

Interpolation of SUSY quantum mechanics

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Abstract

Interpolation of two adjacent Hamiltonians in SUSY quantum mechanics $\mathcal{H}_s \stackrel{\text{def}}{=} (1-s)\mathcal{A}^\dagger\mathcal{A} + s\mathcal{A}\mathcal{A}^\dagger$, $0 \leq s \leq 1$ is discussed together with related operators. For a wide variety of shape-invariant degree one quantum mechanics and their ‘discrete’ counterparts, the interpolation Hamiltonian is also shape-invariant, that is it takes the same form as the original Hamiltonian with shifted coupling constant(s).

1 Introduction

The factorisation method [1] or the so-called super-symmetric (SUSY) quantum mechanics [2, 3] is a well-established tool for investigating degree one quantum mechanics including their ‘discrete’ counterparts [4]. In its essence the SUSY quantum mechanics asserts that a factorised Hamiltonian $\mathcal{H} \stackrel{\text{def}}{=} \mathcal{A}^\dagger\mathcal{A}$ and its reversed order (SUSY-partner) Hamiltonian $\mathcal{H}_r \stackrel{\text{def}}{=} \mathcal{A}\mathcal{A}^\dagger$ are *iso-spectral* except for the ground state. Let us denote by ϕ_n the eigenfunction of \mathcal{H} :

$$\mathcal{H}\phi_n = \mathcal{E}_n\phi_n, \quad n = 0, 1, 2, \dots \quad \mathcal{E}_0 < \mathcal{E}_1 < \dots \quad (1.1)$$

Then $\tilde{\phi}_n \stackrel{\text{def}}{=} \mathcal{A}\phi_n$ is also an eigenfunction of \mathcal{H}_r with the same eigenvalue

$$\mathcal{H}_r\tilde{\phi}_n = \mathcal{E}_n\tilde{\phi}_n, \quad n = 1, 2, \dots, \quad (1.2)$$

except for ϕ_0 , which is annihilated by \mathcal{A}

$$\mathcal{A}\phi_0 = 0 \quad (\Rightarrow \mathcal{H}\phi_0 = 0, \quad \mathcal{E}_0 = 0). \quad (1.3)$$

In various contexts of quantum physics, one encounters quite often a situation [5] in which an interpolation of the two super-symmetric partner Hamiltonians

$$\mathcal{H}_s \stackrel{\text{def}}{=} (1-s)\mathcal{A}^\dagger\mathcal{A} + s\mathcal{A}\mathcal{A}^\dagger, \quad 0 \leq s \leq 1, \quad (1.4)$$

or an operator closely related with it, plays an important role. For a wide class of *shape-invariant* Hamiltonians [4, 6, 7], we show in this paper that the interpolation also *retains shape-invariance*. That is the interpolating Hamiltonian (1.4) has the same form as the original Hamiltonian with shifted coupling constant(s) and a shifted ground state energy.

This paper is organised as follows. In section two the basic facts and notation of shape-invariant quantum mechanics are recapitulated. In section three the assertion of the shape-invariant interpolation is demonstrated for various examples of shape-invariant potentials in ordinary quantum mechanical systems [3, 6, 7]. They have the classical orthogonal polynomials, the Hermite, Laguerre and Jacobi polynomials as a part of the eigenfunctions. In section four, we demonstrate the assertion, in a slightly different form, for various ‘discrete’ quantum mechanical systems [4, 7] which have the Askey-Wilson, Wilson, continuous dual Hahn, continuous Hahn and Meixner-Pollaczek polynomials [8, 9] as a part of the eigenfunctions. These polynomials belong to the Askey scheme of hypergeometric orthogonal polynomials and they are the deformations of the Jacobi, Laguerre and Hermite polynomials. In section five we apply the method presented in the previous section to the ordinary quantum mechanical systems. This gives another type of interpolation of the ordinary SUSY quantum mechanics. The final section is for a summary and comments.

2 Shape-Invariant Quantum Mechanics

Here we discuss only the degree one quantum mechanics. A shape-invariant [6] quantum mechanical system consists of a series of isospectral (and factorised) Hamiltonians $\{\mathcal{H}(\boldsymbol{\lambda})\}$ parametrised by (a set of) parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$:

$$\mathcal{H}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}(\boldsymbol{\lambda})^\dagger \mathcal{A}(\boldsymbol{\lambda}), \quad \mathcal{H}(\boldsymbol{\lambda})\phi_n(x; \boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})\phi_n(x; \boldsymbol{\lambda}), \quad n = 0, 1, 2, \dots \quad (2.1)$$

Shape invariance simply means that the reversed order Hamiltonian

$$\mathcal{H}_r(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger, \quad (2.2)$$

takes the same form (shape) as the original Hamiltonian with ‘shifted’ parameters:

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger = \mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta})^\dagger \mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}), \quad (2.3)$$

where $\boldsymbol{\delta}$ is the shift of the parameter and $\mathcal{E}_1(\boldsymbol{\lambda})$ is the increase of the ground state energy. Note that the entire energy spectrum is determined by shape-invariance, since $\mathcal{E}_{n+1}(\boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda}) + \mathcal{E}_1(\boldsymbol{\lambda} + n\boldsymbol{\delta})$, which implies $\{\mathcal{E}_n\}$ are determined in terms of \mathcal{E}_1 , $\mathcal{E}_n(\boldsymbol{\lambda}) = \sum_{k=0}^{n-1} \mathcal{E}_1(\boldsymbol{\lambda} + k\boldsymbol{\delta})$ [4, 7].

With this notation, our assertion goes as follows:

$$\mathcal{H}_s \stackrel{\text{def}}{=} (1-s)\mathcal{H} + s\mathcal{H}_r = \mathcal{H}(\boldsymbol{\lambda}') + \Delta\mathcal{E}(\boldsymbol{\lambda}, s), \quad (2.4)$$

in which the new coupling constants $\boldsymbol{\lambda}'$ satisfy the boundary condition

$$\text{for } s = 0 : \boldsymbol{\lambda}' = \boldsymbol{\lambda}, \quad \text{and for } s = 1 : \boldsymbol{\lambda}' = \boldsymbol{\lambda} + \boldsymbol{\delta}. \quad (2.5)$$

The constant part $\Delta\mathcal{E}(\boldsymbol{\lambda}, s)$ is the shift of the ground state energy. It also satisfies the boundary condition

$$\text{for } s = 0 : \Delta\mathcal{E}(\boldsymbol{\lambda}, 0) = 0, \quad \text{and for } s = 1 : \Delta\mathcal{E}(\boldsymbol{\lambda}, 1) = \mathcal{E}_1(\boldsymbol{\lambda}). \quad (2.6)$$

Here $\mathcal{E}_1(\boldsymbol{\lambda})$ is the last term in (2.3).

