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Multi-Particle Quasi Exactly Solvable Difference Equations

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Abstract

Several explicit examples of *multi-particle quasi exactly solvable* 'discrete' quantum mechanical Hamiltonians are derived by deforming the well-known exactly solvable multi-particle Hamiltonians, the Ruijsenaars-Schneider-van Diejen systems. These are difference analogues of the quasi exactly solvable multi-particle systems, the quantum Inozemtsev systems obtained by deforming the well-known exactly solvable Calogero-Sutherland systems. They have a finite number of exactly calculable eigenvalues and eigenfunctions. This paper is a multi-particle extension of the recent paper by one of the authors on deriving quasi exactly solvable *difference* equations of single degree of freedom.

1 Introduction

Recently a recipe to obtain a quasi exactly solvable *difference* equation from an exactly solvable difference equation is developed by one of the present authors [1]. In the present paper we apply the recipe to obtain multi-particle quasi exactly solvable difference equations. A quantum mechanical system is called Quasi Exactly Solvable (QES), if a finite number of eigenvalues and the corresponding eigenfunctions can be determined exactly [2]. Since the number of exactly solvable states can be chosen as large as wanted, a QES system could be used as a good alternative to an exactly solvable system for theoretical as well as practical purposes. Many examples of QES systems of single degree of freedom have been known for some time, whereas the known examples of those with many degrees of freedom are rather limited in their structure [3, 4]. They are all obtained by deforming Calogero-Sutherland systems [5, 6, 7], the exactly solvable Hamiltonian dynamics based on root systems. Among them those based on the BC type root systems (rational and trigonometric) and on the A type root systems (trigonometric) can contain an arbitrary number of particles. Therefore the examples of multi-particle QES are infinite in number. In the present paper we derive several examples of multi-particle QES *difference* equations by deforming Ruijsenaars-Schneider van Diejen (RSvD) systems [8, 9, 10, 11], which are the difference equation analogues of Calogero-Sutherland systems. To be more precise, we derive multi-particle QES difference equations by deforming RSvD systems based on BC (rational and trigonometric) and A (trigonometric) type root systems. Thus the examples of multi-particle QES difference equations are now infinite in number. The Hamiltonian of RSvD systems has two kinds of interaction terms, the 'single particle interaction' part and the 'multi-particle interaction' part. The latter is kept intact and the former, the 'single particle interaction' part is *deformed* according to the recipe given in the recent paper [1].

This paper is organised as follows. In section 2 the Hamiltonian of the rational BC type theory is derived and the finite dimensional invariant polynomial space is identified. The trigonometric BC type theory is explained in section 3. The Hamiltonian of the trigonometric A type theory and the finite dimensional invariant polynomial space are derived in section 4. The final section is for a short summary and comments.

2 Rational BC type theory

The first multi-particle QES Hamiltonian is a simple deformation of the rational Ruijsenaars-Schneider-van Diejen (RSvD) [8, 9] system based on the BC_n root system. Here n is the rank of the root system as well as the degree of freedom with coordinates and conjugate momenta:

$$x \stackrel{\text{def}}{=} (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \quad p \stackrel{\text{def}}{=} (p_1, p_2, \ldots, p_n).$$

As always, the momentum operator is realised as a differential operator $p_j = -i\partial_j = -i\partial/\partial x_j$. In 'discrete' quantum mechanical Hamiltonians the momentum operators appear in exponentiated forms $e^{\pm p_j} = e^{\mp i\partial_j}$, in contrast to the ordinary quantum mechanics, in which the momentum operators appear as polynomials. Thus their action on the wavefunction is a finite shift in the imaginary direction:

$$e^{\pm p_j}\psi(x) = \psi(x_1, \ldots, x_{j-1}, x_j \mp i, x_{j+1}, \ldots, x_n),$$

leading to *difference* Schrödinger equations. The quasi exactly solvable rational Hamiltonian has the following general form [1, 12, 11]:

$$\mathcal{H} \stackrel{\text{def}}{=} \sum_{j=1}^{n}\Big(\sqrt{V_j}\, e^{p_j}\sqrt{V_j^*} + \sqrt{V_j^*}\, e^{-p_j}\sqrt{V_j} - V_j - V_j^*\Big) + \alpha_{\mathcal{M}}, \quad \mathcal{M} \in \mathbb{N}, \tag{2.1}$$

$$= \sum_{j=1}^{n}\Big(\sqrt{V_j}\, e^{-i\partial_j}\sqrt{V_j^*} + \sqrt{V_j^*}\, e^{i\partial_j}\sqrt{V_j} - V_j - V_j^*\Big) + \alpha_{\mathcal{M}}. \tag{2.2}$$

