2}{=} 2(A \circ Y_2, Z_4) \stackrel{4.7}{=} -(Y_2, Z_4) \stackrel{4.8}{=} 0.$$

$$(A, X_7) \equiv -2(A, Y_3 \circ Z_4) \stackrel{2.2}{=} -2(A \circ Y_3, Z_4) \stackrel{4.7}{=} (Y_3, Z_4) \stackrel{4.8}{=} 0.$$

$$4.10 \text{ Proposition. } 2A \circ Y_i = -Y_i, \quad 2A \circ Z_i = Z_i \text{ for } i=0, 1, \dots, 7.$$

Hence we have $(Y_i, Z_j) = 0$ for $i, j = 0, 1, \dots, 7$ by 3.4.

Proof. For $i=0$, they are by definition or proved by 4.2. For $i \neq 0$, $2A \circ Y_i \equiv -4A \circ (X_i \circ Z_0) \stackrel{3.3}{=} 4X_i \circ (A \circ Z_0) \stackrel{4.2}{=} 2X_i \circ Z_0 \equiv -Y_i$, $2A \circ Z_i \equiv -4A \circ (X_i \circ Y_0) \stackrel{3.3}{=} 4X_i \circ (A \circ Y_0) \equiv -2X_i \circ Y_0 \equiv Z_i$.

$$4.11 \text{ Proposition. } A, X_i \in \mathfrak{S}_{01}, \quad Y_i, Z_i \in \mathfrak{S}_{23} \text{ for } i = 0, 1, \dots, 7.$$

Proof. About A, X_i , they are by definition or by 4.6, 4.9. Therefore $Y_i \equiv -2X_i \circ Z_0 \in \mathfrak{S}_{23}$, $Z_i \equiv -2X_i \circ Y_0 \in \mathfrak{S}_{23}$ by 3.1.

$$4.12 \text{ Proposition } (X_i, X_i) = (Y_i, Y_i) = (Z_i, Z_i) = 2 \text{ for } i = 0, 1, \dots, 7.$$

$$(X_i, X_j) = (Y_i, Y_j) = (Z_i, Z_j) \text{ for } i, j = 0, 1, \dots, 7.$$

Proof. Since $(X_0, X_0) \equiv 2$, $(Y_0, Y_0) \equiv 2$, $Z_0 \equiv 2X_0 \circ Y_0$, we have $(Z_0, Z_0) = 2$ by 3.10. Similarly since $(X_2, X_2) \equiv 2$, $(Y_0, Y_0) \equiv 2$, $Z_2 \equiv -2X_2 \circ Y_0$ and $(Z_0, Z_0) = 2$, $(X_1, X_1) \equiv 2$, $Y_1 \equiv -2Z_0 \circ X_1$, we have $(Z_2, Z_2) = (Y_1, Y_1) = 2$ by 3.10. Since $(A, A) \equiv 2$, $(Y_1, Y_1) = (Z_2, Z_2) = 2$, $2A \circ Y_1 \stackrel{4.3}{=} -Y_1$, $2A \circ Z_2 \stackrel{4.4}{=} Z_2$, $X_3 \equiv -2Y_1 \circ Z_2$, we have $(X_3, X_3) \stackrel{3.10}{=} 2$. By the similar method, we have $(X_5, X_5) = (X_6, X_6) = (X_7, X_7) = 2$. Next for $i, j \neq 0$, $(Y_i, Y_j) \equiv 4(Z_0 \circ X_i, Z_0 \circ X_j) \stackrel{2.2}{=} 4(Z_0 \circ (Z_0 \circ X_i), Y_j) \stackrel{3.8}{=} (X_i, X_j)$. To prove for $i = 0, j \neq 0$, we need

$$4.13 \quad Y_0 = 2X_0 \circ Z_0.$$

In fact, $2X_0 \circ Z_0 \equiv 4X_0 \circ (X_0 \circ Y_0) \stackrel{3.3}{=} Y_0$. Now, $(Y_0, Y_j) \stackrel{4.13}{=} -4(Z_0 \circ X_0, Z_0 \circ X_j) \stackrel{2.2}{=} -4(Z_0 \circ (Z_0 \circ X_0), X_j) \stackrel{3.8}{=} -(X_0, X_j)$. (This is 0 by the following II, hence this is equal to (X_0, X_j)). Remark that the proof is never the vicious circle !). For $i = j = 0$, $(X_0, X_0) \equiv 2$, $(Y_0, Y_0) \equiv 2$ by definition. About (Z_i, Z_j) , we can treat as (Y_i, Y_j) .
II. $(X_0, X_1) = (X_0, X_2) = (X_0, X_4) = 0$ by definition.

