

Discrete Quantum Mechanics

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Abstract

A comprehensive review of the discrete quantum mechanics with the pure imaginary shifts and the real shifts is presented in parallel with the corresponding results in the ordinary quantum mechanics. The main subjects to be covered are the factorised Hamiltonians, the general structure of the solution spaces of the Schrödinger equation (Crum's theorem and its modification), the shape invariance, the exact solvability in the Schrödinger picture as well as in the Heisenberg picture, the creation/annihilation operators and the dynamical symmetry algebras, the unified theory of exact and quasi-exact solvability based on the sinusoidal coordinates, the infinite families of new orthogonal (the exceptional) polynomials. Two new infinite families of orthogonal polynomials, the X_ℓ Meixner-Pollaczek and the X_ℓ Meixner polynomials are reported.

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1 Introduction

In the discrete quantum mechanics (dQM), developed by the present authors, the time-independent Schrödinger equation is a second order *difference* equation instead of a second order *differential* equation in the ordinary quantum mechanics (oQM). The basic framework

of quantum mechanics, such as the probability interpretation, the Hilbert space structure, the continuous real time, etc. is unchanged and the only difference is the discrete forms of various Hamiltonians. The discrete quantum mechanics can be considered as a generalisation (deformation) of the ordinary QM in the sense that a differential equation can be obtained from a difference equation in an appropriate limit. The generalisation is also consistent with Heisenberg's idea that a certain fundamental length would appear in the theory at some stage of descending the microscopic ladder. This generalisation has turned out to be extremely fruitful. Most of the concepts and methods of oQM can be easily transplanted to dQM, offering a unified platform for the systematic understanding of various orthogonal polynomials satisfying second order difference equations with the pure *imaginary shifts* (idQM) as well as the *real shifts* (rdQM). Sometimes these polynomials are said to have the *bi-spectral property* [1]. They satisfy two types of spectral conditions, the difference eigenvalue (Schrödinger) equation (2.23) and the three term recurrence relations (2.139). Those polynomials satisfying the difference equations with the real shifts are also called the orthogonal polynomials of a *discrete variable* [2]. They contain most of the (q)-hypergeometric orthogonal polynomials belonging to the Askey scheme [2, 3, 4, 5, 6]. The power of the discrete QM is demonstrated most eloquently in the discovery of various infinite families of *new orthogonal (exceptional) polynomials*. In less than two years after the discovery of the infinite family of the exceptional Laguerre and Jacobi polynomials [7, 8, 9], the generic counterparts in dQM, the exceptional Wilson, Askey-Wilson, Racah and q -Racah polynomials are constructed by the present authors [10, 11]. These new orthogonal polynomials are expected to play a significant role in mathematics, mathematical physics and related disciplines. The exceptional Jacobi polynomials provide infinitely many global solutions of Fuchsian differential equations with more than three and arbitrarily many regular singularities [9], although the locations of the extra singularities are rigidly specified.

This topical review of dQM provides a comprehensive overview of the subjects related to the *exactly and quasi-exactly solvable* quantum particle dynamics [12]–[26]. Due to the length constraints, we have to concentrate on the systems of single degree of freedom, which is the most basic and best established part of the theory. The subjects to be covered in this article are the adaptation to dQM of the methods and concepts of oQM accumulated over 80 years after the birth of quantum mechanics. After the general setting of the discrete quantum mechanics with the pure imaginary shifts (idQM) as well as the real shifts (rdQM), we start

with the *factorised* Hamiltonians and the Schrödinger equations [12, 20, 21]. The general structure of the solution spaces is explored by the intertwining relations and Crum's theorem together with its modification [27]–[29], [24]–[26]. Exact solvability in the Schrödinger picture is explained by the *shape invariance* [30, 31, 12, 20, 21]. The *generic eigenvalue formula*, *unified Rodrigues formulas* and the *forward/backward shift operators* are deduced. The solvability in the Heisenberg picture is derived based on the *closure relation* between the *sinusoidal coordinate* and the Hamiltonian [14, 15]. The *creation/annihilation* operators are introduced and their connection with the *three term recurrence relations* of the orthogonal polynomials is emphasised. The *dynamical symmetry algebra* generated by the Hamiltonian and the creation/annihilation operators are also established for all the solvable systems in the Heisenberg picture [21, 23]. This includes the dynamical realisation of the q -oscillator algebras [19]. For the rdQM, the *dual pair* of orthogonal polynomials (the Leonard pair [32, 33]) and the related Askey-Wilson algebras [34] are explored. The unified theory of exactly and quasi-exactly solvable dQM is another major theme of this review. Based on the fundamental properties of various sinusoidal coordinates, a simple recipe to construct a solvable dQM Hamiltonian is provided [23]. It generates all known exactly solvable theories and many new ones. As a byproduct it also gives a recipe of constructing quasi-exactly solvable dQM Hamiltonians. The final major subject of the review is the new (exceptional) orthogonal polynomials [7]–[11],[35]–[38].

Throughout the review we stress the similarity and differences among oQM, idQM and rdQM. As far as possible we adopt the same notation and present one all-inclusive formula for the three categories, oQM, idQM and rdQM. The differences are exhibited by explicit examples. We usually adopt three examples from each group denoted by the name of the corresponding eigenpolynomials with an abbreviation. For the oQM, they are the Hermite (H), the Laguerre (L) and the Jacobi (J) polynomials. For the idQM, we choose the special case of the Meixner-Pollaczek (MP) (the parameter ϕ is fixed to $\phi = \frac{\pi}{2}$), the Wilson (W) and the Askey-Wilson (AW) polynomials. For the rdQM, the Meixner (M), the Racah (R) and the q -Racah (qR) polynomials are selected. In some cases, we present only one representative from each group. They are usually the most generic ones, J, AW and qR or the simplest ones H, MP and M. For the full line-ups, the original references should be consulted [20, 21, 23]. We will provide cross references to the original articles with the section or the equation numbers.

This paper is organised as follows. In section two various concepts and methods in dQM are explained. These are: (a) the factorised Hamiltonians (2.5)–(2.12) & the ‘Schrödinger equations’ for the eigenpolynomials (2.23)–(2.27) together with the groundstate eigenfunctions (2.52), (2.55) as the orthogonality weight functions (2.49)–(2.50). (b) the intertwining relations connecting various Hamiltonians (2.58), (2.65)–(2.66), and eigenfunctions (2.61), (2.68), (2.73). This is a brief summary of Crum’s theory in our language. Its deformation à la Krein-Adler is also mentioned. (c) shape invariance (2.105) and the full energy spectrum (2.106) and the unified Rodrigues type formula (2.107) for all the eigenfunctions. (d) solvability in the Heisenberg picture (2.144) and the closure relation (2.142). The formulas for the creation/annihilation operators (2.147)–(2.148) are given. (e) dual closure relation (2.182) together with its connection with the Askey-Wilson algebra. In section three we provide the essence of the unified theory of exact and quasi-exact solvability in dQM. In section four the new (exceptional) orthogonal polynomials satisfying second order differential (difference) equations are explored. The exceptional Meixner-Pollaczek and the exceptional Meixner polynomials are new results. The final section is for a summary and comments. Basic symbols and definitions are listed in Appendix. A substantially shorter version of the present review is published recently [39]. We apologise to all the authors whose good works could not be referred to in the review due to the lack of space.

2 Discrete Quantum Mechanics

The dynamical variables of one-dimensional QM are the coordinate x and its conjugate momentum p , which is realised as a differential operator $p = -i\hbar\frac{d}{dx} \equiv -i\hbar\partial_x$. Hereafter we adopt the convention $\hbar = 1$. The dQM is a generalisation of oQM in which the Schrödinger equation is a difference equation instead of differential in the oQM [12]–[14], [20, 21]. In other words, the Hamiltonian contains the momentum operator in exponentiated forms $e^{\pm\beta p}$ which work as shift operators on the wavefunction

$$e^{\pm\beta p}\psi(x) = \psi(x \mp i\beta). \quad (2.1)$$

For the two choices of the parameter β , either *real* or pure *imaginary*, we have two types of dQM; with (i) pure imaginary shifts $\beta = \gamma \in \mathbb{R}_{\neq 0}$ (idQM), or (ii) real shifts $\beta = i$ (rdQM), respectively. In the case of idQM, $\psi(x \mp i\gamma)$, we require the wavefunction and potential functions etc to be *analytic* functions of x with their domains including the real axis or

a part of it on which the dynamical variable x is defined. In contrast, in the rdQM, the difference equation gives constraints on wavefunctions only on equally spaced lattice points. Then we choose, after proper rescaling, the variable x to take value of non-negative *integers*, with the total number either finite ($N + 1$) or infinite ($x_{\max} = N$ or ∞). To sum up, the dynamical variable x of the one dimensional dQM takes continuous or discrete values:

$$\text{idQM} : x \in \mathbb{R}, \quad x \in (x_1, x_2) ; \quad \text{rdQM} : x \in \mathbb{Z}_{\geq 0}, \quad x \in [0, x_{\max}]. \quad (2.2)$$

Here x_1, x_2 may be finite, $-\infty$ or $+\infty$. Correspondingly, the inner product of the wavefunctions has the following form:

$$\text{idQM} : (f, g) = \int_{x_1}^{x_2} f(x)^* g(x) dx ; \quad \text{rdQM} : (f, g) = \sum_{x=0}^{x_{\max}} f(x)^* g(x), \quad (2.3)$$

and the norm of $f(x)$ is $\|f\| = \sqrt{(f, f)}$.

We will consider the Hamiltonians having a finite (rdQM with finite N) or semi-infinite number of *discrete energy levels only*:

$$0 = \mathcal{E}(0) < \mathcal{E}(1) < \mathcal{E}(2) < \dots . \quad (2.4)$$

The additive constant of the Hamiltonian is so chosen that the groundstate energy vanishes. That is, the Hamiltonian is *positive semi-definite*. It is a well known theorem in linear algebra that any positive semi-definite hermitian matrix can be factorised as a product of a certain matrix, say \mathcal{A} , and its hermitian conjugate \mathcal{A}^\dagger . As we will see shortly, the Hamiltonians we consider always have factorised forms in one-dimension as well as in higher dimensions.

2.1 Factorised Hamiltonian

The Hamiltonian we consider has a simple factorised form [40]

$$\mathcal{H} \stackrel{\text{def}}{=} \mathcal{A}^\dagger \mathcal{A} \quad \text{or} \quad \mathcal{H} \stackrel{\text{def}}{=} \sum_{j=1}^D \mathcal{A}_j^\dagger \mathcal{A}_j \quad \text{in } D \text{ dimensions.} \quad (2.5)$$

The operators \mathcal{A} and \mathcal{A}^\dagger in one dimensional QM are:

$$\text{oQM} : \quad \mathcal{A} \stackrel{\text{def}}{=} \frac{d}{dx} - \frac{dw(x)}{dx}, \quad \mathcal{A}^\dagger = -\frac{d}{dx} - \frac{dw(x)}{dx}, \quad w(x) \in \mathbb{R}, \quad \phi_0(x) = e^{w(x)}, \quad (2.6)$$

$$\mathcal{H} = p^2 + U(x), \quad U(x) \stackrel{\text{def}}{=} (\partial_x w(x))^2 + \partial_x^2 w(x), \quad (2.7)$$

$$\begin{aligned} \text{idQM: } \mathcal{A} &\stackrel{\text{def}}{=} i(e^{\frac{\gamma}{2}p}\sqrt{V^*(x)} - e^{-\frac{\gamma}{2}p}\sqrt{V(x)}), \quad \gamma \in \mathbb{R}_{\neq 0}, \\ \mathcal{A}^\dagger &= -i(\sqrt{V(x)}e^{\frac{\gamma}{2}p} - \sqrt{V^*(x)}e^{-\frac{\gamma}{2}p}), \quad V(x), V^*(x) \in \mathbb{C}, \end{aligned} \quad (2.8)$$

$$\mathcal{H} = \sqrt{V(x)}e^{\gamma p}\sqrt{V^*(x)} + \sqrt{V^*(x)}e^{-\gamma p}\sqrt{V(x)} - V(x) - V^*(x), \quad (2.9)$$

$$\text{rdQM: } \mathcal{A} \stackrel{\text{def}}{=} \sqrt{B(x)} - e^\partial \sqrt{D(x)}, \quad \mathcal{A}^\dagger = \sqrt{B(x)} - \sqrt{D(x)}e^{-\partial}, \quad (2.10)$$

$$D(x) > 0 \text{ (for } x > 0), \quad D(0) = 0,$$

$$B(x) > 0 \text{ (for } x \geq 0), \quad B(N) = 0 \text{ (for the finite case),} \quad (2.11)$$

$$\mathcal{H} = -\sqrt{B(x)}e^\partial \sqrt{D(x)} - \sqrt{D(x)}e^{-\partial} \sqrt{B(x)} + B(x) + D(x). \quad (2.12)$$

Here $w(x)$ is called *prepotential*. The function $V^*(x)$ in idQM is an analytic function of x obtained from $V(x)$ by the $*$ -operation, which is defined as follows. If $f(x) = \sum_n a_n x^n$, $a_n \in \mathbb{C}$, then $f^*(x) \stackrel{\text{def}}{=} \sum_n a_n^* x^n$, in which a_n^* is the complex conjugation of a_n . Obviously $f^{**}(x) = f(x)$ and $f(x)^* = f^*(x^*)$. If a function satisfies $f^* = f$, then it takes real values on the real line. To the best of our knowledge, the factorised Hamiltonian approach to dQM is our invention. The existing factorisation methods for dQM are not for the Hamiltonian \mathcal{H} , but for Hamiltonian minus an eigenvalue $\mathcal{H} - \lambda(n)$ [41, 42]. The condition $D(0) = 0$ in rdQM (2.11) is necessary for the term $\psi(-1)$ not to appear in $\mathcal{H}\psi(0)$. Likewise $B(N) = 0$ is necessary for the finite case. Roughly speaking these correspond to the regular singularities in oQM. The Hamiltonians of idQM and rdQM can be written in a unified notation:

$$\text{dQM: } \mathcal{H} = \varepsilon \left(\sqrt{V_+(x)} e^{\beta p} \sqrt{V_-(x)} + \sqrt{V_-(x)} e^{-\beta p} \sqrt{V_+(x)} - V_+(x) - V_-(x) \right), \quad (2.13)$$

$$\text{idQM: } \beta = \gamma, \quad \varepsilon = 1, \quad V_+(x) = V(x), \quad V_-(x) = V^*(x),$$

$$\text{rdQM: } \beta = i, \quad \varepsilon = -1, \quad V_+(x) = B(x), \quad V_-(x) = D(x). \quad (2.14)$$

Multi-particle exactly solvable systems can be constructed in a similar way. The prepotential approach is also useful in Calogero-Moser systems [43] in oQM [44, 45]:

$$\text{oQM: } \mathcal{A}_j \stackrel{\text{def}}{=} \frac{\partial}{\partial x_j} - \frac{\partial w(x)}{\partial x_j}, \quad \mathcal{A}_j^\dagger = -\frac{\partial}{\partial x_j} - \frac{\partial w(x)}{\partial x_j} \quad (j = 1, \dots, D), \quad \phi_0(x) = e^{w(x)}. \quad (2.15)$$

We also adopt the factorised Hamiltonian for the discrete analogues of the Calogero-Moser systems:

$$\begin{aligned} \text{idQM: } \mathcal{A}_j &\stackrel{\text{def}}{=} i(e^{\frac{\gamma}{2}p_j}\sqrt{V_j^*(x)} - e^{-\frac{\gamma}{2}p_j}\sqrt{V_j(x)}), \quad V_j(x), V_j^*(x) \in \mathbb{C}, \\ \mathcal{A}_j^\dagger &= -i(\sqrt{V_j(x)}e^{\frac{\gamma}{2}p_j} - \sqrt{V_j^*(x)}e^{-\frac{\gamma}{2}p_j}) \quad (j = 1, \dots, D), \end{aligned} \quad (2.16)$$

which is markedly different from the approach of Ruijsenaars et al [46]. The multiparticle version of rdQM is now being developed.

The Schrödinger equation

$$\mathcal{H}\phi_n(x) = \mathcal{E}(n)\phi_n(x) \quad (n = 0, 1, 2, \dots), \quad 0 = \mathcal{E}(0) < \mathcal{E}(1) < \mathcal{E}(2) < \dots, \quad (2.17)$$

is a second order differential (oQM) or difference equation (dQM) and the *groundstate wavefunction* $\phi_0(x)$ is determined as a zero mode of the operator \mathcal{A} (\mathcal{A}_j) which is a first order equation:

$$\mathcal{A}\phi_0(x) = 0 \quad (\mathcal{A}_j\phi_0(x) = 0, \quad j = 1, \dots, D) \quad \Rightarrow \quad \mathcal{H}\phi_0(x) = 0. \quad (2.18)$$

By definition all the eigenfunctions are required to have a finite positive norm:

$$0 < (\phi_n, \phi_n) < \infty. \quad (2.19)$$

Throughout this paper we look for the eigenfunctions in a factorised form:

$$\phi_n(x) = \phi_0(x)P_n(\eta(x)), \quad (2.20)$$

in which $P_n(\eta(x))$ is a degree n (except for those discussed in §4) polynomial in the *sinusoidal coordinate* $\eta(x)$. However, some basic results, such as the (modified) Crum's theorem in §2.2–2.3 and the shape invariance in §2.4 hold without this assumption (2.20). The explicit forms of the eigenfunctions will be given for various examples (2.30)–(2.39). Here we will briefly introduce their general property for dQM. The sinusoidal coordinate $\eta(x)$ in dQM is a *monotone increasing function* of x with the initial (boundary) condition

$$\text{dQM : } \quad \eta(0) = 0. \quad (2.21)$$

With this, we also normalise the eigenpolynomials $\{P_n(\eta)\}$ in rdQM as:

$$\text{rdQM : } \quad P_n(0) = 1 \quad (n = 0, 1, \dots). \quad (2.22)$$

In the multi-particle case n will be a multi-index. Here we require $\phi_0(x)$ to be chosen real and positive for the physical values of x , which is always guaranteed in oQM. One important distinction between a differential and a difference equation is the *uniqueness* of the solution. For a linear differential equation, the solution is unique when the initial conditions are specified. In contrast, a solution of a difference Schrödinger equation (2.17) multiplied by

any periodic function of a period $i\beta$ ($i\gamma$ for idQM, 1 for rdQM) is another solution. In the case of idQM, this non-uniqueness is removed by requiring the finite norm condition for the groundstate wavefunction ϕ_0 , $(\phi_0, \phi_0) < \infty$ and the hermiticity (self-adjointness) condition of the Hamiltonian. For details, see [21] Appendix A. For the rdQM, the periodic ambiguity of period 1 is harmless since the values of eigenfunctions on the integer lattice points only count for the inner products, etc. It is also important to realise that the periodic ambiguity does not appear in the polynomial solutions of (2.23).