Here $\alpha_{\mathcal{M}}$ is a compensation term indexed by a natural number $\mathcal{M}$, to be specified shortly in (2.5) and (2.7). The potential function V_j has the following general form:

$$V_j(x) \stackrel{\text{def}}{=} w(x_j)\prod_{\substack{k=1\\k\neq j}}^{n}\prod_{\varepsilon=\pm1} v(x_j + \varepsilon x_k), \tag{2.3}$$

in which the 'multi-particle interaction' part v is *not deformed*

$$v(y) \stackrel{\text{def}}{=} 1 - i\,\frac{g}{y}, \quad g > 0. \tag{2.4}$$

Whereas the 'single particle interaction' part w allows *two types of deformation* for QES, corresponding to the linear and quadratic polynomial deformations introduced in section 3.1 of [1]:

$$\text{Type I} \;:\quad w(y) \stackrel{\text{def}}{=} (a_2 + iy)w_0(y), \quad \alpha_{\mathcal{M}}(x) \stackrel{\text{def}}{=} \mathcal{M}\sum_{j=1}^{n} x_j^2, \tag{2.5}$$

$$\text{Type II} :\quad w(y) \stackrel{\text{def}}{=} (a_1 + iy)(a_2 + iy)w_0(y), \tag{2.6}$$

$$\alpha_{\mathcal{M}}(x) \stackrel{\text{def}}{=} \mathcal{M}\Big(\mathcal{M} - 1 + \sum_{\alpha=1}^{6} a_\alpha + 2(n-1)g\Big)\sum_{j=1}^{n} x_j^2, \tag{2.7}$$

with a common undeformed $w_0(y)$ [12, 11]

$$w_0(y) \stackrel{\text{def}}{=} \frac{\prod_{\alpha=3}^{6}(a_\alpha + iy)}{2iy(2iy+1)}, \quad a_\alpha > 0. \tag{2.8}$$

As a single particle dynamics, the above w_0 corresponds to the deformation of the harmonic oscillator with the centrifugal potential. The corresponding eigenfunctions are the Wilson polynomials [12, 13].

The main part of the Hamiltonian is factorised [1, 12, 11]:

$$\mathcal{H} = \sum_{j=1}^{n} \mathcal{A}_j^\dagger \mathcal{A}_j + \alpha_{\mathcal{M}}, \tag{2.9}$$

$$\mathcal{A}_j \stackrel{\text{def}}{=} -i\left(e^{-\frac{i}{2}\partial_j}\sqrt{V_j^*} - e^{\frac{i}{2}\partial_j}\sqrt{V_j}\right), \quad \mathcal{A}_j^\dagger = i\left(\sqrt{V_j}\, e^{-\frac{i}{2}\partial_j} - \sqrt{V_j^*}\, e^{\frac{i}{2}\partial_j}\right), \tag{2.10}$$

exhibiting the hermiticity (self-adjointness) of the Hamiltonian.

The *pseudo ground state* wavefunction ϕ_0 is defined as the one annihilated by all the $\mathcal{A}_j$ operators:

$$\mathcal{A}_j \phi_0 = 0, \quad j = 1, \ldots, n. \tag{2.11}$$

It is given by

$$\text{Type I} \ : \quad \phi_0(x) \stackrel{\text{def}}{=} \left| \prod_{j=1}^{n} \frac{\prod_{\alpha=2}^{6} \Gamma(a_\alpha + i x_j)}{\Gamma(2 i x_j)} \cdot \prod_{1 \le j < k \le n} \prod_{\varepsilon = \pm 1} \frac{\Gamma(g + i(x_j + \varepsilon x_k))}{\Gamma(i(x_j + \varepsilon x_k))} \right|, \tag{2.12}$$

$$\text{Type II} : \quad \phi_0(x) \stackrel{\text{def}}{=} \left| \prod_{j=1}^{n} \frac{\prod_{\alpha=1}^{6} \Gamma(a_\alpha + i x_j)}{\Gamma(2 i x_j)} \cdot \prod_{1 \le j < k \le n} \prod_{\varepsilon = \pm 1} \frac{\Gamma(g + i(x_j + \varepsilon x_k))}{\Gamma(i(x_j + \varepsilon x_k))} \right|, \tag{2.13}$$