If our assertion (2.4) holds, the eigenfunctions and eigenvalues of \mathcal{H}_s are given by

$$\mathcal{H}_s \phi_n(x; \boldsymbol{\lambda}') = (\mathcal{E}_n(\boldsymbol{\lambda}') + \Delta\mathcal{E}(\boldsymbol{\lambda}, s)) \phi_n(x; \boldsymbol{\lambda}'), \quad n = 0, 1, 2, \dots \quad (2.7)$$

Note that $\boldsymbol{\lambda}'$ depend on s .

3 Ordinary Quantum Mechanics

The ordinary quantum mechanics is quite conveniently described by the *prepotential* $W(x; \boldsymbol{\lambda}) \in \mathbb{R}$ which parametrises the ground state wavefunction

$$\phi_0(x; \boldsymbol{\lambda}) \propto e^{W(x; \boldsymbol{\lambda})}. \quad (3.1)$$

This is possible because the ground state wavefunction does not have nodes and can be chosen real. Then the factorised Hamiltonian reads

$$\mathcal{H} = \mathcal{A}(\boldsymbol{\lambda})^\dagger \mathcal{A}(\boldsymbol{\lambda}) = -\frac{d^2}{dx^2} + \left(\frac{dW(x; \boldsymbol{\lambda})}{dx} \right)^2 + \frac{d^2W(x; \boldsymbol{\lambda})}{dx^2}, \quad (3.2)$$

$$\mathcal{A}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -\frac{d}{dx} + \frac{dW(x; \boldsymbol{\lambda})}{dx}, \quad \mathcal{A}(\boldsymbol{\lambda})^\dagger = \frac{d}{dx} + \frac{dW(x; \boldsymbol{\lambda})}{dx}. \quad (3.3)$$

Here we have adopted the unit system $m = \hbar = 1$ and the overall normalisation $\mathcal{H} = -\frac{d^2}{dx^2} + \dots$. Obviously the ground state is annihilated by the operator \mathcal{A}

$$\mathcal{A}(\boldsymbol{\lambda})\phi_0(x; \boldsymbol{\lambda}) = 0 \quad (\Rightarrow \mathcal{H}\phi_0(x; \boldsymbol{\lambda}) = 0, \quad \mathcal{E}_0(\boldsymbol{\lambda}) = 0). \quad (3.4)$$

We assume the shape-invariance (2.3). In this form the interpolating Hamiltonian \mathcal{H}_s reads simply

$$\mathcal{H}_s = -\frac{d^2}{dx^2} + \left(\frac{dW(x; \boldsymbol{\lambda})}{dx} \right)^2 + (1 - 2s) \frac{d^2W(x; \boldsymbol{\lambda})}{dx^2} = \mathcal{H} - 2s \frac{d^2W(x; \boldsymbol{\lambda})}{dx^2}. \quad (3.5)$$

Let us consider the known typical examples of shape-invariant potentials. All of them satisfy our assertion (2.4).

3.1 harmonic oscillator: a trivial example

The harmonic oscillator is a trivial example with $W(x; \omega) = -\omega x^2/2$. This simply leads to the same Hamiltonian with the shifted ground state energy

$$\mathcal{H}_s = \mathcal{H} + \Delta\mathcal{E}(\omega, s), \quad \Delta\mathcal{E}(\omega, s) = 2s\omega. \quad (3.6)$$

3.2 harmonic oscillator with a centrifugal barrier potential

This is the one-body case of the Calogero model [10] with

$$\boldsymbol{\lambda} = (\omega, g), \quad \boldsymbol{\delta} = (0, 1), \quad \mathcal{E}_n(\omega, g) = 4n, \quad n = 0, 1, 2, \dots, \quad (3.7)$$

$$W(x; \omega, g) = -\omega x^2/2 + g \log x, \quad \omega > 0, \quad g \geq 1/2, \quad x > 0, \quad (3.8)$$

$$\mathcal{A}(\omega, g) = -\frac{d}{dx} - \omega x + \frac{g}{x}, \quad \mathcal{A}(\omega, g)^\dagger = \frac{d}{dx} - \omega x + \frac{g}{x}. \quad (3.9)$$

The original and the interpolating Hamiltonians read

$$\mathcal{H}(\omega, g) = -\frac{d^2}{dx^2} + \omega^2 x^2 + \frac{g(g-1)}{x^2} - \omega(1+2g), \quad (3.10)$$

$$\mathcal{H}_s = -\frac{d^2}{dx^2} + \omega^2 x^2 + \frac{g(g+2s-1)}{x^2} - \omega(1+2g-2s). \quad (3.11)$$

It is obvious that the interpolating Hamiltonian \mathcal{H}_s has the same form as the old (3.10) with the coupling constant g replaced by g' :

$$g' = (1 + \sqrt{1 + 4g(g + 2s - 1)})/2 \quad (3.12)$$

together with the shift of the ground state energy $\Delta\mathcal{E}(\omega, g, s) = 2\omega(s + g' - g)$.

3.3 $1/\sin^2 x$ potential or symmetric Pöschl-Teller potential

This is the one-body case of the well-known Sutherland model [11]. The prepotential, the Hamiltonian and other data are:

$$\lambda = g, \quad \delta = 1, \quad \mathcal{E}_n(g) = n(n + 2g), \quad n = 0, 1, 2, \dots, \quad (3.13)$$

$$W(x; g) = g \log \sin x, \quad g \geq 1/2, \quad 0 < x < \pi, \quad (3.14)$$

$$\mathcal{A}(g) = -\frac{d}{dx} + g \cot x, \quad \mathcal{A}(g)^\dagger = \frac{d}{dx} + g \cot x, \quad (3.15)$$

$$\mathcal{H} = -\frac{d^2}{dx^2} + \frac{g(g-1)}{\sin^2 x} - g^2. \quad (3.16)$$

The interpolating Hamiltonian reads simply

$$\mathcal{H}_s = -\frac{d^2}{dx^2} + \frac{g(g + 2s - 1)}{\sin^2 x} - g^2. \quad (3.17)$$

It is obvious that the interpolating Hamiltonian \mathcal{H}_s has the same form as the old (3.16) with the coupling constant g replaced by g' as in the previous case (3.12) together with the shift of the ground state energy $\Delta\mathcal{E}(g, s) = g'^2 - g^2$.