$$4.14 \quad 2X_0 \circ Y_1 = -Z_1.$$

In fact, $2X_0 \circ Y_1 \equiv -4X_0 \circ (X_1 \circ Z_0) \stackrel{3.3}{=} 4X_1 \circ (X_0 \circ Z_0) \stackrel{4.13}{=} 2X_1 \circ Y_0 \equiv -Z_1$.

$$(X_0, X_3) \equiv -2(X_0, Y_1 \circ Z_2) \stackrel{2.2}{=} -2(X_0 \circ Y_1, Z_2) \stackrel{4.14}{=} (Z_1, Z_2) \stackrel{4.12}{=} (X_1, X_2) \equiv 0.$$

$$(X_0, X_5) \equiv -2(X_0, Y_1 \circ Z_4) \stackrel{2.2}{=} -2(X_0 \circ Y_1, Z_4) \stackrel{4.14}{=} (Z_1, Z_4) \stackrel{4.12}{=} (X_1, X_4) \equiv 0.$$

$$4.15 \quad 2X_0 \circ Y_2 = -Z_2, \quad 2X_0 \circ Y_3 = -Z_3.$$

Since $(X_0, X_2) \equiv 0$ and $(X_0, X_3) = 0$ has been shown, the proof of 4.15 is similar as 4.14.

$$(X_0, X_6) \equiv 2(X_0, Y_2 \circ Z_4) \stackrel{2.2}{=} 2(X_0 \circ Y_2, Z_4) \stackrel{4.15}{=} -(Z_2, Z_4) \stackrel{4.12}{=} -(X_2, X_4) \equiv 0.$$

$$(X_0, X_7) \equiv -2(X_0, Y_3 \circ Z_4) \stackrel{2.2}{=} -2(X_0 \circ Y_3, Z_4) \stackrel{4.15}{=} (Z_3, Z_4) \stackrel{4.12}{=} (X_3, X_4) \equiv 0.$$

4.16 **Proposition.** (1) For $i = 1, 2, \dots, 7$, we have

$$2X_0 \circ Y_i = 2X_i \circ Y_0 = -Z_i, \quad 2Y_0 \circ Z_i = 2Y_i \circ Z_0 = -X_i, \quad 2Z_0 \circ X_i = 2Z_i \circ X_0 = -Y_i.$$

$$(2) \quad 2X_0 \circ Y_0 = Z_0, \quad 2Y_0 \circ Z_0 = X_0, \quad 2Z_0 \circ X_0 = Y_0.$$

$$(3) \quad 2X_i \circ Y_i = -Z_0, \quad 2Y_i \circ Z_i = -X_0, \quad 2Z_i \circ X_i = -Y_0 \text{ for } i = 1, 2, \dots, 7.$$

Proof. (1) $2X_0 \circ Y_i \equiv -4X_0 \circ (X_i \circ Z_0) \stackrel{3.3}{=} 4X_i \circ (X_0 \circ Z_0) \stackrel{4.13}{=} 2X_i \circ Y_0 \equiv -Z_i$. $2Y_0 \circ Z_i \equiv -4Y_0 \circ (Y_0 \circ X_i) \stackrel{3.8}{=} -X_i \stackrel{3.8}{=} -4Z_0 \circ (Z_0 \circ X_i) \equiv 2Z_0 \circ Y_i$. $2X_0 \circ Z_i \equiv -4X_0 \circ (X_i \circ Y_0) \stackrel{3.3}{=} 4X_i \circ (X_0 \circ Y_0) \equiv 2X_i \circ Z_0 \equiv -Y_i$. (2) $2Y_0 \circ Z_0 \equiv 2Y_0 \circ (Y_0 \circ X_0) \stackrel{3.8}{=} X_0$. The others are proved by definition or by 4.13. (3) $2X_i \circ Y_i \stackrel{(1)}{=} -4X_i \circ (X_i \circ Z_0) \stackrel{3.3}{=} -Z_0$. The others are proved analogously.