in which an abbreviation $|f| \stackrel{\text{def}}{=} \sqrt{f f^*}$ is used. It is obvious that these ϕ_0 are square integrable in the principal Weyl chamber of BC_n:

$$\int_{PW} \phi_0(x)^2 d^n x < \infty, \qquad PW \stackrel{\text{def}}{=} \bigl\{ x \in \mathbb{R}^n \bigm| x_1 > x_2 > \cdots > x_n > 0 \bigr\}, \tag{2.14}$$

and they have no node or singularity in PW. By the similarity transformation in terms of the above pseudo ground state wavefunction $\phi_0(x)$, we obtain a Hamiltonian $\tilde{\mathcal{H}}$ leading to a finite difference eigenvalue equation with rational potentials (2.3)–(2.8) [1, 12]:

$$\tilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0^{-1} \circ \mathcal{H} \circ \phi_0 = \sum_{j=1}^{n} \Bigl(V_j(x)(e^{-i\partial_j} - 1) + V_j(x)^*(e^{i\partial_j} - 1) \Bigr) + \alpha_{\mathcal{M}}(x), \tag{2.15}$$

$$\mathcal{H}\phi = \mathcal{E}\phi, \quad \phi(x) = \phi_0(x) P_{\mathcal{M}}(x) \quad \Longleftrightarrow \quad \tilde{\mathcal{H}} P_{\mathcal{M}} = \mathcal{E} P_{\mathcal{M}}. \tag{2.16}$$

In the *undeformed* limit, *i.e.* $w = w_0$ and $\alpha_{\mathcal{M}} = 0$, the theory is the exactly solvable rational BC_n RSvD [9] system with a Hamiltonian $\tilde{\mathcal{H}}_0$. The corresponding ϕ_0 becomes the true ground state wavefunction. The exact solvability means that $\tilde{\mathcal{H}}_0$ maps a Weyl-invariant polynomial in $\{x_j\}$ into another of the same degree. The BC_n Weyl-invariant polynomials in $\{x_j\}$ are simply symmetric (under any permutation $j \leftrightarrow k$) polynomials in $\{x_j^2\}$. For later convenience, let us introduce a monomial symmetric polynomial

$$m_\lambda(\{y_j\}) = m_{(\lambda_1,\dots,\lambda_n)}(y_1,\dots,y_n) \stackrel{\text{def}}{=} \sum_{(l_1,\dots,l_n)} y_1^{l_1}\cdots y_n^{l_n}, \tag{2.17}$$

where the summation with respect to $(l_1,\dots,l_n)$ is taken over all distinct permutations of $\lambda \stackrel{\text{def}}{=} (\lambda_1,\dots,\lambda_n)$.

In the *deformed* theory, it is straightforward to demonstrate that $\tilde{\mathcal{H}}$ maps a symmetric polynomial in $\{x_j^2\}$ of degree equal or less than $\mathcal{M}$ into another:

$$\tilde{\mathcal{H}}\,\mathcal{V}_{\mathcal{M}} \subseteq \mathcal{V}_{\mathcal{M}}, \qquad \dim \mathcal{V}_{\mathcal{M}} = \binom{\mathcal{M}+n}{n}, \tag{2.18}$$

$$\mathcal{V}_{\mathcal{M}} \stackrel{\text{def}}{=} \text{Span}\big[m_{(l_1,\dots,l_n)}(\{x_j^2\}) \bigm| 0 \le l_j \le \mathcal{M},\ 1 \le j \le n\big]. \tag{2.19}$$

This establishes the quasi exact solvability. The proof that an invariant polynomial is mapped into another goes almost parallel with the undeformed theory. One simply has to verify the vanishing of the residues of the simple poles at $q_j = \pm q_k, 0, \pm i/2$. The other step is to show that a symmetric polynomial in $\{x_j^2\}$ with degree m ($\le \mathcal{M}-1$) is mapped to another with degree $m+1$, whereas a symmetric polynomial of degree $\mathcal{M}$ remains with the same degree. This part goes almost the same as in the single particle case shown in [1], since it is caused by the deformation of w, the single particle interaction part. The present examples are the difference equation version of the QES theory called rational BC Inozemtsev systems discussed in section 6 of Sasaki-Takasaki paper [3].

