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Non-polynomial extensions of solvable potentials à la Abraham-Moses

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Abstract

Abraham-Moses transformations, besides Darboux transformations, are well-known procedures to generate extensions of solvable potentials in one-dimensional quantum mechanics. Here we present the explicit forms of infinitely many *seed solutions* for adding eigenstates at *arbitrary real energy* through the Abraham-Moses transformations for typical solvable potentials, *e.g.* the radial oscillator, the Darboux-Pöschl-Teller and some others. These seed solutions are simple generalisations of the *virtual state wavefunctions*, which are obtained from the eigenfunctions by discrete symmetries of the potentials. The virtual state wavefunctions have been an essential ingredient for constructing *multi-indexed* Laguerre and Jacobi polynomials through multiple Darboux-Crum transformations. In contrast to the Darboux transformations, the virtual state wavefunctions generate non-polynomial extensions of solvable potentials through the Abraham-Moses transformations.

1 Introduction

In order to extend solvable potentials in one-dimensional quantum mechanics [1, 2, 3], two methods are well-known; the Darboux transformation [4, 5] and the Abraham-Moses transformation [6]. The latter, about 30 years old, does not seem to have been well exploited compared with the former, which is known for about 120 years and has seen remarkable developments brought about by the multi-indexed orthogonal polynomials [7, 8] and the exceptional orthogonal polynomials [9]–[15] generated in terms of seed solutions called *virtual* *state wavefunctions*. They are obtained from the eigenfunctions by the discrete symmetries of the Hamiltonian [7].

In this paper, we assert that these virtual state wavefunctions and their generalisations can also be used for the Abraham-Moses transformations for adding finitely many eigenstates at *arbitrary real energy*. We present the explicit forms of various seed solutions for typical solvable potentials; the radial oscillator, the Darboux-Pöschl-Teller potential and some others. These will bring immensely rich applications of the Abraham-Moses transformations. The harmonic oscillator potential has been discussed in the original paper [6] and in many others [16]–[20].

Historically, the Abraham-Moses transformations have been introduced and discussed in connection with the formulation of the inverse scattering theory [21]. However, like the Darboux transformations, as a map relating one Hamiltonian system to another including the proper solutions, the Abraham-Moses transformations can be formulated totally algebraically, without recourse to the inverse scattering theory, so long as the boundary conditions of various solutions are well specified. The key idea, as stressed by many authors, is that the Wronskian of two solutions $\mathrm{W}[\varphi, \psi]$ can be expressed as an integral from one boundary (2.24); an essential ingredient of the Abraham-Moses transformations.

The present paper is organised as follows. In section two, the basic formulas of the Abraham-Moses transformations are recapitulated for introducing necessary notation and for self-containedness. They are presented algebraically, without making use of the inverse scattering theory formulation. Starting from one state adding transformation in § 2.1, the multiple sates addition formulas are given in § 2.2. The multiple states adding process, starting from a set of M non-normalisable seed solutions $\{\varphi_j\}$ and ending up as many orthonormal vectors $(\varphi_j^{(M)}, \varphi_k^{(M)}) = \delta_{j\,k}$ $(j, k = 1, \ldots, M)$, (2.35), can be considered as a good example of an *orthonormalisation procedure of non-normalisable vectors*. The one state deletion is presented in § 2.3 as an inverse process of one state addition. The multi-states deletion is commented on briefly. Various remarks and comments on Abraham-Moses transformations are listed in § 2.4 including a note on the relation between Darboux transformations and Abraham-Moses transformations. Section three is the main body of the paper. Starting from the two well-known solvable potentials, the radial oscillator and the Darboux-Pöschl-Teller potential, the familiar virtual state wavefunctions are introduced in § 3.1. They are polynomial type wavefunctions. For the Darboux-Pöschl-Teller potential, the total number of addable eigenstates is limited by the parameters of the starting Hamiltonian. In § 3.1.1 their "degrees" are changed to real numbers by rewriting the Laguerre and Jacobi polynomials as (confluent) hypergeometric functions. The other genres of seed solutions are also given there. The seed solutions for other solvable potentials, the Morse potential, etc, are given in § 3.2. The final section is for a summary and discussions.

2 Multiple Abraham-Moses Transformations

Here we first recapitulate the essence of the Abraham-Moses transformations [6] for adding one bound state with an *arbitrary real* energy in § 2.1. By repeating the one state additions, the multiple Abraham-Moses transformations are realised in § 2.2. We briefly discuss one and multiple state deletions in § 2.3. The addition and deletion are shown to be the inverse processes of each other. The other properties are discussed in § 2.4.

In contrast to the original and most of the subsequent publications on the Abraham-Moses transformations [6, 18, 16], our derivations are purely algebraic without recourse to the inverse scattering method [21]. This is partly because some important quantum mechanical systems are defined in finite intervals, for which the inverse scattering method is inadequate. The main reason is the clarity of the presentation. Like the Darboux-Crum transformations [4, 5, 22], most salient features of the Abraham-Moses transformations can be better understood algebraically.

The starting point is the general quantum mechanics in one dimension defined in an interval $x_1 < x < x_2$ with a smooth potential $U(x) \in \mathbb{R}$. The system has an infinite (or a finite) number of discrete eigenstates. For simplicity we assume vanishing groundstate energy:

$$\mathcal{H} = -\frac{d^2}{dx^2} + U(x), \tag{2.1}$$

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x) \quad (n \in \mathbb{Z}_{\geq 0} \text{ or } 0 \leq n \leq n_{\max}), \quad 0 = \mathcal{E}_0 < \mathcal{E}_1 < \mathcal{E}_2 < \cdots, \tag{2.2}$$

$$(\phi_m, \phi_n) \stackrel{\text{def}}{=} \int_{x_1}^{x_2} dx\, \phi_m(x)\phi_n(x) = h_n\delta_{m\,n}, \quad 0 < h_n < \infty. \tag{2.3}$$

In quantum mechanics, another requirement is built in. That is, the momentum operator $p = -i\hbar\partial_x$ ($i \equiv \sqrt{-1}$) must be hermitian. This simply means that the boundary terms in partial integration should vanish. We require the following boundary conditions on the eigenfunctions:

$$\lim_{x\to x_1}\frac{\phi_n(x)^2}{x-x_1}=0,\quad \lim_{x\to x_2}\frac{\phi_n(x)^2}{x_2-x}=0\quad (n=0,1,\ldots).\tag{2.4}$$

An appropriate modification is needed when $x_2=+\infty$ and/or $x_1=-\infty$. Throughout this paper we adopt the convention that all the wavefunctions are real. We will not discuss the scattering state wavefunctions.

Let $\{\varphi_j(x),\tilde{\mathcal{E}}_j\}$ $(j=1,2,\ldots,M)$ be distinct solutions of the original Schrödinger equation (2.1):

$$\mathcal{H}\varphi_j(x)=\tilde{\mathcal{E}}_j\varphi_j(x)\quad (\tilde{\mathcal{E}}_j\in\mathbb{R}\ ;\ j=1,2,\ldots,M),\tag{2.5}$$

to be called *seed* solutions. In this paper we consider such seed solutions that are square non-integrable at one boundary:

$$\text{Type I}:\quad \lim_{x\to x_1}\frac{\varphi_j(x)^2}{x-x_1}=0,\qquad \int_{x_2-\epsilon}^{x_2}dx\,\varphi_j(x)^2=\infty,\tag{2.6}$$

$$\text{Type II}:\quad \int_{x_1}^{x_1+\epsilon}dx\,\varphi_j(x)^2=\infty,\quad \lim_{x\to x_2}\frac{\varphi_j(x)^2}{x_2-x}=0.\tag{2.7}$$

Of course this means that $\varphi_j(x)$ is square integrable at x_1 for type I and at x_2 for type II [17]. It should be stressed that there is no type I or type II seed solution belonging to the spectrum of the Hamiltonian $\{\mathcal{E}_n\}$ $(n=0,1,\ldots)$, because of the uniqueness of the solutions of the Schrödinger equation.

2.1 One state addition

Let us introduce the Abraham-Moses transformations for *adding one bound state by using a seed solution with an arbitrary real energy*. For simplicity of the presentation, we will restrict ourselves to utilise the type I seed solutions only. We will comment on the use of the type II and both I and II in § 2.4. For a pair of real functions f and g, which are square integrable at the lower boundary, let us introduce a new function $\langle f,g\rangle$ by integration:

$$\langle f,g\rangle(x)\stackrel{\text{def}}{=}\int_{x_1}^{x}dy f(y)g(y)=\langle g,f\rangle(x),\quad x_1<x<x_2,\tag{2.8}$$

$$\langle f,g\rangle(x_1)=0,\quad \langle f,g\rangle(x_2)=(f,g).\tag{2.9}$$

Note that $\frac{d}{dx}\langle f,g\rangle(x)=f(x)g(x)$.

For a seed solution, say φ_1, with the energy $\tilde{\mathcal{E}}_1$, an Abraham-Moses transformation for adding one bound state with the energy $\tilde{\mathcal{E}}_1$, is defined as follows:

$$\psi(x)\to\psi^{(1)}(x)\stackrel{\text{def}}{=}\psi(x)-\frac{\varphi_1(x)}{1+\langle\varphi_1,\varphi_1\rangle(x)}\times\langle\varphi_1,\psi\rangle(x),\tag{2.10}$$

or simply $$\psi \to \psi^{(1)} \stackrel{\text{def}}{=} \psi - \frac{\varphi_1}{1+\langle\varphi_1,\varphi_1\rangle}\langle\varphi_1,\psi\rangle. \tag{2.11}$$

Here ψ is an arbitrary smooth function of $x \in (x_1, x_2)$ and $\langle\varphi_1,\psi\rangle$ must be well defined at the lower boundary x_1. We have the following:

Proposition 2.1 [6] *Let ψ be a solution of the original Schrödinger equation satisfying the boundary condition*

$$\mathcal{H}\psi = \mathcal{E}\psi, \qquad \mathcal{E} \in \mathbb{R}, \qquad \lim_{x\to x_1}\frac{\psi(x)^2}{x-x_1} = 0. \tag{2.12}$$

Then the function $\psi^{(1)}$ (2.11) satisfies the deformed Schrödinger equation with the same energy:

$$\mathcal{H}^{(1)}\psi^{(1)} = \mathcal{E}\psi^{(1)}, \tag{2.13}$$

$$\mathcal{H}^{(1)} \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + U^{(1)}(x), \quad U^{(1)}(x) \stackrel{\text{def}}{=} U(x) - 2\frac{d^2}{dx^2}\log\bigl(1+\langle\varphi_1,\varphi_1\rangle\bigr). \tag{2.14}$$

The eigenfunctions are mapped to eigenfunctions with the same norm

$$\phi_n \to \phi_n^{(1)}, \quad (\phi_n^{(1)}, \phi_m^{(1)}) = (\phi_n, \phi_m) = h_n\delta_{n\,m}, \tag{2.15}$$

together with the newly created eigenfunction $\varphi_1^{(1)}$, which has a unit norm:

$$\varphi_1 \to \varphi_1^{(1)} = \frac{\varphi_1}{1+\langle\varphi_1,\varphi_1\rangle}, \quad (\varphi_1^{(1)}, \varphi_1^{(1)}) = 1, \quad (\varphi_1^{(1)}, \phi_n^{(1)}) = 0. \tag{2.16}$$

