General Solution of Trusses

(*A Solution by Matrix Method*)

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1. Introduction

Theories of analysis of trusses by matrix methods have been developed and applied by many authors. The present method is one of displacement methods, in which the relation between joint displacements of a member and its member force is determined by a geometrical equation of deformation and then the unknown quantities, that is, joint displacements are determined by the joint method of truss analysis. This method can judge whether the structure is stable or not and can solve all the cases whether the structure may be statically determinate or not, plane or space, simple or compound or complex type, and under various loading conditions, that is, joint loads, heat loads and so on. So this method can be said to be a general one in truss analysis.

First the equations of this theory will be derived and then typical numerical examples of plane and space trusses under various loading conditions will be explained in this report.

2. Development of the Method

This method can be applied to both plane and space trusses and here for concise explanation, we'll develop about plane trusses. We assume the position

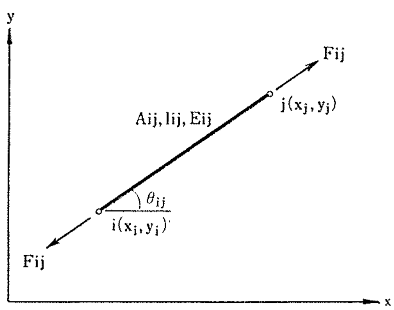

Fig. 1. A Member of a Truss.

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of each joint i, j of an arbitrary member i-j in a truss to be (x_i, y_i), (x_j, y_j) respectively (Fig. 1). The x-y axes are located as shown in the figure. Assume the member length to be l_{ij}, then

$$l^2_{ij} = (x_j - x_i)^2 + (y_j - y_i)^2. \tag{1}$$

Differentiating this, we have

$$2l_{ij}\, dl_{ij} = 2(x_j - x_i)(dx_j - dx_i) + 2(y_j - y_i)(dy_j - dy_i), \tag{2}$$

or

$$dl_{ij} = (x_j - x_i)(dx_j - dx_i)/l_{ij} + (y_j - y_i)(dy_j - dy_i)/l_{ij}. \tag{3}$$

Here expressing the displacement in the x direction u for dx and in the y direction v for dy, we obtain

$$dl_{ij} = (x_j - x_i)(u_j - u_i)/l_{ij} + (y_j - y_i)(v_j - v_i)/l_{ij}. \tag{4}$$

We again assume the sectional area and Young's modulus of the member to be A_{ij} and E_{ij}, respectively, then from Hooke's law, the relation between member force F_{ij} and member deformation dl_{ij} becomes

$$dl_{ij} = F_{ij}l_{ij}/A_{ij}E_{ij}. \tag{5}$$

From Eqs. (4) and (5),

$$F_{ij} = A_{ij}E_{ij}/l_{ij}\{(x_j - x_i)(u_j - u_i)/l_{ij} + (y_j - y_i)(v_j - v_i)/l_{ij}\}. \tag{6}$$

There are as many numbers of F_{ij}'s as that of members in the whole structure and we express this group of F_{ij}'s by the column matrix $\{F\}$. Similarly there are as many numbers of joint displacements as the double of the number of joints and we express these, using column matrix, $\{u\}$. The relation between $\{F\}$ and $\{u\}$ is expressed in matrix notation,

$$\{F\} = [S]\{u\}, \tag{7}$$

in which $[S]$ is determined by Eq. (6) and of the order of rows equal to the number of the members and of columns equal to the double number of the joints. External joint force at each joint, including reactions at supports, is resolved into P_{ix} : x component in the i-th joint, and P_{iy} : y component in the i-th joint. These external joint forces are grouped into column matrix $\{P\}$.

The equilibrium equations at the i-th joint by the joint method are

$$P_{ix} = -\sum_j F_{ij}\cos\theta_{ij}, \tag{8}$$

$$P_{iy} = -\sum_j F_{ij}\sin\theta_{ij}. \tag{9}$$

These equations are expressed in matrix notation

$$\{P\} = [A]\{F\}. \tag{10}$$

Substituting Eq. (7) into Eq. (10), we have

$$\{P\} = [A]\{F\} = [A][S]\{u\} = [K]\{u\}, \tag{11}$$

where $[K] = [A][S]$.

Joint displacements $\{u\}$ are partitioned into $\{u_\alpha\}$ and $\{u_\beta\}$, where $\{u_\alpha\}$ are unknown joint displacements and $\{u_\beta\}$ are known joint displacements, that is, constraints. Correspondingly external joint force $\{P\}$ is grouped into $\{P_\alpha\}$ and $\{P_\beta\}$, where $\{P_\alpha\}$ are known joint forces and $\{P_\beta\}$ unknown reactions at the supports. Consequently the elements of $[K]$ matrix are interchanged so as to be consistent with the original matrix equations. Then Eq. (11) are rewritten as follows:

$$\left\{\begin{array}{c} P_\alpha \\ \hdashline P_\beta \end{array}\right\} = \left[\begin{array}{c:c} K_{\alpha\alpha} & K_{\alpha\beta} \\ K_{\beta\alpha} & K_{\beta\beta} \end{array}\right] \left\{\begin{array}{c} u_\alpha \\ \hdashline u_\beta \end{array}\right\}, \tag{12}$$

from which

$$\{P_\alpha\} = [K_{\alpha\alpha}]\{u_\alpha\} + [K_{\alpha\beta}]\{u_\beta\}, \tag{13}$$

$$\{P_\beta\} = [K_{\beta\alpha}]\{u_\alpha\} + [K_{\beta\beta}]\{u_\beta\}. \tag{14}$$

From Eq. (13) we obtain

$$\{u_\alpha\} = [K_{\alpha\alpha}{}^{-1}](\{P_\alpha\} - [K_{\alpha\beta}]\{u_\beta\}). \tag{15}$$

In inverting $[K_{\alpha\alpha}]$, if the structure is unstable, $[K_{\alpha\alpha}]$ will be singular and $[K_{\alpha\alpha}{}^{-1}]$ cannot be obtained, from which we can know whether the structure is stable or not. Substituting $\{u_\alpha\}$ in Eq. (15) into Eq. (14) we have unknown reactions $\{P_\beta\}$.

3. Fabrication Errors and Temperature Change

When some of the members undergo temperature change Δt 's or fabrication errors δl_{ij}'s, Eq. (5) is rewritten as follows:

$$dl_{ij} = F_{ij}l_{ij}/A_{ij}E_{ij} + \alpha_{ij}\Delta t\cdot l_{ij} + \delta l_{ij}, \tag{5'}$$

$$F_{ij} = E_{ij}A_{ij}/l_{ij}(dl_{ij} - \alpha_{ij}\Delta t\cdot l_{ij} - \delta l_{ij}).$$

The second and third terms on the right hand side of above equation are summed into Δ_{ij} and

$$F_{ij} = E_{ij}A_{ij}dl_{ij}/l_{ij} - \Delta_{ij}E_{ij}A_{ij}/l_{ij}.$$

Substituting Eq. (4) into the equation above, we have

$$F_{ij} = E_{ij}A_{ij}/l_{ij}\{(x_j - x_i)(u_j - u_i)/l_{ij} + (y_j - y_i)(v_j - v_i)/l_{ij}\} \\ - \Delta_{ij}E_{ij}A_{ij}/l_{ij}. \tag{6'}$$

Expressing Eq. (6′) in matrix notation, we have

$$\{F\} = [S]\{u\} - \{\Delta\}. \tag{7'}$$

Substituting Eq. (7′) into equilibrium equation $\{P\} = [A]\{F\}$, we have

$$P = [A]([S]\{u\} - \{\Delta\}) = [A][S]\{u\} - [A]\{\Delta\} \\ = [K]\{u\} - [A]\{\Delta\}, \tag{11'}$$

or

$$\{P\} + [A]\{\Delta\} = [K]\{u\}.$$

$\{u\}$ can be determined by this equation in the same way as before. After determined, $\{u\}$ is substituted into Eq. (7′) and member forces $\{F\}$ can be determined.

4. Illustrative Examples

To show how the method is applied to actual trusses, four numerical examples will be illustrated here.

4.1 A Plane Truss

A plane truss shown in Fig. 2 undergoes following loading conditions.

1) a 1 kip load at joint 2 2) a 1 kip load at joint 4
3) a 1 kip load at joint 3
4) a fabrication error in member 2–5 of 1/8 in. too long
5) a settlement of the roller support at joint 6 of 1/4 in.