It is important to stress that the second order equation for $P_n(\eta(x))$ is *square root free*:

$$\tilde{\mathcal{H}}P_n(\eta(x)) = \mathcal{E}(n)P_n(\eta(x)). \quad (2.23)$$

In other words, the similarity transformed Hamiltonian $\tilde{\mathcal{H}}$ in terms of the groundstate wavefunction $\phi_0(x)$ has a much simpler form than the original Hamiltonian \mathcal{H} :

$$\tilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0(x)^{-1} \circ \mathcal{H} \circ \phi_0(x), \quad (2.24)$$

$$\text{oQM: } \tilde{\mathcal{H}} = -\frac{d^2}{dx^2} - 2\frac{dw(x)}{dx} \frac{d}{dx}, \quad (2.25)$$

$$\text{dQM: } \tilde{\mathcal{H}} = \varepsilon \left(V_+(x)(e^{\beta p} - 1) + V_-(x)(e^{-\beta p} - 1) \right) \quad (2.26)$$

$$= \begin{cases} V(x)(e^{\gamma p} - 1) + V^*(x)(e^{-\gamma p} - 1) & : \text{idQM} \\ B(x)(1 - e^\vartheta) + D(x)(1 - e^{-\vartheta}) & : \text{rdQM} \end{cases}. \quad (2.27)$$

For all the examples discussed in this section, $\tilde{\mathcal{H}}$ is *lower triangular*

$$\tilde{\mathcal{H}}\eta(x)^n = \mathcal{E}(n)\eta(x)^n + \text{lower orders in } \eta(x), \quad (2.28)$$

in the special basis

$$1, \eta(x), \eta(x)^2, \dots, \eta(x)^n, \dots,$$

spanned by the sinusoidal coordinate $\eta(x)$. This situation is expressed as

$$\tilde{\mathcal{H}}\mathcal{V}_n \subseteq \mathcal{V}_n, \quad \mathcal{V}_n \stackrel{\text{def}}{=} \text{Span}[1, \eta(x), \dots, \eta(x)^n]. \quad (2.29)$$

Here are some explicit examples. For the oQM the *prepotential* $w(x)$ determines the potential $U(x)$ of the Hamiltonian $\mathcal{H} = p^2 + U(x)$, $U(x) \stackrel{\text{def}}{=} (\partial_x w(x))^2 + \partial_x^2 w(x)$:

$$\begin{aligned} \text{H: } w(x) &= -\frac{1}{2}x^2, & -\infty < x < \infty, \\ U(x) &= x^2 - 1, & \eta(x) &= x, \end{aligned} \quad (2.30)$$

$$\begin{aligned} \text{L : } w(x) &= -\frac{1}{2}x^2 + g \log x, & g > 0, \quad 0 < x < \infty, \\ U(x) &= x^2 + \frac{g(g-1)}{x^2} - 1 - 2g, & \eta(x) &= x^2, \end{aligned} \quad (2.31)$$

$$\begin{aligned} \text{J : } w(x) &= g \log \sin x + h \log \cos x, & g > 0, \quad h > 0, \quad 0 < x < \frac{\pi}{2}, \\ U(x) &= \frac{g(g-1)}{\sin^2 x} + \frac{h(h-1)}{\cos^2 x} - (g+h)^2, & \eta(x) &= \cos 2x. \end{aligned} \quad (2.32)$$

Let us note that $x = 0$ for L and $x = 0, \frac{\pi}{2}$ for J are the *regular singular points* of the Fuchsian differential equations. The monodromy at the regular singular point is determined by the *characteristic exponent* ρ :

$$M_\rho = e^{2\pi i \rho}. \quad (2.33)$$

The corresponding exponents are $\rho = g, 1 - g$ for L and $\rho = g, 1 - g$ and $\rho = h, 1 - h$ for J. For the idQM ($0 < q < 1$):

$$\text{MP : } V(x) = a + ix, \quad a > 0, \quad -\infty < x < \infty, \quad \gamma = 1, \quad \eta(x) = x, \quad (2.34)$$

$$\begin{aligned} \text{W : } V(x) &= \frac{\prod_{j=1}^4 (a_j + ix)}{2ix(2ix + 1)}, & \text{Re}(a_j) > 0, \quad 0 < x < \infty, \quad \gamma = 1, \\ \eta(x) &= x^2, & \{a_1^*, a_2^*, a_3^*, a_4^*\} &= \{a_1, a_2, a_3, a_4\} \quad (\text{as a set}), \end{aligned} \quad (2.35)$$

$$\begin{aligned} \text{AW : } V(x) &= \frac{\prod_{j=1}^4 (1 - a_j e^{ix})}{(1 - e^{2ix})(1 - q e^{2ix})}, & |a_j| < 1, \quad 0 < x < \pi, \quad \gamma = \log q, \\ \eta(x) &= 1 - \cos x, & \{a_1^*, a_2^*, a_3^*, a_4^*\} &= \{a_1, a_2, a_3, a_4\} \quad (\text{as a set}). \end{aligned} \quad (2.36)$$

For the rdQM ($0 < q < 1$):

$$\begin{aligned} \text{M : } B(x) &= \frac{c}{1-c}(x + \beta), \quad D(x) = \frac{1}{1-c}x, \quad \beta > 0, \quad 0 < c < 1, \\ \eta(x) &= x, \quad x_{\max} = \infty, \end{aligned} \quad (2.37)$$

$$\begin{aligned} \text{R : } B(x) &= -\frac{(x+a)(x+b)(x+c)(x+d)}{(2x+d)(2x+1+d)}, \\ D(x) &= -\frac{(x+d-a)(x+d-b)(x+d-c)x}{(2x-1+d)(2x+d)}, \quad \tilde{d} \stackrel{\text{def}}{=} a+b+c-d-1, \\ a &= -N, \quad a+b > d > 0, \quad 0 < c < 1+d, \\ \eta(x) &= x(x+d), \quad x_{\max} = N, \end{aligned} \quad (2.38)$$

$$\begin{aligned} q\text{R : } B(x) &= -\frac{(1-aq^x)(1-bq^x)(1-cq^x)(1-dq^x)}{(1-dq^{2x})(1-dq^{2x+1})}, \\ D(x) &= -\tilde{d} \frac{(1-a^{-1}dq^x)(1-b^{-1}dq^x)(1-c^{-1}dq^x)(1-q^x)}{(1-dq^{2x-1})(1-dq^{2x})}, \quad \tilde{d} \stackrel{\text{def}}{=} abcd^{-1}q^{-1}, \\ a &= q^{-N}, \quad 0 < ab < d < 1, \quad qd < c < 1, \end{aligned}$$

$$\eta(x) = (q^{-x} - 1)(1 - dq^x), \quad x_{\max} = N. \quad (2.39)$$

The R and q R cases admit four different parametrisations specified by $\epsilon, \epsilon' = \pm 1$, see [20]. Here we present $\epsilon = \epsilon' = 1$ case only. It should be emphasised that the sinusoidal coordinates $\eta(x)$ in rdQM in general depends on the parameters in contradistinction to those of the oQM or idQM.

The eigenfunctions of these examples have the form (2.20). The eigenpolynomials $P_n(\eta(x))$ are the following (for notation, see [6]):

oQM :

$$\text{H : } P_n(\eta(x)) = H_n(x) \stackrel{\text{def}}{=} (2x)^n {}_2F_0 \left(\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2} \\ - \end{matrix} \middle| -\frac{1}{x^2} \right), \quad (2.40)$$

$$\text{L : } P_n(\eta(x)) = L_n^{(g-\frac{1}{2})}(x^2) \stackrel{\text{def}}{=} \frac{(g+\frac{1}{2})_n}{n!} {}_1F_1 \left(\begin{matrix} -n \\ g+\frac{1}{2} \end{matrix} \middle| x^2 \right), \quad (2.41)$$

$$\text{J : } P_n(\eta(x)) = P_n^{(g-\frac{1}{2}, h-\frac{1}{2})}(\cos 2x) \stackrel{\text{def}}{=} \frac{(g+\frac{1}{2})_n}{n!} {}_2F_1 \left(\begin{matrix} -n, n+g+h \\ g+\frac{1}{2} \end{matrix} \middle| \frac{1-\cos 2x}{2} \right), \quad (2.42)$$

idQM :

$$\text{MP : } P_n(\eta(x)) = P_n^{(a)}(x; \frac{\pi}{2}) \stackrel{\text{def}}{=} \frac{(2a)_n}{n!} i^n {}_2F_1 \left(\begin{matrix} -n, a+ix \\ 2a \end{matrix} \middle| 2 \right), \quad (2.43)$$

$$\begin{aligned} \text{W : } P_n(\eta(x)) &= W_n(x^2; a_1, a_2, a_3, a_4) \quad (b_1 \stackrel{\text{def}}{=} a_1 + a_2 + a_3 + a_4) \\ &\stackrel{\text{def}}{=} (a_1 + a_2)_n (a_1 + a_3)_n (a_1 + a_4)_n {}_4F_3 \left(\begin{matrix} -n, n+b_1-1, a_1+ix, a_1-ix \\ a_1+a_2, a_1+a_3, a_1+a_4 \end{matrix} \middle| 1 \right), \end{aligned} \quad (2.44)$$

$$\begin{aligned} \text{AW : } P_n(\eta(x)) &= p_n(\cos x; a_1, a_2, a_3, a_4 | q) \quad (b_4 \stackrel{\text{def}}{=} a_1 a_2 a_3 a_4) \\ &\stackrel{\text{def}}{=} a_1^{-n} (a_1 a_2, a_1 a_3, a_1 a_4; q)_n {}_4\phi_3 \left(\begin{matrix} q^{-n}, b_4 q^{n-1}, a_1 e^{ix}, a_1 e^{-ix} \\ a_1 a_2, a_1 a_3, a_1 a_4 \end{matrix} \middle| q; q \right), \end{aligned} \quad (2.45)$$

rdQM :

$$\text{M : } P_n(\eta(x)) = M_n(x; \beta, c) \stackrel{\text{def}}{=} {}_2F_1 \left(\begin{matrix} -n, -x \\ \beta \end{matrix} \middle| 1 - c^{-1} \right), \quad (2.46)$$

$$\text{R : } P_n(\eta(x)) = R_n(x(x+d); a-1, \tilde{d}-a, c-1, d-c) \stackrel{\text{def}}{=} {}_4F_3 \left(\begin{matrix} -n, n+\tilde{d}, -x, x+d \\ a, b, c \end{matrix} \middle| 1 \right), \quad (2.47)$$

$$\text{qR : } P_n(\eta(x)) = R_n(q^{-x} + dq^x; aq^{-1}, \tilde{d}a^{-1}, cq^{-1}, dc^{-1} | q) \stackrel{\text{def}}{=} {}_4\phi_3 \left(\begin{matrix} q^{-n}, \tilde{d}q^n, q^{-x}, dq^x \\ a, b, c \end{matrix} \middle| q; q \right). \quad (2.48)$$

Obviously, the square of the groundstate wavefunction $\phi_0(x)^2$ provides the positive definite orthogonality weight function for the polynomials:

$$\text{oQM, idQM : } \int_{x_1}^{x_2} \phi_0(x)^2 P_n(\eta(x)) P_m(\eta(x)) dx = h_n \delta_{nm}, \quad (2.49)$$

$$\text{rdQM : } \sum_{x=0}^{x_{\max}} \phi_0(x)^2 P_n(\eta(x)) P_n(\eta(x)) = \frac{1}{d_n^2} \delta_{nm}. \quad (2.50)$$

Let us emphasise that the weight function, or $\phi_0(x)$ is determined as a solution of a first order differential (difference) equation (2.18), without recourse to a *moment problem*. This situation becomes crucially important when various deformations of orthogonal polynomials are considered. For the oQM, the weight function is simply given by the prepotential $\phi_0(x)^2 = e^{2w(x)}$ and the normalisation constants are

$$h_n = \begin{cases} 2^n n! \sqrt{\pi} & : \text{H} \\ \frac{1}{2n!} \Gamma(n + g + \frac{1}{2}) & : \text{L} \\ \frac{\Gamma(n + g + \frac{1}{2}) \Gamma(n + h + \frac{1}{2})}{2n! (2n + g + h) \Gamma(n + g + h)} & : \text{J} \end{cases}. \quad (2.51)$$

The explicit forms of the squared groundstate wavefunction (weight function) $\phi_0(x)^2$ and the normalisation constants h_n for the above examples in pure imaginary shifts dQM are:

$$\phi_0(x)^2 = \begin{cases} \Gamma(a + ix) \Gamma(a - ix) & : \text{MP} \\ (\Gamma(2ix) \Gamma(-2ix))^{-1} \prod_{j=1}^4 \Gamma(a_j + ix) \Gamma(a_j - ix) & : \text{W} \\ (e^{2ix}; q)_\infty (e^{-2ix}; q)_\infty \prod_{j=1}^4 ((a_j e^{ix}; q)_\infty (a_j e^{-ix}; q)_\infty)^{-1} & : \text{AW} \end{cases}, \quad (2.52)$$

$$h_n = \begin{cases} 2\pi (2^{2a} n!)^{-1} \Gamma(n + 2a) & : \text{MP} \\ 2\pi n! (n + b_1 - 1)_n \prod_{1 \leq i < j \leq 4} \Gamma(n + a_i + a_j) \cdot \Gamma(2n + b_1)^{-1} & : \text{W} \\ 2\pi (b_4 q^{n-1}; q)_n (b_4 q^{2n}; q)_\infty (q^{n+1}; q)_\infty^{-1} \prod_{1 \leq i < j \leq 4} (a_i a_j q^n; q)_\infty^{-1} & : \text{AW} \end{cases}. \quad (2.53)$$

For the rdQM, the zero-mode equation $\mathcal{A}\phi_0(x) = 0$ (2.18) is a two term recurrence relation, which can be solved elementarily by using the boundary condition (2.11):

$$\phi_0(x)^2 = \prod_{y=0}^{x-1} \frac{B(y)}{D(y+1)}, \quad \phi_0(0) = 1. \quad (2.54)$$

The explicit forms of $\phi_0(x)^2$ and d_n^2 , which are related by duality, are:

$$\phi_0(x)^2 = \begin{cases} \frac{(\beta)_x c^x}{x!} & : \text{M} \\ \frac{(a, b, c, d)_x}{(1 + d - a, 1 + d - b, 1 + d - c, 1)_x} \frac{2x + d}{d} & : \text{R} \\ \frac{(a, b, c, d; q)_x}{(a^{-1} dq, b^{-1} dq, c^{-1} dq, q; q)_x} \frac{1 - dq^{2x}}{1 - d} & : q\text{R} \end{cases}, \quad (2.55)$$

$$d_n^2 = \begin{cases} \frac{(\beta)_n c^n}{n!} \times (1-c)^\beta & : \text{M} \\ \frac{(a, b, c, \tilde{d})_n}{(1 + \tilde{d} - a, 1 + \tilde{d} - b, 1 + \tilde{d} - c, 1)_n} \frac{2n + \tilde{d}}{\tilde{d}} & : \text{R} \\ \times \frac{(-1)^N (1 + d - a, 1 + d - b, 1 + d - c)_N}{(\tilde{d} + 1)_N (d + 1)_{2N}} & \\ \frac{(a, b, c, \tilde{d}; q)_n}{(a^{-1} \tilde{d}q, b^{-1} \tilde{d}q, c^{-1} \tilde{d}q, q; q)_n} \frac{1 - \tilde{d}q^{2n}}{d^n} & : q\text{R} \\ \times \frac{(-1)^N (a^{-1} dq, b^{-1} dq, c^{-1} dq; q)_N \tilde{d}^N q^{\frac{1}{2}N(N+1)}}{(\tilde{d}q; q)_N (dq; q)_{2N}} & \end{cases} \quad (2.56)$$

It should be stressed that $\phi_0(x)^2$ for the rdQM (2.54), when continued analytically in x , vanish on the integer points outside the defining domain, i.e., $x \in \mathbb{Z}_{<0}$ and $x \in \mathbb{Z}_{>x_{\max}}$.

To the best of our knowledge, the first example of a factorised idQM Hamiltonian was mentioned in 2001 eq.(4.28) of [47] for the Meixner-Pollaczek polynomial. But it was not considered as a general principle.

2.2 Intertwining Relations: Crum's Theorem

The general structure of the intertwining relations works equally well for the oQM as well as the dQM. Let us denote by $\mathcal{H}^{[0]}$ the original factorised Hamiltonian and by $\mathcal{H}^{[1]}$ its partner (associated) Hamiltonian obtained by changing the order of \mathcal{A}^\dagger and \mathcal{A} :

$$\mathcal{H}^{[0]} \stackrel{\text{def}}{=} \mathcal{A}^\dagger \mathcal{A}, \quad \mathcal{H}^{[1]} \stackrel{\text{def}}{=} \mathcal{A} \mathcal{A}^\dagger. \quad (2.57)$$

One simple and most important consequence of the factorised Hamiltonian (2.5) is the *intertwining relations*:

$$\mathcal{A} \mathcal{H}^{[0]} = \mathcal{A} \mathcal{A}^\dagger \mathcal{A} = \mathcal{H}^{[1]} \mathcal{A}, \quad \mathcal{A}^\dagger \mathcal{H}^{[1]} = \mathcal{A}^\dagger \mathcal{A} \mathcal{A}^\dagger = \mathcal{H}^{[0]} \mathcal{A}^\dagger, \quad (2.58)$$

which are equally valid in the oQM and the dQM. The pair of Hamiltonians $\mathcal{H}^{[0]}$ and $\mathcal{H}^{[1]}$ are essentially *iso-spectral* and their eigenfunctions $\{\phi_n^{[0]}(x)\}$ and $\{\phi_n^{[1]}(x)\}$ are related by the Darboux-Crum transformations [48, 27]:

$$\mathcal{H}^{[0]} \phi_n^{[0]}(x) = \mathcal{E}(n) \phi_n^{[0]}(x) \quad (n = 0, 1, \dots), \quad \mathcal{A} \phi_0^{[0]}(x) = 0, \quad (2.59)$$

$$\mathcal{H}^{[1]} \phi_n^{[1]}(x) = \mathcal{E}(n) \phi_n^{[1]}(x) \quad (n = 1, 2, \dots), \quad (2.60)$$

$$\phi_n^{[1]}(x) = \mathcal{A} \phi_n^{[0]}(x), \quad \phi_n^{[0]}(x) = \frac{\mathcal{A}^\dagger}{\mathcal{E}(n)} \phi_n^{[1]}(x) \quad (n = 1, 2, \dots), \quad (2.61)$$

$$(\phi_n^{[1]}, \phi_m^{[1]}) = \mathcal{E}(n) (\phi_n, \phi_m) \quad (n, m = 1, 2, \dots). \quad (2.62)$$

The partner Hamiltonian $\mathcal{H}^{[1]}$ has the lowest eigenvalue $\mathcal{E}(1)$. If the groundstate energy $\mathcal{E}(1)$ is subtracted from the partner Hamiltonian $\mathcal{H}^{[1]}$, it is again positive semi-definite and can be factorised in terms of new operators $\mathcal{A}^{[1]}$ and $\mathcal{A}^{[1]\dagger}$:

$$\mathcal{H}^{[1]} = \mathcal{A}^{[1]\dagger} \mathcal{A}^{[1]} + \mathcal{E}(1), \quad \mathcal{A}^{[1]} \phi_1^{[1]}(x) = 0. \quad (2.63)$$

It should be stressed that in rdQM with a finite N , the size of the Hamiltonian decreases by one, since the lowest eigenstate is removed, see [26].

By changing the orders of $\mathcal{A}^{[1]\dagger}$ and $\mathcal{A}^{[1]}$, a new Hamiltonian $\mathcal{H}^{[2]}$ is defined:

$$\mathcal{H}^{[2]} \stackrel{\text{def}}{=} \mathcal{A}^{[1]} \mathcal{A}^{[1]\dagger} + \mathcal{E}(1). \quad (2.64)$$

These two Hamiltonians are intertwined by $\mathcal{A}^{[1]}$ and $\mathcal{A}^{[1]\dagger}$:

$$\mathcal{A}^{[1]}(\mathcal{H}^{[1]} - \mathcal{E}(1)) = \mathcal{A}^{[1]} \mathcal{A}^{[1]\dagger} \mathcal{A}^{[1]} = (\mathcal{H}^{[2]} - \mathcal{E}(1)) \mathcal{A}^{[1]}, \quad (2.65)$$

$$\mathcal{A}^{[1]\dagger}(\mathcal{H}^{[2]} - \mathcal{E}(1)) = \mathcal{A}^{[1]\dagger} \mathcal{A}^{[1]} \mathcal{A}^{[1]\dagger} = (\mathcal{H}^{[1]} - \mathcal{E}(1)) \mathcal{A}^{[1]\dagger}. \quad (2.66)$$

The iso-spectrality of the two Hamiltonians $\mathcal{H}^{[1]}$ and $\mathcal{H}^{[2]}$ and the relationship among their eigenfunctions follow as before:

$$\mathcal{H}^{[2]} \phi_n^{[2]}(x) = \mathcal{E}(n) \phi_n^{[2]}(x) \quad (n = 2, 3, \dots), \quad (2.67)$$

$$\phi_n^{[2]}(x) = \mathcal{A}^{[1]} \phi_n^{[1]}(x), \quad \phi_n^{[1]}(x) = \frac{\mathcal{A}^{[1]\dagger}}{\mathcal{E}(n) - \mathcal{E}(1)} \phi_n^{[2]}(x) \quad (n = 2, 3, \dots), \quad (2.68)$$

$$(\phi_n^{[2]}, \phi_m^{[2]}) = (\mathcal{E}(n) - \mathcal{E}(1)) (\phi_n^{[1]}, \phi_m^{[1]}) \quad (n, m = 2, 3, \dots), \quad (2.69)$$

$$\mathcal{H}^{[2]} = \mathcal{A}^{[2]\dagger} \mathcal{A}^{[2]} + \mathcal{E}(2), \quad \mathcal{A}^{[2]} \phi_2^{[2]}(x) = 0. \quad (2.70)$$

This process can go on indefinitely by successively deleting the lowest lying energy level:

$$\mathcal{H}^{[s]} \stackrel{\text{def}}{=} \mathcal{A}^{[s-1]} \mathcal{A}^{[s-1]\dagger} + \mathcal{E}(s-1) = \mathcal{A}^{[s]\dagger} \mathcal{A}^{[s]} + \mathcal{E}(s), \quad (2.71)$$

$$\mathcal{H}^{[s]} \phi_n^{[s]}(x) = \mathcal{E}(n) \phi_n^{[s]}(x) \quad (n = s, s+1, \dots), \quad \mathcal{A}^{[s]} \phi_s^{[s]}(x) = 0, \quad (2.72)$$

$$\phi_n^{[s]}(x) = \mathcal{A}^{[s-1]} \phi_n^{[s-1]}(x), \quad \phi_n^{[s-1]}(x) = \frac{\mathcal{A}^{[s-1]\dagger}}{\mathcal{E}(n) - \mathcal{E}(s-1)} \phi_n^{[s]}(x) \quad (n = s, s+1, \dots), \quad (2.73)$$

$$(\phi_n^{[s]}, \phi_m^{[s]}) = (\mathcal{E}(n) - \mathcal{E}(s-1)) (\phi_n^{[s-1]}, \phi_m^{[s-1]}) \quad (n, m = s, s+1, \dots). \quad (2.74)$$