It should be stressed that the seed solution φ_1 is not square integrable $(\varphi_1, \varphi_1) = \infty$ and its overall scale is immaterial. The normalisation part of the Proposition is a simple consequence of the transformation form (2.11). The transformed seed solution has the form:

$$\bigl(\varphi_1^{(1)}\bigr)^2 = \frac{\varphi_1^2}{(1+\langle\varphi_1,\varphi_1\rangle)^2} = -\frac{d}{dx}\left(\frac{1}{1+\langle\varphi_1,\varphi_1\rangle}\right).$$

By integrating the above expression from x_1 to x_2, we obtain

$$(\varphi_1^{(1)}, \varphi_1^{(1)}) = -\left[\frac{1}{1+\langle\varphi_1,\varphi_1\rangle}\right]_{x_1}^{x_2} = 1 - \frac{1}{1+(\varphi_1,\varphi_1)} = 1. \tag{2.17}$$

Likewise we have

$$f^{(1)}g^{(1)} = fg - \frac{d}{dx}\left(\frac{\langle\varphi_1, f\rangle\langle\varphi_1, g\rangle}{1+\langle\varphi_1,\varphi_1\rangle}\right), \tag{2.18}$$

for arbitrary smooth functions f and g with well-defined $\langle\varphi_1, f\rangle$ and $\langle\varphi_1, g\rangle$. Taking $f = \phi_n$, $g = \phi_m$ and integrating from x_1 to x, we obtain

$$\langle\phi_n^{(1)}, \phi_m^{(1)}\rangle = \langle\phi_n, \phi_m\rangle - \frac{\langle\varphi_1, \phi_n\rangle\langle\varphi_1, \phi_m\rangle}{1+\langle\varphi_1,\varphi_1\rangle}. \tag{2.19}$$

At the upper boundary $x = x_2$, we obtain (2.15). The other orthogonality relation $(\phi_n^{(1)}, \varphi_1^{(1)})$ $= 0$ (2.16) can be shown in a similar way.

Next we note ($f' = \frac{df}{dx}$)

$$\big(\varphi_1^{(1)}\big)' = \frac{\varphi_1'}{1 + \langle\varphi_1, \varphi_1\rangle} - \frac{\varphi_1^3}{(1 + \langle\varphi_1, \varphi_1\rangle)^2} \;\Rightarrow\; \varphi_1\big(\varphi_1^{(1)}\big)' = \varphi_1'\varphi_1^{(1)} - \varphi_1^2\big(\varphi_1^{(1)}\big)^2, \tag{2.20}$$

which simplifies the expression of the deformed potential

$$U^{(1)} = U - 2\bigg(\frac{\varphi_1^2}{1 + \langle\varphi_1, \varphi_1\rangle}\bigg)' = U - 2\big(\varphi_1\varphi_1^{(1)}\big)' = U - 2\Big(2\varphi_1'\varphi_1^{(1)} - \varphi_1^2\big(\varphi_1^{(1)}\big)^2\Big). \tag{2.21}$$

Then it is straightforward to show the deformed Schrödinger equation for $\varphi_1^{(1)}$:

$$\begin{aligned}\big(\varphi_1^{(1)}\big)'' &= \frac{\varphi_1''}{1 + \langle\varphi_1, \varphi_1\rangle} - \frac{4\varphi_1'\varphi_1^2}{(1 + \langle\varphi_1, \varphi_1\rangle)^2} + \frac{2\varphi_1^5}{(1 + \langle\varphi_1, \varphi_1\rangle)^3} \\ &= (U - \tilde{\mathcal{E}}_1)\varphi_1^{(1)} - 4\varphi_1'\big(\varphi_1^{(1)}\big)^2 + 2\varphi_1^2\big(\varphi_1^{(1)}\big)^3 = (U^{(1)} - \tilde{\mathcal{E}}_1)\varphi_1^{(1)}.\end{aligned} \tag{2.22}$$

The deformed Schrödinger equation for $\psi^{(1)}$ can be shown as follows:

$$\begin{aligned}\psi^{(1)} &= \psi - \varphi_1^{(1)}\langle\varphi_1, \psi\rangle \;\Rightarrow\; \big(\psi^{(1)}\big)' = \psi' - \big(\varphi_1^{(1)}\big)'\langle\varphi_1, \psi\rangle - \varphi_1^{(1)}\varphi_1\psi, \\ \big(\psi^{(1)}\big)'' &= \psi'' - \big(\varphi_1^{(1)}\big)''\langle\varphi_1, \psi\rangle - 2\big(\varphi_1^{(1)}\big)'\varphi_1\psi - \varphi_1^{(1)}(\varphi_1'\psi + \varphi_1\psi') \\ &= (U - \mathcal{E})\psi - (U^{(1)} - \tilde{\mathcal{E}}_1)\varphi_1^{(1)}\langle\varphi_1, \psi\rangle - 4\varphi_1'\varphi_1^{(1)}\psi + 2\varphi_1^2\big(\varphi_1^{(1)}\big)^2\psi \\ &\quad - \varphi_1^{(1)}(\varphi_1\psi' - \varphi_1'\psi) \\ &= (U^{(1)} - \mathcal{E})\psi^{(1)} - \varphi_1^{(1)}\Big(\mathrm{W}[\varphi_1, \psi] - (\tilde{\mathcal{E}}_1 - \mathcal{E})\langle\varphi_1, \psi\rangle\Big).\end{aligned} \tag{2.23}$$

Here $\mathrm{W}[\varphi_1, \psi]$ is the Wronskian, $\mathrm{W}[\varphi_1, \psi] \stackrel{\text{def}}{=} \varphi_1\psi' - \varphi_1'\psi$, satisfying

$$\big(\mathrm{W}[\varphi_1, \psi]\big)' = (\tilde{\mathcal{E}}_1 - \mathcal{E})\varphi_1\psi \;\Rightarrow\; \mathrm{W}[\varphi_1, \psi] = (\tilde{\mathcal{E}}_1 - \mathcal{E})\int_{x_1}^{x} dy\varphi(y)\psi(y) = (\tilde{\mathcal{E}}_1 - \mathcal{E})\langle\varphi_1, \psi\rangle, \tag{2.24}$$

because of the boundary conditions (2.4), (2.6) and (2.12). This proves the deformed Schrödinger equation for $\psi^{(1)}$.

There is one important exceptional situation when $\mathcal{E} = \tilde{\mathcal{E}}_1$, *i.e.* at the newly added eigenenergy level. In this case, $\mathrm{W}[\varphi_1, \psi] = \text{constant} \neq 0$ (otherwise $\psi^{(1)} \propto \varphi_1^{(1)}$) and $\psi^{(1)}$ is no longer a solution of the deformed Schrödinger equation.

2.2 Multiple states addition

It is now obvious that the new Hamiltonian $\mathcal{H}^{(1)}$ has the eigenspectrum $\{\tilde{\mathcal{E}}_1, \mathcal{E}_n\}$, and the corresponding eigenfunctions $\{\varphi_1^{(1)}, \phi_n^{(1)}\}$ $(n = 0, 1, \ldots)$, together with the seed solution $\{\varphi_j^{(1)}\}$

with energy $\{\tilde{\mathcal{E}}_j\}$ ($j=2,3,\ldots,M$), assuming that they also satisfy the boundary conditions. By picking up another seed solution, say $\varphi_2^{(1)}$, one can define another Abraham-Moses transformation:

$$
\begin{aligned}
\varphi_2^{(1)} \to \varphi_2^{(2)} &\stackrel{\text{def}}{=} \frac{\varphi_2^{(1)}}{1+\langle\varphi_2^{(1)},\varphi_2^{(1)}\rangle},\\
\psi^{(1)} \to \psi^{(2)} &\stackrel{\text{def}}{=} \psi^{(1)} - \varphi_2^{(2)}\langle\varphi_2^{(1)},\psi^{(1)}\rangle,\\
\mathcal{H}^{(1)} \to \mathcal{H}^{(2)} &\stackrel{\text{def}}{=} \mathcal{H}^{(1)} - 2\frac{d^2}{dx^2}\log\bigl(1+\langle\varphi_2^{(1)},\varphi_2^{(1)}\rangle\bigr),\\
\mathcal{H}^{(1)}\psi^{(1)} = \mathcal{E}\psi^{(1)} &\to \mathcal{H}^{(2)}\psi^{(2)} = \mathcal{E}\psi^{(2)}.
\end{aligned}
$$

This step can go on as many as the number of the prepared seed solutions, so long as the seed functions satisfy the boundary conditions. Let us use the M seed solutions $\{\varphi_j\}$ (2.5) in the order $j=1,2,\ldots,M$. At the K-th step, the transformation reads:

$$
\varphi_K^{(K-1)} \to \varphi_K^{(K)} \stackrel{\text{def}}{=} \frac{\varphi_K^{(K-1)}}{1+\langle\varphi_K^{(K-1)},\varphi_K^{(K-1)}\rangle}, \tag{2.25}
$$

$$
\psi^{(K-1)} \to \psi^{(K)} \stackrel{\text{def}}{=} \psi^{(K-1)} - \varphi_K^{(K)}\langle\varphi_K^{(K-1)},\psi^{(K-1)}\rangle, \tag{2.26}
$$

$$
\mathcal{H}^{(K-1)} \to \mathcal{H}^{(K)} \stackrel{\text{def}}{=} \mathcal{H}^{(K-1)} - 2\frac{d^2}{dx^2}\log\bigl(1+\langle\varphi_K^{(K-1)},\varphi_K^{(K-1)}\rangle\bigr), \tag{2.27}
$$

$$
\mathcal{H}^{(K-1)}\psi^{(K-1)} = \mathcal{E}\psi^{(K-1)} \to \mathcal{H}^{(K)}\psi^{(K)} = \mathcal{E}\psi^{(K)}, \tag{2.28}
$$

together with the orthogonality conditions of the eigenfunctions

$$
(\phi_n^{(K)},\phi_m^{(K)}) = (\phi_n,\phi_m) = h_n\delta_{n\,m} \quad (n,m=0,1,\ldots), \tag{2.29}
$$

$$
(\phi_n^{(K)},\varphi_j^{(K)}) = 0 \quad (n=0,1,\ldots,\,;\, j=1,\ldots,K), \tag{2.30}
$$

$$
(\varphi_j^{(K)},\varphi_k^{(K)}) = \delta_{j\,k} \quad (j,k=1,\ldots,K). \tag{2.31}
$$

The last formula (2.31) means that the multiple Abraham-Moses transformations could be interpreted as *orthonormalisation of non-normalisable vectors* $\{\varphi_j\}$. Indeed the formula (2.31) is independent of the fact that the functions $\{\varphi_j\}$ are the solutions of the Schrödinger equation (cf. (2.18), (2.42)).