3.4 soliton potential or the symmetric Rosen-Morse potential

As is well-known $-g(g+1)/\cosh^2 x$ potential is *reflectionless* for integer coupling constant g , corresponding to the KdV soliton. It has a finite number $1 + [g]'$ (the greatest integer not equal or exceeding g) of bound states:

$$\lambda = g, \quad \delta = -1, \quad \mathcal{E}_n(g) = n(2g - n), \quad n = 0, 1, \dots, [g]', \quad (3.18)$$

$$W(x; g) = -g \log \cosh x, \quad g > 0, \quad -\infty < x < \infty, \quad (3.19)$$

$$\mathcal{A}(g) = -\frac{d}{dx} - g \tanh x, \quad \mathcal{A}(g)^\dagger = \frac{d}{dx} - g \tanh x, \quad (3.20)$$

$$\mathcal{H} = -\frac{d^2}{dx^2} - \frac{g(g+1)}{\cosh^2 x} + g^2. \quad (3.21)$$

The one-parameter Hamiltonian reads simply

$$\mathcal{H}_s = -\frac{d^2}{dx^2} - \frac{g(g-2s+1)}{\cosh^2 x} + g^2. \quad (3.22)$$

It is obvious that the new Hamiltonian \mathcal{H}_s has the same form as the old (3.21) with the coupling constant g replaced by g'' (for $g > 2s - 1$ case):

$$g'' = (-1 + \sqrt{1 + 4g(g+1-2s)})/2 \quad (3.23)$$

together with the shift of the ground state energy $\Delta\mathcal{E}(g, s) = g^2 - g''^2$.

3.5 Morse potential

The prepotential, the Hamiltonian and other data of the Morse potential read

$$\boldsymbol{\lambda} = (g, \mu), \quad \boldsymbol{\delta} = (-1, 0), \quad \mathcal{E}_n(g, \mu) = n(2g - n), \quad n = 0, 1, \dots, [g]', \quad (3.24)$$

$$W(x; g, \mu) = gx - \mu e^x, \quad g, \mu > 0, \quad -\infty < x < \infty, \quad (3.25)$$

$$\mathcal{A}(g, \mu) = -\frac{d}{dx} + g - \mu e^x, \quad \mathcal{A}(g, \mu)^\dagger = \frac{d}{dx} + g - \mu e^x, \quad (3.26)$$

$$\mathcal{H} = -\frac{d^2}{dx^2} + \mu^2 e^{2x} - \mu(2g+1)e^x + g^2. \quad (3.27)$$

The interpolating Hamiltonian reads simply

$$\mathcal{H}_s = -\frac{d^2}{dx^2} + \mu^2 e^{2x} - \mu(2g - 2s + 1)e^x + g^2. \quad (3.28)$$

This has the same form as the old one with the coupling constant g replaced by g' (for $g > s$ case):

$$g' = g - s. \quad (3.29)$$

The shift of the ground state energy is $\Delta\mathcal{E}(g, \mu, s) = g^2 - g'^2$.

3.6 hyperbolic symmetric top

This system is obtained by the interchange $\sinh \leftrightarrow \cosh$ from the hyperbolic analog of the symmetric top problem:

$$\boldsymbol{\lambda} = (g, \mu), \quad \boldsymbol{\delta} = (-1, 0), \quad \mathcal{E}_n(g, \mu) = n(2g - n), \quad n = 0, 1, \dots, [g]', \quad (3.30)$$

$$W(x; g, \mu) = -g \log \cosh x - \mu \arctan \sinh x, \quad g, \mu > 0, \quad -\infty < x < \infty, \quad (3.31)$$

$$\mathcal{A}(g, \mu) = -\frac{d}{dx} - g \tanh x - \frac{\mu}{\cosh x}, \quad \mathcal{A}(g, \mu)^\dagger = \frac{d}{dx} - g \tanh x - \frac{\mu}{\cosh x}, \quad (3.32)$$

$$\mathcal{H} = -\frac{d^2}{dx^2} + \frac{\mu^2 - g(g+1) + \mu(2g+1) \sinh x}{\cosh^2 x} + g^2. \quad (3.33)$$

The interpolating Hamiltonian reads simply

$$\mathcal{H}_s = -\frac{d^2}{dx^2} + \frac{\mu^2 - g(g - 2s + 1) + \mu(2g - 2s + 1) \sinh x}{\cosh^2 x} + g^2. \quad (3.34)$$

This has the same form as the old one with the coupling constants (g, μ) replaced by (g', μ') , which are determined by

$$\mu'^2 - g'(g' + 1) = \mu^2 - g(g - 2s + 1), \quad \mu'(2g' + 1) = \mu(2g - 2s + 1). \quad (3.35)$$

The shift of the ground state energy is $\Delta\mathcal{E}(g, \mu, s) = g^2 - g'^2$. For the boundary values $s = 0$ and $s = 1$, $(g', \mu') = (g - s, \mu)$ give a solution of (3.35).

3.7 other examples

There are several more examples of shape-invariant quantum mechanics [3]. But the situation is the same as either the symmetric Pöschl-Teller or the symmetric Rosen-Morse potential case, with the change of the coupling constant (3.12), (3.23). We present some of them briefly.

3.7.1 Pöschl-Teller potential

This is the one-body case of the Sutherland model of BC type:

$$\boldsymbol{\lambda} = (g, h), \quad \boldsymbol{\delta} = (1, 1), \quad \mathcal{E}_n(g, h) = 4n(n + g + h), \quad n = 0, 1, 2, \dots, \quad (3.36)$$

$$W(x; g, h) = g \log \sin x + h \log \cos x, \quad g, h \geq 1/2, \quad 0 < x < \pi/2, \quad (3.37)$$

$$\mathcal{H} = -\frac{d^2}{dx^2} + \frac{g(g-1)}{\sin^2 x} + \frac{h(h-1)}{\cos^2 x} - (g+h)^2, \quad (3.38)$$

$$\mathcal{H}_s = -\frac{d^2}{dx^2} + \frac{g(g+2s-1)}{\sin^2 x} + \frac{h(h+2s-1)}{\cos^2 x} - (g+h)^2. \quad (3.39)$$

The coupling constant g is replaced by g' as (3.12) and h by h' similarly. The shift of the ground state energy is $\Delta\mathcal{E}(g, h, s) = (g' + h')^2 - (g + h)^2$.