The quantities in the s -th step are defined by those in the $(s-1)$ -st step: ($s \geq 1$)

$$\text{oQM} : \quad w^{[s]}(x) \stackrel{\text{def}}{=} \log |\phi_s^{[s]}(x)|, \quad (2.75)$$

$$\mathcal{A}^{[s]} \stackrel{\text{def}}{=} \partial_x - \partial_x w^{[s]}(x), \quad \mathcal{A}^{[s]\dagger} = -\partial_x - \partial_x w^{[s]}(x), \quad (2.76)$$

$$\text{idQM} : V^{[s]}(x) \stackrel{\text{def}}{=} \sqrt{V^{[s-1]}(x - i\frac{\gamma}{2})V^{[s-1]*}(x - i\frac{\gamma}{2})} \frac{\phi_s^{[s]}(x - i\gamma)}{\phi_s^{[s]}(x)}, \quad (2.77)$$

$$\begin{aligned} \mathcal{A}^{[s]} &\stackrel{\text{def}}{=} i(e^{\frac{\gamma}{2}p}\sqrt{V^{[s]*}(x)} - e^{-\frac{\gamma}{2}p}\sqrt{V^{[s]}(x)}), \\ \mathcal{A}^{[s]\dagger} &= -i(\sqrt{V^{[s]}(x)}e^{\frac{\gamma}{2}p} - \sqrt{V^{[s]*}(x)}e^{-\frac{\gamma}{2}p}), \end{aligned} \quad (2.78)$$

$$\text{rdQM} : B^{[s]}(x) \stackrel{\text{def}}{=} \sqrt{B^{[s-1]}(x+1)D^{[s-1]}(x+1)} \frac{\phi_s^{[s]}(x+1)}{\phi_s^{[s]}(x)}, \quad (2.79)$$

$$D^{[s]}(x) \stackrel{\text{def}}{=} \sqrt{B^{[s-1]}(x)D^{[s-1]}(x)} \frac{\phi_s^{[s]}(x-1)}{\phi_s^{[s]}(x)}, \quad (2.80)$$

$$\mathcal{A}^{[s]} \stackrel{\text{def}}{=} \sqrt{B^{[s]}(x)} - e^\partial \sqrt{D^{[s]}(x)}, \quad \mathcal{A}^{[s]\dagger} = \sqrt{B^{[s]}(x)} - \sqrt{D^{[s]}(x)} e^{-\partial}. \quad (2.81)$$

The eigenfunctions at the s -th step have succinct *determinant forms* in terms of the Wronskian (oQM) and the Casoratian (dQM) [27, 49, 24, 26]: ($n \geq s \geq 0$)

$$\text{oQM} : W[f_1, \dots, f_m](x) \stackrel{\text{def}}{=} \det\left(\frac{d^{j-1}f_k(x)}{dx^{j-1}}\right)_{1 \leq j, k \leq m} \quad (\text{Wronskian}), \quad (2.82)$$

$$\phi_n^{[s]}(x) = \frac{W[\phi_0, \phi_1, \dots, \phi_{s-1}, \phi_n](x)}{W[\phi_0, \phi_1, \dots, \phi_{s-1}](x)}, \quad (2.83)$$

$$\text{idQM} : W_\gamma[f_1, \dots, f_m](x) \stackrel{\text{def}}{=} i^{\frac{1}{2}m(m-1)} \det\left(f_k(x + i\frac{m+1-2j}{2}\gamma)\right)_{1 \leq j, k \leq m} \quad (\text{Casoratian}), \quad (2.84)$$

$$\phi_n^{[s]}(x) = \prod_{j=0}^{s-1} \sqrt{V^{[j]}(x + i\frac{s-j}{2}\gamma)} \cdot \frac{W_\gamma[\phi_0, \phi_1, \dots, \phi_{s-1}, \phi_n](x)}{W_\gamma[\phi_0, \phi_1, \dots, \phi_{s-1}](x - i\frac{\gamma}{2})}, \quad (2.85)$$

$$\text{rdQM} : W_C[f_1, \dots, f_m](x) \stackrel{\text{def}}{=} \det\left(f_k(x + j - 1)\right)_{1 \leq j, k \leq m} \quad (\text{Casoratian}), \quad (2.86)$$

$$\phi_n^{[s]}(x) = (-1)^s \prod_{k=0}^{s-1} \sqrt{B^{[k]}(x)} \cdot \frac{W_C[\phi_0, \phi_1, \dots, \phi_{s-1}, \phi_n](x)}{W_C[\phi_0, \phi_1, \dots, \phi_{s-1}](x+1)} \quad (2.87)$$

$$= (-1)^s \prod_{k=0}^{s-1} \sqrt{D^{[k]}(x + s - k)} \cdot \frac{W_C[\phi_0, \phi_1, \dots, \phi_{s-1}, \phi_n](x)}{W_C[\phi_0, \phi_1, \dots, \phi_{s-1}](x)}. \quad (2.88)$$

The norm of the s -th step eigenfunctions have a simple uniform expression:

$$(\phi_n^{[s]}, \phi_m^{[s]}) = \prod_{j=0}^{s-1} (\mathcal{E}(n) - \mathcal{E}(j)) \cdot (\phi_n, \phi_m). \quad (2.89)$$

This situation of the Crum's theorem is illustrated in Fig. 1.

A quantum mechanical system with a factorised Hamiltonian $\mathcal{H} = \mathcal{A}^\dagger \mathcal{A}$ together with the associated one $\mathcal{H}^{[1]} = \mathcal{A} \mathcal{A}^\dagger$ is sometimes called a 'supersymmetric' QM [50, 51]. This

appears rather a misnomer, since as we have shown the factorised form is generic and it implies no extra symmetry. The iso-spectrality is shared by all the associated Hamiltonians, not merely by the first two. In this connection, the transformation of mapping the s -th to the $(s + 1)$ -st associated Hamiltonian is sometimes called susy transformation. It is also known as the Darboux or Darboux-Crum transformation. Those covering multi-steps are sometimes called ‘higher derivative’ or ‘nonlinear’ or ‘ \mathcal{N} -fold’ susy transformations [52, 53, 54].

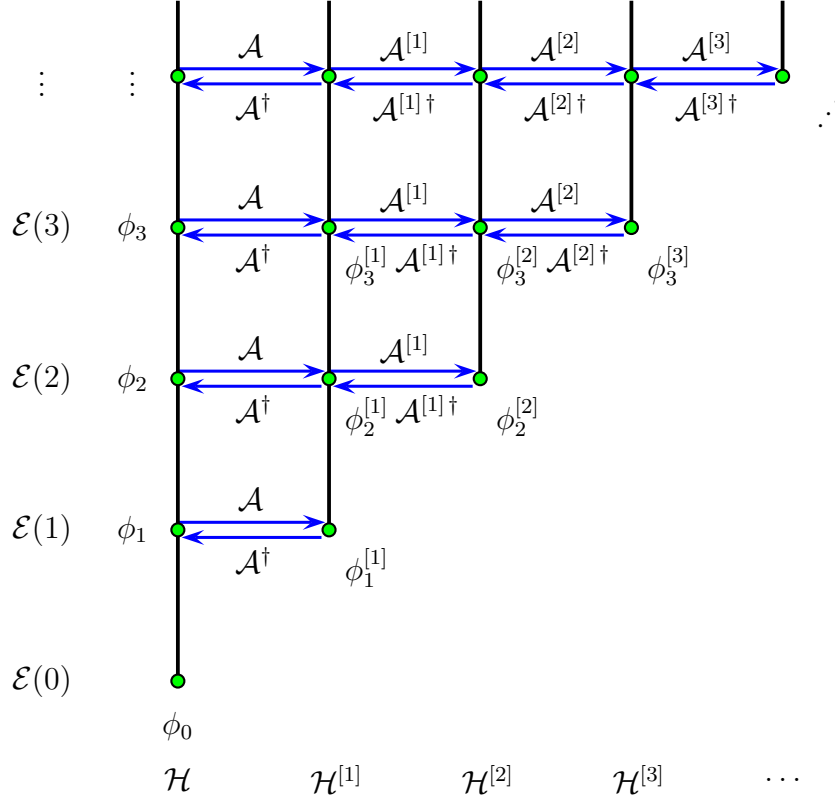


Figure 1: Schematic picture of the Crum's theorem

2.3 Modified Crum's Theorem

Crum's theorem provides a new iso-spectral Hamiltonian system by deleting successively the lowest lying levels from the original Hamiltonian systems \mathcal{H} and $\{\phi_n(x)\}$. The modification of Crum's theorem by Krein-Adler [28, 29] is achieved by deleting a finite number of eigenstates indexed by a set of non-negative distinct integers¹ $\mathcal{D} \stackrel{\text{def}}{=} \{d_1, d_2, \dots, d_\ell\} \subset \mathbb{Z}_{\geq 0}^\ell$, satisfying certain conditions to be specified later. After the deletion, the new groundstate has the label

¹ Although this notation d_j conflicts with the notation of the normalisation constant d_n in (2.50), we think this does not cause any confusion because the latter appears as $\frac{1}{d_n^2} \delta_{nm}$.

μ given by

$$\mu \stackrel{\text{def}}{=} \min\{n \mid n \in \mathbb{Z}_{\geq 0} \setminus \mathcal{D}\}. \quad (2.90)$$

Corresponding to (2.75)–(2.88), the new iso-spectral Hamiltonian system is [25, 26]:

$$\bar{\mathcal{H}} \stackrel{\text{def}}{=} \bar{\mathcal{A}}^\dagger \bar{\mathcal{A}} + \mathcal{E}(\mu), \quad \bar{\mathcal{A}} \bar{\phi}_\mu(x) = 0, \quad (2.91)$$

$$\bar{\mathcal{H}} \bar{\phi}_n(x) = \mathcal{E}(n) \bar{\phi}_n(x) \quad (n \in \mathbb{Z}_{\geq 0} \setminus \mathcal{D}), \quad (2.92)$$

$$\text{oQM: } \bar{\mathcal{H}} = p^2 + \bar{U}(x), \quad \bar{U}(x) = U(x) - 2\partial_x^2 \left(\log W[\phi_{d_1}, \phi_{d_2}, \dots, \phi_{d_\ell}](x) \right), \quad (2.93)$$

$$\bar{\mathcal{A}} \stackrel{\text{def}}{=} \partial_x - \partial_x \bar{w}(x), \quad \bar{\mathcal{A}}^\dagger = -\partial_x - \partial_x \bar{w}(x), \quad \bar{w}(x) \stackrel{\text{def}}{=} \log |\bar{\phi}_\mu(x)|, \quad (2.94)$$

$$\bar{\phi}_n(x) \stackrel{\text{def}}{=} \frac{W[\phi_{d_1}, \phi_{d_2}, \dots, \phi_{d_\ell}, \phi_n](x)}{W[\phi_{d_1}, \phi_{d_2}, \dots, \phi_{d_\ell}](x)}, \quad (2.95)$$

$$\begin{aligned} \text{idQM: } \bar{V}(x) &\stackrel{\text{def}}{=} \sqrt{V(x - i\frac{\ell}{2}\gamma) V^*(x - i\frac{\ell+2}{2}\gamma)} \\ &\quad \times \frac{W_\gamma[\phi_{d_1}, \dots, \phi_{d_\ell}](x + i\frac{\gamma}{2})}{W_\gamma[\phi_{d_1}, \dots, \phi_{d_\ell}](x - i\frac{\gamma}{2})} \frac{W_\gamma[\phi_{d_1}, \dots, \phi_{d_\ell}, \phi_\mu](x - i\gamma)}{W_\gamma[\phi_{d_1}, \dots, \phi_{d_\ell}, \phi_\mu](x)}, \end{aligned} \quad (2.96)$$

$$\begin{aligned} \bar{\mathcal{A}} &\stackrel{\text{def}}{=} i \left(e^{\frac{\gamma}{2}p} \sqrt{\bar{V}^*(x)} - e^{-\frac{\gamma}{2}p} \sqrt{\bar{V}(x)} \right), \\ \bar{\mathcal{A}}^\dagger &= -i \left(\sqrt{\bar{V}(x)} e^{\frac{\gamma}{2}p} - \sqrt{\bar{V}^*(x)} e^{-\frac{\gamma}{2}p} \right), \end{aligned} \quad (2.97)$$

$$\begin{aligned} F(x) &\stackrel{\text{def}}{=} \frac{\sqrt{\prod_{j=0}^{\ell-1} V(x + i\frac{\ell-2j}{2}\gamma) V^*(x - i\frac{\ell-2j}{2}\gamma)}}{W_\gamma[\phi_{d_1}, \dots, \phi_{d_\ell}](x - i\frac{\gamma}{2}) W_\gamma[\phi_{d_1}, \dots, \phi_{d_\ell}](x + i\frac{\gamma}{2})}, \\ \bar{\phi}_n(x) &\stackrel{\text{def}}{=} \sqrt{F(x)} W_\gamma[\phi_{d_1}, \dots, \phi_{d_\ell}, \phi_n](x), \end{aligned} \quad (2.98)$$

$$\begin{aligned} \text{rdQM: } \bar{B}(x) &= \sqrt{B(x + \ell) D(x + \ell + 1)} \frac{W_C[\phi_{d_1}, \dots, \phi_{d_\ell}](x)}{W_C[\phi_{d_1}, \dots, \phi_{d_\ell}](x + 1)} \\ &\quad \times \frac{W_C[\phi_{d_1}, \dots, \phi_{d_\ell}, \phi_\mu](x + 1)}{W_C[\phi_{d_1}, \dots, \phi_{d_\ell}, \phi_\mu](x)}, \end{aligned} \quad (2.99)$$

$$\begin{aligned} \bar{D}(x) &= \sqrt{B(x - 1) D(x)} \frac{W_C[\phi_{d_1}, \dots, \phi_{d_\ell}](x + 1)}{W_C[\phi_{d_1}, \dots, \phi_{d_\ell}](x)} \\ &\quad \times \frac{W_C[\phi_{d_1}, \dots, \phi_{d_\ell}, \phi_\mu](x - 1)}{W_C[\phi_{d_1}, \dots, \phi_{d_\ell}, \phi_\mu](x)}, \end{aligned} \quad (2.100)$$

$$\bar{\mathcal{A}} \stackrel{\text{def}}{=} \sqrt{\bar{B}(x)} - e^\partial \sqrt{\bar{D}(x)}, \quad \bar{\mathcal{A}}^\dagger = \sqrt{\bar{B}(x)} - \sqrt{\bar{D}(x)} e^{-\partial}, \quad (2.101)$$

$$\begin{aligned} F(x) &\stackrel{\text{def}}{=} \frac{\sqrt{\prod_{k=1}^{\ell} B(x + k - 1) D(x + k)}}{W_C[\phi_{d_1}, \dots, \phi_{d_\ell}](x) W_C[\phi_{d_1}, \dots, \phi_{d_\ell}](x + 1)}, \\ \bar{\phi}_n(x) &= (-1)^\ell \sqrt{F(x)} W_C[\phi_{d_1}, \dots, \phi_{d_\ell}, \phi_n](x), \end{aligned} \quad (2.102)$$

and

$$\text{oQM, dQM : } (\bar{\phi}_n, \bar{\phi}_m) = \prod_{j=1}^{\ell} (\mathcal{E}(n) - \mathcal{E}(d_j)) \cdot (\phi_n, \phi_m) \quad (n, m \in \mathbb{Z}_{\geq 0} \setminus \mathcal{D}). \quad (2.103)$$

It should be emphasised that the Hamiltonian $\bar{\mathcal{H}}$ as well as the eigenfunction $\{\bar{\phi}_n(x)\}$ are symmetric with respect to d_1, \dots, d_ℓ , and thus they are independent of the order of $\{d_j\}$. In order to guarantee the positivity of the norm (2.103) of all the eigenfunctions $\{\bar{\phi}_n(x)\}$ of the modified Hamiltonian, the set of deleted energy levels $\mathcal{D} = \{d_1, \dots, d_\ell\}$ must satisfy the necessary and sufficient conditions [28, 29]

$$\prod_{j=1}^{\ell} (m - d_j) \geq 0, \quad m \in \mathbb{Z}_{\geq 0}. \quad (2.104)$$

And the Hamiltonian $\bar{\mathcal{H}}$ is non-singular under this condition. The Crum's theorem in § 2.2 corresponds to the choice $\{d_1, d_2, \dots, d_\ell\} = \{0, 1, \dots, \ell - 1\}$.

Starting from an exactly solvable Hamiltonian, one can construct infinitely many variants of exactly solvable Hamiltonians and their eigenfunctions by Adler's and García et al's methods [25]. The resulting systems are, however, not shape invariant, even if the starting system is. For the dQM with real shifts, see the recent work [26] and a related work [55].

2.4 Shape Invariance

Shape invariance [30] is a sufficient condition for the exact solvability in the Schrödinger picture. Combined with Crum's theorem [27], or the factorisation method [40] or the so-called supersymmetric quantum mechanics [31, 51], the totality of the discrete eigenvalues and the corresponding eigenfunctions can be easily obtained. It was shown by the present authors that the concept of shape invariance worked equally well in the dQM with the pure imaginary shifts [12, 21] as well as the real shifts [20], providing quantum mechanical explanation of the solvability of the Askey scheme of hypergeometric orthogonal polynomials in general. Throughout this review, we concentrate on the simple situations that the entire spectra are discrete. When continuous spectra exist, however, shape invariance and Crum's theorem fail to provide the full set of eigenvalues and eigenfunctions.

In many cases the Hamiltonian contains some parameter(s), $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$. Here we write the parameter dependence explicitly, $\mathcal{H}(\boldsymbol{\lambda})$, $\mathcal{A}(\boldsymbol{\lambda})$, $\mathcal{E}(n; \boldsymbol{\lambda})$, $\phi_n(x; \boldsymbol{\lambda})$, $P_n(\eta(x; \boldsymbol{\lambda}); \boldsymbol{\lambda})$

etc, since it is the central issue. The shape invariance condition with a suitable choice of parameters is

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger = \kappa\mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta})^\dagger\mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}(1; \boldsymbol{\lambda}), \quad (2.105)$$

where κ is a real positive parameter and $\boldsymbol{\delta}$ is the shift of the parameters. In other words $\mathcal{H}^{[0]}$ and $\mathcal{H}^{[1]}$ have the same shape, only the parameters are shifted by $\boldsymbol{\delta}$. The s -th step Hamiltonian $\mathcal{H}^{[s]}$ in §2.2 is $\mathcal{H}^{[s]} = \kappa^s\mathcal{H}(\boldsymbol{\lambda} + s\boldsymbol{\delta}) + \mathcal{E}(s; \boldsymbol{\lambda})$. The energy spectrum and the excited state wavefunction are determined by the data of the groundstate wavefunction $\phi_0(x; \boldsymbol{\lambda})$ and the energy of the first excited state $\mathcal{E}(1; \boldsymbol{\lambda})$ as follows [31, 12, 20, 21]:

$$\mathcal{E}(n; \boldsymbol{\lambda}) = \sum_{s=0}^{n-1} \kappa^s \mathcal{E}(1; \boldsymbol{\lambda}^{[s]}), \quad \boldsymbol{\lambda}^{[s]} \stackrel{\text{def}}{=} \boldsymbol{\lambda} + s\boldsymbol{\delta}, \quad (2.106)$$

$$\phi_n(x; \boldsymbol{\lambda}) \propto \mathcal{A}(\boldsymbol{\lambda}^{[0]})^\dagger \mathcal{A}(\boldsymbol{\lambda}^{[1]})^\dagger \mathcal{A}(\boldsymbol{\lambda}^{[2]})^\dagger \dots \mathcal{A}(\boldsymbol{\lambda}^{[n-1]})^\dagger \phi_0(x; \boldsymbol{\lambda}^{[n]}). \quad (2.107)$$

The above formula for the eigenfunctions $\phi_n(x; \boldsymbol{\lambda})$ can be considered as the *universal Rodrigues formula* for the Askey scheme of hypergeometric polynomials and their q -analogues. For the explicit forms of the Rodrigues type formula for each polynomial, one only has to substitute the explicit forms of the operator $\mathcal{A}(\boldsymbol{\lambda})$ and the groundstate wavefunction $\phi_0(x; \boldsymbol{\lambda})$. For the nine explicit examples given in (2.30)–(2.39), it is straightforward to verify the shape invariance conditions (2.105) and the energy (2.106) and the eigenfunction (2.107) formulas.

In the case of a finite number of bound states, e.g. the Morse potential, the eigenvalue has a maximum at a certain level n , $\mathcal{E}(n; \boldsymbol{\lambda})$. Beyond that level the formula (2.106) ceases to work and the Rodrigues formula (2.107) does not provide a square integrable eigenfunctions, although ϕ_m ($m > n$) continues to satisfy the Schrödinger equation with $\mathcal{E}(m; \boldsymbol{\lambda})$.

The above shape invariance condition (2.105) is equivalent to the following conditions:

$$\text{oQM: } (\partial_x w(x; \boldsymbol{\lambda}))^2 - \partial_x^2 w(x; \boldsymbol{\lambda}) = (\partial_x w(x; \boldsymbol{\lambda} + \boldsymbol{\delta}))^2 + \partial_x^2 w(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}(1; \boldsymbol{\lambda}), \quad \kappa = 1, \quad (2.108)$$

$$\text{idQM: } V(x - i\frac{\gamma}{2}; \boldsymbol{\lambda})V^*(x - i\frac{\gamma}{2}; \boldsymbol{\lambda}) = \kappa^2 V(x; \boldsymbol{\lambda} + \boldsymbol{\delta})V^*(x - i\gamma; \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (2.109)$$

$$V(x + i\frac{\gamma}{2}; \boldsymbol{\lambda}) + V^*(x - i\frac{\gamma}{2}; \boldsymbol{\lambda}) = \kappa(V(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) + V^*(x; \boldsymbol{\lambda} + \boldsymbol{\delta})) - \mathcal{E}(1; \boldsymbol{\lambda}), \quad (2.110)$$

$$\text{rdQM: } B(x + 1; \boldsymbol{\lambda})D(x + 1; \boldsymbol{\lambda}) = \kappa^2 B(x; \boldsymbol{\lambda} + \boldsymbol{\delta})D(x + 1; \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (2.111)$$

$$B(x; \boldsymbol{\lambda}) + D(x + 1; \boldsymbol{\lambda}) = \kappa(B(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) + D(x; \boldsymbol{\lambda} + \boldsymbol{\delta})) + \mathcal{E}(1; \boldsymbol{\lambda}). \quad (2.112)$$

For the idQM, the first condition (2.109) is multiplicative. If V_1, V_1^*, κ_1 and V_2, V_2^*, κ_2 satisfy the condition independently, then $V = V_1V_2, V^* = V_1^*V_2^*, \kappa = \kappa_1\kappa_2$ also satisfies it.