3 Trigonometric BC type theory

The next example is a QES deformation of the trigonometric BC_n RSvD system. Because of the periodicity of the trigonometric potential, we introduce a slightly different notation for the dynamical variables:

$$\theta \stackrel{\text{def}}{=} (\theta_1,\theta_2,\dots,\theta_n) \in \mathbb{R}^n, \quad 0 < \theta_j < \pi, \quad x_j = \cos\theta_j, \quad z_j = e^{i\theta_j}. \tag{3.1}$$

The dynamical variables are θ. We denote $D_j \stackrel{\text{def}}{=} z_j \frac{d}{dz_j}$. Then q^{D_j} is a q-shift operator,

$$q^{D_j} f(z) = f(z_1,\dots,z_{j-1}, q z_j, z_{j+1},\dots,z_n),$$

with $0 < q < 1$. The quasi exactly solvable trigonometric Hamiltonian has the following general form [1, 12, 11]:

$$\mathcal{H} \stackrel{\text{def}}{=} \sum_{j=1}^{n} \Big(\sqrt{V_j}\, q^{D_j} \sqrt{V_j^*} + \sqrt{V_j^*}\, q^{-D_j} \sqrt{V_j} - V_j - V_j^* \Big) + \alpha_{\mathcal{M}}. \tag{3.2}$$

The compensation term $\alpha_{\mathcal{M}}$ is given in (3.5). The potential function V_j consists of the 'single particle interaction' part w and the 'multi-particle interaction' part v:

$$V_j(z) \stackrel{\text{def}}{=} w(z_j) \prod_{\substack{k=1 \\ k\neq j}}^{n} \prod_{\varepsilon=\pm 1} v(z_j z_k^{\varepsilon}), \tag{3.3}$$

$$v(y) \stackrel{\text{def}}{=} \frac{1-a_0 y}{1-y}, \quad w(y) \stackrel{\text{def}}{=} (1-a_1 y) w_0(y), \quad w_0(y) \stackrel{\text{def}}{=} \frac{\prod_{\alpha=2}^{5}(1-a_\alpha y)}{(1-y^2)(1-qy^2)}, \tag{3.4}$$

$$\alpha_{\mathcal{M}}(z) \stackrel{\text{def}}{=} (q^{\mathcal{M}}-1) q^{-1} a_0^{2(n-1)} a_1 a_2 a_3 a_4 a_5 \sum_{j=1}^{n} \big(z_j + z_j^{-1} \big). \tag{3.5}$$

As before, the 'multi-particle interaction' part v is not deformed but the 'single particle interaction' part w is multiplicatively deformed by a linear term from w_0. As a single particle dynamics, the above w_0 corresponds to the deformation of the Pöschl-Teller potential. The corresponding eigenfunctions are the Askey-Wilson polynomials [12, 13].

The main part of the Hamiltonian is factorised [1, 12, 11]:

$$\mathcal{H} = \sum_{j=1}^{n} \mathcal{A}_j^{\dagger} \mathcal{A}_j + \alpha_{\mathcal{M}}, \tag{3.6}$$

$$\mathcal{A}_j \stackrel{\text{def}}{=} i \Big(q^{\frac{1}{2}D_j} \sqrt{V_j^*} - q^{-\frac{1}{2}D_j} \sqrt{V_j} \Big), \quad \mathcal{A}_j^{\dagger} = -i \Big(\sqrt{V_j}\, q^{\frac{1}{2}D_j} - \sqrt{V_j^*}\, q^{-\frac{1}{2}D_j} \Big), \tag{3.7}$$

and the *pseudo ground state* wavefunction ϕ_0

$$\phi_0(z) \stackrel{\text{def}}{=} \left| \prod_{j=1}^{n} \frac{(z_j^2\,;q)_\infty}{\prod_{\alpha=1}^{5}(a_\alpha z_j\,;q)_\infty} \cdot \prod_{1\le j<k\le n} \prod_{\varepsilon=\pm 1} \frac{(z_j z_k^{\varepsilon}\,;q)_\infty}{(a_0 z_j z_k^{\varepsilon}\,;q)_\infty} \right|, \tag{3.8}$$