III. $(X_1, X_2) = (X_1, X_4) = 0$ by definition.

$$(X_1, X_3) \equiv -2(X_1, Y_1 \circ Z_2) \stackrel{2.2}{=} -2(X_1 \circ Y_1, Z_2) \stackrel{4.16}{=} (Z_0, Z_2) \stackrel{4.12}{=} (X_0, X_2) \equiv 0.$$

$$(X_1, X_5) \equiv -2(X_1, Y_1 \circ Z_4) \stackrel{2.2}{=} -2(X_1 \circ Y_1, Z_4) \stackrel{4.16}{=} (Z_0, Z_4) \stackrel{4.12}{=} (X_0, X_4) \equiv 0.$$

$$4.17 \quad 2X_1 \circ Y_2 = -Z_3 = -2X_2 \circ Y_1, \quad 2X_1 \circ Y_3 = Z_2 = -2X_3 \circ Y_1.$$

In fact, $2X_1 \circ Y_2 \equiv -4X_1 \circ (X_2 \circ Z_0) \stackrel{3.3}{=} 4X_2 \circ (X_1 \circ Z_0) \equiv -2X_2 \circ Y_1 \stackrel{4.16}{=} 4(Y_0 \circ Z_2) \circ Y_1 \stackrel{3.6}{=} -4(Y_1 \circ Z_2) \circ Y_0 \equiv 2X_3 \circ Y_0 \stackrel{4.16}{=} -Z_3$. $2X_1 \circ Y_3 \stackrel{4.16}{=} -4X_1 \circ (Z_0 \circ X_3) \stackrel{3.3}{=} 4X_3 \circ (Z_0 \circ X_1) \equiv -2X_3 \circ Y_1 \stackrel{3.6}{=} 4(Y_1 \circ Z_2) \circ Y_1 \stackrel{3.6}{=} Z_2$.

$$(X_1, X_6) \equiv 2(X_1, Y_2 \circ Z_4) \stackrel{2.2}{=} 2(X_1 \circ Y_2, Z_4) \stackrel{4.17}{=} -(Z_3, Z_4) \stackrel{4.12}{=} -(X_3, X_4) \equiv 0.$$

$$(X_1, X_7) \equiv -2(X_1, Y_3 \circ Z_4) \stackrel{2.2}{=} -2(X_1 \circ Y_3, Z_4) \stackrel{4.17}{=} -(Z_2, Z_4) \stackrel{4.12}{=} -(X_2, X_4) \equiv 0.$$

IV. $(X_2, X_3) \equiv -2(X_2, Y_1 \circ Z_2) \stackrel{2.2}{=} -2(X_2 \circ Y_1, Z_2) \stackrel{4.16}{=} (Y_0, Y_1) \stackrel{4.12}{=} (X_0, X_1) \equiv 0$.

$$(X_2, X_4) \equiv 0.$$

$$(X_2, X_5) \equiv -2(X_2, Y_1 \circ Z_4) \stackrel{2,2}{=} -2(X_2 \circ Y_1, Z_4) \stackrel{4,17}{=} -(Z_3, Z_4) \stackrel{4,12}{=} -(X_3, X_4) \equiv 0.$$

$$(X_2, X_6) \equiv 2(X_2, Y_2 \circ Z_4) \stackrel{2,2}{=} 2(X_2 \circ Y_2, Z_4) \stackrel{4,16}{=} -(Z_0, Z_4) \stackrel{4,12}{=} 0.$$

$$4.18 \quad 2X_2 \circ Y_3 = -Z_1 = -2X_8 \circ Y_2.$$

$$\text{In fact, } 2X_2 \circ Y_8 \equiv -4X_2 \circ (Z_0 \circ X_8) \equiv 8X_2 \circ (Z_0 \circ (X_1 \circ Z_2)) \stackrel{3,6}{=} -8X_2 \circ (Z_2 \circ (Y_1 \circ Z_0)) \stackrel{4,16}{=} 4X_2 \circ (Z_2 \circ X_1) \stackrel{3,3}{=} -4X_1 \circ (Z_2 \circ X_2) \stackrel{4,16}{=} 2X_1 \circ Y_0 \stackrel{4,16}{=} -Z_1. \quad 2X_8 \circ Y_2 \equiv -4X_8 \circ (Z_0 \circ X_2) \stackrel{3,3}{=} 4X_2 \circ (Z_0 \circ X_8) \equiv -2X_2 \circ Y_8 \stackrel{4,18}{=} Z_1.$$