The second conditions (2.110) provides a substantial constraint. The situation is the same in rdQM.

It is straightforward to verify the shape invariance for the nine examples (2.30)–(2.39) in §2.1 with the following data:

$$\text{oQM : H : } \boldsymbol{\lambda} = \phi \text{ (null), } \boldsymbol{\delta} = \phi, \quad \kappa = 1, \quad \mathcal{E}(n; \boldsymbol{\lambda}) = 2n, \quad (2.113)$$

$$\text{L : } \boldsymbol{\lambda} = g, \quad \boldsymbol{\delta} = 1, \quad \kappa = 1, \quad \mathcal{E}(n; \boldsymbol{\lambda}) = 4n, \quad (2.114)$$

$$\text{J : } \boldsymbol{\lambda} = (g, h), \quad \boldsymbol{\delta} = (1, 1), \quad \kappa = 1, \quad \mathcal{E}(n; \boldsymbol{\lambda}) = 4n(n + g + h). \quad (2.115)$$

It should be stressed that the above shape invariant transformation $\boldsymbol{\lambda} \rightarrow \boldsymbol{\lambda} + \boldsymbol{\delta}$, $\mathcal{H}^{[s]} \rightarrow \mathcal{H}^{[s+1]}$ for L and J, that is, $g \rightarrow g + 1$, $h \rightarrow h + 1$, *preserves the monodromy* (2.33) at the regular singularities.

$$\text{idQM : MP : } \boldsymbol{\lambda} = a, \quad \boldsymbol{\delta} = \frac{1}{2}, \quad \kappa = 1, \quad \mathcal{E}(n; \boldsymbol{\lambda}) = 2n, \quad (2.116)$$

$$\begin{aligned} \text{W : } \boldsymbol{\lambda} &= (a_1, a_2, a_3, a_4), \quad \boldsymbol{\delta} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \quad \kappa = 1, \\ \mathcal{E}(n; \boldsymbol{\lambda}) &= 4n(n + b_1 - 1), \end{aligned} \quad (2.117)$$

$$\begin{aligned} \text{AW : } q^\lambda &= (a_1, a_2, a_3, a_4), \quad \boldsymbol{\delta} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \quad \kappa = q^{-1}, \\ \mathcal{E}(n; \boldsymbol{\lambda}) &= (q^{-n} - 1)(1 - b_4 q^{n-1}), \end{aligned} \quad (2.118)$$

$$\text{rdQM : M : } \boldsymbol{\lambda} = (\beta, c), \quad \boldsymbol{\delta} = (1, 0), \quad \kappa = 1, \quad \mathcal{E}(n; \boldsymbol{\lambda}) = n, \quad (2.119)$$

$$\text{R : } \boldsymbol{\lambda} = (a, b, c, d), \quad \boldsymbol{\delta} = (1, 1, 1, 1), \quad \kappa = 1, \quad \mathcal{E}(n; \boldsymbol{\lambda}) = 4n(n + \tilde{d}), \quad (2.120)$$

$$\begin{aligned} \text{qR : } q^\lambda &= (a, b, c, d), \quad \boldsymbol{\delta} = (1, 1, 1, 1), \quad \kappa = q^{-1}, \\ \mathcal{E}(n; \boldsymbol{\lambda}) &= (q^{-n} - 1)(1 - \tilde{d} q^{n-1}), \end{aligned} \quad (2.121)$$

where $q^{(\lambda_1, \lambda_2, \dots)} = (q^{\lambda_1}, q^{\lambda_2}, \dots)$. For more complete lists of exactly solvable oQM see [40, 51] and for idQM and rdQM see [21] and [20]. It is well known that the solutions of Schrödinger equations provide those of the corresponding Fokker-Planck equations [56], which describe the time evolution of probability density functions. This connection can be readily generalised to discrete Schrödinger equations, which would correspond to, for example, discretised stochastic processes, the Markov chains. The birth and death processes [5, 57] are the best known examples. It is interesting to point out that all the exactly solvable rdQM examples in [20] and their modifications [26, 11] provide also *exactly solvable birth and death processes* [58].

The simplest example of rdQM, the Charlier polynomial, has no shiftable parameter. Its shape invariance relation $\mathcal{A}\mathcal{A}^\dagger - \mathcal{A}^\dagger\mathcal{A} = 1$ gives another realisation of the oscillator

algebra. Likewise, the simplest example of idQM, the q -Hermite (q H) polynomial has no parameter other than q . It is obtained as a special case of the Askey-Wilson (AW) (2.36) by setting $a_j = 0$, $j = 1, \dots, 4$. Like the harmonic oscillator (Hermite) case, it is shape invariant and the shape invariance relation ($\kappa = q^{-1}$) (2.105) becomes the q -oscillator algebra $\mathcal{A}\mathcal{A}^\dagger - q^{-1}\mathcal{A}^\dagger\mathcal{A} = q^{-1} - 1$ itself [19].

The shape invariance and the Crum's theorem imply that $\phi_n(x; \boldsymbol{\lambda})$ and $\phi_{n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta})$ are mapped to each other by the operators $\mathcal{A}(\boldsymbol{\lambda})$ and $\mathcal{A}(\boldsymbol{\lambda})^\dagger$:

$$\mathcal{A}(\boldsymbol{\lambda})\phi_n(x; \boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})\phi_{n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \frac{1}{\sqrt{B(0; \boldsymbol{\lambda})}} & : \text{rdQM} \end{cases}, \quad (2.122)$$

$$\mathcal{A}(\boldsymbol{\lambda})^\dagger\phi_{n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) = b_{n-1}(\boldsymbol{\lambda})\phi_n(x; \boldsymbol{\lambda}) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \sqrt{B(0; \boldsymbol{\lambda})} & : \text{rdQM} \end{cases}. \quad (2.123)$$

Here the constants $f_n(\boldsymbol{\lambda})$ and $b_{n-1}(\boldsymbol{\lambda})$ depend on the normalisation of $\phi_n(x; \boldsymbol{\lambda})$ but their product does not. It gives the energy eigenvalue,

$$\mathcal{E}(n; \boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})b_{n-1}(\boldsymbol{\lambda}). \quad (2.124)$$

The factor $\sqrt{B(0; \boldsymbol{\lambda})}$ for rdQM is introduced for later convenience. For our choice of $\phi_n(x; \boldsymbol{\lambda})$ and $P_n(\eta(x; \boldsymbol{\lambda}); \boldsymbol{\lambda})$, the data for $f_n(\boldsymbol{\lambda})$ and $b_{n-1}(\boldsymbol{\lambda})$ are:

$$\text{oQM} : f_n(\boldsymbol{\lambda}) = \begin{cases} 2n & : \text{H} \\ -2 & : \text{L} \\ -2(n+g+h) & : \text{J} \end{cases}, \quad b_{n-1}(\boldsymbol{\lambda}) = \begin{cases} 1 & : \text{H} \\ -2n & : \text{L, J} \end{cases}, \quad (2.125)$$

$$\text{idQM} : f_n(\boldsymbol{\lambda}) = \begin{cases} 2 & : \text{MP} \\ -n(n+b_1-1) & : \text{W} \\ q^{\frac{n}{2}}(q^{-n}-1)(1-b_4q^{n-1}) & : \text{AW} \end{cases}, \quad b_{n-1}(\boldsymbol{\lambda}) = \begin{cases} n & : \text{MP} \\ -1 & : \text{W} \\ q^{-\frac{n}{2}} & : \text{AW} \end{cases}, \quad (2.126)$$

$$\text{rdQM} : f_n(\boldsymbol{\lambda}) = \mathcal{E}(n; \boldsymbol{\lambda}), \quad b_{n-1}(\boldsymbol{\lambda}) = 1. \quad (2.127)$$

By removing the groundstate contributions, the forward and backward shift operators acting on the polynomial eigenfunctions, $\mathcal{F}(\boldsymbol{\lambda})$ and $\mathcal{B}(\boldsymbol{\lambda})$, are introduced:

$$\mathcal{F}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x; \boldsymbol{\lambda} + \boldsymbol{\delta})^{-1} \circ \mathcal{A}(\boldsymbol{\lambda}) \circ \phi_0(x; \boldsymbol{\lambda}) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \sqrt{B(0; \boldsymbol{\lambda})} & : \text{rdQM} \end{cases} \quad (2.128)$$

$$= \begin{cases} c_{\mathcal{F}} \frac{d}{d\eta} & : \text{oQM} \\ i\varphi(x)^{-1}(e^{\frac{\gamma}{2}p} - e^{-\frac{\gamma}{2}p}) & : \text{idQM} \\ B(0; \boldsymbol{\lambda})\varphi(x; \boldsymbol{\lambda})^{-1}(1 - e^\partial) & : \text{rdQM} \end{cases}, \quad (2.129)$$

$$\mathcal{B}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x; \boldsymbol{\lambda})^{-1} \circ \mathcal{A}(\boldsymbol{\lambda})^\dagger \circ \phi_0(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \frac{1}{\sqrt{B(0; \boldsymbol{\lambda})}} & : \text{rdQM} \end{cases} \quad (2.130)$$

$$= \begin{cases} -4c_{\mathcal{F}}^{-1}c_2(\eta)\left(\frac{d}{d\eta} + \frac{c_1(\eta, \boldsymbol{\lambda})}{c_2(\eta)}\right) & : \text{oQM} \\ -i(V(x; \boldsymbol{\lambda})e^{\frac{\gamma}{2}p} - V^*(x; \boldsymbol{\lambda})e^{-\frac{\gamma}{2}p})\varphi(x) & : \text{idQM} \\ \frac{1}{B(0; \boldsymbol{\lambda})}(B(x; \boldsymbol{\lambda}) - D(x; \boldsymbol{\lambda})e^{-\theta})\varphi(x; \boldsymbol{\lambda}) & : \text{rdQM} \end{cases}, \quad (2.131)$$

where $c_{\mathcal{F}}$, $c_1(\eta, \boldsymbol{\lambda})$ and $c_2(\eta)$ are

$$c_{\mathcal{F}} \stackrel{\text{def}}{=} \begin{cases} 1 & : \text{H} \\ 2 & : \text{L} \\ -4 & : \text{J} \end{cases}, \quad c_1(\eta, \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} -\frac{1}{2} & : \text{H} \\ g + \frac{1}{2} - \eta & : \text{L} \\ h - g - (g + h + 1)\eta & : \text{J} \end{cases}, \quad c_2(\eta) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{4} & : \text{H} \\ \eta & : \text{L} \\ 1 - \eta^2 & : \text{J} \end{cases}, \quad (2.132)$$

and the auxiliary functions $\varphi(x)$ are

$$\text{idQM: } \varphi(x) = \begin{cases} 1 & : \text{MP} \\ 2x & : \text{W} \\ 2 \sin x & : \text{AW} \end{cases}, \quad \text{rdQM: } \varphi(x; \boldsymbol{\lambda}) = \begin{cases} 1 & : \text{M} \\ \frac{2x + d + 1}{d + 1} & : \text{R} \\ \frac{q^{-x} - dq^{x+1}}{1 - dq} & : \text{qR} \end{cases}. \quad (2.133)$$

Then the above relations (2.122)–(2.123) become

$$\mathcal{F}(\boldsymbol{\lambda})\check{P}_n(x; \boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})\check{P}_{n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (2.134)$$

$$\mathcal{B}(\boldsymbol{\lambda})\check{P}_{n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) = b_{n-1}(\boldsymbol{\lambda})\check{P}_n(x; \boldsymbol{\lambda}), \quad (2.135)$$

where we have used the notation

$$\check{P}_n(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_n(\eta(x; \boldsymbol{\lambda}); \boldsymbol{\lambda}). \quad (2.136)$$

Corresponding to (2.5), the forward and backward shift operators give a factorisation of the similarity transformed Hamiltonian (2.24),

$$\tilde{\mathcal{H}}(\boldsymbol{\lambda}) = \mathcal{B}(\boldsymbol{\lambda})\mathcal{F}(\boldsymbol{\lambda}). \quad (2.137)$$

2.5 Solvability in the Heisenberg Picture

As is well known the Heisenberg operator formulation is central to quantum field theory. The creation/annihilation operators of the harmonic oscillators are the cornerstones of modern quantum physics. However, until recently, it had been generally conceived that the Heisenberg operator solutions are intractable. Here we show that most of the shape invariant dQM Hamiltonian systems are exactly solvable in the Heisenberg picture, too [14, 15]. To be more precise, the Heisenberg operator of the sinusoidal coordinate $\eta(x)$

$$e^{it\mathcal{H}}\eta(x)e^{-it\mathcal{H}} \quad (2.138)$$

can be evaluated in a closed form. It is well known that any orthogonal polynomials satisfy the *three term recurrence relations* [3, 5]

$$\eta P_n(\eta) = A_n P_{n+1}(\eta) + B_n P_n(\eta) + C_n P_{n-1}(\eta) \quad (n \geq 0), \quad (2.139)$$

with $P_{-1}(\eta) = 0$. Here the coefficients A_n , B_n and C_n are real and $A_{n-1}C_n > 0$ ($n \geq 1$). Conversely all the polynomials starting with degree 0 and satisfy the above three term recurrence relations are orthogonal (Favard's theorem [59]). These relations can also be considered as an eigenvalue equation, in which η is the eigenvalue. For the polynomials in rdQM with the universal normalisation $P_n(0) = 1$ (2.22), the coefficients of the three term recurrence relations are restricted by the condition

$$\text{rdQM : } B_n = -(A_n + C_n) \quad (n = 0, 1, \dots). \quad (2.140)$$

For the factorised quantum mechanical eigenfunctions (2.20), these relations mean

$$\eta(x)\phi_n(x) = A_n\phi_{n+1}(x) + B_n\phi_n(x) + C_n\phi_{n-1}(x) \quad (n \geq 0). \quad (2.141)$$

In other words, the operator $\eta(x)$ acts like a creation operator which sends the eigenstate n to $n + 1$ as well as like an annihilation operator, which maps an eigenstate n to $n - 1$. This fact combined with the well known result that the annihilation/creation operators of the harmonic oscillator are the positive/negative frequency part of the Heisenberg operator solution for the coordinate x is the starting point of this subsection. As will be shown below the *sinusoidal coordinate* $\eta(x)$ undergoes sinusoidal motion (2.144), whose frequencies depend on the energy. Thus it is not harmonic in general. To the best of our knowledge, the sinusoidal coordinate was first introduced in a rather broad sense for general (not necessarily solvable) potentials as a useful means for coherent state research by Nieto and Simmons [60].

The sufficient condition for the closed form expression of the Heisenberg operator (2.138) is the *closure relation*

$$[\mathcal{H}, [\mathcal{H}, \eta(x)]] = \eta(x) R_0(\mathcal{H}) + [\mathcal{H}, \eta(x)] R_1(\mathcal{H}) + R_{-1}(\mathcal{H}). \quad (2.142)$$

Here the coefficients $R_i(y)$ are polynomials in y . It is easy to see that the cubic commutator $[\mathcal{H}, [\mathcal{H}, [\mathcal{H}, \eta(x)]]] \equiv (\text{ad } \mathcal{H})^3 \eta(x)$ is reduced to $\eta(x)$ and $[\mathcal{H}, \eta(x)]$ with \mathcal{H} depending coefficients:

$$(\text{ad } \mathcal{H})^3 \eta(x) = [\mathcal{H}, \eta(x)] R_0(\mathcal{H}) + [\mathcal{H}, [\mathcal{H}, \eta(x)]] R_1(\mathcal{H})$$

$$= \eta(x) R_0(\mathcal{H})R_1(\mathcal{H}) + [\mathcal{H}, \eta(x)] (R_1(\mathcal{H})^2 + R_0(\mathcal{H})) + R_{-1}(\mathcal{H})R_1(\mathcal{H}), \quad (2.143)$$

in which the definition $(\text{ad } \mathcal{H})X \stackrel{\text{def}}{=} [\mathcal{H}, X]$ is used. It is trivial to see that all the higher commutators $(\text{ad } \mathcal{H})^n \eta(x)$ can also be reduced to $\eta(x)$ and $[\mathcal{H}, \eta(x)]$ with \mathcal{H} depending coefficients. The second order closure (2.142) simply reflects the Schrödinger equation, which is a second order differential or difference equation. Thus we arrive at

$$\begin{aligned} e^{it\mathcal{H}}\eta(x)e^{-it\mathcal{H}} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} (\text{ad } \mathcal{H})^n \eta(x) \\ &= [\mathcal{H}, \eta(x)] \frac{e^{i\alpha_+(\mathcal{H})t} - e^{i\alpha_-(\mathcal{H})t}}{\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})} - R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1} \\ &\quad + (\eta(x) + R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1}) \frac{-\alpha_-(\mathcal{H})e^{i\alpha_+(\mathcal{H})t} + \alpha_+(\mathcal{H})e^{i\alpha_-(\mathcal{H})t}}{\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})}. \end{aligned} \quad (2.144)$$

This simply means that $\eta(x)$ oscillates sinusoidally with two energy-dependent “frequencies” $\alpha_{\pm}(\mathcal{H})$ given by

$$\alpha_{\pm}(\mathcal{H}) = \frac{1}{2}(R_1(\mathcal{H}) \pm \sqrt{R_1(\mathcal{H})^2 + 4R_0(\mathcal{H})}), \quad (2.145)$$

$$\alpha_+(\mathcal{H}) + \alpha_-(\mathcal{H}) = R_1(\mathcal{H}), \quad \alpha_+(\mathcal{H})\alpha_-(\mathcal{H}) = -R_0(\mathcal{H}). \quad (2.146)$$

The energy spectrum is determined by the over-determined recursion relations $\mathcal{E}(n+1) = \mathcal{E}(n) + \alpha_+(\mathcal{E}(n))$ and $\mathcal{E}(n-1) = \mathcal{E}(n) + \alpha_-(\mathcal{E}(n))$ with $\mathcal{E}(0) = 0$. It should be stressed that for the known spectra $\{\mathcal{E}(n)\}$ determined by the shape invariance, the quantity inside the square root in the definition of $\alpha_{\pm}(\mathcal{H})$ (2.145) for each n :

$$R_1(\mathcal{E}(n))^2 + 4R_0(\mathcal{E}(n))$$

becomes a complete square and the the above two conditions are consistent. For oQM, the Hamiltonian and the sinusoidal coordinate satisfying the closure relation (2.142) are classified and then the eigenfunctions have the factorised form (2.20) [14]. For dQM we assume (2.20). The *annihilation* and *creation* operators $a^{(\pm)}$ are extracted from this exact Heisenberg operator solution:

$$e^{it\mathcal{H}}\eta(x)e^{-it\mathcal{H}} = a^{(+)}e^{i\alpha_+(\mathcal{H})t} + a^{(-)}e^{i\alpha_-(\mathcal{H})t} - R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1}, \quad (2.147)$$

$$\begin{aligned} a^{(\pm)} &\stackrel{\text{def}}{=} \pm \left([\mathcal{H}, \eta(x)] - (\eta(x) + R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1})\alpha_{\mp}(\mathcal{H}) \right) (\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H}))^{-1} \\ &= \pm (\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H}))^{-1} \left([\mathcal{H}, \eta(x)] + \alpha_{\pm}(\mathcal{H})(\eta(x) + R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1}) \right), \end{aligned} \quad (2.148)$$

$$a^{(+)\dagger} = a^{(-)}, \quad a^{(+)}\phi_n(x) = A_n\phi_{n+1}(x), \quad a^{(-)}\phi_n(x) = C_n\phi_{n-1}(x). \quad (2.149)$$

It should be stressed that the annihilation operator and creation operators are hermitian conjugate of each other and they act on the eigenstate (2.149). Simple commutation relations

$$[\mathcal{H}, a^{(\pm)}] = a^{(\pm)}\alpha_{\pm}(\mathcal{H}), \quad (2.150)$$

follow from (2.148) and (2.142). Commutation relations of $a^{(\pm)}$ are expressed in terms of the coefficients of the three term recurrence relation by (2.149):

$$\begin{aligned} a^{(-)}a^{(+)}\phi_n &= A_n C_{n+1}\phi_n, & a^{(+)}a^{(-)}\phi_n &= C_n A_{n-1}\phi_n, \\ \Rightarrow [a^{(-)}, a^{(+)}]\phi_n &= (A_n C_{n+1} - A_{n-1} C_n)\phi_n. \end{aligned} \quad (2.151)$$

These relations simply mean the operator relations

$$a^{(-)}a^{(+)} = f(\mathcal{H}), \quad a^{(+)}a^{(-)} = g(\mathcal{H}), \quad (2.152)$$

in which f and g are analytic functions of \mathcal{H} explicitly given for each example. In other words, \mathcal{H} and $a^{(\pm)}$ form a so-called quasi-linear algebra [61]. Various *dynamical symmetry algebras* associated with exactly solvable QM, including the q -oscillator algebra, were explicitly identified in [20, 21]. It should be stressed that the situation is quite different from those of the wide variety of proposed annihilation/creation operators for various quantum systems [62], most of which were introduced within the framework of ‘algebraic theory of coherent states’, without exact solvability. In all such cases there is no guarantee for symmetry relations like (2.152). The explicit form of the annihilation operator (2.148) allows us to define the *coherent state* as its eigenvector, $a^{(-)}\psi(\alpha, x) = \alpha\psi(\alpha, x)$, $\alpha \in \mathbb{C}$. See [14, 21] for various coherent states.