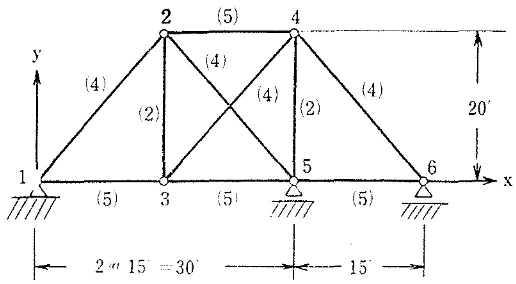


Fig. 2. Plane Truss
Numbers in brackets = sectional area (in^2)
E = 30000 ksi

(a) Construction of $[S]$ matrix

The basic equation, from which the equation $\{F\} = \{S\}\{u\}$ is derived, is shown here again, that is, Eq. (6):

$$F_{ij} = A_{ij}E_{ij}/l_{ij}\{(x_j - x_i)(u_j - u_i)/l_{ij} + (y_j - y_i)(v_j - v_i)/l_{ij}\}.$$

If we put $A_{ij}E_{ij}/l_{ij}$ into k_{ij}, $(x_j - x_i)/l_{ij}$ into λ_{ij} and $(y_j - y_i)/l_{ij}$ into μ_{ij}, Eq. (6) becomes

$$F_{ij} = k_{ij}\{\lambda_{ij}(u_j - u_i) + \mu_{ij}(v_j - v_i)\}$$

Applying this equation to all the members in the truss, we have [S] matrix as shown in Table 1. The numerical values of Table 1 are calculated and shown in Table 2.

Table 1. [S] matrix

	u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	u_5	v_5	u_6	v_6
F_{12}	$-k_{12}\lambda_{12}$	$-k_{12}\mu_{12}$	$k_{12}\lambda_{12}$	$k_{12}\mu_{12}$	0	0	0	0	0	0	0	0
F_{13}	$-k_{13}\lambda_{13}$	$-k_{13}\mu_{13}$	0	0	$k_{13}\lambda_{13}$	$k_{13}\mu_{13}$	0	0	0	0	0	0
F_{23}	0	0	$-k_{23}\lambda_{23}$	$-k_{23}\mu_{23}$	$k_{23}\lambda_{23}$	$k_{23}\mu_{23}$	0	0	0	0	0	0
F_{24}	0	0	$-k_{24}\lambda_{24}$	$-k_{24}\mu_{24}$	0	0	$k_{24}\lambda_{24}$	$k_{24}\mu_{24}$	0	0	0	0
F_{25}	0	0	$-k_{25}\lambda_{25}$	$-k_{25}\mu_{25}$	0	0	0	0	$k_{25}\lambda_{25}$	$k_{25}\mu_{25}$	0	0
F_{34}	0	0	0	0	$-k_{34}\lambda_{34}$	$-k_{34}\mu_{34}$	$k_{34}\lambda_{34}$	$k_{34}\mu_{34}$	0	0	0	0
F_{35}	0	0	0	0	$-k_{35}\lambda_{35}$	$-k_{35}\mu_{35}$	0	0	$k_{35}\lambda_{35}$	$k_{35}\mu_{35}$	0	0
F_{45}	0	0	0	0	0	0	$-k_{45}\lambda_{45}$	$-k_{45}\mu_{45}$	$k_{45}\lambda_{45}$	$k_{45}\mu_{45}$	0	0
F_{46}	0	0	0	0	0	0	$-k_{46}\lambda_{46}$	$-k_{46}\mu_{46}$	0	0	$k_{46}\lambda_{46}$	$k_{46}\mu_{46}$
F_{56}	0	0	0	0	0	0	0	0	$-k_{56}\lambda_{56}$	$-k_{56}\mu_{56}$	$k_{56}\lambda_{56}$	$k_{56}\mu_{56}$

Table 2. Numerical values of [S] matrix (kip/in)

	u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	u_5	v_5	u_6	v_6
F_{12}	−240	−320	240	320	0	0	0	0	0	0	0	0
F_{13}	−833.3	0	0	0	833.3	0	0	0	0	0	0	0
F_{23}	0	0	0	250	0	−250	0	0	0	0	0	0
F_{24}	0	0	−833.3	0	0	0	833.3	0	0	0	0	0
F_{25}	0	0	−240	320	0	0	0	0	240	−320	0	0
F_{34}	0	0	0	0	−240	−320	240	320	0	0	0	0
F_{35}	0	0	0	0	−833.3	0	0	0	833.3	0	0	0
F_{45}	0	0	0	0	0	0	0	250	0	−250	0	0
F_{46}	0	0	0	0	0	0	−240	320	0	0	240	−320
F_{56}	0	0	0	0	0	0	0	0	0	0	0	0

(b) Construction of [A] matrix

[A] matrix is constructed by making equilibrium equations in x and y directions at each joint. Force diagram at each joint is shown in Fig. 3. For example, from the equilibrium condition at joint 1 (see Fig. 4), we have

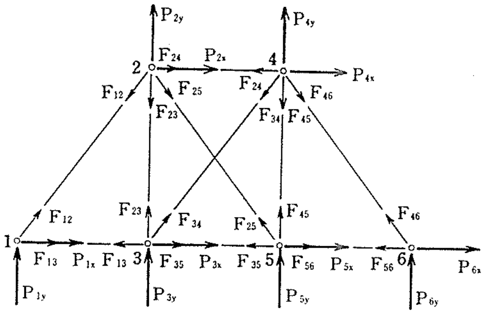


Fig. 3. Force Diagram at Each Joint

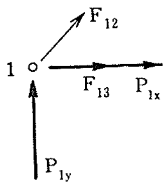


Fig. 4. Force Diagram at 1

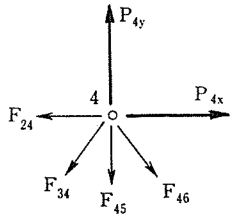


Fig. 5. Force Diagram at 4

$$P_{1x} = F_{12}(-\cos\theta_{12}) + F_{13}(-\cos\theta_{13}),$$

$$P_{1y} = F_{12}(-\sin\theta_{12}) + F_{13}(-\sin\theta_{13}).$$

And again from the equilibrium equation at joint 4 (see Fig. 5), we have

$$P_{4x} = F_{24}\cos\theta_{24} + F_{34}\cos\theta_{34} + F_{45}(-\cos\theta_{45}) + F_{46}(-\cos\theta_{46}),$$

$$P_{4y} = F_{24}\sin\theta_{24} + F_{34}\sin\theta_{34} + F_{45}(-\sin\theta_{45}) + F_{46}(-\sin\theta_{46}).$$

From the equilibrium equations for all joints, we can make $[A]$ matrix, which is shown in Table 3.

Table 3. $[A]$ matrix

	F_{12}	F_{13}	F_{23}	F_{24}	F_{25}	F_{34}	F_{35}	F_{45}	F_{46}	F_{56}
P_{1x}	$-\lambda_{12}$	$-\lambda_{13}$	0	0	0	0	0	0	0	0
P_{1y}	$-\mu_{12}$	$-\mu_{13}$	0	0	0	0	0	0	0	0
P_{2x}	λ_{12}	0	$-\lambda_{23}$	$-\lambda_{24}$	$-\lambda_{25}$	0	0	0	0	0
P_{2y}	μ_{12}	0	$-\mu_{23}$	$-\mu_{24}$	$-\mu_{25}$	0	0	0	0	0
P_{3x}	0	λ_{13}	λ_{23}	0	0	$-\lambda_{34}$	$-\lambda_{35}$	0	0	0
P_{3y}	0	μ_{13}	μ_{23}	0	0	$-\mu_{34}$	$-\mu_{35}$	0	0	0
P_{4x}	0	0	0	λ_{24}	0	λ_{34}	0	$-\lambda_{45}$	$-\lambda_{46}$	0
P_{4y}	0	0	0	μ_{24}	0	μ_{34}	0	$-\mu_{45}$	$-\mu_{46}$	0
P_{5x}	0	0	0	0	λ_{25}	0	λ_{35}	λ_{45}	0	$-\lambda_{56}$
P_{5y}	0	0	0	0	μ_{25}	0	μ_{35}	μ_{45}	0	$-\mu_{56}$
P_{6x}	0	0	0	0	0	0	0	0	λ_{46}	λ_{56}
P_{6y}	0	0	0	0	0	0	0	0	μ_{46}	μ_{56}

$[A]$ matrix can be constructed more systematically from another point of view : from the equilibrium condition, $[A]$ matrix was constructed rowwise in the preceeding way. Now a member force F_{ij} is associated with the equilibrium equations of joint i and j. In Fig. 6, F_{ij} has such component as

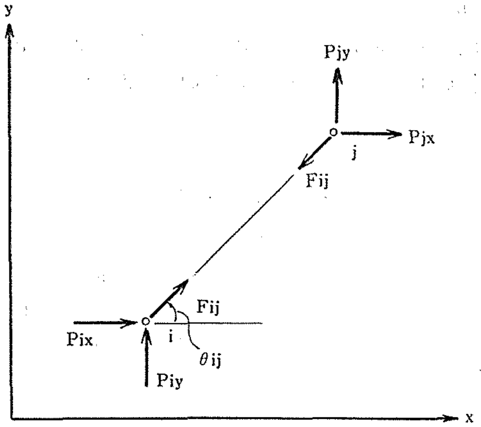


Fig. 6. Member i-j

$$-\cos\theta_{ij}F_{ij} \text{ in } P_{ix} \text{ direction}$$
$$-\sin\theta_{ij}F_{ij} \text{ in } P_{iy} \text{ direction}$$
$$\cos\theta_{ij}F_{ij} \text{ in } P_{jx} \text{ direction}$$
$$\sin\theta_{ij}F_{ij} \text{ in } P_{jy} \text{ direction}$$

For example, the column of F_{ij} in $[A]$ matrix has four non-zero elements, that is, $-\cos\theta_{ij}$, $-\sin\theta_{ij}$, $\cos\theta_{ij}$ and $\sin\theta_{ij}$ in P_{ix}, P_{iy}, P_{jx} and P_{jy} rows, respectively. This latter method is more convenient in making $[A]$ matrix in a computer.