3.7.2 Rosen-Morse potential

The prepotential, the Hamiltonian, etc. are:

$$\boldsymbol{\lambda} = (g, \mu), \quad \boldsymbol{\delta} = (-1, 0), \quad (3.40)$$

$$\mathcal{E}_n(g, \mu) = \frac{\mu^2}{g^2} + g^2 - \left(\frac{\mu^2}{(g-n)^2} + (g-n)^2 \right), \quad n = 0, 1, \dots, [g - \sqrt{|\mu|}]', \quad (3.41)$$

$$W(x; g, \mu) = -\frac{\mu}{g}x - g \log \cosh x, \quad g > 0, \quad -g^2 < \mu < g^2, \quad -\infty < x < \infty, \quad (3.42)$$

$$\mathcal{H} = -\frac{d^2}{dx^2} + 2\mu \tanh x - \frac{g(g+1)}{\cosh^2 x} + \frac{\mu^2}{g^2} + g^2, \quad (3.43)$$

$$\mathcal{H}_s = -\frac{d^2}{dx^2} + 2\mu \tanh x - \frac{g(g-2s+1)}{\cosh^2 x} + \frac{\mu^2}{g^2} + g^2. \quad (3.44)$$

The coupling constant g is replaced by g'' as (3.23) (for $g > 2s - 1$ case) but μ is unchanged. The shift of the ground state energy is $\Delta\mathcal{E}(g, \mu, s) = \mu^2/g^2 + g^2 - (\mu^2/g''^2 + g''^2)$.

3.7.3 Coulomb potential with the centrifugal barrier

The prepotential, the Hamiltonian, etc. are:

$$\boldsymbol{\lambda} = (g, \mu), \quad \boldsymbol{\delta} = (1, 0), \quad \mathcal{E}_n(g, \mu) = \frac{\mu^2}{g^2} - \frac{\mu^2}{(g+n)^2}, \quad n = 0, 1, 2, \dots, \quad (3.45)$$

$$W(x; g, \mu) = -\frac{\mu}{g}x + g \log x, \quad g \geq \frac{1}{2}, \quad \mu > 0, \quad x > 0, \quad (3.46)$$

$$\mathcal{H} = -\frac{d^2}{dx^2} - \frac{2\mu}{x} + \frac{g(g-1)}{x^2} + \frac{\mu^2}{g^2}, \quad (3.47)$$

$$\mathcal{H}_s = -\frac{d^2}{dx^2} - \frac{2\mu}{x} + \frac{g(g+2s-1)}{x^2} + \frac{\mu^2}{g^2}. \quad (3.48)$$

The coupling constant g is replaced by g' as (3.12) but μ is unchanged. The shift of the ground state energy is $\Delta\mathcal{E}(g, \mu, s) = \mu^2/g^2 - \mu^2/g'^2$.

4 Discrete Quantum Mechanics

The corresponding result for discrete quantum mechanics takes a slightly different form. Let us start with the brief introduction of the general setting of the discrete quantum mechanics. For more details we refer to [4, 7, 12].

The Hamiltonian of discrete quantum mechanics has a generic form

$$\mathcal{H} = \sqrt{V(x; \boldsymbol{\lambda})} e^{-i\partial_x} \sqrt{V(x; \boldsymbol{\lambda})^*} + \sqrt{V(x; \boldsymbol{\lambda})^*} e^{i\partial_x} \sqrt{V(x; \boldsymbol{\lambda})} - V(x; \boldsymbol{\lambda}) - V(x; \boldsymbol{\lambda})^*, \quad (4.1)$$

for various potential functions $V(x; \boldsymbol{\lambda})$, which are in general complex. The corresponding Schrödinger equation is a difference equation in stead of a differential equation. It is also factorised, $\mathcal{H} = \mathcal{A}^\dagger \mathcal{A}$, with

$$\mathcal{A} = \mathcal{A}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -i \left(e^{-\frac{i}{2}\partial_x} \sqrt{V(x; \boldsymbol{\lambda})^*} - e^{\frac{i}{2}\partial_x} \sqrt{V(x; \boldsymbol{\lambda})} \right), \quad (4.2)$$

$$\mathcal{A}^\dagger = \mathcal{A}(\boldsymbol{\lambda})^\dagger = i \left(\sqrt{V(x; \boldsymbol{\lambda})} e^{-\frac{i}{2}\partial_x} - \sqrt{V(x; \boldsymbol{\lambda})^*} e^{\frac{i}{2}\partial_x} \right). \quad (4.3)$$

As in the ordinary quantum mechanics cases, the ground state wavefunction ϕ_0 is annihilated by the operator \mathcal{A} :

$$\mathcal{A}(\boldsymbol{\lambda})\phi_0(x; \boldsymbol{\lambda}) = 0 \quad (\Rightarrow \mathcal{H}\phi_0(x; \boldsymbol{\lambda}) = 0, \quad \mathcal{E}_0(\boldsymbol{\lambda}) = 0). \quad (4.4)$$

We introduce a similarity transformed Hamiltonian $\tilde{\mathcal{H}}(\boldsymbol{\lambda})$ in terms of the ground state wavefunction $\phi_0 = \phi_0(x; \boldsymbol{\lambda})$:

$$\tilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0^{-1} \circ \mathcal{H} \circ \phi_0 = V(x; \boldsymbol{\lambda})e^{-i\partial_x} + V(x; \boldsymbol{\lambda})^*e^{i\partial_x} - V(x; \boldsymbol{\lambda}) - V(x; \boldsymbol{\lambda})^*. \quad (4.5)$$

It acts on the polynomial part of the eigenfunction

$$\phi_n(x; \boldsymbol{\lambda}) \propto \phi_0(x; \boldsymbol{\lambda})P_n(\eta(x); \boldsymbol{\lambda}), \quad (4.6)$$

$$\tilde{\mathcal{H}}P_n(\eta(x); \boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})P_n(\eta(x); \boldsymbol{\lambda}). \quad (4.7)$$

Here $P_n(\eta; \boldsymbol{\lambda})$ is a polynomial of degree n in variable η , and $\eta(x)$ is a real function of x . For all the examples given in this section, the eigenfunctions have this form.