The excited state wavefunctions $\{\phi_n(x)\}$ are obtained by the successive action of the creation operator $a^{(+)}$ on the groundstate wavefunction $\phi_0(x)$. This is the exact solvability in the Heisenberg picture.

The data for the three examples in oQM (2.30)–(2.32) are:

$$\text{H} : \quad R_1(y) = 0, \quad R_0(y) = 4, \quad R_{-1}(y) = 0, \quad (2.153)$$

$$\text{L} : \quad R_1(y) = 0, \quad R_0(y) = 16, \quad R_{-1}(y) = -8(y + 2g + 1), \quad (2.154)$$

$$\text{J} : \quad R_1(y) = 8, \quad R_0(y) = 16(y + (g + h)^2 - 1), \quad R_{-1}(y) = 16(g - h)(g + h - 1). \quad (2.155)$$

The data for the three examples in idQM (2.34)–(2.36) are:

$$\text{MP} : \quad R_1(y) = 0, \quad R_0(y) = 4, \quad R_{-1}(y) = 0, \quad (2.156)$$

$$\begin{aligned}
\text{W : } R_1(y) &= 2, \quad R_0(y) = 4y + b_1(b_1 - 2), \quad R_{-1}(y) = -2y^2 + (b_1 - 2b_2)y + (2 - b_1)b_3, \\
b_2 &\stackrel{\text{def}}{=} \sum_{1 \leq j < k \leq 4} a_j a_k, \quad b_3 \stackrel{\text{def}}{=} \sum_{1 \leq j < k < l \leq 4} a_j a_k a_l,
\end{aligned} \tag{2.157}$$

$$\begin{aligned}
\text{AW : } R_1(y) &= (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 y', \quad R_0(y) = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 (y'^2 - (1 + q^{-1})^2 b_4), \\
R_{-1}(y) &= \frac{1}{2} (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 ((b_1 + q^{-1} b_3) y' - (1 + q^{-1})(b_3 + q^{-1} b_1 b_4)) - R_0(y), \\
y' &\stackrel{\text{def}}{=} y + 1 + q^{-1} b_4, \quad b_1 \stackrel{\text{def}}{=} \sum_{j=1}^4 a_j, \quad b_3 \stackrel{\text{def}}{=} \sum_{1 \leq j < k < l \leq 4} a_j a_k a_l.
\end{aligned} \tag{2.158}$$

The closure relation and the creation/annihilation operators of the MP polynomials were obtained in 2001 in [47]. The data for the three examples in rdQM (2.37)–(2.39) are:

$$\text{M : } R_1(y) = 0, \quad R_0(y) = 1, \quad R_{-1}(y) = -\frac{1+c}{1-c}y - \frac{\beta c}{1-c}, \tag{2.159}$$

$$\begin{aligned}
\text{R : } R_1(y) &= 2, \quad R_0(y) = 4y + \tilde{d}^2 - 1, \\
R_{-1}(y) &= 2y^2 + (2(ab + bc + ca) - (1 + d)(1 + \tilde{d}))y + abc(\tilde{d} - 1),
\end{aligned} \tag{2.160}$$

$$\begin{aligned}
\text{qR : } R_1(y) &= (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 y', \quad R_0(y) = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 (y'^2 - (q^{-\frac{1}{2}} + q^{\frac{1}{2}})^2 \tilde{d}), \quad y' \stackrel{\text{def}}{=} y + 1 + \tilde{d}, \\
R_{-1}(y) &= (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 \left((1 + d)y'^2 - (a + b + c + d + \tilde{d} + (ab + bc + ca)q^{-1})y' \right. \\
&\quad \left. + (1 - a)(1 - b)(1 - c)(1 - \tilde{d}q^{-1}) \right. \\
&\quad \left. + (a + b + c - 1 - d\tilde{d} + (ab + bc + ca)q^{-1})(1 + \tilde{d}) \right).
\end{aligned} \tag{2.161}$$

For oQM, the necessary and sufficient condition for the existence of the sinusoidal coordinate satisfying the closure relation (2.142) is analysed in Appendix A of [14]. It was shown that such systems constitute a sub-group of the shape invariant oQM. We also mention that exact Heisenberg operator solutions for independent sinusoidal coordinates as many as the degree of freedom were derived for the Calogero systems based on any root system [16]. These are novel examples of infinitely many multi-particle Heisenberg operator solutions.

There were attempts to relate q -oscillator algebras to the difference equation of the q -Hermite polynomials [63]. None of them is based on a Hamiltonian, thus hermiticity is not manifest and the logic for factorization is unclear. There are several derivations of annihilation/creation operators together with the dynamical symmetry algebras (including the q -oscillator) acting on the polynomials.

2.6 Dual Polynomials in rdQM

The *dual* polynomials are an important concept in the theory of orthogonal polynomials of a discrete variable [2], that is the orthogonal polynomials appearing in rdQM. We show that the dual polynomials arise naturally as the solutions of the original eigenvalue problem, the Schrödinger equation (2.17) or (2.23), obtained in an alternative way [20]. We do believe this derivation of the duality is more intuitive than the existing ones [32, 33]. The Hamiltonian \mathcal{H} (2.12) for rdQM is a real symmetric *tri-diagonal* (Jacobi) matrix of a finite ($x_{\max} = N$) or infinite ($x_{\max} = \infty$) dimension:

$$\mathcal{H} = (\mathcal{H}_{x,y})_{0 \leq x,y \leq x_{\max}}, \quad \mathcal{H}_{x,y} = \mathcal{H}_{y,x}, \quad (2.162)$$

$$\mathcal{H}_{x,y} = -\sqrt{B(x)D(x+1)} \delta_{x+1,y} - \sqrt{B(x-1)D(x)} \delta_{x-1,y} + (B(x) + D(x)) \delta_{x,y}. \quad (2.163)$$

It is well-known that the spectrum of a Jacobi matrix is *simple* (2.4), that is no degeneracy. The factor \mathcal{A}^\dagger (2.10) is a lower triangular matrix with the diagonal and sub-diagonal entries only, and \mathcal{A} is upper triangular having the diagonal and super-diagonal entries only. Throughout this paper we adopt the (standard) convention that the $(0,0)$ element of the matrix is at the upper left corner. The similarity transformed Hamiltonian $\tilde{\mathcal{H}}$ (2.27) is again tri-diagonal:

$$\tilde{\mathcal{H}} = (\tilde{\mathcal{H}}_{x,y})_{0 \leq x,y \leq x_{\max}}, \quad \tilde{\mathcal{H}}_{x,y} = B(x)(\delta_{x,y} - \delta_{x+1,y}) + D(x)(\delta_{x,y} - \delta_{x-1,y}). \quad (2.164)$$

The similarity transformed eigenvalue problem $\tilde{\mathcal{H}} v(x) = \mathcal{E} v(x)$ (2.23) can be rewritten into an explicit matrix form with the change of the notation $v(x) \rightarrow {}^t(Q_0, Q_1, \dots, Q_x, \dots)$

$$\sum_{y=0}^{x_{\max}} \tilde{\mathcal{H}}_{x,y} Q_y = \mathcal{E} Q_x \quad (x = 0, 1, \dots, x_{\max}). \quad (2.165)$$

Because of the tri-diagonality of $\tilde{\mathcal{H}}$, the above eigenvalue equations are in fact the three term recurrence relations for $\{Q_x\}$ as polynomials in \mathcal{E} :

$$\mathcal{E} Q_x(\mathcal{E}) = B(x)(Q_x(\mathcal{E}) - Q_{x+1}(\mathcal{E})) + D(x)(Q_x(\mathcal{E}) - Q_{x-1}(\mathcal{E})) \quad (x = 0, 1, \dots, x_{\max}). \quad (2.166)$$

Starting with the boundary (initial) condition $Q_0 = 1$, $Q_x(\mathcal{E})$ is determined as a degree x polynomial in \mathcal{E} . It is easy to see

$$Q_x(0) = 1 \quad (x = 0, 1, \dots, x_{\max}). \quad (2.167)$$

When \mathcal{E} is replaced by the actual value of the n -th eigenvalue $\mathcal{E}(n)$ (2.28) in $Q_x(\mathcal{E})$, we obtain the explicit form of the eigenvector

$$\sum_{y=0}^{x_{\max}} \tilde{\mathcal{H}}_{x,y} Q_y(\mathcal{E}(n)) = \mathcal{E}(n) Q_x(\mathcal{E}(n)) \quad (x = 0, 1, \dots, x_{\max}). \quad (2.168)$$

In the finite dimensional case, $\{Q_0(\mathcal{E}), \dots, Q_N(\mathcal{E})\}$ are determined by (2.166) for $x = 0, \dots, N - 1$. The last equation

$$\mathcal{E} Q_N(\mathcal{E}) = D(N)(Q_N(\mathcal{E}) - Q_{N-1}(\mathcal{E})), \quad (2.169)$$

is the degree $N + 1$ algebraic equation (characteristic equation) for the determination of all the eigenvalues $\{\mathcal{E}(n)\}$.

We now have two expressions (polynomials) for the eigenvectors of the problem (2.23) belonging to the eigenvalue $\mathcal{E}(n)$; $P_n(\eta(x))$ and $Q_x(\mathcal{E}(n))$. Due to the simplicity of the spectrum of the Jacobi matrix, they must be equal up to a multiplicative factor α_n ,

$$P_n(\eta(x)) = \alpha_n Q_x(\mathcal{E}(n)) \quad (x = 0, 1, \dots, x_{\max}),$$

which turns out to be unity because of the boundary (initial) condition at $x = 0$ (2.21), (2.22), (2.167) (or at $n = 0$ (2.18), (2.22), (2.167)); ($n_{\max} \stackrel{\text{def}}{=} x_{\max}$)

$$P_n(\eta(0)) = P_n(0) = 1 = Q_0(\mathcal{E}(n)), \quad (n = 0, 1, \dots, n_{\max}), \quad (2.170)$$

$$P_0(\eta(x)) = 1 = Q_x(0) = Q_x(\mathcal{E}(0)), \quad (x = 0, 1, \dots, x_{\max}). \quad (2.171)$$

We have established that two polynomials, $\{P_n(\eta)\}$ and its *dual* polynomial $\{Q_x(\mathcal{E})\}$, coincide at the integer lattice points:

$$P_n(\eta(x)) = Q_x(\mathcal{E}(n)) \quad (n = 0, 1, \dots, n_{\max}; x = 0, 1, \dots, x_{\max}). \quad (2.172)$$

The completeness relation, which is dual to the orthogonality relation (2.50),

$$\sum_{n=0}^{n_{\max}} d_n^2 P_n(\eta(x)) P_n(\eta(y)) = \sum_{n=0}^{n_{\max}} d_n^2 Q_x(\mathcal{E}(n)) Q_y(\mathcal{E}(n)) = \frac{1}{\phi_0(x)^2} \delta_{x,y}, \quad (2.173)$$

is now understood as the orthogonality relation of the dual polynomials $Q_x(\mathcal{E})$, and the previous normalisation constant d_n^2 is now the orthogonality measure.

The real symmetric (hermitian) matrix $\mathcal{H}_{x,y}$ (2.163) can be expressed in terms of the complete set of the eigenvalues and the corresponding normalised eigenvectors

$$\mathcal{H}_{x,y} = \sum_{n=0}^{n_{\max}} \mathcal{E}(n) \hat{\phi}_n(x) \hat{\phi}_n(y), \quad (2.174)$$

$$\hat{\phi}_n(x) = d_n \phi_0(x) P_n(\eta(x)) = d_n \phi_0(x) Q_x(\mathcal{E}(n)). \quad (2.175)$$

The very fact that it is tri-diagonal can be easily verified by using the difference equation for the polynomial $P_n(\eta(x))$ or the three term recurrence relations for $Q_x(\mathcal{E}(n))$.

Here is a list of the dual correspondence:

$$x \leftrightarrow n, \quad \eta(x) \leftrightarrow \mathcal{E}(n), \quad \eta(0) = 0 \leftrightarrow \mathcal{E}(0) = 0, \quad (2.176)$$

$$B(x) \leftrightarrow -A_n, \quad D(x) \leftrightarrow -C_n, \quad \frac{\phi_0(x)}{\phi_0(0)} \leftrightarrow \frac{d_n}{d_0}. \quad (2.177)$$

The functions $B(x)$ and $D(x)$ govern the difference equation for the polynomials $P_n(\eta)$, the solution of which requires the knowledge of the sinusoidal coordinate $\eta(x)$. The same quantities $B(x)$ and $D(x)$ specify the three term recurrence of the dual polynomials $Q_x(\mathcal{E})$ without the knowledge of the spectrum $\mathcal{E}(n)$. It is required for them to be the eigenvectors of the eigenvalue problem (2.23). Likewise, A_n and C_n in (2.139) specify the polynomials $P_n(\eta)$ without the knowledge of the sinusoidal coordinate. As for the dual polynomial $Q_x(\mathcal{E}(n))$, A_n and C_n provide the difference equation (in n), the solution of which needs the explicit form of $\mathcal{E}(n)$. Let us stress that it is the eigenvalue problem (2.23) with the specific Hamiltonian (2.12) that determines the polynomials $P_n(\eta)$ and their dual $Q_x(\mathcal{E})$, the spectrum $\mathcal{E}(n)$, the sinusoidal coordinate $\eta(x)$ and the orthogonality measures $\phi_0(x)^2$ and d_n^2 .

By comparing the explicit Heisenberg operator solution for $\eta(x)$ with the three term recurrence relations (2.139), the coefficients A_n and C_n are determined:

$$A_0 = R_{-1}(0)R_0(0)^{-1}, \quad C_0 = 0, \quad (2.178)$$

$$A_n = \frac{R_{-1}(\mathcal{E}(n)) + \eta(1)(\mathcal{E}(n) - B(0))\alpha_+(\mathcal{E}(n))}{\alpha_+(\mathcal{E}(n))(\alpha_+(\mathcal{E}(n)) - \alpha_-(\mathcal{E}(n)))} \quad (n \geq 1), \quad (2.179)$$

$$C_n = \frac{R_{-1}(\mathcal{E}(n)) + \eta(1)(\mathcal{E}(n) - B(0))\alpha_-(\mathcal{E}(n))}{\alpha_-(\mathcal{E}(n))(\alpha_-(\mathcal{E}(n)) - \alpha_+(\mathcal{E}(n)))} \quad (n \geq 1). \quad (2.180)$$

There is an important relation among the four important quantities:

$$A_0 \mathcal{E}(1) + B(0) \eta(1) = 0. \quad (2.181)$$

For the details of the derivation, see (4.45)–(4.53) of [20].

2.7 Dual Closure Relation in dQM

The *dual closure relation* has the same form as the closure relation (2.142) with the roles of the Hamiltonian \mathcal{H} and the sinusoidal coordinate $\eta(x)$ interchanged [20]:

$$[\eta, [\eta, \mathcal{H}]] = \mathcal{H} R_0^{\text{dual}}(\eta) + [\eta, \mathcal{H}] R_1^{\text{dual}}(\eta) + R_{-1}^{\text{dual}}(\eta), \quad (2.182)$$

in which

$$R_1^{\text{dual}}(\eta(x)) = (\eta(x - i\beta) - \eta(x)) + (\eta(x + i\beta) - \eta(x)), \quad (2.183)$$

$$R_0^{\text{dual}}(\eta(x)) = -(\eta(x - i\beta) - \eta(x))(\eta(x + i\beta) - \eta(x)), \quad (2.184)$$

$$R_{-1}^{\text{dual}}(\eta(x)) = \varepsilon(V_+(x) + V_-(x))R_0^{\text{dual}}(\eta(x)). \quad (2.185)$$

The dual closure relation is the characteristic feature shared by all the ‘Hamiltonians’ $\tilde{\mathcal{H}}$ which map a polynomial in $\eta(x)$ into another. Therefore its dynamical contents are not so constraining as the closure relation, *except for* the rdQM exactly solvable case, where the closure relation and the dual closure relations are on the same footing and they form a dynamical symmetry algebra which is sometimes called the Askey-Wilson algebra [34, 64, 33, 20].

2.8 Bochner’s Theorem

In 1884 [65], Routh showed that polynomials satisfying the three term recurrence relations and a second order differential equation were one of the *classical* polynomials, the Hermite, Laguerre, Jacobi and Bessel. Later in 1929 [66], Bochner classified all polynomial solutions to second order Sturm-Liouville operators with polynomial coefficients and arrived at the same conclusions. See §20.1 of [5] for more details. This was a kind of No-Go theorem in oQM, since it declared that no essentially new exactly solvable oQM could be achieved as the solutions of the ordinary Schrödinger equation. Thus avoiding Bochner’s theorem was one of the strongest motivations for the introduction of the discrete quantum mechanics, whose difference Schrödinger equations were not constrained by the theorem and provided various exactly solvable examples [20, 21]. As shown in §4, the new orthogonal (exceptional (X_ℓ) Laguerre and Jacobi) polynomials [67, 68, 7, 8] were discovered in an attempt to evade the restrictions of the theorem by allowing the polynomials to start at degree $\ell \geq 1$. The new orthogonal polynomials in dQM, the exceptional (X_ℓ) Wilson, Askey-Wilson, Racah

and q -Racah polynomials satisfying second order difference equations, were constructed by the present authors in less than two years.

Here we present Bochner's theorem for dQM. As in the original Bochner's theorem, the conditions for the lowest three degrees $n = 0, 1$ and 2 are essential to determine the constraints. Reformulation of Bochner's theorem for second order difference equations was pursued by several authors. In [69], it was shown (in our language) that if orthogonal polynomials in $\eta(x) = \cos x$ satisfy q -difference equations (2.23), then the conditions characterise the Askey-Wilson polynomials. In [70], it was shown that if the q -difference equations (2.23) have polynomial solutions, then the sinusoidal coordinate $\eta(x)$ is at most q -quadratic and that the polynomials are *at most* Askey-Wilson polynomials. In these papers, the distinction between the pure imaginary and the real shifts is blurred.

The starting point is the difference equation for the polynomials (2.23) with (2.26):

$$\begin{aligned} \varepsilon V_+(x)(P_n(\eta(x - i\beta)) - P_n(\eta(x))) + \varepsilon V_-(x)(P_n(\eta(x + i\beta)) - P_n(\eta(x))) \\ = \mathcal{E}(n)P_n(\eta(x)) \quad (n = 0, 1, \dots), \end{aligned} \quad (2.186)$$

and the three term recurrence relations (2.139). The above equation is trivially satisfied for $n = 0$, since $\mathcal{E}(0) = 0$. Two relations for $n = 1$ and $n = 2$ are linear equations

$$\begin{pmatrix} P_1(\eta(x - i\beta)) - P_1(\eta(x)) & P_1(\eta(x + i\beta)) - P_1(\eta(x)) \\ P_2(\eta(x - i\beta)) - P_2(\eta(x)) & P_2(\eta(x + i\beta)) - P_2(\eta(x)) \end{pmatrix} \begin{pmatrix} V_+(x) \\ V_-(x) \end{pmatrix} = \begin{pmatrix} \varepsilon^{-1}\mathcal{E}(1)P_1(\eta(x)) \\ \varepsilon^{-1}\mathcal{E}(2)P_2(\eta(x)) \end{pmatrix},$$

which determine the potential functions $V_+(x)$ and $V_-(x)$ uniquely. By using the explicit forms of P_1 and P_2 ,

$$A_0P_1(\eta) = \eta - B_0, \quad A_0A_1P_2(\eta) = (\eta - B_0)(\eta - B_1) - A_0C_1, \quad (2.187)$$

derived from the three term recurrence relations (2.139), the above linear equation gives

$$V_{\pm}(x) = \frac{S_2 + S_1\eta(x \pm i\beta)}{(\eta(x \mp i\beta) - \eta(x))(\eta(x \mp i\beta) - \eta(x \pm i\beta))}, \quad (2.188)$$

in which S_1 and S_2 are a linear and a quadratic polynomial in $\eta(x)$, respectively:

$$\begin{aligned} S_1 &= -\varepsilon^{-1}\mathcal{E}(1)A_0P_1(\eta(x)), \\ S_2 &= \varepsilon^{-1}\mathcal{E}(2)A_0A_1P_2(\eta(x)) - \varepsilon^{-1}\mathcal{E}(1)A_0P_1(\eta(x))(A_0P_1(\eta(x)) - B_1). \end{aligned} \quad (2.189)$$

These V_{\pm} have essentially the same forms as those which will be presented in §3. If the symmetric shift-addition property of the sinusoidal coordinate $\eta(x)$ (3.4) is satisfied, the

above expressions for $V_{\pm}(x)$ (2.188) are equivalent to V_{\pm} in (3.2)-(3.3) with $L = 2$. Moreover when the symmetric shift-multiplication property (3.5) is satisfied, these potential functions give the exactly solvable models. For rdQM, the sinusoidal coordinates satisfying (3.4) can be classified into five types (3.15)–(3.19) [20] and they also satisfy (3.5).