The numerical values of $[A]$ matrix are shown in Table 4.

Table 4. Numerical values of $[A]$ matrix

	F_{12}	F_{13}	F_{23}	F_{24}	F_{25}	F_{34}	F_{35}	F_{45}	F_{46}	F_{56}
P_{1x}	−0.6	−1.0	0	0	0	0	0	0	0	0
P_{1y}	−0.8	0	0	0	0	0	0	0	0	0
P_{2x}	0.6	0	0	−1.0	−0.6	0	0	0	0	0
P_{2y}	0.8	0	1.0	0	0.8	0	0	0	0	0
P_{3x}	0	1.0	0	0	0	−0.6	−1.0	0	0	0
P_{3y}	0	0	−1.0	0	0	−0.8	0	0	0	0
P_{4x}	0	0	0	1.0	0	0.6	0	0	−0.6	0
P_{4y}	0	0	0	0	0	0.8	0	1.0	0.8	0
P_{5x}	0	0	0	0	0.6	0	1.0	0	0	−1.0
P_{5y}	0	0	0	0	−0.8	0	0	−1.0	0	0
P_{6x}	0	0	0	0	0	0	0	0	0.6	1.0
P_{6y}	0	0	0	0	0	0	0	0	−0.8	0

(c) Construction of [K] matrix

After constructing [S] and [A] matrcıes, [K] matrix is obtained by matrix multiplication of [A] and [S]. The result of matrix multiplication is shown in Table 5.

Table 5. [K] matrix (kip/in)

	u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	u_5	v_5	u_6	v_6
P_{1x}	977.3	192	−144	−192	−833.3	0	0	0	0	0	0	0
P_{1y}	192	256	−192	−256	0	0	0	0	0	0	0	0
P_{2x}	−144	−192	1121.3	0	0	0	−833.3	0	−144	192	0	0
P_{2y}	−192	−256	0	762	0	−250	0	0	192	−256	0	0
P_{3x}	−833.3	0	0	0	1810.6	192	−144	−192	−833.3	0	0	0
P_{3y}	0	0	0	−250	192	506	−192	−256	0	0	0	0
P_{4x}	0	0	−833.3	0	−144	−192	1121.3	0	0	0	−144	192
P_{4y}	0	0	0	0	−192	−256	0	762	0	−250	192	−256
P_{5x}	0	0	−144	192	−833.3	0	0	0	1810.6	−192	−833.3	0
P_{5y}	0	0	192	−256	0	0	0	−250	−192	506	0	0
P_{6x}	0	0	0	0	0	0	−144	192	−833.3	0	977.3	−192
P_{6y}	0	0	0	0	0	0	192	−256	0	0	−192	256

In this example u_1, v_1, v_5 and v_6 of joint displacements are constrained, so unknown joint displacements $\{u_\alpha\}$ and known joint displacements $\{u_\beta\}$ are as follows :

$$\{u_\alpha\}=\begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ u_6 \end{Bmatrix}, \qquad \{u_\beta\}=\begin{Bmatrix} u_1 \\ v_1 \\ v_5 \\ v_6 \end{Bmatrix},$$

while known external joint forces $\{P_\alpha\}$ and unknown reactions at the supports $\{P_\beta\}$ are

$$\{P_\alpha\}=\begin{Bmatrix} P_{2x} \\ P_{2y} \\ P_{3x} \\ P_{3y} \\ P_{4x} \\ P_{4y} \\ P_{5x} \\ P_{6x} \end{Bmatrix}, \qquad \{P_\beta\}=\begin{Bmatrix} P_{1x} \\ P_{1y} \\ P_{5y} \\ P_{6y} \end{Bmatrix}.$$

Table 6. $K_{\alpha\alpha}$, $K_{\alpha\beta}$, $K_{\beta\alpha}$ and $K_{\beta\beta}$ matricies

	$K_{\alpha\alpha}$								$K_{\alpha\beta}$			
	u_2	v_2	u_3	v_3	u_4	v_4	u_5	u_6	u_1	v_1	v_5	v_6
P_{2x}	1121.3	0	0	0	−833.3	0	−144	0	−144	−192	192	0
P_{2y}	0	762	0	−250	0	0	192	0	−192	−256	−256	0
P_{3x}	0	0	1810.6	192	−144	−192	−833.3	0	−833.3	0	0	0
P_{3y}	0	−250	192	506	−192	−256	0	0	0	0	0	0
P_{4x}	−833.3	0	−144	−192	1121.3	0	0	−144	0	0	0	192
P_{4y}	0	0	−192	−256	0	762	0	192	0	0	−250	−256
P_{5x}	−144	192	−833.3	0	0	0	1810.6	−833.3	0	0	−192	0
P_{6x}	0	0	0	0	−144	192	−833.3	977.3	0	0	0	−192
	$K_{\beta\alpha}$								$K_{\beta\beta}$			
	u_2	v_2	u_3	v_3	u_4	v_4	u_5	u_6	u_1	v_1	v_5	v_6
P_{1x}	−144	−192	−833.3	0	0	0	0	0	977.3	192	0	0
P_{1y}	−192	−256	0	0	0	0	0	0	192	256	0	0
P_{5y}	192	−256	0	0	0	−250	−192	0	0	0	506	0
P_{6y}	0	0	0	0	192	−256	0	−192	0	0	0	256

Table 7. $\{P_\alpha\}$ matrix (kip)

	LC1	LC2	LC3	LC4	LC5
P_{2x}	0	0	0	0	0
P_{2y}	−1	0	0	0	0
P_{3x}	0	0	0	0	0
P_{3y}	0	0	−1	0	0
P_{4x}	0	0	0	0	0
P_{4y}	0	−1	0	0	0
P_{5x}	0	0	0	0	0
P_{6x}	0	0	0	0	0

Table 8. $\{u_\beta\}$ matrix (in)

	LC1	LC2	LC3	LC4	LC5
u_1	0	0	0	0	0
v_1	0	0	0	0	0
v_5	0	0	0	0	0
v_6	0	0	0	0	−0.25

Table 9. $\{\Delta\}$ for LC4 (in)

Δ_{12}	0
Δ_{13}	0
Δ_{23}	0
Δ_{24}	0
Δ_{25}	0.125
Δ_{34}	0
Δ_{35}	0
Δ_{45}	0
Δ_{46}	0
Δ_{56}	0

$[K]$ matrix is partitioned into $[K_{\alpha\alpha}]$, $[K_{\alpha\beta}]$, $[K_{\beta\alpha}]$ and $[K_{\beta\beta}]$, which are shown in Table 6.

(d) Calculation of $\{u\}$ and $\{F\}$

$\{P_\alpha\}$, $\{u_\beta\}$ and $\{\Delta\}$ are shown in Tables 7, 8 and 9 for the loading conditions given above. Using these data, we can obtain $\{u\}$ and $\{F\}$ from the following equations :

$$\{u_\alpha\} = [K_{\alpha\alpha}{}^{-1}](\{P_\alpha\} - [K_{\alpha\beta}]\{u_\beta\}), \tag{16}$$

$$\{F\} = [S]\{u\} - \{\Delta\}. \tag{17}$$

Table 10. Joint displacements (10^{-3} in)

	LC1		LC2		LC3		LC4		LC5	
	u	v	u	v	u	v	u	v	u	v
Joint 1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Joint 2	0.066	−1.984	−0.066	−0.568	−0.732	−1.454	−56.12	58.17	54.02	2.403
Joint 3	0.446	−1.454	0.142	−1.375	0.461	−3.978	−3.706	18.47	−9.889	−6.352
Joint 4	−0.045	−0.568	−0.170	−1.928	−1.088	−1.374	−39.77	6.757	75.77	−79.14
Joint 5	0.772	0.0	0.466	0.0	0.591	0.0	1.520	0.0	−17.81	0.0
Joint 6	0.763	0.0	0.751	0.0	0.614	0.0	−5.891	0.0	−37.58	−250.0

Table 11. Member forces (kip)

	LC1	LC2	LC3	LC4	LC5
F_{12}	−0.619	−0.198	−0.641	5.147	13.73
F_{13}	0.371	0.119	0.385	−3.088	−8.241
F_{23}	−0.133	0.202	0.631	9.924	2.189
F_{24}	−0.092	−0.086	−0.296	13.62	18.12
F_{25}	−0.465	−0.054	−0.148	−17.55	−16.47
F_{34}	0.166	−0.252	0.461	−12.41	−2.736
F_{35}	0.272	−0.270	0.108	4.355	−6.599
F_{45}	−0.142	−0.482	−0.344	1.689	−19.79
F_{46}	0.012	−0.396	−0.032	10.29	27.47
F_{56}	−0.007	0.237	0.019	−6.176	−16.48

The results are shown in Tables 10 and 11.