We assume that the system is shape-invariant (2.3). Next let us introduce the similarity transformation of the reversed order Hamiltonian $\mathcal{H}_r(\boldsymbol{\lambda}) = \mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger$:

$$\begin{aligned} \tilde{\mathcal{H}}_r &\stackrel{\text{def}}{=} \phi_{r0}^{-1} \circ \mathcal{H}_r \circ \phi_{r0} \\ &= V(x; \boldsymbol{\lambda} + \boldsymbol{\delta})e^{-i\partial_x} + V(x; \boldsymbol{\lambda} + \boldsymbol{\delta})^*e^{i\partial_x} - V(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) - V(x; \boldsymbol{\lambda} + \boldsymbol{\delta})^* + \mathcal{E}_1(\boldsymbol{\lambda}), \end{aligned} \quad (4.8)$$

with respect to its ground state wavefunction $\phi_{r0} = \phi_{r0}(x; \boldsymbol{\lambda})$ defined by

$$\mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta})\phi_{r0}(x; \boldsymbol{\lambda}) = 0 \quad (\Rightarrow \phi_{r0}(x; \boldsymbol{\lambda}) \propto \phi_0(x; \boldsymbol{\lambda} + \boldsymbol{\delta})). \quad (4.9)$$

We introduce the one parameter family of (similarity transformed) Hamiltonians:

$$\tilde{\mathcal{H}}_s \stackrel{\text{def}}{=} (1-s)\tilde{\mathcal{H}} + s\tilde{\mathcal{H}}_r \quad (4.10)$$

$$= V_s(x; \boldsymbol{\lambda})e^{-i\partial_x} + V_s(x; \boldsymbol{\lambda})^*e^{i\partial_x} - V_s(x; \boldsymbol{\lambda}) - V_s(x; \boldsymbol{\lambda})^* + s\mathcal{E}_1(\boldsymbol{\lambda}), \quad (4.11)$$

which depends on the interpolated potential function $V_s(x; \boldsymbol{\lambda})$:

$$V_s(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} (1-s)V(x; \boldsymbol{\lambda}) + sV(x; \boldsymbol{\lambda} + \boldsymbol{\delta}). \quad (4.12)$$

Since both $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}_r$ act on the space of polynomials, their sum is meaningful and acts on the same space.

Our assertion is that $V_s(x; \boldsymbol{\lambda})$ is simply the same potential function with shifted coupling constants $\boldsymbol{\lambda}'$:

$$V_s(x; \boldsymbol{\lambda}) = V(x; \boldsymbol{\lambda}'). \quad (4.13)$$

Then we have

$$\tilde{\mathcal{H}}_s = \tilde{\mathcal{H}}(\boldsymbol{\lambda}') + \tilde{\Delta}\mathcal{E}(\boldsymbol{\lambda}, s), \quad \tilde{\Delta}\mathcal{E}(\boldsymbol{\lambda}, s) = s\mathcal{E}_1(\boldsymbol{\lambda}). \quad (4.14)$$

Recall that $\boldsymbol{\lambda}'$ depend on s . The eigenfunctions and eigenvalues of $\tilde{\mathcal{H}}_s$ are given by

$$\tilde{\mathcal{H}}_s P_n(\eta(x); \boldsymbol{\lambda}') = (\mathcal{E}_n(\boldsymbol{\lambda}') + s\mathcal{E}_1(\boldsymbol{\lambda})) P_n(\eta(x); \boldsymbol{\lambda}'), \quad n = 0, 1, 2, \dots \quad (4.15)$$

By similarity transformation inversely with respect to $\phi_{s0} = \phi_{s0}(x; \boldsymbol{\lambda})$ defined by

$$\mathcal{A}(\boldsymbol{\lambda}') \phi_{s0}(x; \boldsymbol{\lambda}) = 0 \quad \implies \phi_{s0}(x; \boldsymbol{\lambda}) \propto \phi_0(x; \boldsymbol{\lambda}'),$$

we introduce $\check{\mathcal{H}}_s$ as

$$\check{\mathcal{H}}_s \stackrel{\text{def}}{=} \phi_{s0} \circ \tilde{\mathcal{H}}_s \circ \phi_{s0}^{-1} \quad (4.16)$$

$$\begin{aligned} &= (1-s) \frac{\phi_0(x; \boldsymbol{\lambda}')}{\phi_0(x; \boldsymbol{\lambda})} \circ \mathcal{H} \circ \frac{\phi_0(x; \boldsymbol{\lambda})}{\phi_0(x; \boldsymbol{\lambda}')} + s \frac{\phi_0(x; \boldsymbol{\lambda}')}{\phi_0(x; \boldsymbol{\lambda} + \boldsymbol{\delta})} \circ \mathcal{H}_r \circ \frac{\phi_0(x; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\phi_0(x; \boldsymbol{\lambda}')} \\ &= \mathcal{H}(\boldsymbol{\lambda}') + s\mathcal{E}_1(\boldsymbol{\lambda}). \end{aligned} \quad (4.17)$$

This one parameter family of Hamiltonians $\check{\mathcal{H}}_s$ interpolates \mathcal{H} ($s = 0$) and \mathcal{H}_r ($s = 1$). Its eigenfunctions and eigenvalues are given by

$$\check{\mathcal{H}}_s \phi_n(x; \boldsymbol{\lambda}') = (\mathcal{E}_n(\boldsymbol{\lambda}') + s\mathcal{E}_1(\boldsymbol{\lambda})) \phi_n(x; \boldsymbol{\lambda}'), \quad n = 0, 1, 2, \dots \quad (4.18)$$

From this our claim follows

the interpolated (similarity transformed) Hamiltonian $\check{\mathcal{H}}_s$ ($\tilde{\mathcal{H}}_s$) ($0 \leq s \leq 1$) also describes the same *shape-invariant* and *exactly solvable* system with shifted *coupling constants*.

This claim can be easily verified for each known shape-invariant discrete quantum mechanics given below. We will denote them by the name of the polynomials $\{P_n(\eta)\}$ constituting the eigenfunctions.

4.1 Meixner-Pollaczek polynomial

The Meixner-Pollaczek polynomial [4, 9, 13] is a one-parameter deformation of the Hermite polynomial. Therefore the corresponding discrete quantum mechanics is also called a *deformed harmonic oscillator*. Its potential function and other data are:

$$\boldsymbol{\lambda} = \lambda, \quad \boldsymbol{\delta} = 1/2, \quad \mathcal{E}_n(\lambda) = 2n, \quad n = 0, 1, 2, \dots, \quad (4.19)$$

$$V(x; \lambda) = \lambda + ix, \quad \lambda > 0, \quad -\infty < x < \infty. \quad (4.20)$$

The increase of the ground state energy $\mathcal{E}_1(\boldsymbol{\lambda}) = 2$ is independent of the coupling constant as for the ordinary harmonic oscillator. It is rather trivial to verify our assertion:

$$V_s(x; \lambda) = \lambda + s/2 + ix, \quad (4.21)$$

and

$$\lambda' = \lambda + s/2, \quad \widetilde{\Delta}\mathcal{E}(\lambda; s) = 2s. \quad (4.22)$$