3 Unified Theory of Exactly Solvable dQM

In this section we present a simple theory of constructing exactly solvable ‘Hamiltonians’ in dQM based on the two required properties of the sinusoidal coordinates (3.4)-(3.5). The general strategy is to construct the similarity transformed ‘Hamiltonian’ $\tilde{\mathcal{H}}$ (2.26) in such a way that it maps a polynomial in $\eta(x)$ into another:

$$\tilde{\mathcal{H}}\mathcal{V}_n \subseteq \mathcal{V}_{n+L-2} \subset \mathcal{V}_{\infty} \quad (n \in \mathbb{Z}_{\geq 0}), \quad (3.1)$$

where L is a fixed positive integer. Here \mathcal{V}_n is defined by (2.29) and $\mathcal{V}_{\infty} \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \mathcal{V}_n$. When $L = 2$, the above relation (3.1) is simply the *lower triangularity* of the ‘Hamiltonian’ $\tilde{\mathcal{H}}$, leading to exact solvability.

In the following we will take the similarity transformed Hamiltonian $\tilde{\mathcal{H}}$ (2.24) instead of \mathcal{H} as the starting point. That is, we reverse the argument and construct directly the ‘Hamiltonian’ $\tilde{\mathcal{H}}$ (2.26) based on the *sinusoidal coordinate* $\eta(x)$. This section is a brief review of [23].

3.1 Potential Functions

The general form of the ‘Hamiltonian’ $\tilde{\mathcal{H}}$ mapping a polynomial in $\eta(x)$ into another is achieved by the following form of the potential functions $V_{\pm}(x)$:

$$V_{\pm}(x) = \frac{\tilde{V}_{\pm}(x)}{(\eta(x \mp i\beta) - \eta(x))(\eta(x \mp i\beta) - \eta(x \pm i\beta))}, \quad (3.2)$$

$$\tilde{V}_{\pm}(x) = \sum_{\substack{k,l \geq 0 \\ k+l \leq L}} v_{k,l} \eta(x)^k \eta(x \mp i\beta)^l, \quad (3.3)$$

where L is a natural number indicating the degree of $\eta(x)$ in $\tilde{V}_{\pm}(x)$ and $v_{k,l}$ are real constants, with the constraint $\sum_{k+l=L} v_{k,l}^2 \neq 0$. It is important that the same $v_{k,l}$ appears in both $\tilde{V}_{\pm}(x)$. The ‘Hamiltonian’ $\tilde{\mathcal{H}}$ with the above $V_{\pm}(x)$ maps a degree n polynomial in $\eta(x)$ to a degree

$n + L - 2$ polynomial. This can be shown elementarily based on the two basic properties of the sinusoidal coordinates called the symmetric shift-addition property:

$$\eta(x - i\beta) + \eta(x + i\beta) = (2 + r_1^{(1)})\eta(x) + r_{-1}^{(2)}, \quad (3.4)$$

and the symmetric shift-multiplication property:

$$\eta(x - i\beta)\eta(x + i\beta) = (\eta(x) - \eta(-i\beta))(\eta(x) - \eta(i\beta)), \quad (3.5)$$

together with $\eta(x) \neq \eta(x - i\beta) \neq \eta(x + i\beta) \neq \eta(x)$. Here $r_1^{(1)}$ and $r_{-1}^{(2)}$ are real parameters. In fact these parameters also appear in the three functions R_0 , R_1 and R_{-1} in the closure relation (2.142):

$$R_1(y) = r_1^{(1)}y + r_1^{(0)}, \quad R_0(y) = r_0^{(2)}y^2 + r_0^{(1)}y + r_0^{(0)}, \quad R_{-1}(y) = r_{-1}^{(2)}y^2 + r_{-1}^{(1)}y + r_{-1}^{(0)}. \quad (3.6)$$

Here are the lists of the known sinusoidal coordinates satisfying the above two conditions (3.4)-(3.5). There are eight sinusoidal coordinates for the idQM:

$$(i) : \quad \eta(x) = x, \quad -\infty < x < \infty, \quad \gamma = 1, \quad (3.7)$$

$$(ii) : \quad \eta(x) = x^2, \quad 0 < x < \infty, \quad \gamma = 1, \quad (3.8)$$

$$(iii) : \quad \eta(x) = 1 - \cos x, \quad 0 < x < \pi, \quad \gamma \in \mathbb{R}_{\neq 0}, \quad (3.9)$$

$$(iv) : \quad \eta(x) = \sin x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad \gamma \in \mathbb{R}_{\neq 0}, \quad (3.10)$$

$$(v) : \quad \eta(x) = 1 - e^{-x}, \quad -\infty < x < \infty, \quad \gamma \in \mathbb{R}_{\neq 0}, \quad (3.11)$$

$$(vi) : \quad \eta(x) = e^x - 1, \quad -\infty < x < \infty, \quad \gamma \in \mathbb{R}_{\neq 0}, \quad (3.12)$$

$$(vii) : \quad \eta(x) = \cosh x - 1, \quad 0 < x < \infty, \quad \gamma \in \mathbb{R}_{\neq 0}, \quad (3.13)$$

$$(viii) : \quad \eta(x) = \sinh x, \quad -\infty < x < \infty, \quad \gamma \in \mathbb{R}_{\neq 0}, \quad (3.14)$$

and five sinusoidal coordinates for the rdQM: ($0 < q < 1$)

$$(i)' : \quad \eta(x) = x, \quad (3.15)$$

$$(ii)' : \quad \eta(x) = \epsilon'x(x + d), \quad \epsilon' = \begin{cases} 1 & \text{for } d > -1 \\ -1 & \text{for } d < -N \end{cases}, \quad (3.16)$$

$$(iii)' : \quad \eta(x) = 1 - q^x, \quad (3.17)$$

$$(iv)' : \quad \eta(x) = q^{-x} - 1, \quad (3.18)$$

$$(v)' : \quad \eta(x) = \epsilon'(q^{-x} - 1)(1 - dq^x), \quad \epsilon' = \begin{cases} 1 & \text{for } d < q^{-1} \\ -1 & \text{for } d > q^{-N} \end{cases}. \quad (3.19)$$

As shown in detail in §4C of [20], the above five sinusoidal coordinates for rdQM (3.15)–(3.19) exhaust all the solutions of (3.4)–(3.5) up to a multiplicative factor. On the other hand, those for idQM (i)–(viii) (3.7)–(3.14) are merely typical examples satisfying all the postulates for the sinusoidal coordinate (3.4)–(3.5) and the extra one used for the shape invariance, (3.50) of [23].

The essential part of the formula (3.2) is the denominators. They have the same form as the generic formula for the coefficients of the three term recurrence relations of the orthogonal polynomials, (2.179) and (2.180) (see (4.52) and (4.53) in [20]). The translation rules are the *duality* correspondence itself, (2.176)–(2.177) (see also (3.14)–(3.18) in [20]):

$$\begin{aligned} \mathcal{E}(n) &\rightarrow \eta(x), & -A_n &\rightarrow V_+(x), & -C_n &\rightarrow V_-(x), \\ \alpha_+(\mathcal{E}(n)) &\rightarrow \eta(x - i\beta) - \eta(x), & \alpha_-(\mathcal{E}(n)) &\rightarrow \eta(x + i\beta) - \eta(x). \end{aligned} \quad (3.20)$$

Some of the parameters $v_{k,l}$ in (3.3) are redundant. It is sufficient to keep $v_{k,l}$ with $l = 0, 1$. The remaining $2L + 1$ parameters $v_{k,l}$ ($k + l \leq L$, $l = 0, 1$) are independent, with one of which corresponds to the overall normalisation of the Hamiltonian. See § II of [23] for the actual proof of the property (3.1).

The $L = 2$ case is exactly solvable. This corresponds to the general hypergeometric equations having at most degree two polynomial coefficients. Since the Hamiltonian of the polynomial space $\tilde{\mathcal{H}}$ is expressed as an upper triangular matrix, its eigenvalues and eigenvectors are easily obtained explicitly. For the solutions of a full quantum mechanical problem, however, one needs the square-integrable groundstate wavefunction $\phi_0(x)$ (2.18), which is essential for the existence of the Hamiltonian \mathcal{H} and the verification of its hermiticity. These conditions would usually restrict the ranges of the parameters $v_{0,0}, \dots, v_{2,0}$. It is easy to verify that the explicit examples of the potential functions $V(x)$, $V^*(x)$ and $B(x)$, $D(x)$ in (2.34)–(2.39) are simply reproduced by proper choices of the parameters $\{v_{k,l}\}$. It should be stressed that the above form of the potential function (3.2)–(3.3) provides a *unified proof* of the shape invariance relation (2.105), the closure relation (2.142) and the dual closure relation (2.182) in the $\tilde{\mathcal{H}}$ scheme, see [23] for more details. This is in good contrast with the various explicit examples presented in the preceding section. The two solution methods, the shape invariance and the closure relation are verified for each example.

3.2 Askey-Wilson Algebra

Here we briefly comment on the Askey-Wilson algebra, which are generated by the closure plus the dual closure relations. By simply expanding the double commutators in the closure (2.142) and the dual closure (2.182) relations, we obtain two cubic relations generated by the two operators \mathcal{H} and η :

$$\mathcal{H}^2\eta - (2 + r_1^{(1)})\mathcal{H}\eta\mathcal{H} + \eta\mathcal{H}^2 - r_1^{(0)}(\mathcal{H}\eta + \eta\mathcal{H}) - r_0^{(0)}\eta = r_{-1}^{(2)}\mathcal{H}^2 + r_{-1}^{(1)}\mathcal{H} + r_{-1}^{(0)}, \quad (3.21)$$

$$\eta^2\mathcal{H} - (2 + r_1^{(1)})\eta\mathcal{H}\eta + \mathcal{H}\eta^2 - r_{-1}^{(2)}(\eta\mathcal{H} + \mathcal{H}\eta) + \eta(-i\beta)\eta(i\beta)\mathcal{H} = r_1^{(0)}\eta^2 + r_{-1}^{(1)}\eta + \varepsilon v_{0,0}. \quad (3.22)$$

From its structure, the closure relation is at most linear in η and at most quadratic in \mathcal{H} . So the l.h.s. of (3.21) has terms containing one factor of η and the r.h.s, none. It is simply $R_{-1}(\mathcal{H})$. Likewise, the l.h.s. of (3.22) has terms containing one factor of \mathcal{H} and the r.h.s, none. It is simply $R_{-1}^{\text{dual}}(\eta)$. In (3.22), $\eta(-i\beta)\eta(i\beta)$ is just a real number, not an operator. These have the same form as the so-called Askey-Wilson algebra, which has many different expressions [34, 64, 61, 33]. While the Askey-Wilson algebra has no inherent structure, the closure relation (2.142) has the right structure to lead to the Heisenberg operator solution for $\eta(x)$, whose positive and negative frequency parts are the annihilation-creation operators [14, 20, 21]. It is the Hamiltonian and the annihilation-creation operators that form the *dynamical symmetry algebra* of the system [20, 21], not the closure or dual-closure relations, nor the Askey-Wilson algebra relations.

3.3 Quasi-Exact Solvability in dQM

The unified theory is general enough to generate *quasi-exactly solvable* Hamiltonians in the same manner. The quasi-exact solvability means, in contrast to the exact solvability, that only a finite number of energy eigenvalues and the corresponding eigenfunctions can be obtained exactly. Many examples are known in the oQM [71, 72, 73], but only a few are known in dQM in spite of the proposal that the $sl(2, R)$ algebra characterisation of quasi-exact solvability could be extended to difference Schrödinger equations [72]. The unified theory also incorporates the known examples of quasi-exactly solvable Hamiltonians in dQM [17, 18, 22]. A new type of quasi-exactly solvable Hamiltonians is constructed. The present approach reveals the common structure underlying the exactly and quasi-exactly solvable theories, in particular, the important roles played by the sinusoidal coordinates.

The higher $L \geq 3$ cases are obviously non-solvable. Among them, the tame non-solvability of $L = 3$ and 4 can be made quasi-exactly solvable (QES) by adding suitable compensation terms. This is a simple generalisation of the method of Sasaki & Takasaki [74] for multi-particle QES in the oQM. For a given positive integer M , let us try to find a QES ‘Hamiltonian’ $\tilde{\mathcal{H}}$, or more precisely its modification $\tilde{\mathcal{H}}'$, having an invariant polynomial subspace \mathcal{V}_M :

$$\tilde{\mathcal{H}}'\mathcal{V}_M \subseteq \mathcal{V}_M. \quad (3.23)$$

For $L = 3$, $\tilde{\mathcal{H}}'$ is defined by adding one single compensation term of degree one

$$\tilde{\mathcal{H}}' \stackrel{\text{def}}{=} \tilde{\mathcal{H}} - e_0(M)\eta(x), \quad (3.24)$$

and we have achieved the quasi-exact solvability $\tilde{\mathcal{H}}'\mathcal{V}_M \subseteq \mathcal{V}_M$. Known discrete QES examples belong to this class [18, 22].

For $L = 4$ case, $\tilde{\mathcal{H}}'$ is defined by adding a linear and a quadratic in $\eta(x)$ compensation terms to the Hamiltonian $\tilde{\mathcal{H}}$:

$$\tilde{\mathcal{H}}' \stackrel{\text{def}}{=} \tilde{\mathcal{H}} - e_0(M)\eta(x)^2 - e_1(M)\eta(x), \quad (3.25)$$

and one condition between $v_{3,1}$ and $v_{4,0}$ is imposed. Then we have $\tilde{\mathcal{H}}'\mathcal{V}_M \subseteq \mathcal{V}_M$. This type of QES theory is new. For explicit forms of $e_0(M)$ and $e_1(M)$, see [23].

4 New Orthogonal Polynomials

In this section we present a brief review of the recent hot topic, the discovery of infinitely many new orthogonal polynomials satisfying second order differential or difference equations. They are obtained as the main parts of eigenfunctions of exactly solvable quantum mechanics in all the three categories, oQM, idQM and rdQM. They are the exceptional (X_ℓ) Laguerre (XL) and Jacobi (XJ) polynomials in oQM, the exceptional Meixner-Pollaczek (XMP), Wilson (XW) and Askey-Wilson (XAW) polynomials in idQM and the exceptional Meixner (XM), Racah (XR) and q -Racah (XqR) polynomials in rdQM. The XMP and XM are *new results*. For any positive integer ℓ these X_ℓ polynomials have degrees $\ell + n$, $n = 0, 1, 2, \dots$ and are orthogonal with each other (4.43), (4.50), (4.55) with respect to explicitly given positive definite weight functions. Thus they do not satisfy the three term recurrence relations (2.139) and therefore the restrictions of Bochner’s theorem and its discrete counterparts

in §2.8 do not apply. As will be explained later, the quantum mechanical Hamiltonians for these exceptional polynomials are deformations of those for the original polynomials in terms of a degree ℓ eigenpolynomial with twisted parameters. The deformation terms satisfy the same type of equations as the original polynomials, but with the twisted parameters. The parameters are so chosen that the deformed system retain the shape invariance, too. The general scheme of the deformation is common for various exceptional polynomials as shown in detail below. The original system corresponds to $\ell = 0$.

It should be stressed that the X_ℓ Jacobi polynomials provide infinitely many global solutions of Fuchsian differential equations having $3 + \ell$ regular singularities [9].

4.1 Deforming Polynomials

In order to introduce new orthogonal polynomials, we prepare the deforming polynomials. The deforming polynomials $\xi_\ell(\eta)$ are eigenpolynomials $P_\ell(\eta)$ with twisted parameters (or twisted coordinate) and they are crucial for the construction of new orthogonal polynomials. Two identities relating ξ_ℓ with $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda} + \boldsymbol{\delta}$, (4.3)–(4.4), (4.15)–(4.16) and (4.25)–(4.26), will play important roles in the derivation of various results.

For oQM there are two types of the exceptional Jacobi polynomials, called XJ1 and XJ2. Their confluent limits give the exceptional Laguerre polynomials, XL1 and XL2. The deforming polynomials are:

$$\xi_\ell(\eta; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} P_\ell(-\eta; \boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}) & : \text{XL1} \\ P_\ell(\eta; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta})), \quad \mathbf{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} -\lambda_1 - 1 & : \text{XL2} \\ (\lambda_1, -\lambda_2 - 1) & : \text{XJ1} \\ (-\lambda_1 - 1, \lambda_2) & : \text{XJ2} \end{cases} \end{cases} \quad (4.1)$$

Explicitly they are

$$\xi_\ell(\eta; \boldsymbol{\lambda}) = \begin{cases} L_\ell^{(g+\ell-\frac{3}{2})}(-\eta) & : \text{XL1} \\ L_\ell^{(-g-\ell-\frac{1}{2})}(\eta) & : \text{XL2} \end{cases}, \quad \xi_\ell(\eta; \boldsymbol{\lambda}) = \begin{cases} P_\ell^{(g+\ell-\frac{3}{2}, -h-\ell-\frac{1}{2})}(\eta) & : \text{XJ1} \\ P_\ell^{(-g-\ell-\frac{1}{2}, h+\ell-\frac{3}{2})}(\eta) & : \text{XJ2} \end{cases} \quad (4.2)$$

The two identities relating $\xi_\ell(\eta; \boldsymbol{\lambda})$ and $\xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})$ are:

$$d_1(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})\xi_\ell(\eta; \boldsymbol{\lambda}) + d_2(\eta)\partial_\eta\xi_\ell(\eta; \boldsymbol{\lambda}) = d_1(\boldsymbol{\lambda})\xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (4.3)$$

$$d_3(\boldsymbol{\lambda}, \ell)\xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta}) + \frac{c_2(\eta)}{d_2(\eta)}\partial_\eta\xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta}) = d_3(\boldsymbol{\lambda} + \ell\boldsymbol{\delta}, \ell)\xi_\ell(\eta; \boldsymbol{\lambda}), \quad (4.4)$$

where $d_1(\boldsymbol{\lambda})$, $d_2(\eta)$ and $d_3(\boldsymbol{\lambda}, \ell)$ are given by

$$d_1(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 1 & : \text{XL1} \\ g + \frac{1}{2} & : \text{XL2, XJ2} \\ h + \frac{1}{2} & : \text{XJ1} \end{cases}, \quad d_2(\eta) \stackrel{\text{def}}{=} \begin{cases} 1 & : \text{XL1} \\ -\eta & : \text{XL2} \\ \mp(1 \pm \eta) & : \text{XJ1/XJ2} \end{cases}, \quad (4.5)$$

$$d_3(\boldsymbol{\lambda}, \ell) \stackrel{\text{def}}{=} \begin{cases} g + \ell - \frac{1}{2} & : \text{XL1, XJ1} \\ 1 & : \text{XL2} \\ h + \ell - \frac{1}{2} & : \text{XJ2} \end{cases}. \quad (4.6)$$

Note that these two identities, (4.3) and (4.4), imply the differential equation for the deforming polynomial,

$$c_2(\eta) \partial_\eta^2 \xi_\ell(\eta; \boldsymbol{\lambda}) + \tilde{c}_1(\eta, \boldsymbol{\lambda}, \ell) \partial_\eta \xi_\ell(\eta; \boldsymbol{\lambda}) = -\frac{1}{4} \tilde{\mathcal{E}}(\ell; \boldsymbol{\lambda}) \xi_\ell(\eta; \boldsymbol{\lambda}), \quad (4.7)$$

$$\tilde{c}(\eta, \boldsymbol{\lambda}, \ell) \stackrel{\text{def}}{=} \begin{cases} c_1(-\eta, \boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}) & : \text{XL1} \\ c_1(\eta, \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta})) & : \text{XL2, XJ1, XJ2} \end{cases}, \quad (4.8)$$

$$\tilde{\mathcal{E}}(\ell; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} -\mathcal{E}(\ell; \boldsymbol{\lambda}) & : \text{XL1} \\ \mathcal{E}(\ell; \mathbf{t}(\boldsymbol{\lambda})) & : \text{XL2, XJ1, XJ2} \end{cases}, \quad (4.9)$$

which is to be compared with the differential equation for the eigenpolynomial (2.23),

$$c_2(\eta) \partial_\eta^2 P_n(\eta; \boldsymbol{\lambda}) + c_1(\eta, \boldsymbol{\lambda}) \partial_\eta P_n(\eta; \boldsymbol{\lambda}) = -\frac{1}{4} \mathcal{E}(n; \boldsymbol{\lambda}) P_n(\eta; \boldsymbol{\lambda}). \quad (4.10)$$

Note that $\mathcal{E}(\ell; \mathbf{t}(\boldsymbol{\lambda})) = \mathcal{E}(\ell; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}))$. The deforming polynomial $\xi_\ell(\eta; \boldsymbol{\lambda})$ has the same sign in the orthogonality domain, that is $(0, \infty)$ for L and $(-1, 1)$ for J, see (2.39), (2.40) of [35] and (3.2) of [38].