$$(X_2, Y_7) \equiv -2(X_2, Y_8 \circ Z_4) \stackrel{2,2}{=} -2(X_2 \circ Y_8, Z_4) \stackrel{4,18}{=} (Z_1, Z_4) \stackrel{4,12}{=} 0.$$

$$V. (X_3, X_4) \equiv 0.$$

$$(X_3, X_5) \equiv -2(X_3, Y_1 \circ Z_4) \stackrel{2,2}{=} -2(X_3 \circ Y_1, Z_4) \stackrel{4,17}{=} (Z_2, Z_4) \stackrel{4,12}{=} 0.$$

$$(X_3, X_6) \equiv 2(X_3, Y_2 \circ Z_4) \stackrel{2,2}{=} 2(X_3 \circ Y_2, Z_4) \stackrel{4,18}{=} (Z_1, Z_4) \stackrel{4,12}{=} 0.$$

$$(X_3, X_7) \equiv -2(X_3, Y_3 \circ Z_4) \stackrel{2,2}{=} -2(X_3 \circ Y_3, Z_4) \stackrel{4,16}{=} (Z_0, Z_4) \stackrel{4,12}{=} 0.$$

$$VI. (X_4, X_5) \equiv -2(X_4, Y_1 \circ Z_4) \stackrel{2,2}{=} -2(X_4 \circ Y_1, Z_4) \stackrel{4,16}{=} (Y_0, Y_1) \stackrel{4,12}{=} 0.$$

$$(X_4, X_6) \equiv 2(X_4, Y_2 \circ Z_4) \stackrel{2,2}{=} 2(X_4 \circ Y_2, Z_4) \stackrel{4,16}{=} -(Y_0, Y_2) \stackrel{4,12}{=} 0.$$

$$(X_4, X_7) \equiv -2(X_4, Y_3 \circ Z_4) \stackrel{2,2}{=} -2(X_4 \circ Y_3, Z_4) \stackrel{4,16}{=} (Y_0, Y_3) \stackrel{4,12}{=} 0.$$

$$VII. (X_5, X_6) \equiv -4(Y_1 \circ Z_4, Y_2 \circ Z_4) \stackrel{2,2}{=} -4((Y_1 \circ Z_4) \circ Z_4, Z_2) \stackrel{3,6}{=} -(Y_1, Y_2) \stackrel{4,12}{=} 0.$$

$$(X_5, X_7) \equiv 4(Y_1 \circ Z_4, Y_3 \circ Z_4) \stackrel{2,2}{=} 4((Y_1 \circ Z_4) \circ Z_4, Y_3) \stackrel{3,6}{=} (Y_1, Y_3) \stackrel{4,12}{=} 0.$$

$$VIII. (X_6, X_7) \equiv -4(Y_2 \circ Z_4, Y_3 \circ Z_4) \stackrel{2,2}{=} -4((Y_2 \circ Z_4) \circ Z_4, Y_3) \stackrel{3,6}{=} -(Y_2, Y_3) \stackrel{4,12}{=} 0.$$

4.19 **Lemma.** For $i, j=1, 2, \dots, 7, i \neq j$, we have

$$X_i \circ Y_j = -X \circ Y_i, \quad Y_i \circ Z_j = -Y_j \circ Z_i, \quad Z_i \circ X_j = -Z_j \circ X_i.$$

$$\text{Proof. } X_i \circ Y_j \equiv -2X_i \circ (X_j \circ Z_0) \stackrel{3,3}{=} 2X_j \circ (X_i \circ Z_0) \equiv -X_j \circ Y_i, \quad Y_i \circ Z_j \stackrel{4,16}{=} -2Y_i \circ (Y_j \circ X_0) \stackrel{3,6}{=} 2Y_j \circ (Y_i \circ X_0) \stackrel{4,19}{=} -2Y_j \circ Z_i, \quad Z_i \circ X_j \equiv -2(Y_0 \circ X_i) \circ X_j \stackrel{3,3}{=} 2(Y_0 \circ X_j) \circ X_i \equiv -Z_j \circ X_i.$$

4.20 **Lemma.** For $i, j, k=0, 1, \dots, 7$, the following conditions are equivalent.