The Abraham-Moses transformation for adding one eigenstate (2.11)–(2.14) involves one integration. It is naturally expected that the K-fold Abraham-Moses transformation would require K-fold integrals. It turns out that all the higher integrals can be partially integrated and only simple integrals remain. Let us define an $M\times M$ symmetric and positive definite matrix $\mathcal{F}$ depending on the seed solutions $\{\varphi_j\}$ $(j=1,\ldots,M)$ as follows:

$$\mathcal{F}(x) \equiv \mathcal{F}[\varphi_1,\ldots,\varphi_M](x), \quad (\mathcal{F})_{j\,k} \stackrel{\text{def}}{=} \delta_{j\,k} + \langle \varphi_j, \varphi_k \rangle \quad (j,k=1,\ldots,M). \tag{2.32}$$

For any $M \times M$ matrix $\mathcal{G}$, let us denote by $\mathcal{G}_K$ its $K \times K$ submatrix consisting of $(\mathcal{G})_{j\,k}$ $(j,k=1,\ldots,K)$. Because of the positive definiteness of $\mathcal{F}_K$, the inverse $\mathcal{F}_K^{-1}$ is always well-defined. In terms of $\mathcal{F}_K$ $(K=1,\ldots,M)$, we have the following:

Proposition 2.2 [18] *Repeating the one eigenstate adding Abraham-Moses transformations* (2.11)–(2.14) *M-times based on the seed solutions $\{\varphi_j\}$, $j=1,\ldots,M$ in this order, the Hamiltonian $\mathcal{H}^{(M)}$ and the corresponding eigenfunctions $\{\phi_n^{(M)}\}$, $\{\varphi_j^{(M)}\}$ can be expressed in the following simple form:*

$$\mathcal{H}^{(M)} = \mathcal{H} - 2\frac{d^2}{dx^2}\log\det\bigl(\mathcal{F}_M(x)\bigr), \tag{2.33}$$

$$\phi_n^{(M)}(x) = \phi_n(x) - \sum_{j,k=1}^{M} \varphi_j(x)\bigl(\mathcal{F}_M^{-1}(x)\bigr)_{j\,k}\langle \varphi_k, \phi_n\rangle(x) \quad (n=0,1,\ldots), \tag{2.34}$$

$$\varphi_j^{(M)}(x) = \sum_{k=1}^{M}\bigl(\mathcal{F}_M^{-1}(x)\bigr)_{j\,k}\varphi_k(x), \quad (\varphi_j^{(M)}, \varphi_k^{(M)}) = \delta_{j\,k} \quad (j,k=1,\ldots,M), \tag{2.35}$$

provided that all the intermediate seed solutions satisfy the boundary conditions.

Obviously $M=1$ quantities, $\mathcal{H}^{(1)}$ (2.14), $\{\phi_n^{(1)}\}$ (2.11) and $\{\varphi_1^{(1)}\}$ (2.16) have these forms.

It is rather amusing to verify $M=2$ formulas. From (2.19), we obtain

$$\langle \varphi_2^{(1)}, \varphi_2^{(1)}\rangle = \langle \varphi_2, \varphi_2\rangle - \frac{\langle \varphi_1, \varphi_2\rangle\langle \varphi_1, \varphi_2\rangle}{1+\langle \varphi_1, \varphi_1\rangle},$$

which means that

$$\begin{aligned}\bigl(1+\langle \varphi_1,\varphi_1\rangle\bigr)\bigl(1+\langle \varphi_2^{(1)},\varphi_2^{(1)}\rangle\bigr) &= \bigl(1+\langle \varphi_1,\varphi_1\rangle\bigr)\bigl(1+\langle \varphi_2,\varphi_2\rangle\bigr) - \langle \varphi_1,\varphi_2\rangle\langle \varphi_1,\varphi_2\rangle \\ &= (\mathcal{F}_2)_{1\,1}\times(\mathcal{F}_2)_{2\,2} - (\mathcal{F}_2)_{1\,2}\times(\mathcal{F}_2)_{2\,1} = \det(\mathcal{F}_2).\end{aligned}$$

This proves the potential formula for $U^{(2)}$. Likewise (2.19) gives

$$\langle \varphi_2^{(1)}, \phi_n^{(1)}\rangle = \langle \varphi_2, \phi_n\rangle - \frac{\langle \varphi_1, \varphi_2\rangle\langle \varphi_1, \phi_n\rangle}{1+\langle \varphi_1, \varphi_1\rangle}.$$

This gives an explicit expression of the eigenfunction $\phi_n^{(2)}$ as a linear combination of terms $\varphi_j\langle \varphi_k, \phi_n\rangle$ $(j,k=1,2)$:

$$\phi_n^{(2)} = \phi_n^{(1)} - \frac{\varphi_2^{(1)}\langle \varphi_2^{(1)}, \phi_n^{(1)}\rangle}{1+\langle \varphi_2^{(1)}, \varphi_2^{(1)}\rangle}$$

$$= \phi_n - \frac{\varphi_1\langle\varphi_1,\phi_n\rangle}{1+\langle\varphi_1,\varphi_1\rangle}$$
$$- \left(\varphi_2 - \frac{\varphi_1\langle\varphi_1,\varphi_2\rangle}{1+\langle\varphi_1,\varphi_1\rangle}\right) \times \left(\langle\varphi_2,\phi_n\rangle - \frac{\langle\varphi_1,\varphi_2\rangle\langle\varphi_1,\phi_n\rangle}{1+\langle\varphi_1,\varphi_1\rangle}\right) \times \frac{1}{1+\langle\varphi_2^{(1)},\varphi_2^{(1)}\rangle}.$$

It is indeed trivial to verify that the coefficient of the term $-\varphi_j\langle\varphi_k,\phi_n\rangle\det(\mathcal{F}_2)^{-1}$ is the co-factor of the matrix element $(\mathcal{F}_2)_{j\,k}$ $(j,k=1,2)$. This proves the eigenfunction formula (2.34) for $M=2$. The added eigenfunction formula (2.35) for $M=2$ can be verified in a similar manner.

In order to prove Proposition 2.2 inductively, we need the following Lemma, with the correspondence $A_n \leftrightarrow \mathcal{F}_K$, $A_{n-1} \leftrightarrow \mathcal{F}_{K-1}$, $a_{j\,k} \leftrightarrow \mathcal{F}_{j\,k} = \delta_{j\,k} + \langle\varphi_j,\varphi_k\rangle$. The Lemma can be proven elementarily by using the cofactor expansion theorem once or twice.

Lemma 2.3 *For an arbitrary regular matrix $A_n = (a_{j\,k})_{1\le j,k\le n}$ and its regular submatrix $A_{n-1} = (a_{j\,k})_{1\le j,k\le n-1}$, the following relations hold*

(i)
$$\left(A_n^{-1}\right)_{n\,n} = \frac{\det(A_{n-1})}{\det(A_n)}, \tag{2.36}$$

(i′)
$$\frac{\det(A_n)}{\det(A_{n-1})} = a_{n\,n} - \sum_{j,k=1}^{n-1} a_{n\,j}\left(A_{n-1}^{-1}\right)_{j\,k}a_{k\,n}, \tag{2.37}$$

(ii) $1\le j\le n-1$,
$$\left(A_{n-1}^{-1}\right)_{j\,n} = -\frac{\det(A_{n-1})}{\det(A_n)}\sum_{k=1}^{n-1}\left(A_{n-1}^{-1}\right)_{j\,k}a_{k\,n}, \tag{2.38}$$

(ii′) $1\le k\le n-1$,
$$\left(A_{n-1}^{-1}\right)_{n\,k} = -\frac{\det(A_{n-1})}{\det(A_n)}\sum_{j=1}^{n-1} a_{n\,j}\left(A_{n-1}^{-1}\right)_{j\,k}, \tag{2.39}$$

(iii) $1\le j,k\le n-1$,
$$\left(A_n^{-1}\right)_{j\,k} = \left(A_{n-1}^{-1}\right)_{j\,k} + \frac{\det(A_{n-1})}{\det(A_n)}\sum_{l,m=1}^{n-1}\left(A_{n-1}^{-1}\right)_{j\,l}a_{l\,n}a_{n\,m}\left(A_{n-1}^{-1}\right)_{m\,k}. \tag{2.40}$$

Supposing Proposition 2.2 is true up to $K-1$, we will show that it is true for K. For an arbitrary smooth function f with well-defined $\langle\varphi_j,f\rangle$, the $(K-1)$-th transformed function $f^{(K-1)}$ has the form
$$f^{(K-1)} = f - \sum_{j,k=1}^{K-1}\varphi_j\left(\mathcal{F}_{K-1}^{-1}\right)_{j\,k}\langle\varphi_k,f\rangle. \tag{2.41}$$

For such f and g, we have
$$f^{(K-1)}g^{(K-1)} = fg - \sum_{j,k=1}^{K-1}\varphi_j f\left(\mathcal{F}_{K-1}^{-1}\right)_{j\,k}\langle\varphi_k,g\rangle - \sum_{j,k=1}^{K-1}\varphi_j g\left(\mathcal{F}_{K-1}^{-1}\right)_{j\,k}\langle\varphi_k,f\rangle$$

$$+\sum_{j,k,l,m=1}^{K-1}\varphi_j\varphi_l\big(\mathcal{F}_{K-1}^{-1}\big)_{j\,k}\big(\mathcal{F}_{K-1}^{-1}\big)_{l\,m}\langle\varphi_k,f\rangle\langle\varphi_m,g\rangle$$

$$=fg-\frac{d}{dx}\bigg(\sum_{k,m=1}^{K-1}\langle\varphi_k,f\rangle\big(\mathcal{F}_{K-1}^{-1}\big)_{k\,m}\langle\varphi_m,g\rangle\bigg), \tag{2.42}$$

where we have used

$$\sum_{j,l=1}^{K-1}\varphi_j\varphi_l\big(\mathcal{F}_{K-1}^{-1}\big)_{j\,k}\big(\mathcal{F}_{K-1}^{-1}\big)_{l\,m}=\sum_{j,l=1}^{K-1}\frac{d}{dx}\Big((\mathcal{F}_{K-1})_{j\,l}\Big)\cdot\big(\mathcal{F}_{K-1}^{-1}\big)_{j\,k}\big(\mathcal{F}_{K-1}^{-1}\big)_{l\,m}=-\frac{d}{dx}\big(\mathcal{F}_{K-1}^{-1}\big)_{k\,m}.$$

Thus we obtain

$$\langle f^{(K-1)},g^{(K-1)}\rangle=\langle f,g\rangle-\sum_{j,k=1}^{K-1}\langle\varphi_j,f\rangle\big(\mathcal{F}_{K-1}^{-1}\big)_{j\,k}\langle\varphi_k,g\rangle. \tag{2.43}$$

The transformation is generated by $\varphi_K^{(K-1)}$,

$$\varphi_K^{(K-1)}=\varphi_K-\sum_{j,k=1}^{K-1}\varphi_j\big(\mathcal{F}_{K-1}^{-1}\big)_{j\,k}\langle\varphi_k,\varphi_K\rangle, \tag{2.44}$$

and (2.43) leads to

$$1+\langle\varphi_K^{(K-1)},\varphi_K^{(K-1)}\rangle=1+\langle\varphi_K,\varphi_K\rangle-\sum_{j,k=1}^{K-1}\langle\varphi_j,\varphi_K\rangle\big(\mathcal{F}_{K-1}^{-1}\big)_{j\,k}\langle\varphi_k,\varphi_K\rangle$$

$$=\frac{\det(\mathcal{F}_K)}{\det(\mathcal{F}_{K-1})}=\frac{1}{\big(\mathcal{F}_K^{-1}\big)_{K\,K}}. \tag{2.45}$$

In the second equality, Lemma (i) and (i′) are used. This proves the change of the potentials (2.33) of Proposition 2.2.