is obtained as the common zero-mode of all $\mathcal{A}_j$ operators $\mathcal{A}_j \phi_0 = 0$, $j = 1, \ldots, n$. Here the standard notation $(a\,;q)_\infty \stackrel{\text{def}}{=} \prod_{n=0}^{\infty}(1-aq^n)$ and $|f| \stackrel{\text{def}}{=} \sqrt{ff^*}$ are used. It is obvious that ϕ_0 has no zero or singularity in the principal Weyl alcove of BC_n

$$PW_T \stackrel{\text{def}}{=} \big\{ \theta \in \mathbb{R}^n \bigm| \pi > \theta_1 + \theta_2 > \theta_1 > \theta_2 > \cdots > \theta_n > 0 \big\}, \tag{3.9}$$

so long as the parameters are restricted to

$$-1 < a_\alpha < 1, \quad \alpha = 0, 1, \ldots, 5. \tag{3.10}$$

Then the square integrability of ϕ_0, $\int_{PW_T}\phi_0^2 d^n\theta<\infty$ is trivial. By the similarity transformation in terms of the above pseudo ground state wavefunction $\phi_0(z)$, we obtain a Hamiltonian $\tilde{\mathcal{H}}$ leading to a finite difference eigenvalue equation with rational potentials in z and z^{-1} (3.3)–(3.4) [1, 12]:

$$\tilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0^{-1}\circ\mathcal{H}\circ\phi_0=\sum_{j=1}^{n}\Big(V_j(z)(q^{D_j}-1)+V_j(z)^*(q^{-D_j}-1)\Big)+\alpha_{\mathcal{M}}(z), \tag{3.11}$$

$$\mathcal{H}\phi=\mathcal{E}\phi,\quad \phi(z)=\phi_0(z)P_{\mathcal{M}}(z)\quad\Longleftrightarrow\quad \tilde{\mathcal{H}}P_{\mathcal{M}}=\mathcal{E}P_{\mathcal{M}}. \tag{3.12}$$

In the *undeformed* limit, *i.e.* $w=w_0$ and $\alpha_{\mathcal{M}}=0$, the theory is the exactly solvable trigonometric BC_n RSvD [9] system with a Hamiltonian $\tilde{\mathcal{H}}_0$. The corresponding ϕ_0 becomes the true ground state wavefunction. The exact solvability means that $\tilde{\mathcal{H}}_0$ maps a Weyl-invariant polynomial in $\{x_j=\cos\theta_j=\frac{1}{2}(z_j+z_j^{-1})\}$ into another of the same degree. The BC_n Weyl-invariant polynomials in $\{x_j\}$ are simply symmetric (under any permutation $j\leftrightarrow k$) polynomials in $\{x_j\}$ or in $\{z_j+z_j^{-1}\}$. The eigenfunctions of $\tilde{\mathcal{H}}_0$ are the BC type Jack polynomials [14].

In the *deformed* theory, it is straightforward to demonstrate that $\tilde{\mathcal{H}}$ maps a Weyl-invariant polynomial in $\{x_j\}$ of degree equal or less than $\mathcal{M}$ into another:

$$\tilde{\mathcal{H}}\,\mathcal{V}_{\mathcal{M}}\subseteq\mathcal{V}_{\mathcal{M}},\qquad\qquad \dim\mathcal{V}_{\mathcal{M}}=\binom{\mathcal{M}+n}{n}, \tag{3.13}$$

$$\mathcal{V}_{\mathcal{M}}\stackrel{\text{def}}{=}\text{Span}\big[m_{(l_1,\ldots,l_n)}(\{z_j+z_j^{-1}\})\ \big|\ 0\le l_j\le\mathcal{M},\ 1\le j\le n\big]. \tag{3.14}$$

This establishes the quasi exact solvability. The proof that an invariant polynomial is mapped into another goes almost parallel with the undeformed theory. One simply has to verify the vanishing of the residues of the simple poles at $z_j=z_k^{\pm1},\pm1,\pm q^{1/2},\pm q^{-1/2}$. The other step is to show that a symmetric polynomial in $\{x_j\}$ with degree m $(\le\mathcal{M}-1)$ is mapped to another with degree $m+1$, whereas a symmetric polynomial of degree $\mathcal{M}$ remains with the same degree. This part goes almost the same as in the single particle case shown in [1], since it is caused by the deformation of w, the single particle interaction part. The present example is the difference equation version of the QES theory called trigonometric BC type Inozemtsev system discussed in section 7 of Sasaki-Takasaki paper [3].