$$(2) \quad 2X_i \circ Y_j = \pm Z_k. \quad (2) \quad 2Y_i \circ Z_j = \pm X_k. \quad (3) \quad 2Z_i \circ X_j = \pm Y_k.$$

Proof. Only (1) \rightarrow (2) will be proved. In the case $i, j \neq 0, i \neq j, 2Y_i \circ Z_j \equiv -4Y_i \circ (X_j \circ Y_0) \stackrel{3,8}{=} 4Y_0 \circ (X_j \circ Y_i) \stackrel{4,16}{=} -4Y_0 \circ (X_i \circ Y_j) \stackrel{(1)}{=} \mp 2Y_0 \circ Z_k \stackrel{4,16}{=} \pm X_k$. In the other cases, they are shown in 4.16.

5. Proof of theorem

Since E, E_1, A, X_i, Y_i, Z_i for $i=0, 1, \dots, 7$ form a basis of \mathfrak{Z} by 4.10, 4.11, 4.12, we see first that f defined in the theorem is an \mathbf{R} -isomorphism of \mathfrak{Z} . To complete the proof of theorem, we have to show that $f(X \circ Y) = fX \circ fY$ for $X, Y \in \mathfrak{Z}$. To do so, it is sufficient to show this for $E, E_1, E_2 - E_3, F_k^i$ instead of X, Y . Now the multiplication table among them is given as follows. (E is the unit of \mathfrak{Z}).

$$5.1 \quad \begin{aligned} E_1 \circ E_1 &= E_1, & E_1 \circ (E_2 - E_3) &= 0, \\ E_1 \circ F_1^i &= 0, & 2E_1 \circ F_2^i &= F_2^i, & 2E_1 \circ F_3^i &= F_3^i. \end{aligned}$$

- 5.2 $(E_2 - E_3) \circ (E_2 - E_3) = E - E_1,$
 $(E_2 - E_3) \circ F_1^i = 0, \quad 2(E_2 - E_3) \circ F_2^i = -F_2^i, \quad 2(E_2 - E_3) \circ F_3^i = F_3^i.$
- 5.3 $F_1^i \circ F_1^j = \delta_{ij}(E - E_1),$
 $2F_2^i \circ F_2^j = \delta_{ij}(E + E_1 - (E_2 - E_3)), \quad 2F_3^i \circ F_3^j = \delta_{ij}(E + E_1 + (E_2 - E_3)).$
- 5.4

	F_{k+1}^0	F_{k+1}^1	F_{k+1}^2	F_{k+1}^3	F_{k+1}^4	F_{k+1}^5	F_{k+1}^6	F_{k+1}^7
$2F_k^0$	F_{k+2}^0	$-F_{k+2}^1$	$-F_{k+2}^2$	$-F_{k+2}^3$	$-F_{k+2}^4$	$-F_{k+2}^5$	$-F_{k+2}^6$	$-F_{k+2}^7$
$2F_k^1$	F_{k+2}^1	$-F_{k+2}^0$	$-F_{k+2}^3$	F_{k+2}^2	$-F_{k+2}^5$	F_{k+2}^4	$-F_{k+2}^7$	F_{k+2}^6
$2F_k^2$	F_{k+2}^2	F_{k+2}^3	$-F_{k+2}^0$	$-F_{k+2}^1$	F_{k+2}^6	$-F_{k+2}^7$	$-F_{k+2}^4$	F_{k+2}^5
$2F_k^3$	F_{k+2}^3	$-F_{k+2}^2$	F_{k+2}^1	$-F_{k+2}^0$	F_{k+2}^7	$-F_{k+2}^6$	F_{k+2}^5	F_{k+2}^4
$2F_k^4$	F_{k+2}^4	F_{k+2}^5	$-F_{k+2}^6$	$-F_{k+2}^7$	$-F_{k+2}^0$	$-F_{k+2}^1$	F_{k+2}^2	$-F_{k+2}^3$
$2F_k^5$	F_{k+2}^5	$-F_{k+2}^6$	F_{k+2}^7	F_{k+2}^6	F_{k+2}^1	$-F_{k+2}^0$	$-F_{k+2}^3$	$-F_{k+2}^2$
$2F_k^6$	F_{k+2}^6	F_{k+2}^7	F_{k+2}^4	$-F_{k+2}^5$	$-F_{k+2}^2$	F_{k+2}^3	$-F_{k+2}^0$	F_{k+2}^1
$2F_k^7$	F_{k+2}^7	$-F_{k+2}^6$	$-F_{k+2}^5$	$-F_{k+2}^4$	F_{k+2}^3	F_{k+2}^2	$-F_{k+2}^1$	$-F_{k+2}^0$

for $k = 1, 2, 3$ and the lower suffixes are modulo 3.