4.2 continuous Hahn polynomial

The continuous Hahn polynomial with special parameters [4, 9] is a two-parameter deformation of the Hermite polynomial. The potential function and the other data are:

$$\boldsymbol{\lambda} = (a, b), \quad \boldsymbol{\delta} = (1/2, 1/2), \quad \mathcal{E}_n(\boldsymbol{\lambda}) = n(n + 2a + 2b - 1), \quad n = 0, 1, 2, \dots, \quad (4.23)$$

$$V(x; \boldsymbol{\lambda}) = (a + ix)(b + ix), \quad a, b > 0, \quad -\infty < x < \infty. \quad (4.24)$$

These lead to $\boldsymbol{\lambda}' = (a', b')$ determined by

$$a' + b' = a + b + s, \quad a'b' = ab + (a + b)s/2 + s/4 \quad (4.25)$$

and

$$\widetilde{\Delta}\mathcal{E}(\boldsymbol{\lambda}; s) = 2(a + b)s. \quad (4.26)$$

For the boundary values $s = 0$ and $s = 1$, $(a', b') = (a + s/2, b + s/2)$ is a solution of (4.25).

4.3 continuous dual Hahn polynomial

The continuous dual Hahn polynomial [4, 9] is a two-parameter deformation of the Laguerre polynomial. Therefore the corresponding quantum mechanics is a deformation of the potential $x^2 + 1/x^2$, or the one body case of the Calogero model [10] discussed in §3.2. The

potential function and the other data are:

$$\boldsymbol{\lambda} = (a, b, c), \quad \boldsymbol{\delta} = (1/2, 1/2, 1/2), \quad \mathcal{E}_n(\boldsymbol{\lambda}) = n, \quad n = 0, 1, 2, \dots, \quad (4.27)$$

$$V(x; \boldsymbol{\lambda}) = \frac{(a+ix)(b+ix)(c+ix)}{2ix(2ix+1)}, \quad a, b, c > 0, \quad 0 < x < \infty. \quad (4.28)$$

As in the Calogero case, the increase of the ground state energy is independent of the coupling constants and this leads to the linear energy spectrum. The shifted parameters $\boldsymbol{\lambda}' = (a', b', c')$ are determined by

$$a' + b' + c' = a + b + c + 3s/2, \quad (4.29)$$

$$a'b' + b'c' + c'a' = ab + bc + ca + (a + b + c)s + 3s/4, \quad (4.30)$$

$$a'b'c' = abc + (ab + bc + ca)s/2 + (a + b + c)s/4 + s/8 \quad (4.31)$$

and

$$\widetilde{\Delta\mathcal{E}}(\boldsymbol{\lambda}; s) = s. \quad (4.32)$$

For the boundary values $s = 0$ and $s = 1$, $(a', b', c') = (a + s/2, b + s/2, c + s/2)$ is a solution of (4.29)–(4.31).

4.4 Wilson polynomial

The Wilson polynomial [4, 9] is a three-parameter deformation of the Laguerre polynomial. The potential function and the other data are:

$$\boldsymbol{\lambda} = (a, b, c, d), \quad \boldsymbol{\delta} = (1/2, 1/2, 1/2, 1/2), \quad (4.33)$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = n(n + a + b + c + d - 1), \quad n = 0, 1, 2, \dots, \quad (4.34)$$

$$V(x; \boldsymbol{\lambda}) = \frac{(a+ix)(b+ix)(c+ix)(d+ix)}{2ix(2ix+1)}, \quad a, b, c, d > 0, \quad 0 < x < \infty. \quad (4.35)$$

The shifted parameters $\boldsymbol{\lambda}' = (a', b', c', d')$ are determined by

$$a' + b' + c' + d' = a + b + c + d + 2s, \quad (4.36)$$

$$\begin{aligned} a'b' + a'c' + a'd' + b'c' + b'd' + c'd' \\ = ab + ac + ad + bc + bd + cd + 3(a + b + c + d)s/2 + 3s/2, \end{aligned} \quad (4.37)$$

$$a'b'c' + a'b'd' + a'c'd' + b'c'd' \quad (4.38)$$

$$\begin{aligned} = abc + abd + acd + bcd + (ab + ac + ad + bc + bd + cd)s + 3(a + b + c + d)s/4 + s/2, \\ a'b'c'd' = abcd + (abc + abd + acd + bcd)s/2 + (ab + ac + ad + bc + bd + cd)s/4 \\ + (a + b + c + d)s/8 + s/16 \end{aligned} \quad (4.39)$$

and

$$\widetilde{\Delta\mathcal{E}}(\boldsymbol{\lambda}; s) = (a + b + c + d)s. \quad (4.40)$$

For the boundary values $s = 0$ and $s = 1$, $(a', b', c', d') = (a + s/2, b + s/2, c + s/2, d + s/2)$ is a solution of (4.36)–(4.39).

4.5 Askey-Wilson polynomial

The Askey-Wilson polynomial [4, 9] belongs to a different type of shape-invariant discrete quantum mechanical single particle systems. It has a multiplicative shift of parameters rather than the additive shift (2.3) discussed in the rest of this paper.

Following [4, 7, 12], we use variables θ , x and z , which are related as

$$0 \leq \theta \leq \pi, \quad x = \cos \theta, \quad z = e^{i\theta}. \quad (4.41)$$

The dynamical variable is θ and the inner product is $\langle f|g \rangle = \int_0^\pi d\theta f(\theta)^* g(\theta)$. We denote $D \stackrel{\text{def}}{=} z \frac{d}{dz}$. Then q^D is a q -shift operator, $q^D f(z) = f(qz)$. We assume $0 < q < 1$. We note here that

$$\int_0^\pi d\theta = \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}, \quad -i \frac{d}{d\theta} = z \frac{d}{dz} = D. \quad (4.42)$$

The Hamiltonian is

$$\mathcal{H} = \sqrt{V(z; \boldsymbol{\lambda})} q^D \sqrt{V(z; \boldsymbol{\lambda})}^* + \sqrt{V(z; \boldsymbol{\lambda})}^* q^{-D} \sqrt{V(z; \boldsymbol{\lambda})} - V(z; \boldsymbol{\lambda}) - V(z; \boldsymbol{\lambda})^*, \quad (4.43)$$

which is factorized, i.e. $\mathcal{H} = \mathcal{A}^\dagger \mathcal{A}$, with

$$\mathcal{A} = \mathcal{A}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} i \left(q^{\frac{D}{2}} \sqrt{V(z; \boldsymbol{\lambda})}^* - q^{-\frac{D}{2}} \sqrt{V(z; \boldsymbol{\lambda})} \right), \quad (4.44)$$

$$\mathcal{A}^\dagger = \mathcal{A}(\boldsymbol{\lambda})^\dagger = -i \left(\sqrt{V(z; \boldsymbol{\lambda})} q^{\frac{D}{2}} - \sqrt{V(z; \boldsymbol{\lambda})}^* q^{-\frac{D}{2}} \right). \quad (4.45)$$