Here we introduce a simplifying notation

$$\alpha\stackrel{\text{def}}{=}\big(1+\langle\varphi_K^{(K-1)},\varphi_K^{(K-1)}\rangle\big)^{-1}=\big(\mathcal{F}_K^{-1}\big)_{K\,K}=\frac{\det(\mathcal{F}_{K-1})}{\det(\mathcal{F}_K)}. \tag{2.46}$$

The Abraham-Moses transformation on $\varphi_j^{(K-1)}$ gives for $1\le j\le K-1$

$$\varphi_j^{(K)}=\varphi_j^{(K-1)}-\frac{\varphi_K^{(K-1)}\langle\varphi_K^{(K-1)},\varphi_j^{(K-1)}\rangle}{1+\langle\varphi_K^{(K-1)},\varphi_K^{(K-1)}\rangle}. \tag{2.47}$$

By (2.43), the numerator on the right hand side can be evaluated:

$$\langle\varphi_K^{(K-1)},\varphi_j^{(K-1)}\rangle=\langle\varphi_K,\varphi_j\rangle-\sum_{l,m=1}^{K-1}\langle\varphi_l,\varphi_K\rangle\big(\mathcal{F}_{K-1}^{-1}\big)_{l\,m}\langle\varphi_m,\varphi_j\rangle$$

$$= \sum_{l=1}^{K-1} \langle \varphi_l, \varphi_K \rangle \big(\mathcal{F}_{K-1}^{-1}\big)_{l\,j}, \tag{2.48}$$

since $\langle \varphi_m, \varphi_j \rangle = (\mathcal{F}_{K-1})_{m\,j} - \delta_{m\,j}$. We obtain

$$\begin{aligned} \varphi_j^{(K)} &= \sum_{k=1}^{K-1} \big(\mathcal{F}_{K-1}^{-1}\big)_{j\,k} \varphi_k - \alpha\Big(\varphi_K - \sum_{k,m=1}^{K-1} \varphi_k \big(\mathcal{F}_{K-1}^{-1}\big)_{k\,m} \langle \varphi_m, \varphi_K \rangle\Big) \times \Big(\sum_{l=1}^{K-1} \langle \varphi_l, \varphi_K \rangle \big(\mathcal{F}_{K-1}^{-1}\big)_{l\,j}\Big) \\ &= \sum_{k=1}^{K-1} \varphi_k \bigg(\big(\mathcal{F}_{K-1}^{-1}\big)_{j\,k} + \alpha \sum_{l,m=1}^{K-1} \big(\mathcal{F}_{K-1}^{-1}\big)_{l\,j} \big(\mathcal{F}_{K-1}^{-1}\big)_{k\,m} \langle \varphi_l, \varphi_K \rangle \langle \varphi_m, \varphi_K \rangle\bigg) \\ &\quad - \alpha\, \varphi_K \Big(\sum_{k=1}^{K-1} \langle \varphi_k, \varphi_K \rangle \big(\mathcal{F}_{K-1}^{-1}\big)_{k\,j}\Big). \end{aligned} \tag{2.49}$$

By Lemma (ii) and (iii) we arrive at

$$\varphi_j^{(K)} = \sum_{k=1}^{K} \big(\mathcal{F}_K^{-1}\big)_{j\,k} \varphi_k \quad (j = 1, \ldots, K-1).$$

For $j = K$, we obtain directly from (2.47),

$$\varphi_K^{(K)} = \frac{\varphi_K^{(K-1)}}{1 + \langle \varphi_K^{(K-1)}, \varphi_K^{(K-1)} \rangle} = \alpha\Big(\varphi_K - \sum_{j,k=1}^{K-1} \varphi_j \big(\mathcal{F}_{K-1}^{-1}\big)_{j\,k} \langle \varphi_k, \varphi_K \rangle\Big),$$

which gives the desired result through Lemma (i) and (ii′),

$$\varphi_K^{(K)} = \sum_{k=1}^{K} \big(\mathcal{F}_K^{-1}\big)_{K\,k} \varphi_k.$$

We apply the Abraham-Moses transformation to $\phi_n^{(K-1)}$ by using the above seed solution:

$$\phi_n^{(K)} = \phi_n^{(K-1)} - \frac{\varphi_K^{(K-1)} \langle \varphi_K^{(K-1)}, \phi_n^{(K-1)} \rangle}{1 + \langle \varphi_K^{(K-1)}, \varphi_K^{(K-1)} \rangle}. \tag{2.50}$$

From (2.43), we have

$$\langle \varphi_K^{(K-1)}, \phi_n^{(K-1)} \rangle = \langle \varphi_K, \phi_n \rangle - \sum_{m,k=1}^{K-1} \langle \varphi_m, \varphi_K \rangle \big(\mathcal{F}_{K-1}^{-1}\big)_{m\,k} \langle \varphi_k, \phi_n \rangle. \tag{2.51}$$

We obtain, by using Lemma,

$$\phi_n^{(K)} = \phi_n - \sum_{j,k=1}^{K-1} \varphi_j \big(\mathcal{F}_{K-1}^{-1}\big)_{j\,k} \langle \varphi_k, \phi_n \rangle$$

$$
\begin{aligned}
&\quad - \alpha\Big(\varphi_K - \sum_{j,l=1}^{K-1}\varphi_j\big(\mathcal{F}_{K-1}^{-1}\big)_{j\,l}\langle\varphi_l,\varphi_K\rangle\Big)\times\Big(\langle\varphi_K,\phi_n\rangle - \sum_{m,k=1}^{K-1}\langle\varphi_m,\varphi_K\rangle\big(\mathcal{F}_{K-1}^{-1}\big)_{m\,k}\langle\varphi_k,\phi_n\rangle\Big)\\
&= \phi_n - \bigg(\sum_{j,k=1}^{K-1}\varphi_j\langle\varphi_k,\phi_n\rangle\Big(\big(\mathcal{F}_{K-1}^{-1}\big)_{j\,k} + \alpha\sum_{l,m=1}^{K-1}\big(\mathcal{F}_{K-1}^{-1}\big)_{j\,l}\big(\mathcal{F}_{K-1}^{-1}\big)_{m\,k}\langle\varphi_l,\varphi_K\rangle\langle\varphi_m,\varphi_K\rangle\Big)\\
&\qquad - \alpha\sum_{j=1}^{K-1}\varphi_j\langle\varphi_K,\phi_n\rangle\Big(\sum_{k=1}^{K-1}\big(\mathcal{F}_{K-1}^{-1}\big)_{j\,k}\langle\varphi_k,\varphi_K\rangle\Big)\\
&\qquad - \alpha\sum_{j=1}^{K-1}\varphi_K\langle\varphi_j,\phi_n\rangle\Big(\sum_{k=1}^{K-1}\big(\mathcal{F}_{K-1}^{-1}\big)_{k\,j}\langle\varphi_k,\varphi_K\rangle\Big) + \alpha\,\varphi_K\langle\varphi_K,\phi_n\rangle\bigg)\\
&= \phi_n - \sum_{j,k=1}^{K}\varphi_j\big(\mathcal{F}_K^{-1}\big)_{j\,k}\langle\varphi_k,\phi_n\rangle.
\end{aligned}
$$

This concludes the proof of Proposition 2.2.

It is rather easy to show the orthonormality (2.29)–(2.31) based on (2.34)–(2.35) of Proposition 2.2.

2.3 One state deletion

Deleting multiple eigenstates by Darboux transformation is well established by Krein-Adler [22]. By choosing a subset of the original eigenfunctions (2.1)–(2.4) specified by $\mathcal{D} = \{d_1, \ldots, d_M\}$ ($d_j \geq 0$), the deleted system is given by the ratio of Wronskians:

$$
\begin{aligned}
&\mathrm{W}[f_1, f_2, \ldots, f_n](x) \stackrel{\text{def}}{=} \det\Big(\frac{d^{j-1}f_k(x)}{dx^{j-1}}\Big)_{1\leq j,k\leq n},\\
&\psi \to \psi^{[M]} \stackrel{\text{def}}{=} \frac{\mathrm{W}[\phi_{d_1},\phi_{d_2},\ldots,\phi_{d_M},\psi]}{\mathrm{W}[\phi_{d_1},\phi_{d_2},\ldots,\phi_{d_M}]},\\
&\phi_n \to \phi_n^{[M]} \stackrel{\text{def}}{=} \frac{\mathrm{W}[\phi_{d_1},\phi_{d_2},\ldots,\phi_{d_M},\phi_n]}{\mathrm{W}[\phi_{d_1},\phi_{d_2},\ldots,\phi_{d_M}]} \quad (n = 0, 1, \ldots, ;\, n \notin \mathcal{D}),\\
&\mathcal{H}^{[M]}\psi^{[M]} = \mathcal{E}\psi^{[M]}, \quad \mathcal{H}^{[M]}\phi_n^{[M]} = \mathcal{E}_n\phi_n^{[M]},\\
&\mathcal{H}^{[M]} \stackrel{\text{def}}{=} \mathcal{H} - 2\frac{d^2}{dx^2}\log\Big|\mathrm{W}[\phi_{d_1},\phi_{d_2},\ldots,\phi_{d_M}]\Big|,\\
&(\phi_m^{[M]},\phi_n^{[M]}) = \prod_{j=1}^{M}(\mathcal{E}_n - \mathcal{E}_{d_j})\cdot h_n\delta_{m\,n}.
\end{aligned}
$$

In order to guarantee the non-singularity of the potential and the positive definiteness of the norm, the deleted levels must satisfy the conditions [22]:

$$
\prod_{j=1}^{M}(n - d_j) \geq 0 \quad (\forall n \in \mathbb{Z}_{\geq 0}). \tag{2.52}
$$

The conditions mean, in particular, a single state ($M = 1$) cannot be deleted except for the groundstate ϕ_0. By the Abraham-Moses transformations, in contrast, one can delete one and many eigenstates.