By the correspondence f , 5.1–5.3 correspond to the following 5.1*–5.3*.

5.1* $E_1 \circ A = 0, \quad E_1 \circ X_i = 0, \quad 2E_1 \circ Y_i = Y_i, \quad 2E_1 \circ Z_i = Z_i$ (Proposition 4.11).

5.2* $A \circ A \stackrel{2.6}{=} \frac{(A, A)}{2}(E - E_1) = E - E_1, \quad A \circ X_i \stackrel{2.6}{=} \frac{(A, X_i)}{2}(E - E_1) \stackrel{4.1}{=} 0, \quad 2A \circ Y_i = -Y_i, \quad 2A \circ Z_i = Z_i$ (Proposition 4.10).

5.3* $X_i \circ X_j \stackrel{2.6}{=} \frac{(X_i, X_j)}{2}(E - E_1) \stackrel{4.1}{=} \delta_{ij}(E - E_1), \quad 2Y_i \circ Y_j \stackrel{3.9}{=} \frac{(Y_i, Y_j)}{2}(E + E_1 - A) \stackrel{4.12}{=} \delta_{ij}(E + E_1 - A), \quad 2Z_i \circ Z_j \stackrel{3.9}{=} \frac{(Z_i, Z_j)}{2}(E + E_1 + A) \stackrel{4.12}{=} \delta_{ij}(E + E_1 + A).$

Using the lemmas 4.19, 4.20, then it is easily to see that the following calculations show the fact which corresponds to 5.4.

$$5.4^* \quad 2X_0 \circ Y_0 \stackrel{4.16}{=} Z_0, \quad 2X_i \circ Y_i \stackrel{4.16}{=} -Z_0 \quad (i \neq 0).$$

$$I. \quad 2X_1 \circ Y_2 \stackrel{4.17}{=} -Z_3.$$

$$(1) \quad 2X_1 \circ Y_3 \stackrel{4.17}{=} Z_2.$$

$$(2) \quad 2X_1 \circ Y_4 \stackrel{4.20}{=} -Z_5 \text{ (by } 2Y_1 \circ Z_4 \equiv -X_4).$$

$$(3) \quad 2X_1 \circ Y_5 \stackrel{4.19}{=} -2X_5 \circ Y_1 \equiv 4(Y_1 \circ Z_4) \circ Y_1 \stackrel{3.6}{=} Z_4.$$

$$(4) \quad 2X_1 \circ Y_6 \stackrel{4.19}{=} -2X_6 \circ Y_1 \equiv -4(Y_2 \circ Z_4) \circ Y_1 \stackrel{4.19}{=} 4(Y_4 \circ Z_2) \circ Y_1 \stackrel{3.6}{=} -4(Y_1 \circ Z_2) \circ Y_4 \equiv 2X_3 \circ Y_4 \stackrel{4.20}{=} -Z_7 \text{ (by } 2Y_3 \circ Z_4 \equiv -\lambda_7).$$