4 Trigonometric A type theory

The QES deformation of the trigonometric A_{n-1} RS [8] system goes almost parallel with the previous example, or even simpler. For the A-type theory, it is customary to consider A_{n-1} and to embed all the roots in $\mathbb{R}^n$. This is accompanied by the introduction of one more degree of freedom, θ_n and p_n. The genuine A_{n-1} theory corresponds to the relative coordinates and their momenta, and the extra degree of freedom is the center of mass coordinate and its momentum. The Hamiltonian takes the general form (3.2) with the potential function V_j

$$V_j(z) \stackrel{\text{def}}{=} w(z_j) \prod_{\substack{k=1 \\ k\neq j}}^{n} v(z_j z_k^{-1}), \tag{4.1}$$

$$v(y) \stackrel{\text{def}}{=} \frac{1-a_0 y}{1-y}, \quad w(y) \stackrel{\text{def}}{=} 1 - a_1 y, \tag{4.2}$$

$$\alpha_{\mathcal{M}}(z) \stackrel{\text{def}}{=} (q^{\mathcal{M}}-1) a_0^{n-1} a_1 \sum_{j=1}^{n} \bigl(z_j + z_j^{-1}\bigr). \tag{4.3}$$

The undeformed theory, the trigonometric A_{n-1} RS system [8] has $w \equiv 1$ and $\alpha_{\mathcal{M}} = 0$. The main part of the Hamiltonian is factorised as in (3.6) and (3.7). The *pseudo ground state* wavefunction ϕ_0 annihilated by all $\mathcal{A}_j$ reads

$$\phi_0(z) \stackrel{\text{def}}{=} \left| \prod_{j=1}^{n} \frac{1}{(a_1 z_j\,;q)_\infty} \cdot \prod_{1\le j<k\le n} \frac{(z_j z_k^{-1}\,;q)_\infty}{(a_0 z_j z_k^{-1}\,;q)_\infty} \right|, \tag{4.4}$$

which has no node or singularity in the principal Weyl alcove of A_{n-1}:

$$PW_T \stackrel{\text{def}}{=} \bigl\{\theta \in \mathbb{R}^n \bigm| \pi > \theta_1 > \theta_2 > \cdots > \theta_n > 0\bigr\}, \tag{4.5}$$

so long as the parameters are restricted to $-1 < a_0,\, a_1 < 1$. The similarity transformed Hamiltonian $\tilde{\mathcal{H}}$ in terms of ϕ_0 has the same form as (3.11), (3.12).

In the *undeformed* limit, *i.e.* $w \equiv 1$ and $\alpha_{\mathcal{M}} = 0$, the theory is the exactly solvable trigonometric A_{n-1} RS system with a Hamiltonian $\tilde{\mathcal{H}}_0$. The eigenfunctions of $\tilde{\mathcal{H}}_0$ are the well-known Jack polynomials in $\{z_j\}$ [14]. Since all the coefficients of the eigenvalue equation $\tilde{\mathcal{H}}_0\varphi = \mathcal{E}\varphi$ are real, the Jack polynomials in $\{z_j^{-1}\}$ are also eigenfunctions. In other words $\tilde{\mathcal{H}}_0$ maps a symmetric polynomial in $\{z_j\}$ into another of the same degree.

The eigenfunctions of the *deformed* Hamiltonian $\tilde{\mathcal{H}}$ are still symmetric polynomials in $\{z_j\}$ and $\{z_j^{-1}\}$, but truncated to the maximal power of $\mathcal{M}$ for each variable, due to the single particle interaction term w (4.2) and the compensation term (4.3). This form of the compensation term is necessary for the hermiticity of the Hamiltonian. In other words $\tilde{\mathcal{H}}$ has the invariant subspace

$$\tilde{\mathcal{H}}\,\mathcal{V}_{\mathcal{M}} \subseteq \mathcal{V}_{\mathcal{M}}, \qquad \dim \mathcal{V}_{\mathcal{M}} = \binom{2\mathcal{M}+n}{n}, \tag{4.6}$$

$$\mathcal{V}_{\mathcal{M}} \stackrel{\text{def}}{=} \text{Span}\big[m_{(l_1,\ldots,l_n)}(\{z_j\}) \bigm| -\mathcal{M} \leq l_j \leq \mathcal{M},\ 1 \leq j \leq n\big]. \tag{4.7}$$