For idQM, we restrict parameters:

$$\text{XMP} : \ell \text{ even}, \quad (4.11)$$

$$\text{XW} : a_1, a_2 \in \mathbb{R}, \quad \{a_3^*, a_4^*\} = \{a_3, a_4\} \text{ (as a set)}, \quad 0 < a_j < \text{Re } a_k \quad (j = 1, 2; k = 3, 4), \quad (4.12)$$

$$\text{XAW} : a_1, a_2 \in \mathbb{R}, \quad \{a_3^*, a_4^*\} = \{a_3, a_4\} \text{ (as a set)}, \quad 1 > a_j > |a_k| \quad (j = 1, 2; k = 3, 4). \quad (4.13)$$

The deforming polynomials for XMP, XW and XAW are:

$$\xi_\ell(\eta; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_\ell(\eta; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta})), \quad \mathbf{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} -\lambda_1 & : \text{XMP} \\ (-\lambda_1, -\lambda_2, \lambda_3, \lambda_4) & : \text{XW, XAW} \end{cases}. \quad (4.14)$$

The two identities relating $\xi_\ell(\eta(x); \boldsymbol{\lambda})$ and $\xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})$ are [37]:

$$\frac{i}{\varphi(x)} (v_1^*(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) e^{\frac{\gamma}{2}p} - v_1(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) e^{-\frac{\gamma}{2}p}) \xi_\ell(\eta(x); \boldsymbol{\lambda}) = \hat{f}_{\ell,0}(\boldsymbol{\lambda}) \xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (4.15)$$

$$\begin{aligned} & \frac{-i}{\varphi(x)} (v_2(x; \boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}) e^{\frac{\gamma}{2}p} - v_2^*(x; \boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}) e^{-\frac{\gamma}{2}p}) \xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta}) \\ & = \hat{b}_{\ell,0}(\boldsymbol{\lambda}) \xi_\ell(\eta(x); \boldsymbol{\lambda}), \end{aligned} \quad (4.16)$$

where $v_1(x; \boldsymbol{\lambda})$, $v_2(x; \boldsymbol{\lambda})$ are the factors of the potential function $V(x; \boldsymbol{\lambda})$:

$$V(x; \boldsymbol{\lambda}) = -\sqrt{\kappa} \frac{v_1(x; \boldsymbol{\lambda}) v_2(x; \boldsymbol{\lambda})}{\varphi(x) \varphi(x - i\frac{\gamma}{2})}, \quad (4.17)$$

$$v_1(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} i(a+ix) & : \text{XMP} \\ \prod_{j=1}^2 (a_j + ix) & : \text{XW} \\ e^{-ix} \prod_{j=1}^2 (1 - a_j e^{ix}) & : \text{XAW} \end{cases}, \quad v_2(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} i & : \text{XMP} \\ \prod_{j=3}^4 (a_j + ix) & : \text{XW} \\ e^{-ix} \prod_{j=3}^4 (1 - a_j e^{ix}) & : \text{XAW} \end{cases}. \quad (4.18)$$

The constants $\hat{f}_{\ell,n}(\boldsymbol{\lambda})$ and $\hat{b}_{\ell,n}(\boldsymbol{\lambda})$ are given by

$$\hat{f}_{\ell,n}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 2a+n & : \text{XMP} \\ a_1+a_2+n & : \text{XW} \\ -q^{-\frac{n-\ell}{2}}(1-a_1a_2q^n) & : \text{XAW} \end{cases}, \quad \hat{b}_{\ell,n}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 2 & : \text{XMP} \\ a_3+a_4+n+2\ell-1 & : \text{XW} \\ -q^{-\frac{n+\ell}{2}}(1-a_3a_4q^{n+2\ell-1}) & : \text{XAW} \end{cases}. \quad (4.19)$$

These two identities (4.15) and (4.16) imply the difference equation for the deforming polynomial,

$$\begin{aligned} & \left(V(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell-1)\boldsymbol{\delta})) (e^{\gamma p} - 1) + V^*(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell-1)\boldsymbol{\delta})) (e^{-\gamma p} - 1) \right) \xi_{\ell}(\eta(x); \boldsymbol{\lambda}) \\ & = \mathcal{E}(\ell; \mathbf{t}(\boldsymbol{\lambda})) \xi_{\ell}(\eta(x); \boldsymbol{\lambda}), \end{aligned} \quad (4.20)$$

which should be compared with the difference equation for the eigenpolynomial

$$(V(x; \boldsymbol{\lambda})(e^{\gamma p} - 1) + V^*(x; \boldsymbol{\lambda})(e^{-\gamma p} - 1)) P_n(\eta(x); \boldsymbol{\lambda}) = \mathcal{E}(n; \boldsymbol{\lambda}) P_n(\eta(x); \boldsymbol{\lambda}). \quad (4.21)$$

Note that $\mathcal{E}(\ell; \mathbf{t}(\boldsymbol{\lambda})) = \mathcal{E}(\ell; \mathbf{t}(\boldsymbol{\lambda} + (\ell-1)\boldsymbol{\delta}))$. For the appropriate parameter ranges, the deforming polynomial $\xi_{\ell}(\eta(x); \boldsymbol{\lambda})$ has no zero in the rectangular domain $x_1 \leq \text{Re } x \leq x_2$, $|\text{Im } x| \leq |\gamma|$, which is necessary for the hermiticity of the Hamiltonian.

For rdQM, the deforming polynomials for XM, XR and XqR are:

$$\check{\xi}_{\ell}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \xi_{\ell}(\eta(x; \boldsymbol{\lambda} + (\ell-1)\boldsymbol{\delta}); \boldsymbol{\lambda}), \quad \check{P}_n(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_n(\eta(x; \boldsymbol{\lambda}); \boldsymbol{\lambda}), \quad (4.22)$$

$$\check{\xi}_{\ell}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} c^{\ell} \check{P}_{\ell}(-(x+\beta+\ell-1); \boldsymbol{\lambda} + (\ell-1)\boldsymbol{\delta}) & : \text{XM} \\ \check{P}_{\ell}(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell-1)\boldsymbol{\delta})), \quad \mathbf{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (\lambda_4 - \lambda_1, \lambda_4 - \lambda_2, \lambda_3, \lambda_4) & : \text{XR, XqR} \end{cases}, \quad (4.23)$$

which satisfies the normalisation

$$\xi_{\ell}(0; \boldsymbol{\lambda}) = 1. \quad (4.24)$$

The two identities relating $\check{\xi}_{\ell}(x; \boldsymbol{\lambda})$ and $\check{\xi}_{\ell}(x; \boldsymbol{\lambda} + \boldsymbol{\delta})$ are [11]:

$$\frac{1}{\varphi(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}})} \left(v_1^B(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) - v_1^D(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) e^{\partial} \right) \check{\xi}_{\ell}(x; \boldsymbol{\lambda}) = \hat{f}_{\ell,0}(\boldsymbol{\lambda}) \check{\xi}_{\ell}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (4.25)$$

$$\begin{aligned} & \frac{1}{\varphi(x; \boldsymbol{\lambda} + (\ell-1)\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}})} \left(v_2^B(x; \boldsymbol{\lambda} + (\ell-1)\boldsymbol{\delta}) - v_2^D(x; \boldsymbol{\lambda} + (\ell-1)\boldsymbol{\delta}) e^{-\partial} \right) \check{\xi}_{\ell}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) \\ & = \hat{b}_{\ell,0}(\boldsymbol{\lambda}) \check{\xi}_{\ell}(x; \boldsymbol{\lambda}). \end{aligned} \quad (4.26)$$

Here $v_1^B(x; \boldsymbol{\lambda})$, $v_2^B(x; \boldsymbol{\lambda})$, $v_1^D(x; \boldsymbol{\lambda})$, $v_2^D(x; \boldsymbol{\lambda})$ are the factors of the potential functions $B(x; \boldsymbol{\lambda})$ and $D(x; \boldsymbol{\lambda})$:

$$B(x; \boldsymbol{\lambda}) = -\sqrt{\kappa} \frac{v_1^B(x; \boldsymbol{\lambda})v_2^B(x; \boldsymbol{\lambda})}{\varphi(x; \boldsymbol{\lambda} + \tilde{\boldsymbol{\delta}})\varphi(x + \frac{1}{2}; \boldsymbol{\lambda} + \tilde{\boldsymbol{\delta}})}, \quad (4.27)$$

$$D(x; \boldsymbol{\lambda}) = -\sqrt{\kappa} \frac{v_1^D(x; \boldsymbol{\lambda})v_2^D(x; \boldsymbol{\lambda})}{\varphi(x; \boldsymbol{\lambda} + \tilde{\boldsymbol{\delta}})\varphi(x - \frac{1}{2}; \boldsymbol{\lambda} + \tilde{\boldsymbol{\delta}})}, \quad (4.28)$$

$$v_1^B(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} -c^{\frac{1}{2}} & : \text{XM} \\ d^{-1}(x+a)(x+b) & : \text{XR} \\ \frac{q^{-x}}{1-d}(1-aq^x)(1-bq^x) & : \text{XqR} \end{cases}, \quad (4.29)$$

$$v_2^B(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} c^{-\frac{1}{2}}B(x; \boldsymbol{\lambda}) & : \text{XM} \\ d^{-1}(x+c)(x+d) & : \text{XR} \\ \frac{q^{-x}}{1-d}(1-cq^x)(1-dq^x) & : \text{XqR} \end{cases}, \quad (4.30)$$

$$v_1^D(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} -c^{-\frac{1}{2}} & : \text{XM} \\ d^{-1}(x+d-a)(x+d-b) & : \text{XR} \\ \frac{q^{-x}}{1-d}abd^{-1}(1-a^{-1}dq^x)(1-b^{-1}dq^x) & : \text{XqR} \end{cases}, \quad (4.31)$$

$$v_2^D(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} c^{\frac{1}{2}}D(x; \boldsymbol{\lambda}) & : \text{XM} \\ d^{-1}(x+d-c)x & : \text{XR} \\ \frac{q^{-x}}{1-d}c(1-c^{-1}dq^x)(1-q^x) & : \text{XqR} \end{cases}, \quad (4.32)$$

where $\tilde{\boldsymbol{\delta}}$ is

$$\tilde{\boldsymbol{\delta}} \stackrel{\text{def}}{=} \begin{cases} (-1, 0) & : \text{XM} \\ (0, 0, -1, -1) & : \text{XR, XqR} \end{cases}. \quad (4.33)$$

The constants $\hat{f}_{\ell, n}(\boldsymbol{\lambda})$ and $\hat{b}_{\ell, n}(\boldsymbol{\lambda})$ are given by

$$\hat{f}_{\ell, n}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} \frac{1-c}{\sqrt{c}} \frac{\beta+n+2\ell-1}{\beta+\ell-1} & : \text{XM} \\ (a+b-d+n) \frac{c+2\ell+n-1}{c+\ell-1} & : \text{XR} \\ q^{-n}(1-abd^{-1}q^n) \frac{1-cq^{2\ell+n-1}}{1-cq^{\ell-1}} & : \text{XqR} \end{cases}, \quad (4.34)$$

$$\hat{b}_{\ell, n}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} \frac{\sqrt{c}}{1-c}(\beta+\ell-1) & : \text{XM} \\ c+\ell-1 & : \text{XR} \\ 1-cq^{\ell-1} & : \text{XqR} \end{cases}. \quad (4.35)$$

Again these two identities (4.25) and (4.26) imply the difference equation for the deforming polynomial,

$$(\tilde{B}(x; \boldsymbol{\lambda})(1-e^\partial) + \tilde{D}(x; \boldsymbol{\lambda})(1-e^{-\partial}))\check{\xi}_\ell(x; \boldsymbol{\lambda}) = \tilde{\mathcal{E}}(\ell; \boldsymbol{\lambda})\check{\xi}_\ell(x; \boldsymbol{\lambda}), \quad (4.36)$$

$$\tilde{B}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} -D(-(x + \beta + \ell - 1); \boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}) & : \text{XM} \\ B(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta})) & : \text{XR, XqR} \end{cases}, \quad (4.37)$$

$$\tilde{D}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} -B(-(x + \beta + \ell - 1); \boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}) & : \text{XM} \\ D(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta})) & : \text{XR, XqR} \end{cases}, \quad (4.38)$$

$$\tilde{\mathcal{E}}(\ell; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} -\mathcal{E}(\ell; \boldsymbol{\lambda}) & : \text{XM} \\ \mathcal{E}(\ell; \mathbf{t}(\boldsymbol{\lambda})) & : \text{XR, XqR} \end{cases}, \quad (4.39)$$

to be compared with the difference equation for the eigenpolynomial

$$(B(x; \boldsymbol{\lambda})(1 - e^\partial) + D(x; \boldsymbol{\lambda})(1 - e^{-\partial}))\check{P}_n(x; \boldsymbol{\lambda}) = \mathcal{E}(n; \boldsymbol{\lambda})\check{P}_n(x; \boldsymbol{\lambda}). \quad (4.40)$$

Note that $\mathcal{E}(\ell; \mathbf{t}(\boldsymbol{\lambda})) = \mathcal{E}(\ell; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}))$. For the appropriate parameter ranges, the deforming polynomial $\check{\xi}_\ell(x; \boldsymbol{\lambda})$ is positive at integer points $x = 0, 1, \dots, x_{\max}^\ell + 1$ ($x_{\max}^\ell \stackrel{\text{def}}{=} x_{\max} - \ell$).

Note that the range of parameters can be enlarged. For XJ, see [75].

4.2 New Orthogonal Polynomials

In this subsection we present the explicit forms of the exceptional orthogonal polynomials and the orthogonality relations. They are bilinear in the deforming polynomial ξ_ℓ (with $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda} + \boldsymbol{\delta}$) and the eigenpolynomial P_n (with $\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}$).

For oQM, the exceptional polynomials are:

$$P_{\ell,n}(\eta; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{2}{\hat{f}_{\ell,n}(\boldsymbol{\lambda})} \left(d_2(\eta)\xi_\ell(\eta; \boldsymbol{\lambda})\partial_\eta P_n(\eta; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) - d_1(\boldsymbol{\lambda})\xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})P_n(\eta; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \right) \times \begin{cases} \pm 1 & : \text{XL1/XL2} \\ \pm 1 & : \text{XJ1/XJ2} \end{cases}, \quad (4.41)$$

where $\tilde{\boldsymbol{\delta}}$ is

$$\tilde{\boldsymbol{\delta}} \stackrel{\text{def}}{=} \begin{cases} \mp 1 & : \text{XL1/XL2} \\ \mp(1, -1) & : \text{XJ1/XJ2} \end{cases}, \quad (4.42)$$

and $\hat{f}_{\ell,n}(\boldsymbol{\lambda})$ will be given in (4.45). They satisfy the orthogonality relation:

$$\int_{x_1}^{x_2} \psi_\ell(x; \boldsymbol{\lambda})^2 P_{\ell,n}(\eta(x); \boldsymbol{\lambda})P_{\ell,m}(\eta(x); \boldsymbol{\lambda})dx = h_{\ell,n}(\boldsymbol{\lambda})\delta_{nm}, \quad (4.43)$$

where the weight function $\psi_\ell(x; \boldsymbol{\lambda})^2$ will be given in (4.63). The normalisation constant $h_{\ell,n}(\boldsymbol{\lambda})$ is related to $h_n(\boldsymbol{\lambda})$ (2.51) as

$$h_{\ell,n}(\boldsymbol{\lambda}) = \frac{\hat{b}_{\ell,n}(\boldsymbol{\lambda})}{\hat{f}_{\ell,n}(\boldsymbol{\lambda})} h_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) = \frac{\hat{b}_{\ell,n}(\boldsymbol{\lambda}) \hat{f}_{0,n}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})}{\hat{f}_{\ell,n}(\boldsymbol{\lambda}) \hat{b}_{0,n}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})} h_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta}), \quad (4.44)$$

where $\hat{f}_{\ell,n}(\boldsymbol{\lambda})$ and $\hat{b}_{\ell,n}(\boldsymbol{\lambda})$ are given by

$$\hat{f}_{\ell,n}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} 2 \times \begin{cases} -1 & : \text{XL1} \\ n + g + \frac{1}{2} & : \text{XL2} \\ -(n + h + \frac{1}{2}) & : \text{XJ1} \\ n + g + \frac{1}{2} & : \text{XJ2} \end{cases}, \quad \hat{b}_{\ell,n}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} 2 \times \begin{cases} -(n + g + 2\ell - \frac{1}{2}) & : \text{XL1} \\ 1 & : \text{XL2} \\ -(n + g + 2\ell - \frac{1}{2}) & : \text{XJ1} \\ n + h + 2\ell - \frac{1}{2} & : \text{XJ2} \end{cases}. \quad (4.45)$$

The deforming polynomial $\xi_\ell(\eta; \boldsymbol{\lambda})$ has the same sign in the orthogonality domain, that is $(0, \infty)$ for XL and $(-1, 1)$ for XJ.

The two types of the XJ polynomials are the mirror images of each other

$$P_{\ell,n}^{\text{XJ2}}(x; g, h) = (-1)^{\ell+n} P_{\ell,n}^{\text{XJ1}}(-x; h, g), \quad (4.46)$$

due to the parity property of the Jacobi polynomial $P_n^{(\alpha, \beta)}(-x) = (-1)^n P_n^{(\beta, \alpha)}(x)$, see (50)–(54) of [8]. The Laguerre polynomials are known to be obtained from the Jacobi polynomials in a certain limit:

$$\lim_{\beta \rightarrow \infty} P_n^{(\alpha, \pm\beta)}(1 - 2x\beta^{-1}) = L_n^{(\alpha)}(\pm x). \quad (4.47)$$

This connects the XJ polynomials to the XL polynomials as shown in (43), (58) of [8].

For idQM, the exceptional polynomials are [37]:

$$P_{\ell,n}(\eta(x); \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{-i}{\hat{f}_{\ell,n}(\boldsymbol{\lambda})\varphi(x)} \left(v_1(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda}) P_n(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) - v_1^*(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda}) P_n(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \right), \quad (4.48)$$

where $\tilde{\boldsymbol{\delta}}$ is

$$\tilde{\boldsymbol{\delta}} \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2} & : \text{XMP} \\ (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) & : \text{XW, XAW} \end{cases}. \quad (4.49)$$

They satisfy the orthogonality relation:

$$\int_{x_1}^{x_2} \psi_\ell(x; \boldsymbol{\lambda})^2 P_{\ell,n}(\eta(x); \boldsymbol{\lambda}) P_{\ell,m}(\eta(x); \boldsymbol{\lambda}) dx = h_{\ell,n}(\boldsymbol{\lambda}) \delta_{nm}, \quad (4.50)$$

where the weight function $\psi_\ell(x; \boldsymbol{\lambda})^2$ will be given in (4.67). The normalisation constant $h_{\ell,n}(\boldsymbol{\lambda})$ is related to $h_n(\boldsymbol{\lambda})$ (2.53) as

$$h_{\ell,n}(\boldsymbol{\lambda}) = \frac{\hat{b}_{\ell,n}(\boldsymbol{\lambda})}{\hat{f}_{\ell,n}(\boldsymbol{\lambda})} h_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) = \frac{\hat{b}_{\ell,n}(\boldsymbol{\lambda}) \hat{f}_{0,n}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})}{\hat{f}_{\ell,n}(\boldsymbol{\lambda}) \hat{b}_{0,n}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})} h_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta}). \quad (4.51)$$

We have verified by numerical calculation for the parameter ranges in (4.11)–(4.13) that the deforming polynomial $\xi_\ell(\eta(x); \boldsymbol{\lambda})$ has the same sign in the orthogonality domain, that is $x \in (-\infty, \infty)$ for XM, $x \in (0, \infty)$ for XW and $x \in (0, \pi)$ for XAW.

For rdQM, the exceptional polynomials are [11]:

$$\begin{aligned} \check{P}_{\ell,n}(x; \boldsymbol{\lambda}) &\stackrel{\text{def}}{=} P_{\ell,n}(\eta(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}); \boldsymbol{\lambda}) \\ &\stackrel{\text{def}}{=} \frac{1}{\hat{f}_{\ell,n}(\boldsymbol{\lambda})} \frac{1}{\varphi(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}})} \left(v_1^B(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \check{\xi}_{\ell}(x; \boldsymbol{\lambda}) \check{P}_n(x+1; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \right. \\ &\quad \left. - v_1^D(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \check{\xi}_{\ell}(x+1; \boldsymbol{\lambda}) \check{P}_n(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \right), \end{aligned} \quad (4.52)$$

which satisfy the normalisation

$$P_{\ell,n}(0; \boldsymbol{\lambda}) = 1. \quad (4.53)$$

Here $\tilde{\boldsymbol{\delta}}$ is

$$\tilde{\boldsymbol{\delta}} \stackrel{\text{def}}{=} \begin{cases} (-1, 0) & : \text{XM} \\ (0, 0, -1, -1) & : \text{XR, XqR} \end{cases}. \quad (4.54)$$

The orthogonality relation reads ($x_{\max}^{\ell} \stackrel{\text{def}}{=} x_{\max} - \ell$, $n_{\max}^{\ell} \stackrel{\text{def}}{=} x_{\max} - \ell$):

$$\sum_{x=0}^{x_{\max}^{\ell}} \frac{\psi_{\ell}(x; \boldsymbol{\lambda})^2}{\check{\xi}_{\ell}(1; \boldsymbol{\lambda})} \check{P}_{\ell,n}(x; \boldsymbol{\lambda}) \check{P}_{\ell,m}(x; \boldsymbol{\lambda}) = \frac{\delta_{nm}}{d_{\ell,n}(\boldsymbol{\lambda})^2} \quad (n, m = 0, 1, \dots, n_{\max}^{\ell}), \quad (4.55)$$

where the weight function $\psi_{\ell}(x; \boldsymbol{\lambda})^2$ will be given in (4.71). The normalisation constant $d_{\ell,n}(\boldsymbol{\lambda})^2$ is related to $d_n(\boldsymbol{\lambda})^2$ (2.56) as

$$d_{\ell,n}(\boldsymbol{\lambda})^2 = d_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}})^2 \frac{\hat{f}_{\ell,n}(\boldsymbol{\lambda})}{\hat{b}_{\ell,n}(\boldsymbol{\lambda})} \frac{1}{s_{\ell}(\boldsymbol{\lambda})} = d_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})^2 \frac{\hat{f}_{\ell,n}(\boldsymbol{\lambda})}{\hat{b}_{\ell,n}(\boldsymbol{\lambda})} \frac{\hat{b}_{0,n}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})}{\hat{f}_{0,n}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})} \frac{s_0(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})}{s_{\ell}(\boldsymbol{\lambda})}, \quad (4.56)$$

where $s_{\ell}(\boldsymbol{\lambda})$ is

$$s_{\ell}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} \frac{1-c}{c} \frac{1}{\beta + \ell - 1} & : \text{XM} \\ -\frac{(d-a)(d-b)}{(c+\ell-1)(d+\ell)} & : \text{XR} \\ -abd^{-1}q^{\ell} \frac{(1-a^{-1}d)(1-b^{-1}d)}{(1-cq^{\ell-1})(1-dq^{\ell})} & : \text{XqR} \end{cases}. \quad (4.57)$$

We have verified by numerical calculation for the parameter ranges in (2.37)–(2.39) that the deforming polynomial $\check{\xi}_{\ell}(x; \boldsymbol{\lambda})$ has the same sign in the orthogonality domain, that is $x = 0, 1, \dots, x_{\max}^{\ell}$.