$$\begin{aligned}
(5) \quad & 2X_1 \circ Y_7 \stackrel{4,19}{=} -2X_7 \circ Y_1 \equiv 4(Y_3 \circ Z_4) \circ Y_1 \stackrel{4,19}{=} -4(Y_4 \circ Z_3) \circ Y_1 \stackrel{3,6}{=} 4(Y_1 \circ Z_3) \circ Y_4 \stackrel{4,20}{\underset{(1)}{=}} 2X_2 \circ Y_4 \\
& \stackrel{4,20}{=} Z_6 \text{ (by } 2Y_2 \circ Z_4 \equiv X_6). \\
\text{II. (6)} \quad & 2X_2 \circ Y_3 \stackrel{4,18}{=} -Z_1. \\
& (7) \quad 2X_2 \circ Y_4 \stackrel{4,20}{=} Z_6 \text{ (by } 2Y_2 \circ Z_4 \equiv X_6). \\
& (8) \quad 2X_2 \circ Y_5 \stackrel{4,19}{=} -2Y_2 \circ X_5 \equiv 4Y_2 \circ (Y_1 \circ Z_4) \stackrel{3,6}{=} -4Y_1 \circ (Y_2 \circ Z_4) \equiv -2Y_1 \circ X_6 \stackrel{4,19}{=} 2X_1 \circ Y_6 \\
& \stackrel{(4)}{=} -Z_7. \\
& (9) \quad 2X_2 \circ Y_6 \stackrel{4,19}{=} -2Y_2 \circ X_6 \equiv -4Y_2 \circ (Y_2 \circ X_4) \stackrel{3,6}{=} -Z_4. \\
& \quad 2X_2 \circ Y_7 \stackrel{4,19}{=} -2Y_2 \circ X_7 \equiv 4Y_2 \circ (Y_3 \circ Z_4) \stackrel{4,19}{=} -4Y_2 \circ (Y_4 \circ Z_3) \stackrel{3,6}{=} 4Y_4 \circ (Y_2 \circ Z_3) \stackrel{4,20}{\underset{(6)}{=}} -2Y_4 \\
& \circ X_1 \stackrel{(2)}{=} Z_5. \\
\text{III. (10)} \quad & 2X_3 \circ Y_4 \stackrel{4,20}{=} -Z_7 \text{ (by } 2Y_3 \circ Z_4 \equiv -X_7). \\
& (11) \quad 2X_3 \circ Y_5 \stackrel{4,19}{=} -2Y_3 \circ X_5 \equiv 4Y_3 \circ (Y_1 \circ Z_4) \stackrel{3,6}{=} -4Y_1 \circ (Y_3 \circ Z_4) \equiv 2Y_1 \circ X_7 \stackrel{4,19}{=} -2X_1 \circ Y_7 \\
& \stackrel{(5)}{=} -Z_6. \\
& \quad 2X_3 \circ Y_6 \stackrel{4,19}{=} -4(Y_1 \circ Z_2) \circ Y_6 \stackrel{3,6}{=} 4(Y_6 \circ Z_2) \circ Y_1 \stackrel{4,19}{=} -4(Z_6 \circ Y_2) \circ Y_1 \stackrel{4,20}{\underset{(9)}{=}} 2X_4 \circ Y_1 \stackrel{4,19}{=} \\
& -2X_1 \circ Y_4 \stackrel{(2)}{=} Z_5. \\
& \quad 2X_3 \circ Y_7 \stackrel{4,19}{=} -2Y_3 \circ X_7 \equiv 4Y_3 \circ (Y_3 \circ Z_4) \stackrel{3,6}{=} Z_4. \\
\text{IV.} \quad & 2X_4 \circ Y_5 \stackrel{4,20}{\underset{(2)}{=}} -4X_4 \circ (Z_1 \circ X_4) \stackrel{3,3}{=} -Z_1. \\
& \quad 2X_4 \circ Y_6 \stackrel{4,20}{\underset{(7)}{=}} 4X_4 \circ (Z_2 \circ X_4) \stackrel{3,3}{=} Z_2. \\
& \quad 2X_4 \circ Y_7 \stackrel{4,20}{\underset{(10)}{=}} -4X_4 \circ (Z_3 \circ X_4) \stackrel{3,3}{=} -Z_3. \\
\text{V.} \quad & 2X_5 \circ Y_6 \stackrel{4,20}{\underset{(11)}{=}} -4X_5 \circ (Z_3 \circ X_5) \stackrel{3,3}{=} -Z_3. \\
& \quad 2X_5 \circ Y_7 \stackrel{4,20}{\underset{(8)}{=}} -4X_5 \circ (Z_2 \circ X_5) \stackrel{3,3}{=} -Z_2. \\
\text{VI.} \quad & 2X_6 \circ Y_7 \stackrel{4,20}{\underset{(4)}{=}} -4X_6 \circ (Z_1 \circ X_6) \stackrel{3,3}{=} -Z_1.
\end{aligned}$$

At last we can complete the proof of the theorem,

Reference

- [1] Yokota, I., Exceptional Lie Group F_4 and its Representation Rings. Jour. Fac. Sci. Shinshu Univ. Vol. 3, 1968.