This establishes the quasi exact solvability. The proof that a symmetric polynomial is mapped into another goes almost parallel with the undeformed theory. One simply has to verify the vanishing of the residues of the simple poles at $z_j = z_k$. The other step is to show that a symmetric polynomial in $\{z_j\}$ with degree m ($-\mathcal{M}+1 \leq m \leq \mathcal{M}-1$) is mapped to another with degree $m \pm 1$, whereas a symmetric polynomial of degree $\pm\mathcal{M}$ remains with the same degree. This part goes almost the same as in the single particle case shown in [1], since it is caused by the deformation of w, the single particle interaction part. The present example is the difference equation version of the QES theory called trigonometric A type Inozemtsev system discussed in section 8 of Sasaki-Takasaki paper [3].

5 Summary and Comments

Quasi exactly solvable multi-particle difference equations are derived by deforming rational and trigonometric BC type RSvD systems as well as the trigonometric A type RS systems. The method of multi-particle deformations is a simple extension of the single particle case recently developed by one of the authors [1]. These examples are the difference equation analogues of the quasi exactly solvable multi-particle quantum mechanical systems derived by Sasaki-Takasaki [3].

A few comments are in order. As in the single particle cases, the ranges of the parameters can be loosened without losing QES. For example, the six positive parameters in Type II in section 2 (2.6), $a_1, \ldots, a_6$ could be replaced by three complex conjugate pairs with positive real parts. Among the five real parameters $a_1, \ldots, a_5$ in (3.10), four could be replaced by two complex conjugate pairs b_1, b_1^*, b_2, b_2^* with modulus less than unity, $|b_1| < 1$, $|b_2| < 1$.

Let us elaborate on the connection with the trigonometric Inozemtsev systems mentioned at the end of section 3 and 4. In fact the examples in section 3 and 4 reduce to the Inozemtsev systems given in [3] in a certain limit ($q \to 1$). Let us introduce a parameter c, and set $q = e^{-2/c}$, $a_0 = q^g$, $a_1 = 1 - q^{-a/2}$, $\theta = 2\theta^{\text{new}}$. Moreover set $\mathcal{H}^{\text{new}} = (a_2a_3a_4a_5q^{-1})^{-1/2}a_0^{-(n-1)}\mathcal{H}$

and $(a_2,a_3,a_4,a_5)=(q^{g_1},q^{g_2+1/2},-q^{g_1'},-q^{g_2'+1/2})$ for the BC type theory, $\mathcal{H}^{\text{new}}=a_0^{-(n-1)/2}\mathcal{H}$ for the A type theory (see [11]). Then in the $c\to\infty$ limit, the Hamiltonian $\frac{1}{2}c^2\mathcal{H}^{\text{new}}$ reduces to the trigonometric Inozemtsev systems discussed in section 7 and 8 of [3].

The corresponding statement for the rational theory is somehow complicated and it requires a double limit. For the parameters in section 2, let us set $(a_3,a_4,a_5,a_6)=(\frac{c}{\omega}\sqrt{\frac{\omega_1}{a}}+\frac{1}{2a}\frac{\omega_1}{\omega}(\frac{\omega_1}{\omega}-1),-\frac{c}{\omega}\sqrt{\frac{\omega_1}{a}}+\frac{1}{2a}\frac{\omega_1}{\omega}(\frac{\omega_1}{\omega}-1),g_1,g_2+\frac{1}{2})$ and $x_j=cx_j^{\text{new}}$. Moreover set $\tilde{\mathcal{H}}^{\text{new}}=4(a_2a_3a_4)^{-1}\tilde{\mathcal{H}}$ and $a_2=c^2/\omega_1$ for the type I theory, $\tilde{\mathcal{H}}^{\text{new}}=4(a_1a_2a_3a_4)^{-1}\tilde{\mathcal{H}}$ and $a_1=a_2=2c^2/\omega_1$ for the type II theory (see [11]). Then a double limit $\lim_{\omega_1\to\infty}\big(\lim_{c\to\infty}\frac{1}{2}c^2\tilde{\mathcal{H}}^{\text{new}}\big)$ gives the Hamiltonian of the rational Inozemtsev system discussed in section 6 of [3].

It is interesting to note that the weight function $\phi_0^2(x)$ for the polynomial eigenfunctions $\{P_{\mathcal{M}}(x)\}$ is the zero mode (stationary distribution) of the corresponding deformed Fokker-Planck equation [15].