4.2 A Plane Truss

A 1000 lb. load is applied to a plane truss at joint 5 as shown in Fig. 7. We assume the sectional area A to be 1 in^2. and Young's modulus E to be $30(10)^6$

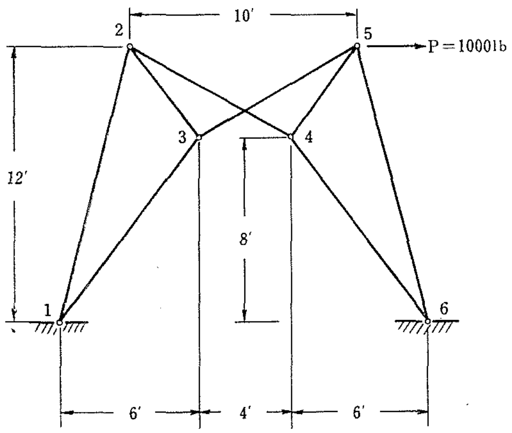


Fig. 7. Plane Truss

psi. for all the member in the truss. In the same way as 4.2 we obtain [S] matrix, [A] matrix, [K] matrix and [$K_{\alpha\alpha}$] matrix, which are shown in Tables 12, 13, 14 and 15. Since in this case u_1, v_1, u_6 and v_6 are constrained, $\{u_\alpha\}$, $\{u_\beta\}$,

Table 12 [S] matrix (kip/ft)

	u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	u_5	v_5	u_6	v_6
F_{12}	−588.2	−2352.9	888.2	2352.9	0	0	0	0	0	0	0	0
F_{13}	−1800	−2400	0	0	1800	2400	0	0	0	0	0	0
F_{23}	0	0	−3600	4800	3600	−4800	0	0	0	0	0	0
F_{24}	0	0	−3230.7	1846.1	0	0	3230.7	−1846.1	0	0	0	0
F_{35}	0	0	0	0	−3230.3	−1845.9	0	0	3230.3	1845.9	0	0
F_{45}	0	0	0	0	0	0	−3600	−4800	3600	4800	0	0
F_{46}	0	0	0	0	0	0	−1800	2400	0	0	1800	−2400
F_{56}	0	0	0	0	0	0	0	0	−588.2	2352.9	588.2	−2352.9

Table 13 [A] matrix

	F_{12}	F_{13}	F_{23}	F_{24}	F_{35}	F_{45}	F_{46}	F_{56}
P_{1x}	−0.242	−0.6	0	0	0	0	0	0
P_{1y}	−0.970	−0.8	0	0	0	0	0	0
P_{2x}	0.242	0	−0.6	−0.868	0	0	0	0
P_{2y}	0.970	0	0.8	0.496	0	0	0	0
P_{3x}	0	0.6	0.6	0	−0.868	0	0	0
P_{3y}	0	0.8	−0.8	0	−0.496	0	0	0
P_{4x}	0	0	0	0.868	0	−0.6	−0.6	0
P_{4y}	0	0	0	−0.496	0	−0.8	0.8	0
P_{5x}	0	0	0	0	0.868	0.6	0	−0.242
P_{5y}	0	0	0	0	0.496	0.8	0	0.970
P_{6x}	0	0	0	0	0	0	0.6	0.242
P_{6y}	0	0	0	0	0	0	−0.8	−0.970

Table 14. [K] matrix

	u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	u_5	v_5	u_6	v_6
P_{1x}	1222.6	2010.6	−142.6	−570.6	−1080	−1440	0	0	0	0	0	0
P_{1y}	2010.6	4202.6	−570.6	−2282.6	−1440	−1920	0	0	0	0	0	0
P_{2x}	−142.6	−570.6	5107.7	−3912.2	−2160.0	2880	−2805.0	1602.9	0	0	0	0
P_{2y}	−570.6	−2282.6	−3912.2	7038.6	2880	−3840	1602.9	−915.9	0	0	0	0
P_{3x}	−1080	−1440	−2160.0	2880	6044.7	162.7	0	0	−2804.7	−1602.7	0	0
P_{3y}	−1440	−1920	2880	−3840	162.7	6675.8	0	0	−1602.7	−915.8	0	0
P_{4x}	0	0	−2805.0	1602.9	0	0	6045.0	−162.9	−2160	−2880	−1080	1440
P_{4y}	0	0	1602.9	−915.9	0	0	−162.9	6675.9	−2880	−3840	1440	−1920
P_{5x}	0	0	0	0	−2804.7	−1602.7	−2160	−2880	5107.4	3912.0	−142.6	570.6
P_{5y}	0	0	0	0	−1602.7	−915.8	−2880	−3840	3912.0	7038.5	570.6	−2282.6
P_{6x}	0	0	0	0	0	0	−1080	1440	−142.6	570.6	1222.6	−2010.6
P_{6y}	0	0	0	0	0	0	1440	−1920	570.6	−2282.6	−2010.6	4202.6

Table 15. $[K_{\alpha\alpha}]$ matrix

	u_2	v_2	u_3	v_3	u_4	v_4	u_5	v_5
P_{2x}	5107.7	−3912.2	−2160.0	2880	−2805.0	1602.9	0	0
P_{2y}	−3912.2	7038.6	2880	−3840	1602.9	−915.9	0	0
P_{3x}	−2160.0	2880	6044.7	162.7	0	0	−2804.7	−1602.7
P_{3y}	2880	−3840	162.7	6675.8	0	0	−1602.7	−915.8
P_{4x}	−2805.0	1602.9	0	0	6045.0	−162.9	−2160	−2880
P_{4y}	1602.9	−915.9	0	0	−162.9	6675.9	−2880	−3840
P_{5x}	0	0	−2804.7	−1602.7	−2160	−2880	5107.4	3912.0
P_{5y}	0	0	−1602.7	−915.8	−2880	−3840	3912.0	7038.5

$\{P_\alpha\}$ and $\{P_\beta\}$ become as follows:

$$\{u_\alpha\} = \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{Bmatrix}, \qquad \{u_\beta\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_6 \\ v_6 \end{Bmatrix},$$

$$\{P_\alpha\} = \begin{Bmatrix} P_{2x} \\ P_{2y} \\ P_{3x} \\ P_{3y} \\ P_{4x} \\ P_{4y} \\ P_{5x} \\ P_{5y} \end{Bmatrix}, \qquad \{P_\beta\} = \begin{Bmatrix} P_{1x} \\ P_{1y} \\ P_{6x} \\ P_{6y} \end{Bmatrix}.$$

From these equations we have joint displacement $\{u\}$ and member force $\{F\}$ (Tables 16, 17).

Table 16. Joint displacement $\{u\}$ ($\times 10^{-4}$ft)

	u	v
Joint 1	0	0
Joint 2	4.880	−2.041
Joint 3	7.707	−0.897
Joint 4	6.907	3.552
Joint 5	10.32	0.664
Joint 6	0	0

Table 17. $\{F\}$ matrix (kip)

member	member force
1 — 2	−193.2
1 — 3	1171.8
2 — 3	468.7
2 — 4	−377.9
3 — 5	1133.7
4 — 5	−156.2
4 — 6	−390.6
5 — 6	−450.9

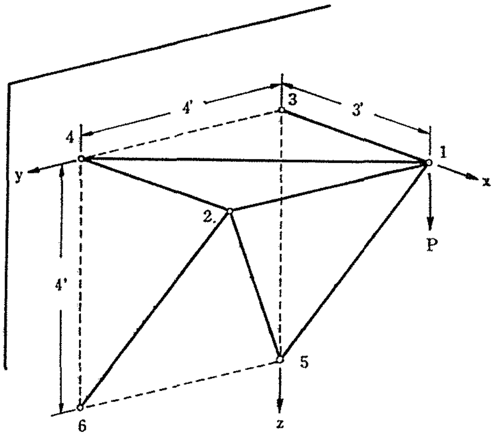


Fig. 8. Space truss

4.3 A Simple Space Truss

A simple space truss shown in Fig. 8 is composed of panels 1234 and 1256 and attached to the vertical wall. The panel 1234 is in a horizontal plane. Loading conditions are as follows:

1) a load $P=1$ kip is applied to joint 1.

2) the temperature of whole structure rises $50°F$ uniformly.

We assume Young's modulus E to be $30(10)^6$ psi. and thermal expansion coefficient α to be $6.5(10)^{-6}(in/in)/°F$ and sectional areas all to be 1 in^2.

For a space truss, we use the extended equation instead of Eq. (6) in obtaining [S] matrix, that is,

$$F_{ij} = A_{ij}E_{ij}/l_{ij}\{(x_j - x_i)(u_j - u_i)/l_{ij} + (y_j - y_i)(v_j - v_i)/l_{ij} + (z_j - z_i)(w_j - w_i)/l_{ij}\} \tag{18}$$

where z_i and z_j are the positions of joints i, j in the z-coordinate and w_i and w_j are the displacements in the z direction.