Here we consider the process of deleting one discrete eigenlevel from the original Hamiltonian system (2.1)–(2.4). Let us denote the eigenfunction to be deleted by ϕ_d. By almost the same calculation as in the case of one state addition, we obtain the following:

Proposition 2.4 [6] *When the eigenfunction ϕ_d has unit norm $(\phi_d, \phi_d) = 1$, the following transformation maps the solution ψ of the original Hamiltonian system to a solution $\psi^{(1)}$ of the deformed Hamiltonian system $\mathcal{H}^{(1)}$* (2.56) *with the same energy:*

$$\phi_d \to \phi_d^{(1)} \stackrel{\text{def}}{=} \frac{\phi_d}{1 - \langle \phi_d, \phi_d \rangle}, \quad \psi \to \psi^{(1)} \stackrel{\text{def}}{=} \psi + \phi_d^{(1)} \langle \phi_d, \psi \rangle, \tag{2.53}$$

$$\phi_n \to \phi_n^{(1)} \stackrel{\text{def}}{=} \phi_n + \phi_d^{(1)} \langle \phi_d, \phi_n \rangle \quad (n = 0, 1, \ldots, ; n \neq d), \tag{2.54}$$

$$\mathcal{H}^{(1)} \psi^{(1)} = \mathcal{E} \psi^{(1)}, \quad \mathcal{H}^{(1)} \phi_n^{(1)} = \mathcal{E}_n \phi_n^{(1)}, \tag{2.55}$$

$$\mathcal{H}^{(1)} \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + U^{(1)}(x), \quad U^{(1)}(x) \stackrel{\text{def}}{=} U(x) - 2\frac{d^2}{dx^2} \log\big(1 - \langle \varphi_1, \varphi_1 \rangle\big). \tag{2.56}$$

The norms of the eigenfunctions are preserved except for $\phi_d^{(1)}$, which becomes non-square integrable. *Thus the eigenstate ϕ_d is deleted:*

$$(\phi_n^{(1)}, \phi_m^{(1)}) = (\phi_n, \phi_m) = h_n \delta_{nm} \quad (n, m \neq d), \quad (\phi_d^{(1)}, \phi_d^{(1)}) = \infty. \tag{2.57}$$

The transformation $\phi_d \to \phi_d^{(1)}$ (2.53)–(2.56) *defines a singular Hamiltonian* $\mathcal{H}^{(1)}$, when ϕ_d *has norm greater than unity* $(\phi_d, \phi_d) > 1$. When ϕ_d's norm is *less than unity* $(\phi_d, \phi_d) < 1$, the new wavefunction $\phi_d^{(1)}$ has a finite norm and the *deletion of the state is not achieved*. If a seed solution φ_d is used in the state deleting transformation (2.54), at a certain point $x \in (x_1, x_2)$, $1 - \langle \varphi_d, \varphi_d \rangle$ vanishes and it leads to a singular Hamiltonian.

The fact that the norm of the eigenfunction to be deleted, ϕ_d, is strictly restricted to unity can be understood easily when we consider that the deletion is indeed the inverse process of the addition, in which all the newly added eigenstates have unit norm, and vice versa.

Let us first add an eigenfunction $\varphi_d^{(1)}$ by using a seed solution φ_d, then delete the created eigenfunction $\varphi_d^{(1)}$:

$$\varphi_d^{(1)} = \frac{\varphi_d}{1 + \langle \varphi_d, \varphi_d \rangle}, \qquad \varphi_d^{(2)} = \frac{\varphi_d^{(1)}}{1 - \langle \varphi_d^{(1)}, \varphi_d^{(1)} \rangle},$$

$$
\begin{aligned}
&\phi_n^{(1)} = \phi_n - \varphi_d^{(1)}\langle\varphi_d, \phi_n\rangle, \qquad \phi_n^{(2)} = \phi_n^{(1)} + \varphi_d^{(2)}\langle\varphi_d^{(1)}, \phi_n^{(1)}\rangle,\\
&\mathcal{H}^{(1)} = \mathcal{H} - 2\frac{d^2}{dx^2}\log\bigl(1 + \langle\varphi_d, \varphi_d\rangle\bigr), \quad \mathcal{H}^{(2)} = \mathcal{H}^{(1)} - 2\frac{d^2}{dx^2}\log\bigl(1 - \langle\varphi_d^{(1)}, \varphi_d^{(1)}\rangle\bigr).
\end{aligned}
$$

It is elementary to show (cf. (2.17), (2.19))

$$
\begin{aligned}
&\langle\varphi_d^{(1)}, \varphi_d^{(1)}\rangle = 1 - \frac{1}{1 + \langle\varphi_d, \varphi_d\rangle} \ \Rightarrow\ \bigl(1 + \langle\varphi_d, \varphi_d\rangle\bigr)\bigl(1 - \langle\varphi_d^{(1)}, \varphi_d^{(1)}\rangle\bigr) = 1,\\
&\langle\varphi_d^{(1)}, \phi_n^{(1)}\rangle = \frac{\langle\varphi_d, \phi_n\rangle}{1 + \langle\varphi_d, \varphi_d\rangle}.
\end{aligned}
$$

These lead, as expected, to:

$$
\begin{aligned}
&\varphi_d^{(2)} = \varphi_d, \quad \mathcal{H}^{(2)} = \mathcal{H},\\
&\phi_n^{(2)} = \phi_n - \varphi_d^{(1)}\langle\varphi_d, \phi_n\rangle + \varphi_d\frac{\langle\varphi_d, \phi_n\rangle}{1 + \langle\varphi_d, \varphi_d\rangle} = \phi_n.
\end{aligned}
$$

Next we work in the opposite direction. We first delete a unit norm eigenstate ϕ_d, $(\phi_d, \phi_d) = 1$, by mapping it to $\phi_d^{(1)}$, which is not square integrable, $(\phi_d^{(1)}, \phi_d^{(1)}) = \infty$. Then we add an eigenstate by using the seed solution $\phi_d^{(1)}$:

$$
\begin{aligned}
&\phi_d^{(1)} = \frac{\phi_d}{1 - \langle\phi_d, \phi_d\rangle}, \qquad \phi_d^{(2)} = \frac{\phi_d^{(1)}}{1 + \langle\phi_d^{(1)}, \phi_d^{(1)}\rangle},\\
&\phi_n^{(1)} = \phi_n + \phi_d^{(1)}\langle\phi_d, \phi_n\rangle, \qquad \phi_n^{(2)} = \phi_n^{(1)} - \phi_d^{(2)}\langle\phi_d^{(1)}, \phi_n^{(1)}\rangle,\\
&\mathcal{H}^{(1)} = \mathcal{H} - 2\frac{d^2}{dx^2}\log\bigl(1 - \langle\phi_d, \phi_d\rangle\bigr), \quad \mathcal{H}^{(2)} = \mathcal{H}^{(1)} - 2\frac{d^2}{dx^2}\log\bigl(1 + \langle\phi_d^{(1)}, \phi_d^{(1)}\rangle\bigr).
\end{aligned}
$$

It is again elementary to show

$$
\begin{aligned}
&\langle\phi_d^{(1)}, \phi_d^{(1)}\rangle = \frac{1}{1 - \langle\phi_d, \phi_d\rangle} - 1 \ \Rightarrow\ \bigl(1 - \langle\phi_d, \phi_d\rangle\bigr)\bigl(1 + \langle\phi_d^{(1)}, \phi_d^{(1)}\rangle\bigr) = 1,\\
&\langle\phi_d^{(1)}, \phi_n^{(1)}\rangle = \frac{\langle\phi_d, \phi_n\rangle}{1 - \langle\phi_d, \phi_d\rangle}.
\end{aligned}
$$

These lead, as expected, to:

$$
\begin{aligned}
&\phi_d^{(2)} = \phi_d, \quad \mathcal{H}^{(2)} = \mathcal{H},\\
&\phi_n^{(2)} = \phi_n + \phi_d^{(1)}\langle\phi_d, \phi_n\rangle - \phi_d\frac{\langle\phi_d, \phi_n\rangle}{1 - \langle\phi_d, \phi_d\rangle} = \phi_n.
\end{aligned}
$$

At the end of this subsection let us present the formulas of multiple eigenstates deletion by using M eigenfunctions $\{\phi_{d_j}\}$, $(\phi_{d_j}, \phi_{d_k}) = \delta_{j\,k}$, $j = 1, \ldots, M$, in this order:

$$
\mathcal{H}^{(M)} = \mathcal{H} - 2\frac{d^2}{dx^2}\log\det\bigl(\bar{\mathcal{F}}_M(x)\bigr), \quad \mathcal{D} \stackrel{\text{def}}{=} \{d_1, \ldots, d_M\}, \tag{2.58}
$$

$$\phi_n^{(M)}(x)=\phi_n(x)+\sum_{j,k=1}^{M}\phi_{d_j}(x)\big(\bar{\mathcal{F}}_M^{-1}(x)\big)_{j\,k}\langle\phi_{d_k},\phi_n\rangle(x)\quad(n=0,1,\ldots,;\,n\notin\mathcal{D}),\tag{2.59}$$

$$\phi_{d_j}^{(M)}(x)=\sum_{k=1}^{M}\big(\bar{\mathcal{F}}_M^{-1}(x)\big)_{j\,k}\phi_{d_k}(x)\quad(j=1,\ldots,M),\tag{2.60}$$

$$\bar{\mathcal{F}}(x)\equiv\bar{\mathcal{F}}[\phi_{d_1},\ldots,\phi_{d_M}](x)=(\bar{\mathcal{F}}_{j,k})_{1\le j,k\le M},\quad(\bar{\mathcal{F}})_{j\,k}\stackrel{\text{def}}{=}\delta_{j\,k}-\langle\phi_{d_j},\phi_{d_k}\rangle.\tag{2.61}$$

These formulas are almost the same as those for the multiple eigenstate addition (2.33)–(2.35) in Proposition 2.2, with $\mathcal{F}$ replaced by $\bar{\mathcal{F}}$ and a plus sign in (2.59) instead of a minus sign in (2.34). The proof goes parallel with the multiple eigenstate addition case. Indeed these formulas are obtained from those for the multiple eigenstate addition (2.33)–(2.35) by changing $\varphi_j\to i\phi_{d_j}$, $\psi\to i\psi$ and $\phi_n\to i\phi_n$ ($n\notin\mathcal{D}$), $i\equiv\sqrt{-1}$.

2.4 Comments on Abraham-Moses transformations

Here are some comments on various aspects of the Abraham-Moses transformations. As for the seed solutions for adding eigenstates (2.5), we have not specified the overall scale of these functions, since there is no standard way of fixing the scale of such non square integrable functions. The very fact that the obtained eigenfunctions have unit norms is independent of such overall scales. As stressed in Abraham-Moses paper [6], one could use one of the original eigenfunctions, ϕ_a with energy $\mathcal{E}_a$, as a seed solution. In this case, $\phi_a^{(1)}=\phi_a/(1+\langle\phi_a,\phi_a\rangle)$ is still an eigenfunction with energy $\mathcal{E}_a$. Its norm is changed to $(\phi_a^{(1)},\phi_a^{(1)})=(\phi_a,\phi_a)/(1+(\phi_a,\phi_a))$.