The ground state ϕ_0 is the function annihilated by \mathcal{A} :

$$\mathcal{A}(\boldsymbol{\lambda}) \phi_0(z; \boldsymbol{\lambda}) = 0 \quad (\Rightarrow \mathcal{H} \phi_0(z; \boldsymbol{\lambda}) = 0, \quad \mathcal{E}_0(\boldsymbol{\lambda}) = 0). \quad (4.46)$$

The potential function and other data are

$$\boldsymbol{\lambda} = (a, b, c, d), \quad -1 < a, b, c, d < 1, \quad abcd < q, \quad (4.47)$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = (q^{-n} - 1)(1 - abcdq^{n-1}), \quad n = 0, 1, 2, \dots, \quad (4.48)$$

$$V(z; \boldsymbol{\lambda}) = \frac{(1 - az)(1 - bz)(1 - cz)(1 - dz)}{(1 - z^2)(1 - qz^2)}. \quad (4.49)$$

The similarity transformed Hamiltonian in terms of the ground state wavefunction $\phi_0 = \phi_0(z; \boldsymbol{\lambda})$ reads

$$\tilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0^{-1} \circ \mathcal{H} \circ \phi_0 = V(z; \boldsymbol{\lambda}) q^D + V(z; \boldsymbol{\lambda})^* q^{-D} - V(z; \boldsymbol{\lambda}) - V(z; \boldsymbol{\lambda})^*. \quad (4.50)$$

The shape-invariance relation reads

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger = q^{-1}\mathcal{A}(q^{\frac{1}{2}}\boldsymbol{\lambda})^\dagger\mathcal{A}(q^{\frac{1}{2}}\boldsymbol{\lambda}) + \mathcal{E}_1(\boldsymbol{\lambda}). \quad (4.51)$$

The similarity transformed reversed order Hamiltonian $\mathcal{H}_r(\boldsymbol{\lambda}) = \mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger$, with respect to its ground state wavefunction $\phi_{r0} = \phi_{r0}(z; \boldsymbol{\lambda})$ ($\mathcal{A}(q^{\frac{1}{2}}\boldsymbol{\lambda})\phi_{r0}(z; \boldsymbol{\lambda}) = 0 \Rightarrow \phi_{r0}(z; \boldsymbol{\lambda}) \propto \phi_0(z; q^{\frac{1}{2}}\boldsymbol{\lambda})$), reads

$$\begin{aligned} \tilde{\mathcal{H}}_r &\stackrel{\text{def}}{=} \phi_{r0}^{-1} \circ \mathcal{H}_r \circ \phi_{r0} \\ &= q^{-1} \left(V(z; q^{\frac{1}{2}}\boldsymbol{\lambda}) q^D + V(z; q^{\frac{1}{2}}\boldsymbol{\lambda})^* q^{-D} - V(z; q^{\frac{1}{2}}\boldsymbol{\lambda}) - V(z; q^{\frac{1}{2}}\boldsymbol{\lambda})^* \right) + \mathcal{E}_1(\boldsymbol{\lambda}). \end{aligned} \quad (4.52)$$

The one parameter family of (similarity transformed) Hamiltonians is introduced as

$$\begin{aligned} \tilde{\mathcal{H}}_s &\stackrel{\text{def}}{=} (1-s)\tilde{\mathcal{H}} + s\tilde{\mathcal{H}}_r \\ &= V_s(z; \boldsymbol{\lambda}) q^D + V_s(z; \boldsymbol{\lambda})^* q^{-D} - V_s(z; \boldsymbol{\lambda}) - V_s(z; \boldsymbol{\lambda})^* + s\mathcal{E}_1(\boldsymbol{\lambda}), \end{aligned} \quad (4.53)$$

which depends on the interpolated potential function $V_s(z; \boldsymbol{\lambda})$:

$$V_s(z; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} (1-s)V(z; \boldsymbol{\lambda}) + sq^{-1}V(z; q^{\frac{1}{2}}\boldsymbol{\lambda}). \quad (4.54)$$

Our assertion is that $V_s(z; \boldsymbol{\lambda})$ is simply the same potential function with shifted coupling constants $\boldsymbol{\lambda}' = (a', b', c', d')$ with an overall scaling factor $\alpha > 0$:

$$V_s(z; \boldsymbol{\lambda}) = \alpha V(z; \boldsymbol{\lambda}'). \quad (4.55)$$

Then we have

$$\tilde{\mathcal{H}}_s = \alpha\tilde{\mathcal{H}}(\boldsymbol{\lambda}') + \widetilde{\Delta\mathcal{E}}(\boldsymbol{\lambda}, s), \quad \widetilde{\Delta\mathcal{E}}(\boldsymbol{\lambda}, s) = s\mathcal{E}_1(\boldsymbol{\lambda}). \quad (4.56)$$

We can define $\tilde{\mathcal{H}}_s$ like as (4.16). Equations like as (4.15) and (4.18) hold also.

The shifted parameters are determined by

$$\alpha = 1 + (q^{-1} - 1)s, \quad (4.57)$$

$$a' + b' + c' + d' = \frac{1 + (q^{-1/2} - 1)s}{1 + (q^{-1} - 1)s} (a + b + c + d), \quad (4.58)$$

$$a'b' + a'c' + a'd' + b'c' + b'd' + c'd' = \frac{1}{1 + (q^{-1} - 1)s} (ab + ac + ad + bc + bd + cd), \quad (4.59)$$

$$a'b'c' + a'b'd' + a'c'd' + b'c'd' = \frac{1 - (1 - q^{1/2})s}{1 + (q^{-1} - 1)s} (abc + abd + acd + bcd), \quad (4.60)$$

$$a'b'c'd' = \frac{1 - (1 - q)s}{1 + (q^{-1} - 1)s} abcd \quad (4.61)$$

and

$$\widetilde{\Delta\mathcal{E}}(\boldsymbol{\lambda}, s) = (q^{-1} - 1)(1 - abcd)s. \quad (4.62)$$

For the boundary values $s = 0$ and $s = 1$, $\alpha = q^{-s}$ and $(a', b', c', d') = q^{s/2}(a, b, c, d)$ is a solution of (4.57)–(4.61).