Similarly in obtaining [A] matrix, the equilibrium equations in three directions must be considered. The direction cosines of member i-j are assumed to be λ_{ij}, μ_{ij}, ν_{ij} then

$$P_{ix} = -\sum_j \lambda_{ij}F_{ij}$$
$$P_{iy} = -\sum_j \mu_{ij}F_{ij}$$
$$P_{iz} = -\sum_j \nu_{ij}F_{ij}$$

Applying these equations to all the members of this truss, thus [S] and [A] matricies are obtained, which are shown in Tables 18 and 19 and their nume-

Table 18. [S] matrix

	u_1	v_1	w_1	u_2	v_2	w_2	u_3	v_3	w_3
F_{12}	$-k_{12}\lambda_{12}$	$-k_{12}\mu_{12}$	$-k_{12}\nu_{12}$	$k_{12}\lambda_{12}$	$k_{12}\mu_{12}$	$k_{12}\nu_{12}$	0	0	0
F_{13}	$-k_{13}\lambda_{13}$	$-k_{13}\mu_{13}$	$-k_{13}\nu_{13}$	0	0	0	$k_{13}\lambda_{13}$	$k_{13}\mu_{13}$	$k_{13}\nu_{13}$
F_{14}	$-k_{14}\lambda_{14}$	$-k_{14}\mu_{14}$	$-k_{14}\nu_{14}$	0	0	0	0	0	0
F_{15}	$-k_{15}\lambda_{51}$	$-k_{15}\mu_{15}$	$-k_{15}\nu_{15}$	0	0	0	0	0	0
F_{24}	0	0	0	$-k_{24}\lambda_{24}$	$-k_{24}\mu_{24}$	$-k_{24}\nu_{24}$	0	0	0
F_{25}	0	0	0	$-k_{25}\lambda_{25}$	$-k_{25}\mu_{25}$	$-k_{25}\nu_{25}$	0	0	0
F_{26}	0	0	0	$-k_{26}\lambda_{26}$	$-k_{26}\mu_{26}$	$-k_{26}\nu_{26}$	0	0	0
	u_4	v_4	w_4	u_5	v_5	w_5	u_6	v_6	w_6
F_{12}	0	0	0	0	0	0	0	0	0
F_{13}	0	0	0	0	0	0	0	0	0
F_{14}	$k_{14}\lambda_{14}$	$k_{14}\mu_{14}$	$k_{14}\nu_{14}$	0	0	0	0	0	0
F_{15}	0	0	0	$k_{15}\lambda_{15}$	$k_{15}\mu_{15}$	$k_{15}\nu_{15}$	0	0	0
F_{24}	$k_{24}\lambda_{24}$	$k_{24}\mu_{24}$	$k_{24}\nu_{24}$	0	0	0	0	0	0
F_{25}	0	0	0	$k_{25}\lambda_{25}$	$k_{25}\mu_{25}$	$k_{25}\nu_{25}$	0	0	0
F_{26}	0	0	0	0	0	0	$k_{26}\lambda_{26}$	$k_{26}\mu_{26}$	$k_{26}\nu_{26}$

Table 19. [A] matrix

	F_{12}	F_{13}	F_{14}	F_{15}	F_{24}	F_{25}	F_{26}
P_{1x}	$-\lambda_{12}$	$-\lambda_{13}$	$-\lambda_{14}$	$-\lambda_{15}$	0	0	0
P_{1y}	$-\mu_{12}$	$-\mu_{13}$	$-\mu_{14}$	$-\mu_{15}$	0	0	0
P_{1z}	$-\nu_{12}$	$-\nu_{13}$	$-\nu_{14}$	$-\nu_{15}$	0	0	0
P_{2x}	λ_{12}	0	0	0	$-\lambda_{24}$	$-\lambda_{25}$	$-\lambda_{26}$
P_{2y}	μ_{12}	0	0	0	$-\mu_{24}$	$-\mu_{25}$	$-\mu_{26}$
P_{2z}	ν_{12}	0	0	0	$-\nu_{24}$	$-\nu_{25}$	$-\nu_{26}$
P_{3x}	0	λ_{13}	0	0	0	0	0
P_{3y}	0	μ_{13}	0	0	0	0	0
P_{3z}	0	ν_{13}	0	0	0	0	0
P_{4x}	0	0	λ_{14}	0	λ_{24}	0	0
P_{4y}	0	0	μ_{14}	0	μ_{24}	0	0
P_{4z}	0	0	ν_{14}	0	ν_{24}	0	0
P_{5x}	0	0	0	λ_{15}	0	λ_{25}	0
P_{5y}	0	0	0	μ_{15}	0	μ_{25}	0
P_{5z}	0	0	0	ν_{15}	0	ν_{25}	0
P_{6x}	0	0	0	0	0	0	λ_{26}
P_{6y}	0	0	0	0	0	0	μ_{26}
P_{6z}	0	0	0	0	0	0	ν_{26}

Table 20. Numerical values of [S] matrix ($\times 10^5$ lb/in)

	u_1	v_1	w_1	u_2	v_2	w_2	u_3	v_3	w_3
F_{12}	0	−6.25	0	0	6.25	0	0	0	0
F_{13}	8.333	0	0	0	0	0	−8.333	0	0
F_{14}	3.0	−4.0	0	0	0	0	0	0	0
F_{15}	3.0	0	−4.0	0	0	0	0	0	0
F_{24}	0	0	0	8.333	0	0	0	0	0
F_{25}	0	0	0	1.829	2.439	−2.439	0	0	0
F_{26}	0	0	0	3.0	0	−4.0	0	0	0

	u_4	v_4	w_4	u_5	v_5	w_5	u_6	v_6	w_6
F_{12}	0	0	0	0	0	0	0	0	0
F_{13}	0	0	0	0	0	0	0	0	0
F_{14}	−3.0	4.0	0	0	0	0	0	0	0
F_{15}	0	0	0	−3.0	0	4.0	0	0	0
F_{24}	−8.333	0	0	0	0	0	0	0	0
F_{25}	0	0	0	−1.829	−2.439	2.439	0	0	0
F_{26}	0	0	0	0	0	0	−3.0	0	4.0

Table 21. Numerical values of [A] matrix

	F_{12}	F_{13}	F_{14}	F_{15}	F_{24}	F_{25}	F_{26}
P_{1x}	0	1	0.6	0.6	0	0	0
P_{1y}	−1	0	−0.8	0	0	0	0
P_{1z}	0	0	0	−0.8	0	0	0
P_{2x}	1	0	0	0	1	0.4685	0.6
P_{2y}	0	0	0	0	0	0.6246	0
P_{2z}	0	0	0	0	0	−0.6246	−0.8
P_{3x}	0	−1	0	0	0	0	0
P_{3y}	0	0	0	0	0	0	0
P_{3z}	0	0	0	0	0	0	0
P_{4x}	0	0	−0.6	0	−1	0	0
P_{4y}	0	0	0.8	0	0	0	0
P_{4z}	0	0	0	0	0	0	0
P_{5x}	0	0	0	−0.6	0	−0.4685	0
P_{5y}	0	0	0	0	0	−0.6246	0
P_{5z}	0	0	0	0.8	0	0.6246	0
P_{6x}	0	0	0	0	0	0	−0.6
P_{6y}	0	0	0	0	0	0	0
P_{6z}	0	0	0	0	0	0	0.8

Table 22. [K] matrix ($\times 10^5$ lb/in)

	u_1	v_1	w_1	u_2	v_2	w_2	u_3	v_3	w_3
P_{1x}	11.933	−2.4	−2.4	0	0	0	−8.333	0	0
P_{1y}	−2.4	9.45	0	0	−6.25	0	0	0	0
P_{1z}	−2.4	0	3.2	0	0	0	0	0	0
P_{2x}	0	0	0	10.99	1.142	−3.542	0	0	0
P_{2y}	0	−6.25	0	1.142	7.773	−1.523	0	0	0
P_{2z}	0	0	0	−3.542	−1.523	4.723	0	0	0
P_{3x}	−8.333	0	0	0	0	0	8.333	0	0
P_{3y}	0	0	0	0	0	0	0	0	0
P_{3z}	0	0	0	0	0	0	0	0	0
P_{4x}	−1.8	2.4	0	−8.333	0	0	0	0	0
P_{4y}	2.4	−3.2	0	0	0	0	0	0	0
P_{4z}	0	0	0	0	0	0	0	0	0
P_{5x}	−1.8	0	2.4	−0.857	−1.142	1.142	0	0	0
P_{5y}	0	0	0	−1.142	−1.523	1.523	0	0	0
P_{5z}	2.4	0	−3.2	1.142	1.523	−1.523	0	0	0
P_{6x}	0	0	0	−1.8	0	2.4	0	0	0
P_{6y}	0	0	0	0	0	0	0	0	0
P_{6z}	0	0	0	2.4	0	−3.2	0	0	0

	u_4	v_4	w_4	u_5	v_5	w_5	u_6	v_6	w_6
P_{1x}	−1.8	2.4	0	−1.8	0	2.4	0	0	0
P_{1y}	2.4	−3.2	0	0	0	0	0	0	0
P_{1z}	0	0	0	2.4	0	−3.2	0	0	0
P_{2x}	−8.333	0	0	−0.857	−1.142	1.142	−1.8	0	2.4
P_{2y}	0	0	0	−1.142	−1.523	1.523	0	0	0
P_{2z}	0	0	0	1.142	1.523	−1.523	2.4	0	−3.2
P_{3x}	0	0	0	0	0	0	0	0	0
P_{3y}	0	0	0	0	0	0	0	0	0
P_{3z}	0	0	0	0	0	0	0	0	0
P_{4x}	10.133	−2.4	0	0	0	0	0	0	0
P_{4y}	−2.4	3.2	0	0	0	0	0	0	0
P_{4z}	0	0	0	0	0	0	0	0	0
P_{5x}	0	0	0	2.657	1.142	−3.542	0	0	0
P_{5y}	0	0	0	1.142	1.523	−1.523	0	0	0
P_{5z}	0	0	0	−3.542	−1.523	4.723	0	0	0
P_{6x}	0	0	0	0	0	0	1.8	0	−2.4
P_{6y}	0	0	0	0	0	0	0	0	0
P_{6z}	0	0	0	0	0	0	−2.4	0	3.2

rical values are shown in Tables 20 and 21.