As for the type II seed solutions (2.7) [17], we have to change the definition of the function $\langle f,g\rangle(x)$ as follows:

$$\langle f,g\rangle(x)\stackrel{\text{def}}{=}-\int_{x_2}^{x}dyf(y)g(y)=\langle g,f\rangle(x),\quad x_1<x<x_2,\tag{2.62}$$

$$\langle f,g\rangle(x_1)=(f,g),\quad\langle f,g\rangle(x_2)=0.\tag{2.63}$$

Then all the formulas in this section are also true when the type II seed solutions only are used.

It is definitely true that one can apply the state adding Abraham-Moses transformations in terms of both type I and II seed solutions in any order, if seed solutions of one type remain seed solutions after transformations by the other type. This depends on the explicit forms of the seed solutions. Let us consider a seed solution φ_2 of type II after the transformation by a seed solution φ_1 of type I:

$$\varphi_2^{(1)} = \varphi_2 - \varphi_1^{(1)}\langle\varphi_1, \varphi_2\rangle.$$

By construction $\varphi_1^{(1)} = \varphi_1/(1 + \langle\varphi_1, \varphi_1\rangle)$ is well behaved on both boundaries. If the integral $\langle\varphi_1, \varphi_2\rangle(x) = \int_{x_1}^{x} dy\varphi_1(y)\varphi_2(y)$ exists on both boundaries or its certain regularisation exists, it is highly likely that $\varphi_2^{(1)}$ can qualify as a type II seed solution. The situation is about the same for a seed solution of type I after the transformation by a type II seed solution.

Even when these mixed multiple transformations are possible, to write down the generic formulas like Proposition 2.2 for such Abraham-Moses transformations is a different matter. In contrast to the multiple Darboux transformations in terms of type I and II virtual state wavefunctions worked out for the radial oscillator, Darboux-Pöschl-Teller and other solvable potentials [7, 23, 24], we are not quite sure if generic formulas exist for the multiple state adding Abraham-Moses transformations in terms of both type I and II seed solutions.

Let us briefly comment on the relation between a Darboux transformation and one state adding Abraham-Moses transformation [19, 20]. Let us first execute a Darboux transformation by picking up a seed solution $(\varphi, \tilde{\mathcal{E}})$ of type I (2.6):

$$\mathcal{H} \to \mathcal{H}^{[1]} \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + U^{[1]}(x), \quad U^{[1]}(x) \stackrel{\text{def}}{=} U(x) - 2\frac{d^2}{dx^2}\log|\varphi|,$$
$$\psi \to \psi^{[1]} \stackrel{\text{def}}{=} \psi' - \frac{\varphi'}{\varphi}\psi, \quad \mathcal{H}^{[1]}\psi^{[1]} = \mathcal{E}\psi^{[1]}.$$

Next we perform a second Darboux transformation in terms of a particular solution of $\mathcal{H}^{[1]}$:

$$\bar{\varphi}^{[1]} \stackrel{\text{def}}{=} \frac{1}{\varphi}(1 + \langle\varphi, \varphi\rangle), \quad \mathcal{H}^{[1]}\bar{\varphi}^{[1]} = \tilde{\mathcal{E}}\bar{\varphi}^{[1]}, \tag{2.64}$$
$$\mathcal{H}^{[1]} \to \mathcal{H}^{[2]} \stackrel{\text{def}}{=} \mathcal{H}^{[1]} - 2\frac{d^2}{dx^2}\log\big|\,\bar{\varphi}^{[1]}\big| = \mathcal{H} - 2\frac{d^2}{dx^2}\log\big(1 + \langle\varphi, \varphi\rangle\big),$$
$$\psi^{[1]} \to \psi^{[2]} \stackrel{\text{def}}{=} \big(\psi^{[1]}\big)' - \frac{(\bar{\varphi}^{[1]})'}{\bar{\varphi}^{[1]}}\psi^{[1]} = (\tilde{\mathcal{E}} - \mathcal{E})\psi - \frac{1}{\bar{\varphi}^{[1]}}\mathrm{W}[\varphi, \psi].$$

Since ψ and φ satisfy the boundary conditions (2.4) and (2.6), the Wronskian $\mathrm{W}[\varphi, \psi]$ can be expressed in terms of an integral $\mathrm{W}[\varphi, \psi] = (\tilde{\mathcal{E}} - \mathcal{E})\langle\varphi, \psi\rangle$ as in (2.24). Thus we arrive at the one state adding Abraham-Moses transformation (2.11)

$$\psi^{[2]} = (\tilde{\mathcal{E}} - \mathcal{E})\Big(\psi - \frac{\varphi}{1 + \langle\varphi, \varphi\rangle}\langle\varphi, \psi\rangle\Big). \tag{2.65}$$

Instead of a seed solution $(\varphi, \tilde{\mathcal{E}})$, an eigenstate $(\phi_d, \mathcal{E}_d)$ and $\bar{\phi}_d^{[1]} \stackrel{\text{def}}{=} \frac{1}{\phi_d}(1 - \langle\phi_d, \phi_d\rangle)$ are used, the one eigenstate deleting Abraham-Moses transformation (2.54) is obtained. The relation between the two seed solutions $\varphi \leftrightarrow \bar{\varphi}^{[1]}$ (2.64) and its many disguises have been discussed by many authors in connection with Abraham-Moses transformations [16].

3 Generalised Virtual State Wavefunctions

Here we present the explicit forms of various seed solutions for some exactly solvable potentials [1, 2], in particular the radial oscillator and the Darboux-Pöschl-Teller potential and a few more. (The harmonic oscillator case has been discussed in the original Abraham-Moses paper [6].) They are necessary in order to carry out the program of 'generating exactly solvable potentials' by adding *a finite number of eigenstates* with *arbitrary energies* through multiple Abraham-Moses transformations. These seed solutions are tentatively called 'generalised virtual state wavefunctions.' The 'virtual state wavefunctions' have been introduced by the present authors [7, 9, 10] and extensively used to generate 'rational or polynomial extensions' of various solvable potentials, through multiple Darboux-Crum transformations [4, 5, 22]. Obtained from the eigenfunctions by a discrete symmetry operation of the original Hamiltonian, these virtual state wavefunctions are of polynomial character and their energies are discretised and negative by restricting the ranges of their degrees. They have been indispensable for the construction of the 'multi-indexed Jacobi and Laguerre polynomials' [7] including various exceptional orthogonal polynomials as the simplest cases [11]–[15]. Since the negative energy condition is irrelevant, these virtual state wavefunctions of type I and II, without any restrictions to their degrees, are bona fide seed solutions for Abraham-Moses transformations, easiest to use in practical applications.

In order to construct *seed solutions of arbitrary real energies*, we generalise the polynomial type virtual state wavefunctions as well as the eigenfunctions to hypergeometric functions (${}_2F_1$ and ${}_1F_1$) type, by making the degree of polynomial type solutions to be a continuous real number. This has been done in our previous paper [25]. See also [26]. For the Darboux-Pöschl-Teller potential, we also report another genre of real seed solutions corresponding to 'complex degrees'.

3.1 Radial oscillator and Darboux-Pöschl-Teller potentials

We first recapitulate the known virtual state wavefunctions of type I and II of the Hamiltonian systems with the radial oscillator and Darboux-Pöschl-Teller potentials. The potentials are

$$U(x)=\begin{cases} x^2+\dfrac{g(g-1)}{x^2}-(1+2g), & x_1=0,\ x_2=\infty,\quad g>\frac{3}{2} \qquad :\mathrm{L}\\ \dfrac{g(g-1)}{\sin^2 x}+\dfrac{h(h-1)}{\cos^2 x}-(g+h)^2, & x_1=0,\ x_2=\frac{\pi}{2},\quad g,\,h>\frac{3}{2} \quad :\mathrm{J}\end{cases}, \tag{3.1}$$

in which L and J stand for the names of their eigenfunctions, the Laguerre and Jacobi polynomials. The parameter ranges are consistent with the restrictions for the eigenfunctions (2.4) and for the seed solutions (2.6), (2.7). The eigenfunctions are factorised into the groundstate eigenfunction and the polynomial in a functions $\eta=\eta(x)$, called *sinusoidal coordinate* [3, 27]:

$$\phi_n(x;\boldsymbol{\lambda})=\phi_0(x;\boldsymbol{\lambda})P_n\big(\eta(x);\boldsymbol{\lambda}\big), \tag{3.2}$$

in which $\boldsymbol{\lambda}$ stands for the parameters, g for L and (g,h) for J. Their explicit forms are

$$\begin{aligned}\mathrm{L}:\quad &\phi_0(x;g)=e^{-\frac{1}{2}x^2}x^g,\quad P_n(\eta;g)=L_n^{(g-\frac{1}{2})}(\eta),\quad \eta(x)=x^2,\\ &\mathcal{E}_n(g)=4n,\quad h_n(g)=\frac{1}{2\,n!}\Gamma(n+g+\tfrac{1}{2}),\end{aligned} \tag{3.3}$$

$$\begin{aligned}\mathrm{J}:\quad &\phi_0(x;g,h)=(\sin x)^g(\cos x)^h,\quad P_n(\eta;g,h)=P_n^{(g-\frac{1}{2},h-\frac{1}{2})}(\eta),\quad \eta(x)=\cos 2x,\\ &\mathcal{E}_n(g,h)=4n(n+g+h),\quad h_n(g,h)=\frac{\Gamma(n+g+\frac{1}{2})\Gamma(n+h+\frac{1}{2})}{2\,n!(2n+g+h)\Gamma(n+g+h)},\end{aligned} \tag{3.4}$$

in which h_n is the normalisation constant of the norm introduced in (2.3).