5 Ordinary Quantum Mechanics Revisited

In the previous section we have considered the one parameter family of (similarity transformed) Hamiltonians (4.10), (4.16). This construction method applies also to the ordinary quantum mechanics studied in section 3.

We introduce a similarity transformed Hamiltonian $\tilde{\mathcal{H}}$ in terms of the ground state wavefunction $\phi_0 = \phi_0(x; \boldsymbol{\lambda}) \propto e^{W(x; \boldsymbol{\lambda})}$:

$$\tilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0^{-1} \circ \mathcal{H} \circ \phi_0 = -\frac{d^2}{dx^2} - 2\frac{dW(x; \boldsymbol{\lambda})}{dx} \frac{d}{dx}. \quad (5.1)$$

Next let us introduce the similarity transformation of the reversed order Hamiltonian (2.2), (2.3) with respect to its ground state wavefunction $\phi_{r0} = \phi_{r0}(x; \boldsymbol{\lambda}) \propto e^{W(x; \boldsymbol{\lambda} + \boldsymbol{\delta})}$:

$$\tilde{\mathcal{H}}_r \stackrel{\text{def}}{=} \phi_{r0}^{-1} \circ \mathcal{H}_r \circ \phi_{r0} = -\frac{d^2}{dx^2} - 2\frac{dW(x; \boldsymbol{\lambda} + \boldsymbol{\delta})}{dx} \frac{d}{dx} + \mathcal{E}_1(\boldsymbol{\lambda}). \quad (5.2)$$

The one parameter family of similarity transformed Hamiltonians is introduced as (4.10):

$$\tilde{\mathcal{H}}_s \stackrel{\text{def}}{=} (1 - s)\tilde{\mathcal{H}} + s\tilde{\mathcal{H}}_r = -\frac{d^2}{dx^2} - 2\frac{dW_s(x; \boldsymbol{\lambda})}{dx} \frac{d}{dx} + s\mathcal{E}_1(\boldsymbol{\lambda}), \quad (5.3)$$

where the interpolated prepotential $W_s(x; \boldsymbol{\lambda})$ is given by

$$W_s(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} (1 - s)W(x; \boldsymbol{\lambda}) + sW(x; \boldsymbol{\lambda} + \boldsymbol{\delta}). \quad (5.4)$$

We do not assume (4.6) in general.

Our assertion is that $W_s(x; \boldsymbol{\lambda})$ is simply the same prepotential with shifted coupling constants $\boldsymbol{\lambda}'$:

$$W_s(x; \boldsymbol{\lambda}) = W(x; \boldsymbol{\lambda}'). \quad (5.5)$$

Then we have

$$\check{\mathcal{H}}_s = \check{\mathcal{H}}(\boldsymbol{\lambda}') + \widetilde{\Delta\mathcal{E}}(\boldsymbol{\lambda}, s), \quad \widetilde{\Delta\mathcal{E}}(\boldsymbol{\lambda}, s) = s\mathcal{E}_1(\boldsymbol{\lambda}). \quad (5.6)$$

Note that $\boldsymbol{\lambda}'$ depend on s . As in (4.16), we define inversely similarity transformed Hamiltonian with respect to $\phi_{s0} = \phi_{s0}(x; \boldsymbol{\lambda}) \propto e^{W_s(x; \boldsymbol{\lambda})} = e^{W(x; \boldsymbol{\lambda}')}$:

$$\begin{aligned} \check{\mathcal{H}}_s &\stackrel{\text{def}}{=} \phi_{s0} \circ \check{\mathcal{H}}_s \circ \phi_{s0}^{-1} \\ &= (1-s) \frac{\phi_0(x; \boldsymbol{\lambda}')}{\phi_0(x; \boldsymbol{\lambda})} \circ \mathcal{H} \circ \frac{\phi_0(x; \boldsymbol{\lambda})}{\phi_0(x; \boldsymbol{\lambda}')} + s \frac{\phi_0(x; \boldsymbol{\lambda}')}{\phi_0(x; \boldsymbol{\lambda} + \boldsymbol{\delta})} \circ \mathcal{H}_r \circ \frac{\phi_0(x; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\phi_0(x; \boldsymbol{\lambda}')} \\ &= \mathcal{H}(\boldsymbol{\lambda}') + s\mathcal{E}_1(\boldsymbol{\lambda}). \end{aligned} \quad (5.7)$$

This one parameter family of Hamiltonians $\check{\mathcal{H}}_s$ interpolates \mathcal{H} ($s = 0$) and \mathcal{H}_r ($s = 1$), and its eigenfunctions and eigenvalues have the same form as those in (4.18). This interpolation is different from the one studied in section 3.

For the quantum mechanical systems studied in section 3, we give a list of $\boldsymbol{\lambda}'$:

$$\S 3.1 \quad : \quad \omega' = \omega, \quad (5.8)$$

$$\S 3.2 \quad : \quad \omega' = \omega, \quad g' = g + s, \quad (5.9)$$

$$\S 3.3 \quad : \quad g' = g + s, \quad (5.10)$$

$$\S 3.4 \quad : \quad g' = g - s, \quad (5.11)$$

$$\S 3.5 \quad : \quad g' = g - s, \quad \mu' = \mu, \quad (5.12)$$

$$\S 3.6 \quad : \quad g' = g - s, \quad \mu' = \mu, \quad (5.13)$$

$$\S 3.7.1 \quad : \quad g' = g + s, \quad h' = h + s, \quad (5.14)$$

$$\S 3.7.2 \quad : \quad g' = g - s, \quad \mu' = \frac{(g-s)(g-1+s)}{g(g-1)} \mu, \quad (5.15)$$

$$\S 3.7.3 \quad : \quad g' = g + s, \quad \mu' = \frac{(g+s)(g+1-s)}{g(g+1)} \mu. \quad (5.16)$$

6 Summary and Comments

Interpolation of the SUSY-partner Hamiltonians are discussed for various shape-invariant quantum mechanics including the discrete ones. For ordinary quantum mechanics we present

two kinds of interpolation, \mathcal{H}_s and $\check{\mathcal{H}}_s$ (or $\tilde{\mathcal{H}}_s$). The shape-invariance is inherited by the interpolating Hamiltonian with shifted coupling constants. The interpolation works at the corresponding eigenfunctions level. For the quantum systems whose eigenfunctions have the form (4.6), the annihilation-creation operators can be constructed [12]. These annihilation-creation operators are also interpolated. The eigenfunctions of annihilation operators are interpreted as coherent states. Therefore these coherent states are also interpolated.

It would be interesting if a similar interpolation idea works for a wider class of quantum mechanics [14] which do not have shape-invariance.

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