$[K]$ matrix is then obtained by $[A][S]$, the result of which is shown in Table 22.

For loading condition 1, $\{P_\alpha\}$, $\{u_\beta\}$, $\{\Delta\}$ are

$$\{P_\alpha\}=\begin{Bmatrix}P_{1x}\\P_{1y}\\P_{1z}\\P_{2x}\\P_{2y}\\P_{2z}\end{Bmatrix}=\begin{Bmatrix}0\\0\\1\\0\\0\\0\end{Bmatrix},\quad \{u_\beta\}=\begin{Bmatrix}u_3\\v_3\\w_3\\u_4\\v_4\\w_4\\u_5\\v_5\\w_5\\u_6\\v_6\\w_6\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{Bmatrix},\quad \{\Delta\}=\begin{Bmatrix}0\\0\\0\\0\\0\\0\\0\end{Bmatrix},$$

and for loading condition 2,

$$\{P_\alpha\}=\begin{Bmatrix}P_{1x}\\P_{1y}\\P_{1z}\\P_{2x}\\P_{2y}\\P_{2z}\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\\0\\0\end{Bmatrix},\quad \{u_\beta\}=\{0\},\quad \{\Delta\}=\begin{Bmatrix}\Delta_{12}\\\Delta_{13}\\\Delta_{14}\\\Delta_{15}\\\Delta_{24}\\\Delta_{25}\\\Delta_{26}\end{Bmatrix}=\frac{AE}{l}\begin{Bmatrix}0.0156\\0.0117\\0.0195\\0.0195\\0.0117\\0.0249\\0.0195\end{Bmatrix}$$

Using these data and from Eqs. (16) and (17), we have $\{u\}$ and $\{F\}$ as shown in Tables 23 and 24.

Table 23. Joint displacements ($\times 10^{-4}$ in)

Joint	LC 1			LC 2		
	u	v	w	u	v	w
Joint 1	8.597	5.050	37.70	126.3	−116.7	−149.0
Joint 2	0	4.334	1.398	117.0	55.83	−188.3
Joint 3	0	0	0	0	0	0
Joint 4	0	0	0	0	0	0
Joint 5	0	0	0	0	0	0
Joint 6	0	0	0	0	0	0

Table 24. Member forces (lb)

Member	LC 1	LC 2
1 — 2	−44.73	1033.9
1 — 3	716.4	775.4
1 — 4	55.92	−1292.4
1 — 5	−1250	0
2 — 4	0	0
2 — 5	71.61	−1655.0
2 — 6	−55.92	1292.4

4.4 A Space Truss of Square Pyramid Type

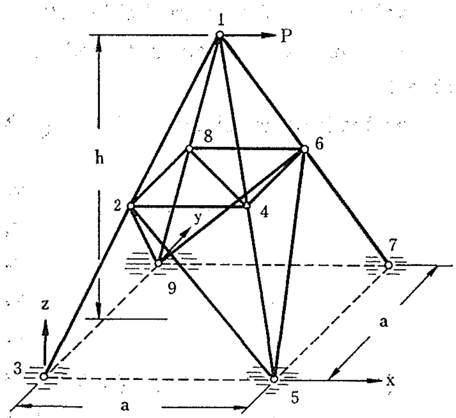


Fig. 9. Square pyramid

The statically indeterminate ball-jointed space truss shown in Fig. 9 has the form of square pyramid with dimensions $a = 10$ ft. and $h = 16$ ft., joints 2, 4, 6, 8 being mid points of the edges 3, 5, 7, 9, respectively. A horizontal force $P = 10$ kips, acting parallel to 3–5, is applied at joint 1 as shown. Bar 4–8 has a length error equal to -0.01 ft., that is, the bar is too short. All the bars have the same cross sectional area $A = 4\ in^2$. and the modulus of elasticity $E = 30(10)^6$ psi.

The procedure of analysis is quite the same as that of 4.2. Applying Eq. (18) to all the member of this truss, we have $[S]$ matrix and again applying Eq. (19), we have $[A]$ matrix, from which $[K]$ matrix is obtained. $\{P_\alpha\}$, $\{u_\beta\}$, $\{\varDelta\}$ are in this case

$$\{P_\alpha\} = \begin{Bmatrix} P_{x1} \\ P_{1y} \\ P_{1z} \\ P_{x2} \\ P_{2y} \\ P_{2z} \\ P_{x4} \\ P_{4y} \\ P_{4z} \\ P_{x6} \\ P_{6y} \\ P_{6z} \\ P_{x8} \\ P_{8y} \\ P_{8z} \end{Bmatrix} = \begin{Bmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \{u_\beta\} = \begin{Bmatrix} u_3 \\ v_3 \\ w_3 \\ u_5 \\ v_5 \\ w_5 \\ u_7 \\ v_7 \\ w_7 \\ u_9 \\ v_9 \\ w_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \{\varDelta\} = \begin{Bmatrix} \varDelta_{12} \\ \varDelta_{14} \\ \varDelta_{16} \\ \varDelta_{18} \\ \varDelta_{23} \\ \varDelta_{24} \\ \varDelta_{25} \\ \varDelta_{28} \\ \varDelta_{29} \\ \varDelta_{45} \\ \varDelta_{46} \\ \varDelta_{48} \\ \varDelta_{56} \\ \varDelta_{67} \\ \varDelta_{68} \\ \varDelta_{69} \\ \varDelta_{89} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -14.1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}.$$

Substituting these data into Eqs. (16) and (17), we have $\{u\}$ and $\{F\}$, which are shown in Tables 25 and 26.

Table 25. Joint displacements ($\times 10^{-2}$ in)

	u	v	w
Joint 1	5.353	0	−1 082
Joint 2	−2.469	−2.469	0.757
Joint 3	0	0	0
Joint 4	−3.116	4.454	−3.743
Joint 5	0	0	0
Joint 6	3.808	3.808	−0.079
Joint 7	0	0	0
Joint 8	4.454	−3.116	−2.070
Joint 9	0	0	0

Table 26. Member forces ($\times 10^3$ lb)

F_{1-2}	14.40	F_{4-5}	−14.40
F_{1-4}	−14.40	F_{4-6}	−12.93
F_{1-6}	−3.090	F_{4-8}	18.29
F_{1-8}	3.090	F_{5-6}	14.55
F_{2-3}	−8.224	F_{6-7}	−25.72
F_{2-4}	−12.93	F_{6-8}	−12.93
F_{2-5}	14.55	F_{6-9}	14.55
F_{2-8}	−12.93	F_{8-9}	3.090
F_{2-9}	14.55		

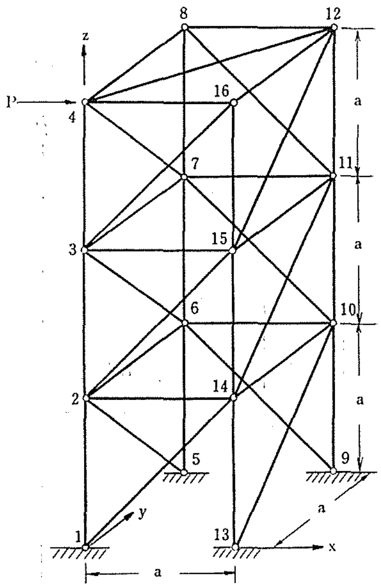


Fig. 10. Tower

4.5 Three Story Tower

A three story space truss of tower type is shown in Fig. 10, to which a horizontal force $P = 1$ ton is applied at joint 1 as shown in the figure.

This truss has as many as 16 joints and 39 bars, but the procedure of analysis is the same as those of 4.3 and 4.4. In this example just a joint load is applied, so we use following equations:

$$\{u_\alpha\} = [K_{\alpha\alpha}{}^{-1}]\{P_\alpha\},$$

$$\{F\} = [S]\{u\}.$$

$\{u\}$ and $\{F\}$ obtained are shown in Tables 27 and 28.