It is obvious that the above potential (3.1) without the constant term ($-(1+2g)$ for L and $-(g+h)^2$ for J) are invariant under the discrete transformation $g\leftrightarrow 1-g$ and/or $h\leftrightarrow 1-h$. The Hamiltonian for L without the constant term changes the sign under $x\to ix$. These are the discrete symmetry transformations mapping the above eigenfunctions to seed solutions of type I and II, which are again *polynomial solutions*. The virtual states wavefunctions for L are:

$$\mathrm{L1}:\quad \tilde{\phi}_{\mathrm{v}}^{\mathrm{I}}(x;g)\stackrel{\text{def}}{=}e^{\frac{1}{2}x^2}x^gL_{\mathrm{v}}^{(g-\frac{1}{2})}\big(-\eta(x)\big),\qquad \tilde{\mathcal{E}}_{\mathrm{v}}^{\mathrm{I}}(g)=-4(g+\mathrm{v}+\tfrac{1}{2})\quad(\mathrm{v}\in\mathbb{Z}_{\geq 0}), \tag{3.5}$$

$$\mathrm{L2}:\quad \tilde{\phi}_{\mathrm{v}}^{\mathrm{II}}(x;g)\stackrel{\text{def}}{=}e^{-\frac{1}{2}x^2}x^{1-g}L_{\mathrm{v}}^{(\frac{1}{2}-g)}\big(\eta(x)\big),\quad \tilde{\mathcal{E}}_{\mathrm{v}}^{\mathrm{II}}(g)=-4(g-\mathrm{v}-\tfrac{1}{2})\quad(\mathrm{v}\in\mathbb{Z}_{\geq 0}). \tag{3.6}$$

The virtual states wavefunctions for J are:

$$\begin{aligned}\mathrm{J1}:\quad &\tilde{\phi}_{\mathrm{v}}^{\mathrm{I}}(x;g,h)\stackrel{\text{def}}{=}(\sin x)^g(\cos x)^{1-h}P_{\mathrm{v}}^{(g-\frac{1}{2},\frac{1}{2}-h)}\big(\eta(x)\big),\\ &\tilde{\mathcal{E}}_{\mathrm{v}}^{\mathrm{I}}(g,h)=-4(g+\mathrm{v}+\tfrac{1}{2})(h-\mathrm{v}-\tfrac{1}{2})\quad(\mathrm{v}\in\mathbb{Z}_{\geq 0}),\end{aligned} \tag{3.7}$$

$$\mathrm{J2}:\quad \tilde{\phi}_{\mathrm{v}}^{\mathrm{II}}(x;g,h)\stackrel{\text{def}}{=}(\sin x)^{1-g}(\cos x)^hP_{\mathrm{v}}^{(\frac{1}{2}-g,h-\frac{1}{2})}\big(\eta(x)\big),$$

$$\tilde{\mathcal{E}}_{\mathrm{v}}^{\mathrm{II}}(g,h) = -4(g-\mathrm{v}-\tfrac{1}{2})(h+\mathrm{v}+\tfrac{1}{2}) \quad (\mathrm{v}\in\mathbb{Z}_{\geq 0}). \tag{3.8}$$

Due to the parity property of the Jacobi polynomial $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$, the two virtual state polynomials for J are related by this parity transformation. It is obvious that the virtual state wavefunctions satisfy the boundary conditions (2.6) and (2.7) and that the type I solutions are not square integrable at the upper boundary x_2 and the type II solutions are not square integrable at the lower boundary x_1. If the two types of the discrete symmetry operations are applied, the resulting solutions are not square integrable at either boundary. They are called pseudo virtual state wavefunctions [24] and they cannot be used for the Abraham-Moses transformations.

At each step of state adding Abraham-Moses transformation, the parameters of the above virtual state wavefunctions describing the boundary conditions change:

$$\mathrm{J1}: \quad h\to h-2, \qquad \mathrm{L2}\ \&\ \mathrm{J2}: \quad g\to g-2. \tag{3.9}$$

These are consistent with the interpretation that Abraham-Moses transformations can be understood as special two-step Darboux transformations. This also means that the total number of addable eigenstates is limited when using the above J1, J2 and L2 seed solutions. As stressed in [10], the L1 case is obtained from J1 by the confluence limit, $h\to\infty$. Thus their boundary conditions are not affected by each Abraham-Moses transformation and the L1 virtual state wavefunctions can be used as many as wanted.

These known virtual state wavefunctions are all of polynomial type and their energies $\tilde{\mathcal{E}}_{\mathrm{v}}$ take only discretised values for integer v, which is the degree of the polynomial. In the next subsection, we generalise the virtual state wavefunctions to take arbitrary real energies. It should be easy for each explicit example of seed solutions to calculate the change of the boundary parameters as above.

3.1.1 generalised virtual state wavefunctions

The strategy for the generalisation is quite simple, as shown in [25] for the type I cases. We rewrite the Laguerre and Jacobi polynomials in terms of (confluent) hypergeometric functions:

$$\begin{aligned} L_n^{(\alpha)}(x) &= \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!} \\ &= \frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)\Gamma(n+1)}\,{}_1F_1\Bigl(\begin{matrix}-n\\ \alpha+1\end{matrix}\Bigm| x\Bigr), \end{aligned} \tag{3.10}$$

$$P_n^{(\alpha,\beta)}(x)=\frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+1)_k\,k!}\Big(\frac{1-x}{2}\Big)^k$$
$$=\frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)\Gamma(n+1)}\,{}_2F_1\Big(\begin{matrix}-n,n+\alpha+\beta+1\\ \alpha+1\end{matrix}\Bigm|\frac{1-x}{2}\Big). \tag{3.11}$$

The expressions in terms of (confluent) hypergeometric functions are valid for any complex number n and satisfy the Laguerre and Jacobi's differential equation, respectively. Since the overall scale of the seed solutions is irrelevant, we will drop the overall factors. The non-polynomial forms of the seed solutions valid for a real number v ($\text{v}\in\mathbb{R}$, $\text{v}\notin\mathbb{Z}_{\geq 0}$) are

$$\text{L1}:\quad \tilde{\phi}^{\text{I}}_{\text{v}}(x;g)\stackrel{\text{def}}{=}e^{\frac12x^2}x^g{}_1F_1\Big(\begin{matrix}-\text{v}\\ g+\frac12\end{matrix}\Bigm|-\eta(x)\Big), \tag{3.12}$$
$$\text{L2}:\quad \tilde{\phi}^{\text{II}}_{\text{v}}(x;g)\stackrel{\text{def}}{=}e^{-\frac12x^2}x^{1-g}{}_1F_1\Big(\begin{matrix}-\text{v}\\ \frac32-g\end{matrix}\Bigm|\eta(x)\Big), \tag{3.13}$$
$$\text{J1}:\quad \tilde{\phi}^{\text{I}}_{\text{v}}(x;g,h)\stackrel{\text{def}}{=}(\sin x)^g(\cos x)^{1-h}{}_2F_1\Big(\begin{matrix}-\text{v},\text{v}+g-h+1\\ g+\frac12\end{matrix}\Bigm|\frac{1-\eta(x)}{2}\Big), \tag{3.14}$$
$$\text{J2}:\quad \tilde{\phi}^{\text{II}}_{\text{v}}(x;g,h)\stackrel{\text{def}}{=}(\sin x)^{1-g}(\cos x)^{h}{}_2F_1\Big(\begin{matrix}-\text{v},\text{v}+h-g+1\\ h+\frac12\end{matrix}\Bigm|\frac{1+\eta(x)}{2}\Big). \tag{3.15}$$

The energy formulas (3.5)–(3.8) are now valid for any real number v. By using the Kummer's transformation formulas,

$${}_1F_1\Big(\begin{matrix}\alpha\\ \beta\end{matrix}\Bigm|x\Big)=e^x{}_1F_1\Big(\begin{matrix}\beta-\alpha\\ \beta\end{matrix}\Bigm|-x\Big), \tag{3.16}$$
$${}_2F_1\Big(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\Bigm|x\Big)=(1-x)^{\gamma-\alpha-\beta}{}_2F_1\Big(\begin{matrix}\gamma-\alpha,\gamma-\beta\\ \gamma\end{matrix}\Bigm|x\Big), \tag{3.17}$$

they can be rewritten [25]. For example,

$$\text{L1 (3.12)}\quad \tilde{\phi}^{\text{I}}_{\text{v}}(x;g)=e^{-\frac12x^2}x^g{}_1F_1\Big(\begin{matrix}g+\frac12+\text{v}\\ g+\frac12\end{matrix}\Bigm|\eta(x)\Big),$$
$$\text{J1 (3.14)}\quad \tilde{\phi}^{\text{I}}_{\text{v}}(x;g,h)=(\sin x)^g(\cos x)^h{}_2F_1\Big(\begin{matrix}g+\frac12+\text{v},h-\frac12-\text{v}\\ g+\frac12\end{matrix}\Bigm|\frac{1-\eta(x)}{2}\Big).$$

The same procedure, *polynomials to (confluent) hypergeometric series*, can be applied to the eigenfunctions to obtain another type of L1, J1 and J2 seed solutions ($\text{v}\in\mathbb{R}$, $\text{v}\notin\mathbb{Z}_{\geq 0}$) :

$$\text{L1}:\quad \phi^{\text{I}}_{\text{v}}(x;g)\stackrel{\text{def}}{=}e^{-\frac12x^2}x^g{}_1F_1\Big(\begin{matrix}-\text{v}\\ g+\frac12\end{matrix}\Bigm|\eta(x)\Big), \tag{3.18}$$
$$\text{J1}:\quad \phi^{\text{I}}_{\text{v}}(x;g,h)\stackrel{\text{def}}{=}(\sin x)^g(\cos x)^{h}{}_2F_1\Big(\begin{matrix}-\text{v},\text{v}+g+h\\ g+\frac12\end{matrix}\Bigm|\frac{1-\eta(x)}{2}\Big), \tag{3.19}$$
$$\text{J2}:\quad \phi^{\text{II}}_{\text{v}}(x;g,h)\stackrel{\text{def}}{=}(\sin x)^{g}(\cos x)^{h}{}_2F_1\Big(\begin{matrix}-\text{v},\text{v}+g+h\\ h+\frac12\end{matrix}\Bigm|\frac{1+\eta(x)}{2}\Big). \tag{3.20}$$

Another generalisation exists for the J1, J2 seed solutions. For certain complex values of v ($B \in \mathbb{R}$, $B \neq 0$) :

$$\begin{cases} \text{J1 (3.14)}: & \mathrm{v} = \frac{1}{2}(h-g-1) + iB \\ \text{J2 (3.15)}: & \mathrm{v} = \frac{1}{2}(g-h-1) + iB \end{cases}, \tag{3.21}$$

$$\begin{cases} \text{J1 (3.19)}: & \mathrm{v} = -\frac{1}{2}(g+h) + iB \\ \text{J2 (3.20)}: & \mathrm{v} = -\frac{1}{2}(g+h) + iB \end{cases}, \tag{3.22}$$

the seed solutions (3.14), (3.15), (3.19) and (3.20) and the corresponding energies (3.4), (3.7) and (3.8) are real:

$$\tilde{\mathcal{E}}_{\mathrm{v}}(g,h) = -\big((g+h)^2 + 4B^2\big) < 0. \tag{3.23}$$

3.2 Other solvable potentials

Among various solvable potentials, some have only *finitely many discrete eigenstates*, which are labeled by the degrees of the polynomial eigenfunctions, $n = 0, 1, \ldots, n_{\max}$. For these, the same polynomial wavefunctions as the eigenfunctions with higher degrees than the highest energy eigenfunction $n > n_{\max}$ provide seed solutions, on the assumption that the boundary condition at one boundary is satisfied [28, 29, 30]. These are called *overshoot eigenfunctions*, [23]. Here we report that the following four potentials have overshoot eigenfunctions.

3.2.1 Morse potential

The system has finitely many discrete eigenstates $0 \le n \le n_{\max} = [h]'$ in the specified parameter range ($[a]'$ denotes the greatest integer not exceeding and not equal to a):

$$\begin{aligned} &U(x;h,\mu) = \mu^2 e^{2x} - \mu(2h+1)e^x + h^2, \quad x_1 = -\infty, \quad x_2 = \infty, \quad h, \mu > 0, \\ &\mathcal{E}_n(h,\mu) = h^2 - (h-n)^2, \quad \eta(x) = e^{-x}, \\ &\phi_n(x;h,\mu) = e^{hx - \mu e^x}(2\mu\eta^{-1})^{-n} L_n^{(2h-2n)}(2\mu\eta^{-1}), \quad h_n(h,\mu) = \frac{\Gamma(2h-n+1)}{(2\mu)^{2h} n!\, 2(h-n)}. \end{aligned}$$

For $n > h$, the overshoot eigenfunctions provide type II seed solutions.