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References

[1] R. Sasaki, "Quasi exactly solvable difference equations," `arXiv:0708.0702[nlin:SI]`, YITP-07-42.

[2] A. G. Ushveridze, "Exact solutions of one- and multi-dimensional Schrödinger equations," Sov. Phys.-Lebedev Inst. Rep. **2** (1988) 54-58; *Quasi-exactly solvable models in quantum mechanics*, (IOP, Bristol, 1994); A. Y. Morozov, A. M. Perelomov, A. A. Roslyi, M. A. Shifman and A. V. Turbiner, "Quasiexactly solvable quantal problems: one-dimensional analog of rational conformal field theories," Int. J. Mod. Phys. A **5** (1990) 803-832; A. V. Turbiner, "Quasi-exactly-soluble problems and sl(2,R) algebra," Comm. Math. Phys. **118** (1988) 467-474.

[3] R. Sasaki and K. Takasaki, "Quantum Inozemtsev model, quasi-exact solvability and $\mathcal{N}$-fold supersymmetry," J. Phys. **A34** (2001) 9533-9553. Corrigendum J. Phys. **A34** (2001) 10335.

[4] A. V. Turbiner, "Quasi-exactly soluble Hamiltonian related to root spaces," J. Nonlinear Math. Phys. **12** Suppl. 1 (2005) 660-675.

[5] F. Calogero, "Solution of the one-dimensional N-body problem with quadratic and/or inversely quadratic pair potentials," J. Math. Phys. **12** (1971) 419-436.

[6] B. Sutherland, "Exact results for a quantum many-body problem in one-dimension. II," Phys. Rev. **A5** (1972) 1372-1376.

[7] S. P. Khastgir, A. J. Pocklington and R. Sasaki, "Quantum Calogero-Moser models: Integrability for all root systems," J. Phys. **A33** (2000) 9033-9064, `arXiv:hep-th/0005277`.

[8] S. N. M. Ruijsenaars and H. Schneider, "A new class of integrable systems and its relation to solitons," Annals Phys. **170** (1986) 370-405; S. N. M. Ruijsenaars, "Complete integrability of relativistic Calogero-Moser systems and elliptic function identities," Comm. Math. Phys. **110** (1987) 191-213.

[9] J. F. van Diejen, "The relativistic Calogero model in an external field," `solv-int/ 9509002`; "Multivariable continuous Hahn and Wilson polynomials related to integrable difference systems," J. Phys. **A28** (1995) L369-L374.

[10] S. Odake and R. Sasaki, "Equilibria of 'discrete' integrable systems and deformations of classical orthogonal polynomials," J. Phys. **A37** (2004) 11841-11876, `arXiv:hep-th/ 0407155`.

[11] S. Odake and R. Sasaki, "Calogero-Sutherland-Moser Systems, Ruijsenaars-Schneider-van Diejen Systems and Orthogonal Polynomials," Prog. Theor. Phys. 114 (2005) 1245-1260, `arXiv:hep-th/0512155`; "Equilibrium Positions and Eigenfunctions of Shape Invariant ('Discrete') Quantum Mechanics," Rokko Lectures in Mathematics (Kobe University) 18 (2005) 85-110, `arXiv:hep-th/0505070`.

[12] S. Odake and R. Sasaki, "Shape invariant potentials in 'discrete' quantum mechanics," J. Nonlinear Math. Phys. **12** Suppl. 1 (2005) 507-521, `arXiv:hep-th/0410102`; "Equilibrium positions, shape invariance and Askey-Wilson polynomials," J. Math. Phys. **46** (2005) 063513 (10 pages), `arXiv:hep-th/0410109`.

[13] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Encyclopedia of mathematics and its applications, Cambridge, (1999); R. Koekoek and R. F. Swarttouw, "The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue," arXiv:math.CA/9602214.

[14] R. P. Stanley, "Some Combinatorial Properties of Jack Symmetric Functions," Adv. Math. **77** (1989) 76-115; I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed., Oxford University Press, (1995).

[15] C.-L. Ho and R. Sasaki, "Deformed Fokker-Planck equations," arXiv:cond-mat/ 0612318; "Deformed multi-variable Fokker-Planck equations," J. Math. Phys. to be published, arXiv:cond-mat/0703291; "Quasi-exactly solvable Fokker-Planck equations," arXiv:0705.0863[cond-mat].