Table 27. Joint displacements (10^{-1}mm)

	u	v	w		u	v	w
Joint 1	0	0	0	Joint 9	0	0	0
Joint 2	4.284	−0.047	0.710	Joint 10	2.219	1.951	−1.195
Joint 3	10.59	−1.197	0.978	Joint 11	6.225	3.100	−1.879
Joint 4	17.99	−3.160	0.758	Joint 12	11.04	3 159	−2.099
Joint 5	0	0	0	Joint 13	0	0	0
Joint 6	1.951	−0.291	1.195	Joint 14	3.598	2.195	−1.662
Joint 7	5.981	−1.417	1.879	Joint 15	9.880	3.321	−2.883
Joint 8	10.82	−3.160	2.099	Joint 16	17.25	3.159	−3.615

Table 28. Member forces (ton)

F_{1-2}	0.745	F_{3-7}	−0.231	F_{6-9}	−0.397	F_{10-14}	−0.256
F_{1-14}	1.017	F_{3-11}	−0.035	F_{6-10}	0.281	F_{11-12}	−0.231
F_{2-3}	0.282	F_{3-15}	−0.744	F_{7-8}	0.231	F_{11-14}	0.362
F_{2-5}	0.397	F_{3-16}	1.087	F_{7-10}	−0.362	F_{11-15}	−0.231
F_{2-6}	−0.256	F_{4-7}	0.327	F_{7-11}	0.256	F_{12-15}	0.327
F_{2-10}	−0.035	F_{4-8}	0	F_{8-11}	−0.326	F_{12-16}	0
F_{2-14}	−0.719	F_{4-12}	−0.327	F_{8-12}	0.231	F_{13-14}	−1.745
F_{2-15}	1.052	F_{4-16}	−0.769	F_{9-10}	−1.254	F_{14-15}	−1.282
F_{3-4}	−0.231	F_{5-6}	1.255	F_{10-11}	−0.718	F_{15-16}	−0.769
F_{3-6}	0.362	F_{6-7}	0.718	F_{10-13}	0.397		

5. A Problem about Stability

In general for a space truss to be stable, it is necessary that the constraints at the supports in at least six or more directions exist.

The truss shown in Fig. 11 has only five constraints under the loading condition given. From mathematical point of view, $[K_{\alpha\alpha}]$ matrix is singular for an unstable structure like this and $[K_{\alpha\alpha}^{-1}]$ cannot be obtained and hence joint displacements cannot be found, from which we can know the structure is unstable.

In fact in numerical calculation, joint displacements were obtained, but their order of magnitude were $10^4 \sim 10^5$ times as large as the dimensions of the structure and we could verify this structure unstable.

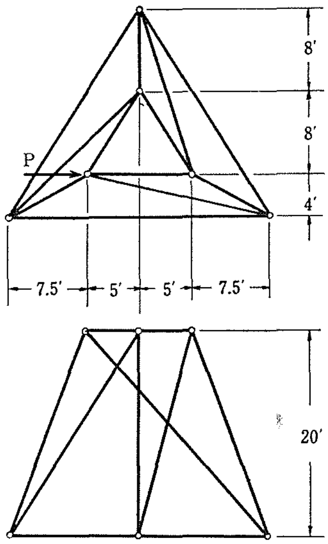


Fig. 11.

6. Conclusion

For the purpose of truss analysis, many kinds of matrix methods have been introduced and our method is one of them. Here we used $\{F\}=[S]\{u\}$: the relation between member forces and joint displacements, and we are sure that using this geometrical condition enabled us to derive this theory so briefly. Applying the equilibrium conditions of joint method to this, we have joint displacements $\{u\}$, member forces $\{F\}$ and reactions at the supports $\{P_\beta\}$ successively. As illustrated in the examples, various conditions can be taken into account in each equation and treated systematically. A computation program based on this theory was made by authors and all numerical calculations in the examples above were done by using this program.

References

1) T. MIYAIRI, "Analysis of Truss by Angle Variation Method," Proceedings of the JSME, Tokai District, 1967.
2) H. MARTIN, "Introduction to Matrix Method of Structural Analysis," 1965, McGraw-Hill Book Co. (Translated into Japanese by M. YOSHIKI).

Appendix

(I) List of Symbols

A = cross sectional area

$[A]$ = statics matrix

E = Young's modulus

F = member force

$\{F\}$ = column of member forces

$[K]$ = stiffness matrix

$\left.\begin{matrix}[K_{\alpha\alpha}]\\ [K_{\alpha\beta}]\\ [K_{\beta\alpha}]\\ [K_{\beta\beta}]\end{matrix}\right\}$ = partitioned stiffness matricies

l = member length

$\{P\}$ = column of external joint loads

$\{P_\alpha\}$ = column of external joint loads at the free joints

$\{P_\beta\}$ = reactions at the supports

$[S]$ = member stiffness matrix

$\{u\}$ = column of joint displacements

$\{u_\alpha\}$ = column of unknown joint displacements

$\{u_\beta\}$ = column of known joint displacements

u = joint displacement in x direction

v = joint displacement in y direction

w = joint displacement in z direction

$\{\ \}$ = column matrix

$[\ ]$ = matrix

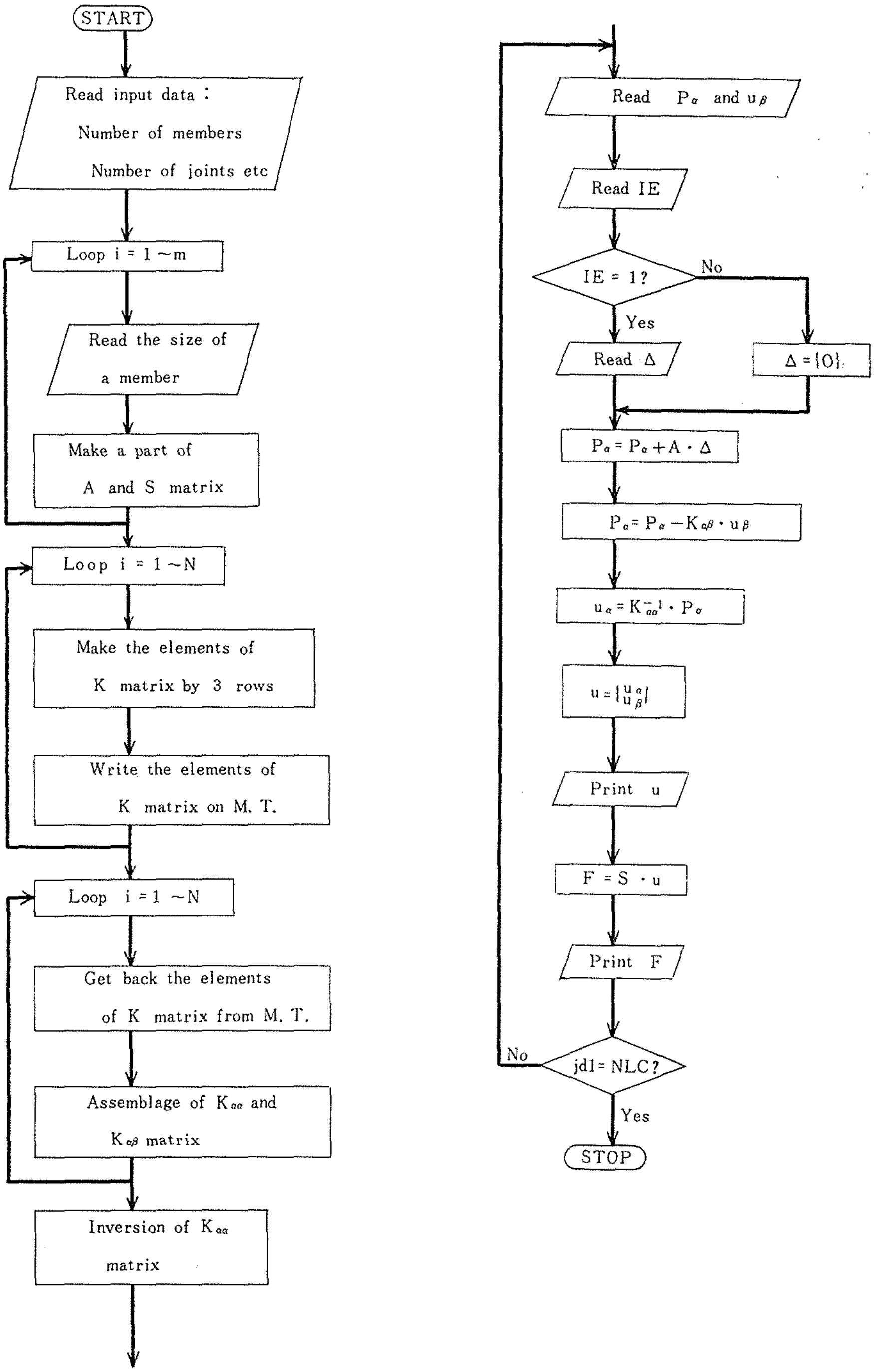
(II) Computer Program

A computer program of truss analysis based on this theory will be shown here. First, notations used in this program and then flow chart and program written in ALGOL will be shown.