3.2.2 Rosen-Morse potential

The system has finitely many discrete eigenstates $0 \le n \le n_{\max} = [h - \sqrt{\mu}]'$ in the specified parameter range:

$$U(x;h,\mu) = -\frac{h(h+1)}{\cosh^2 x} + 2\mu \tanh x + h^2 + \frac{\mu^2}{h^2}, \quad x_1 = -\infty, \quad x_2 = \infty, \quad h > \sqrt{\mu} > 0,$$

$$
\begin{aligned}
&\mathcal{E}_n(h,\mu) = h^2 - (h-n)^2 + \frac{\mu^2}{h^2} - \frac{\mu^2}{(h-n)^2}, \quad \eta(x) = \tanh x,\\
&\phi_n(x;h,\mu) = e^{-\frac{\mu}{h-n}x}(\cosh x)^{-h+n} P_n\big(\eta(x);h,\mu\big),\\
&P_n(\eta;h,\mu) = P_n^{(\alpha_n,\beta_n)}(\eta), \quad \alpha_n = h-n+\frac{\mu}{h-n}, \quad \beta_n = h-n-\frac{\mu}{h-n},\\
&h_n(h,\mu) = \frac{2^{2h-2n}(h-n)\Gamma(h+\frac{\mu}{h-n}+1)\Gamma(h-\frac{\mu}{h-n}+1)}{n!\,\big((h-n)^2-\frac{\mu^2}{(h-n)^2}\big)\Gamma(2h-n+1)}.
\end{aligned}
$$

The overshoot eigenfunctions provide type II seed solutions for $h-\sqrt{\mu} < n < h$ and type I seed solutions for $h < n < h+\sqrt{\mu}$.

3.2.3 Kepler problem in hyperbolic space

This potential is also called Eckart potential. It has finitely many discrete eigenstates $0 \le n \le n_{\max} = [\sqrt{\mu} - g]'$ in the specified parameter range:

$$
\begin{aligned}
&U(x;g,\mu) = \frac{g(g-1)}{\sinh^2 x} - 2\mu\coth x + g^2 + \frac{\mu^2}{g^2}, \quad x_1 = 0, \quad x_2 = \infty, \quad \sqrt{\mu} > g > \frac{3}{2},\\
&\mathcal{E}_n(g,\mu) = g^2 - (g+n)^2 + \frac{\mu^2}{g^2} - \frac{\mu^2}{(g+n)^2}, \quad \eta(x) = \coth x,\\
&\phi_n(x;g,\mu) = e^{-\frac{\mu}{g+n}x}(\sinh x)^{g+n} P_n\big(\eta(x);g,\mu\big),\\
&P_n(\eta;g,\mu) = P_n^{(\alpha_n,\beta_n)}(\eta), \quad \alpha_n = -g-n+\frac{\mu}{g+n}, \quad \beta_n = -g-n-\frac{\mu}{g+n},\\
&h_n(g,\mu) = \frac{(g+n)\Gamma(1-g+\frac{\mu}{g+n})\Gamma(2g+n)}{2^{2g+2n}n!\,\big(\frac{\mu^2}{(g+n)^2}-(g+n)^2\big)\Gamma(g+\frac{\mu}{g+n})}.
\end{aligned}
$$

For $n > \sqrt{\mu} - g$, the overshoot eigenfunctions provide type I seed solutions.

3.2.4 hyperbolic Darboux-Pöschl-Teller potential

This has finitely many discrete eigenstates $0 \le n \le n_{\max} = [\frac{h-g}{2}]'$ in the specified parameter range:

$$
\begin{aligned}
&U(x;g,h) = \frac{g(g-1)}{\sinh^2 x} - \frac{h(h+1)}{\cosh^2 x} + (h-g)^2, \quad x_1 = 0, \quad x_2 = \infty, \quad h > g > \frac{3}{2},\\
&\mathcal{E}_n(g,h) = 4n(h-g-n), \quad \eta(x) = \cosh 2x,\\
&\phi_n(x;g,h) = (\sinh x)^g(\cosh x)^{-h}\, P_n^{(g-\frac{1}{2},-h-\frac{1}{2})}\big(\eta(x)\big),\\
&h_n(g,h) = \frac{\Gamma(n+g+\frac{1}{2})\Gamma(h-g-n+1)}{2\,n!\,(h-g-2n)\Gamma(h-n+\frac{1}{2})}.
\end{aligned}
$$

The overshoot eigenfunctions provide type I seed solutions for $n > \frac{h-g}{2}$.

See [23] for polynomial extensions of known solvable potentials having finitely many discrete eigenfunctions.

3.2.5 seed solutions based on discrete symmetries

In [24] we have examined several well-known exactly solvable potentials and shown that the discrete symmetries of harmonic oscillator, Kepler problem in spherical space, Morse potential, soliton potential, Rosen-Morse potential, hyperbolic symmetric top II, do not provide either type I or II virtual state wavefunctions which could be used as seed solutions for state adding Abraham-Moses transformations.

For the hyperbolic Darboux-Pöschl-Teller potential, Kepler problem in hyperbolic space and Coulomb potential plus the centrifugal barrier, the discrete symmetry produces type I or II virtual state wavefunctions.

Like the trigonometric Darboux-Pöschl-Teller potential, the hyperbolic Darboux-Pöschl-Teller potential has type I and II virtual state wavefunctions obtained by discrete symmetries $h \leftrightarrow -(h+1)$, $g \leftrightarrow 1-g$ from the eigenfunctions and they give seed solutions:

$$
\begin{aligned}
\tilde{\phi}^{\text{I}}_{\text{v}}(x;g,h) &= (\sinh x)^{g}(\cosh x)^{h+1}P_{\text{v}}^{(g-\frac12,h+\frac12)}\bigl(\eta(x)\bigr) \qquad (\text{v}\in\mathbb{Z}_{\geq 0}),\\
\tilde{\mathcal{E}}^{\text{I}}_{\text{v}}(g,h) &= -4(\text{v}+\tfrac12+g)(\text{v}+\tfrac12+h),\\
\tilde{\phi}^{\text{II}}_{\text{v}}(x;g,h) &= (\sinh x)^{1-g}(\cosh x)^{-h}P_{\text{v}}^{(\frac12-g,-h-\frac12)}\bigl(\eta(x)\bigr) \quad (\text{v}\in\mathbb{Z}_{\geq 0},\ \text{v}<\tfrac12(h+g-1)),\\
\tilde{\mathcal{E}}^{\text{II}}_{\text{v}}(g,h) &= -4(\text{v}+\tfrac12-g)(\text{v}+\tfrac12-h).
\end{aligned}
$$

The second example is Kepler problem in hyperbolic space. The virtual state wavefunction is obtained by discrete symmetry $g \leftrightarrow 1-g$ from the eigenfunction:

$$
\tilde{\phi}_{\text{v}}(x;g,\mu) = e^{\frac{\mu}{g-\text{v}-1}x}(\sinh x)^{-g+\text{v}+1}P_{\text{v}}\bigl(\eta(x);1-g,\mu\bigr), \quad \tilde{\mathcal{E}}_{\text{v}}(g,\mu) = \mathcal{E}_{-\text{v}-1}(g,\mu).
$$

For $g-1<\text{v}<g-1+\sqrt{\mu}$ ($\text{v}\in\mathbb{Z}_{\geq 0}$), the above wavefunctions become type II seed solutions [29, 24].

Coulomb potential plus the centrifugal barrier has infinitely many discrete eigenstates in the specified parameter range:

$$
\begin{aligned}
U(x;g) &= \frac{g(g-1)}{x^2} - \frac{2}{x} + \frac{1}{g^2}, \quad x_1 = 0, \quad x_2 = \infty, \quad g > \frac{3}{2},\\
\mathcal{E}_n(g) &= \frac{1}{g^2} - \frac{1}{(g+n)^2}, \quad \eta(x) = x^{-1},
\end{aligned}
$$

$$\phi_n(x;g) = e^{-\frac{x}{g+n}} x^{g+n}\eta^n L_n^{(2g-1)}\big(\tfrac{2}{g+n}\eta^{-1}\big), \quad h_n(g) = \Big(\frac{g+n}{2}\Big)^{2g+2}\frac{4}{n!}\,\Gamma(2g+n).$$

The discrete symmetry $g \leftrightarrow 1-g$ generates the type II seed solutions $\mathrm{v} > g-1$ ($\mathrm{v}\in\mathbb{Z}_{\geq 0}$), [31]:

$$\tilde{\phi}_{\mathrm{v}}(x;g) = e^{\frac{x}{g-\mathrm{v}-1}} x^{1-g+\mathrm{v}}\eta^{\mathrm{v}} L_{\mathrm{v}}^{(1-2g)}\big(\tfrac{2}{1-g+\mathrm{v}}\eta^{-1}\big), \quad \tilde{\mathcal{E}}_{\mathrm{v}}(g) = \mathcal{E}_{-\mathrm{v}-1}(g).$$

It is also possible to generalise the degree n or v to a real number (or certain complex number with real energy) in the above overshoot eigenfunctions or those wavefunctions obtained by discrete symmetry.

4 Summary and discussions

In order to carry out the program of Abraham-Moses [6] to enlarge the list of exactly solvable potentials through extensions by adding a finite number of *eigenstates of arbitrary energies*, one needs proper *seed solutions*. Infinitely many seed solutions of different sorts are presented for some well-known solvable potentials, *e.g.* the radial oscillator, the Darboux-Pöschl-Teller and the Morse potentials, etc. They are the same *virtual state wavefunctions* which have produced the multi-indexed Laguerre and Jacobi polynomials via multiple Darboux transformations, and their straightforward generalisations. There are two types of seed solutions, type I and II, corresponding to the integral transformations starting from the lower and upper boundary points, respectively.

The basic formulas of adding as well as deleting Abraham-Moses transformations are recapitulated. They are presented purely algebraically without the inverse scattering formulation. It is pointed out that the multiple eigenstates addition transformations are a good example of orthonormalisation procedures of non-normalisable vectors.

It would be a good challenge to formulate the difference equation analogues of Abraham-Moses transformations. The theory of difference Schrödinger equations is now well developed as 'discrete quantum mechanics' [32], and most of the orthogonal polynomials of Askey scheme [33, 34], *e.g.* the Askey-Wilson and the q-Racah polynomials, are the eigenfunctions of various solvable models [35, 36]. The discrete analogues of various methods and results of quantum mechanics, including the Heisenberg equation of motion [27], the Darboux transformations [37, 38], and the multi-indexed Askey-Wilson and q-Racah polynomials [39, 40] are already established.

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