(II–1) Notations

N	…… Total number of joints
m	…… Total number of members
NCT	…… Total number of prescribed joint displacements
F	…… Total number of free joint displacements
n	…… Total number of joint displacements $= 3^*N = F + NCT$
$C[i]$	…… Joint number and its component of prescribed joint displacement
$B[i]$	…… Joint numbers with which i-th member is associated
$D[1, i]$	…… One of $B[i]$
$D[2, i]$	…… The other of $B[i]$
ko	…… AE/l (used for other notations later)
E	…… Young's modulus
L, LI	…… Member length and its reciprocal
$s1$, $s2$, $s3$	…… Direction cosines
H	…… A matrix (only non-zero elements are stored)
S	…… S matrix (only non-zero elements are stored)
K	…… $K_{\alpha\alpha}$ matrix
KI	…… A part of K matrix
KAB	…… $K_{\alpha\beta}$ matrix
IE	…… $IE = 1$, problem with initial strain, heat load etc $IE = 0$, problem without initial strain, heat load etc
$i1$, $i2$, $i3$	…… x, y, z coordinates of one joint of i-th member
$j1$, $j2$, $j3$	…… x, y, z coordinates of the other joint of i-th member
PA	…… P_α matrix
ua	…… u_α matrix
ub	…… u_β matrix
U	…… $u = \begin{Bmatrix} u_\alpha \\ u_\beta \end{Bmatrix}$
$jd1$	…… Number corresponding to x component at the j-th joint (Total number of loading conditions done with after some repeated loops, later)
$jd2$	…… Number corresponding to y component at the j-th joint
$jd3$	…… Number corresponding to z component at the j-th joint
del	…… Δ matrix
FO	…… F matrix

(II-2) Flow Chart



(II-3) Program (Space Truss Analysis)

```
begin integer n, m, NCT, F, NLC, N;
      Readinteger(N); Readinteger(m); Readinteger(NLC); Readinteger(NCT);
      n := N*3; F := n-NCT;
begin procedure INVERT(M, A);
 value M; array A; integer M;
 begin integer i, j, k; real pivot, temp;
 for k := 1 step 1 until M do
  begin pivot := 1.0/A[k, k];
   for j := 1 step 1 until M do A[k, j] := A[k, j]* pivot;
   for i := 1 step 1 until M do
    begin temp := A[i, k];
     if i ≠ k and temp ≠ 0.0 then
      for j := 1 step 1 until M do
       A[i, j] := A[i, j] - A[k, j]* temp;
      A[i, k] := - pivot* temp
    end i;
  A[k, k] := pivot
  end k
 end INVERT;

integer i, j, l, q, r, k, v, w, J, jd1, jd2, jd3, IE;
real AR, L, LI, s1, s2, s3, i1, i2, i3, j1, j2, j3, ko, E;
array KI[1 : 3, 1 : n], H[1 : 3, 1 : m], S[1 : m, 1 : 3], K[1 : F, 1 : F],
      KAB[1 : F, 1 : NCT], FO, del[1 : m], PA[1 : F], ua[1 : F], ub[1 : NCT],
      U[1 : n];
integer array B[1 : m], D[1 : 2, 1 : m], C[1 : NCT];
for i := 1 step 1 until NCT do
begin Readinteger(C[i]); C[i] := (fix(C[i]/100) - 1)*3 + C[i] - fix(C[i]/100)*100
end; Readreal(E);

for i := 1 step 1 until m do
begin Readinteger(B[i]); Readreal(AR); Readreal(i1); Readreal(i2);
 Readreal(i3); Readreal(j1); Readreal(j2); Readreal(j3); L := sqrt((j1 - i1)↑2
 + (j2 - i2)↑2 + (j3 - i3)↑2); LI := 1.0/L; ko := AR*E*LI; s1 := (j1 - i1)*LI;
 s2 := (j2 - i2)*LI; s3 := (j3 - i3)*LI; D[1, i] := fix(B[i]/100); D[2, i] := B[i]
 - D[1, i]*100; H[1, i] := - s1; H[2, i] := - s2; H[3, i] := - s3; S[i, 1]
 := - s1*ko; S[i, 2] := - s2*ko; S[i, 3] := - s3*ko;
```

```
end; REWIND(2);

for i := 1 step 1 until N do
 begin for j := 1 step 1 until N do
  begin jd3 := j*3; jd2 := jd3 - 1; jd1 := jd2 - 1;
   for w := 1 step 1 until 3 do
    begin KI[w, jd1] := KI[w, jd2] := KI[w, jd3] := 0.0 end;
   for v := 1 step 1 until m do
    begin if i = j and (D[1, v] = i or D[2, v] = i) then
     for w := 1 step 1 until 3 do
      begin   KI[w, jd3] := KI[w, jd3] + H[w, v]*S[v, 3];
              KI[w, jd2] := KI[w, jd2] + H[w, v]*S[v, 2];
              KI[w, jd1] := KI[w, jd1] + H[w, v]*S[v, 1]
      end else if D[1, v] = i and D[2, v] = j or D[1, v] = j and D[2, v] = i then
     for w := 1 step 1 until 3 do
      begin   KI[w, jd3] := KI[w, jd3] - H[w, v]*S[v, 3];
              KI[w, jd2] := KI[w, jd2] - H[w, v]*S[v, 2];
              KI[w, jd1] := KI[w, jd1] - H[w, v]*S[v, 1]
      end
    end
   end; PUTARRAY(2, KI)
 end;

REWIND(2); w := 0;
for i := 1 step 1 until N do
 begin GETARRAY(2, KI);
  for k := 1 step 1 until 3 do
   begin v := 3*i - 3 + k; for J := 1 step 1 until NCT do if C[J] = v then go
    to END1; w := w + 1; r := 0;
    for j := 1 step 1 until n do
     begin for q := 1 step 1 until NCT do if C[q] = j then
      begin KAB[w, q] := KI[k, j]; go to END2 end;
      r := r + 1; K[w, r] := KI[k, j];
     END2 : end;
   END1 : end
 end;

INVERT(F, K);
```

```
jd1 : = 0;
CALC : CRLF(8); Space(20); Printstring('LOADING CONDITION'); jd1 : = jd1
  +1; Printx(jd1, 1); CRLF(3); Space(10); Printstring('DISPLACEMENTS');
  CRLF; Readarray(PA); Readarray(ub);

Readinteger(IE); if IE = 1 then
begin for i : = 1 step 1 until m do
 begin Readreal(del[i]); if del[i] ≠ 0.0 then
  begin Readreal(AR); Readreal(i1); Readreal(i2); Readreal(i3); Readreal(j1);
   Readreal(j2); Readreal(j3); L : = sqrt((j1 - i1)↑2 + (j2 - i2)↑2 + (j3 - i3)↑2);
   del[i] : = del[i]/L*AR*E
  end
 end; r : = 0;
 for i : = 1 step 1 until N do
  for k : = 1 step 1 until 3 do
   begin l : = (i - 1)*3 + k;
    for J : = 1 step 1 until NCT do if C[J] = l then go to END3; ko : = 0.0;
    for j : = 1 step 1 until m do if D[1, j] = i then ko : = ko + H[k, j]*del[j]
     else if D[2, j] = i then ko : = ko - H[k, j]*del[j];
    r : = r+1; PA[r] : = PA[r] + ko;
   END3 : end
end else for i : = 1 step 1 until m do del[i] : = 0.0;

for i : = 1 step 1 until F do
 begin ko : = 0.0; for j : = 1 step 1 until NCT do ko : = ko + KAB[i, j]*ub[j];
  PA[i] : = PA[i] - ko
 end;

for i : = 1 step 1 until F do
 begin ko : = 0.0;
  for j : = 1 step 1 until F do ko : = ko + K[i, j]*PA[j]; ua[i] : = ko
 end; q : = 0;
for i : = 1 step 1 until n do
 begin for J : = 1 step 1 until NCT do if i = C[J] then
  begin U[i] : = ub[J]; go to END4 end;
  q : = q + 1; U[i] : = ua[q];
 END4 : end;
```

```
for i : = 1 step 1 until n do
 begin if fix((i - 1)/3) - (i - 1)/3 = 0.0 then
  begin Printstring('JOINT'); Printx(entier((i - 1)/3) + 1, 2);
   Space(5); Printstring('X='); Printreal(U[i], 7)
  end else if fix((i - 2)/3) - (i - 2)/3 = 0.0 then
  begin Space(5); Printstring('Y ='); Printreal(U[i], 7); end else
  begin Space(5); Printstring('Z ='); Printreal(U[i], 7); CRLF end
 end;

CRLF(4); Space(10); Printstring('FORCES'); CRLF;
for i : = 1 step 1 until m do
 begin FO[i] : = 0.0;
  for j : = 1 step 1 until 6 do
  FO[i] : = FO[i] + (if j > 3 then -S[i, j - 3]*U[D[2, i]*3 - 6 + j]
                          else S[i, j]*U[D[1, i]*3 - 3 + j]);
  Printstring('MEMBER'); Printinteger(D[1, i]); Printstring('-');
  Printx(D[2, i], 2); Printreal(FO[i] - del[i], 7); CRLF
 end;
 if jd1 ≠ NLC then go to CALC
end
end;
```