The exceptional polynomial $P_{\ell,n}(\eta; \boldsymbol{\lambda})$ is a degree $\ell + n$ polynomial in η but has only n zeros in $(0, \infty)$ for XL, $(-1, 1)$ for XJ, $(\eta(x_1), \eta(x_2))$ for idQM and $(0, \eta(x_{\max}))$ for rdQM. For $\ell = 0$, the expressions of $P_{\ell,n}(\eta; \boldsymbol{\lambda})$ reduce to those of the original polynomials.

These new orthogonal polynomials do not satisfy the ordinary three term recurrence relations (2.139). For the XL and XJ polynomials, a *new type of bi-spectrality* is demonstrated

in §4 of [36]. The bi-spectrality of the other new orthogonal polynomials, XW, XAW, XR and XqR are to be explored. A different type of three term recurrence relations for the XL and XJ polynomials is reported in §10 of [9]. The generating functions $\sum_{n=0}^{\infty} t^n P_{\ell,n}(\eta; \boldsymbol{\lambda})$ for the XL and XJ polynomials are reported in §9 of [9]. For the XL1 and XL2 polynomials, the closed form expression of the *double generating functions*

$$G(s, t, \eta; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} s^{\ell} t^n P_{\ell,n}(\eta; \boldsymbol{\lambda})$$

are given there, too. The zeros of orthogonal polynomials have always attracted the interest of researchers. In [76] the behaviours of the zeros of the XL1, XL2 and XJ2 polynomials $P_{\ell,n}(\eta; \boldsymbol{\lambda})$ are explored as the parameters change. The structure of the extra zeros of the other new orthogonal polynomials, XW, XAW, XR and XqR are also an interesting problem.

4.3 Deformed Hamiltonians

The new orthogonal polynomials are discovered as the main part of the eigenfunctions of quantum mechanical systems in which the original Hamiltonians are deformed in terms of the deforming polynomial ξ_{ℓ} . The parameters are so chosen that the deformed systems retain shape invariance [7, 10, 35, 8, 37, 36, 11].

The deformed Hamiltonian and their eigenfunctions have the following forms:

$$\mathcal{H}_{\ell}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}_{\ell}(\boldsymbol{\lambda})^{\dagger} \mathcal{A}_{\ell}(\boldsymbol{\lambda}), \quad (4.58)$$

$$\mathcal{H}_{\ell}(\boldsymbol{\lambda}) \phi_{\ell,n}(x; \boldsymbol{\lambda}) = \mathcal{E}_{\ell,n}(\boldsymbol{\lambda}) \phi_{\ell,n}(x; \boldsymbol{\lambda}), \quad \mathcal{E}_{\ell,n}(\boldsymbol{\lambda}) = \mathcal{E}(n; \boldsymbol{\lambda} + \ell \boldsymbol{\delta}), \quad (4.59)$$

$$\phi_{\ell,n}(x; \boldsymbol{\lambda}) = \psi_{\ell}(x; \boldsymbol{\lambda}) P_{\ell,n}(\eta(x; \boldsymbol{\lambda} + \ell \boldsymbol{\delta}); \boldsymbol{\lambda}). \quad (4.60)$$

The deformation is achieved additively at the level of the prepotential for oQM and multiplicatively at the level of the potential functions $V(x)$, $V^*(x)$, $B(x)$ and $D(x)$ for dQM. For oQM, operators $\mathcal{A}_{\ell}(\boldsymbol{\lambda})$, $\mathcal{A}_{\ell}(\boldsymbol{\lambda})^{\dagger}$ and $\psi_{\ell}(x; \boldsymbol{\lambda})$ are [7, 8]:

$$\mathcal{A}_{\ell}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{d}{dx} - \partial_x w_{\ell}(x; \boldsymbol{\lambda}), \quad \mathcal{A}_{\ell}(\boldsymbol{\lambda})^{\dagger} = -\frac{d}{dx} - \partial_x w_{\ell}(x; \boldsymbol{\lambda}), \quad (4.61)$$

$$w_{\ell}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} w(x; \boldsymbol{\lambda} + \ell \boldsymbol{\delta}) + \log \frac{\xi_{\ell}(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})}{\xi_{\ell}(\eta(x); \boldsymbol{\lambda})}, \quad (4.62)$$

$$\psi_{\ell}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{\phi_0(x; \boldsymbol{\lambda} + \ell \boldsymbol{\delta})}{\xi_{\ell}(\eta(x); \boldsymbol{\lambda})}, \quad \phi_{\ell,0}(x; \boldsymbol{\lambda}) = e^{w_{\ell}(x; \boldsymbol{\lambda})} = \psi_{\ell}(x; \boldsymbol{\lambda}) \xi_{\ell}(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta}). \quad (4.63)$$

For idQM, they are [10]:

$$\mathcal{A}_{\ell}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} i(e^{\frac{\gamma}{2} p} \sqrt{V_{\ell}^*(x; \boldsymbol{\lambda})} - e^{-\frac{\gamma}{2} p} \sqrt{V_{\ell}(x; \boldsymbol{\lambda})}),$$

$$\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger = -i(\sqrt{V_\ell(x; \boldsymbol{\lambda})} e^{\frac{\gamma}{2}p} - \sqrt{V_\ell^*(x; \boldsymbol{\lambda})} e^{-\frac{\gamma}{2}p}), \quad (4.64)$$

$$V_\ell(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} V(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \frac{\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda}) \xi_\ell(\eta(x - i\gamma); \boldsymbol{\lambda} + \boldsymbol{\delta})}{\xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda}) \xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})}, \quad (4.65)$$

$$V_\ell^*(x; \boldsymbol{\lambda}) = V^*(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \frac{\xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda}) \xi_\ell(\eta(x + i\gamma); \boldsymbol{\lambda} + \boldsymbol{\delta})}{\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda}) \xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})}, \quad (4.66)$$

$$\psi_\ell(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{\phi_0(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})}{\sqrt{\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda}) \xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda})}}. \quad (4.67)$$

For rdQM, they are [11]:

$$\mathcal{A}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \sqrt{B_\ell(x; \boldsymbol{\lambda})} - e^\partial \sqrt{D_\ell(x; \boldsymbol{\lambda})}, \quad \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger = \sqrt{B_\ell(x; \boldsymbol{\lambda})} - \sqrt{D_\ell(x; \boldsymbol{\lambda})} e^{-\partial}, \quad (4.68)$$

$$B_\ell(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} B(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \frac{\check{\xi}_\ell(x; \boldsymbol{\lambda}) \check{\xi}_\ell(x + 1; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\check{\xi}_\ell(x + 1; \boldsymbol{\lambda}) \check{\xi}_\ell(x; \boldsymbol{\lambda} + \boldsymbol{\delta})}, \quad (4.69)$$

$$D_\ell(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} D(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \frac{\check{\xi}_\ell(x + 1; \boldsymbol{\lambda}) \check{\xi}_\ell(x - 1; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\check{\xi}_\ell(x; \boldsymbol{\lambda}) \check{\xi}_\ell(x; \boldsymbol{\lambda} + \boldsymbol{\delta})}, \quad (4.70)$$

$$\psi_\ell(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \sqrt{\frac{\check{\xi}_\ell(1; \boldsymbol{\lambda})}{\check{\xi}_\ell(x; \boldsymbol{\lambda}) \check{\xi}_\ell(x + 1; \boldsymbol{\lambda})}}. \quad (4.71)$$

These potential functions satisfy the boundary conditions $D_\ell(0; \boldsymbol{\lambda}) = 0$ and $B_\ell(N - \ell; \boldsymbol{\lambda}) = 0$, and the functions $\phi_{\ell,0}(x; \boldsymbol{\lambda})$ and $\psi_\ell(x; \boldsymbol{\lambda})$ satisfy the normalisation conditions $\phi_{\ell,0}(0; \boldsymbol{\lambda}) = 1$ and $\psi_\ell(0; \boldsymbol{\lambda}) = 1$.

These deformed systems are shape invariant:

$$\mathcal{A}_\ell(\boldsymbol{\lambda}) \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger = \kappa \mathcal{A}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta})^\dagger \mathcal{A}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}_{\ell,1}(\boldsymbol{\lambda}), \quad (4.72)$$

or equivalently (2.108)–(2.112) with replacements $(w, V, B, D) \rightarrow (w_\ell, V_\ell, B_\ell, D_\ell)$. The shape invariance of these deformed systems is shown analytically by making use of the two identities of the deforming polynomials (4.3)–(4.4), (4.15)–(4.16) and (4.25)–(4.26) [35, 37, 11].

The action of the operators $\mathcal{A}_\ell(\boldsymbol{\lambda})$ and $\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger$ on the eigenfunctions is

$$\mathcal{A}_\ell(\boldsymbol{\lambda}) \phi_{\ell,n}(x; \boldsymbol{\lambda}) = f_{\ell,n}(\boldsymbol{\lambda}) \phi_{\ell,n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \frac{1}{\sqrt{B_\ell(0; \boldsymbol{\lambda})}} & : \text{rdQM} \end{cases}, \quad (4.73)$$

$$\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger \phi_{\ell,n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) = b_{\ell,n-1}(\boldsymbol{\lambda}) \phi_{\ell,n}(x; \boldsymbol{\lambda}) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \sqrt{B_\ell(0; \boldsymbol{\lambda})} & : \text{rdQM} \end{cases}, \quad (4.74)$$

where $f_{\ell,n}(\boldsymbol{\lambda})$ and $b_{\ell,n-1}(\boldsymbol{\lambda})$ are

$$f_{\ell,n}(\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \times \begin{cases} \frac{c}{1-c}(\beta + \ell) & : \text{XM} \\ 1 & : \text{other examples} \end{cases}, \quad (4.75)$$

$$b_{\ell, n-1}(\boldsymbol{\lambda}) = b_{n-1}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \times \begin{cases} \left(\frac{c}{1-c}(\beta + \ell)\right)^{-1} & : \text{XM} \\ 1 & : \text{other examples} \end{cases}. \quad (4.76)$$

The forward shift operator $\mathcal{F}_\ell(\boldsymbol{\lambda})$ and the backward shift operator $\mathcal{B}_\ell(\boldsymbol{\lambda})$ are defined by

$$\mathcal{F}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \psi_\ell(x; \boldsymbol{\lambda} + \boldsymbol{\delta})^{-1} \circ \mathcal{A}_\ell(\boldsymbol{\lambda}) \circ \psi_\ell(x; \boldsymbol{\lambda}) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \sqrt{B_\ell(0; \boldsymbol{\lambda})} & : \text{rdQM} \end{cases}, \quad (4.77)$$

$$\mathcal{B}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \psi_\ell(x; \boldsymbol{\lambda})^{-1} \circ \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger \circ \psi_\ell(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \frac{1}{\sqrt{B_\ell(0; \boldsymbol{\lambda})}} & : \text{rdQM} \end{cases}, \quad (4.78)$$

and their action on the polynomials is $(\check{P}_{\ell, n}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_{\ell, n}(\eta(x); \boldsymbol{\lambda})$ for oQM and idQM)

$$\mathcal{F}_\ell(\boldsymbol{\lambda})\check{P}_{\ell, n}(x; \boldsymbol{\lambda}) = f_{\ell, n}(\boldsymbol{\lambda})\check{P}_{\ell, n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (4.79)$$

$$\mathcal{B}_\ell(\boldsymbol{\lambda})\check{P}_{\ell, n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) = b_{\ell, n-1}(\boldsymbol{\lambda})\check{P}_{\ell, n}(x; \boldsymbol{\lambda}). \quad (4.80)$$

The explicit forms of $\mathcal{F}_\ell(\boldsymbol{\lambda})$ and $\mathcal{B}_\ell(\boldsymbol{\lambda})$ are the following:

oQM:

$$\mathcal{F}_\ell(\boldsymbol{\lambda}) = c_{\mathcal{F}} \frac{\xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\xi_\ell(\eta; \boldsymbol{\lambda})} \left(\frac{d}{d\eta} - \partial_\eta \log \xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta}) \right), \quad (4.81)$$

$$\mathcal{B}_\ell(\boldsymbol{\lambda}) = -4c_{\mathcal{F}}^{-1}c_2(\eta) \frac{\xi_\ell(\eta; \boldsymbol{\lambda})}{\xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})} \left(\frac{d}{d\eta} + \frac{c_1(\eta, \boldsymbol{\lambda} + \ell\boldsymbol{\delta})}{c_2(\eta)} - \partial_\eta \log \xi_\ell(\eta; \boldsymbol{\lambda}) \right), \quad (4.82)$$

idQM:

$$\mathcal{F}_\ell(\boldsymbol{\lambda}) = \frac{i}{\varphi(x)\xi_\ell(\eta(x); \boldsymbol{\lambda})} \left(\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda} + \boldsymbol{\delta})e^{\frac{\gamma}{2}p} - \xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda} + \boldsymbol{\delta})e^{-\frac{\gamma}{2}p} \right), \quad (4.83)$$

$$\begin{aligned} \mathcal{B}_\ell(\boldsymbol{\lambda}) &= \frac{-i}{\xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})} \left(V(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda})e^{\frac{\gamma}{2}p} \right. \\ &\quad \left. - V^*(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})\xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda})e^{-\frac{\gamma}{2}p} \right) \varphi(x), \end{aligned} \quad (4.84)$$

rdQM:

$$\mathcal{F}_\ell(\boldsymbol{\lambda}) = \frac{B(0, \boldsymbol{\lambda} + \ell\boldsymbol{\delta})}{\varphi(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})\check{\xi}_\ell(x + 1; \boldsymbol{\lambda})} \left(\check{\xi}_\ell(x + 1; \boldsymbol{\lambda} + \boldsymbol{\delta}) - \check{\xi}_\ell(x; \boldsymbol{\lambda} + \boldsymbol{\delta})e^\theta \right), \quad (4.85)$$

$$\begin{aligned} \mathcal{B}_\ell(\boldsymbol{\lambda}) &= \frac{1}{B(0; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})} \frac{1}{\check{\xi}_\ell(x; \boldsymbol{\lambda} + \boldsymbol{\delta})} \\ &\quad \times \left(B(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})\check{\xi}_\ell(x; \boldsymbol{\lambda}) - D(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})\check{\xi}_\ell(x + 1; \boldsymbol{\lambda})e^{-\theta} \right) \varphi(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}). \end{aligned} \quad (4.86)$$

4.4 Second Order Equations for the New Polynomials

The new orthogonal polynomials $P_{\ell, n}(\eta; \boldsymbol{\lambda})$ satisfy second order differential or difference equations with *rational coefficients*. Here we list the explicit forms.

The similarity transformed Hamiltonian $\tilde{\mathcal{H}}_\ell(\boldsymbol{\lambda})$ is defined by

$$\tilde{\mathcal{H}}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \psi_\ell(x; \boldsymbol{\lambda})^{-1} \circ \mathcal{H}_\ell(\boldsymbol{\lambda}) \circ \psi_\ell(x; \boldsymbol{\lambda}) = \mathcal{B}_\ell(\boldsymbol{\lambda})\mathcal{F}_\ell(\boldsymbol{\lambda}), \quad (4.87)$$

and its action on the polynomials is

$$\tilde{\mathcal{H}}_\ell(\boldsymbol{\lambda})\check{P}_{\ell,n}(x; \boldsymbol{\lambda}) = \mathcal{E}_{\ell,n}(\boldsymbol{\lambda})\check{P}_{\ell,n}(x; \boldsymbol{\lambda}). \quad (4.88)$$

From (4.79)–(4.80) and (4.75)–(4.76), we have

$$\mathcal{E}_{\ell,n}(\boldsymbol{\lambda}) = f_{\ell,n}(\boldsymbol{\lambda})b_{\ell,n-1}(\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta})b_{n-1}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta}) = \mathcal{E}(n; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}). \quad (4.89)$$

The explicit forms of $\tilde{\mathcal{H}}_\ell(\boldsymbol{\lambda})$ are the following:

oQM:

$$\begin{aligned} \tilde{\mathcal{H}}_\ell(\boldsymbol{\lambda}) = & -4\left(c_2(\eta)\frac{d^2}{d\eta^2} + (c_1(\eta, \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) - 2c_2(\eta)\partial_\eta \log \xi_\ell(\eta; \boldsymbol{\lambda}))\frac{d}{d\eta} \right. \\ & \left. - 2d_2(\eta)d_3(\boldsymbol{\lambda}, \ell)\partial_\eta \log \xi_\ell(\eta; \boldsymbol{\lambda}) - \frac{1}{4}\tilde{\mathcal{E}}_\ell(\boldsymbol{\lambda})\right), \end{aligned} \quad (4.90)$$

idQM:

$$\begin{aligned} \tilde{\mathcal{H}}_\ell(\boldsymbol{\lambda}) = & V(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})\frac{\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda})}{\xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda})}\left(e^{\gamma p} - \frac{\xi_\ell(\eta(x - i\gamma); \boldsymbol{\lambda} + \boldsymbol{\delta})}{\xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})}\right) \\ & + V^*(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})\frac{\xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda})}{\xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda})}\left(e^{-\gamma p} - \frac{\xi_\ell(\eta(x + i\gamma); \boldsymbol{\lambda} + \boldsymbol{\delta})}{\xi_\ell(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})}\right), \end{aligned} \quad (4.91)$$

rdQM:

$$\begin{aligned} \tilde{\mathcal{H}}_\ell(\boldsymbol{\lambda}) = & B(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})\frac{\check{\xi}_\ell(x; \boldsymbol{\lambda})}{\check{\xi}_\ell(x + 1; \boldsymbol{\lambda})}\left(\frac{\check{\xi}_\ell(x + 1; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\check{\xi}_\ell(x; \boldsymbol{\lambda} + \boldsymbol{\delta})} - e^\theta\right) \\ & + D(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta})\frac{\check{\xi}_\ell(x + 1; \boldsymbol{\lambda})}{\check{\xi}_\ell(x; \boldsymbol{\lambda})}\left(\frac{\check{\xi}_\ell(x - 1; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\check{\xi}_\ell(x; \boldsymbol{\lambda} + \boldsymbol{\delta})} - e^{-\theta}\right). \end{aligned} \quad (4.92)$$

For the XL and XJ polynomials in oQM we write concretely [9]:

$$\begin{aligned} \text{XL1} : & \eta\partial_\eta^2 P_{\ell,n}(\eta; \boldsymbol{\lambda}) + \left(g + \ell + \frac{1}{2} - \eta - 2\frac{\eta\partial_\eta \xi_\ell(\eta; \boldsymbol{\lambda})}{\xi_\ell(\eta; \boldsymbol{\lambda})}\right)\partial_\eta P_{\ell,n}(\eta; \boldsymbol{\lambda}) \\ & + \left(2\frac{\eta\partial_\eta \xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\xi_\ell(\eta; \boldsymbol{\lambda})} + n - \ell\right)P_{\ell,n}(\eta; \boldsymbol{\lambda}) = 0, \end{aligned} \quad (4.93)$$

$$\begin{aligned} \text{XL2} : & \eta\partial_\eta^2 P_{\ell,n}(\eta; \boldsymbol{\lambda}) + \left(g + \ell + \frac{1}{2} - \eta - 2\frac{\eta\partial_\eta \xi_\ell(\eta; \boldsymbol{\lambda})}{\xi_\ell(\eta; \boldsymbol{\lambda})}\right)\partial_\eta P_{\ell,n}(\eta; \boldsymbol{\lambda}) \\ & + \left(-2\frac{(g + \frac{1}{2})\partial_\eta \xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\xi_\ell(\eta; \boldsymbol{\lambda})} + n + \ell\right)P_{\ell,n}(\eta; \boldsymbol{\lambda}) = 0, \end{aligned} \quad (4.94)$$

$$\begin{aligned}
\text{XJ1} : & (1 - \eta^2) \partial_\eta^2 P_{\ell,n}(\eta; \boldsymbol{\lambda}) \\
& + \left(h - g - (g + h + 2\ell + 1)\eta - 2 \frac{(1 - \eta^2) \partial_\eta \xi_\ell(\eta; \boldsymbol{\lambda})}{\xi_\ell(\eta; \boldsymbol{\lambda})} \right) \partial_\eta P_{\ell,n}(\eta; \boldsymbol{\lambda}) \\
& + \left(- \frac{2(h + \frac{1}{2})(1 - \eta) \partial_\eta \xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\xi_\ell(\eta; \boldsymbol{\lambda})} \right. \\
& \quad \left. + \ell(\ell + g - h - 1) + n(n + g + h + 2\ell) \right) P_{\ell,n}(\eta; \boldsymbol{\lambda}) = 0,
\end{aligned} \tag{4.95}$$

$$\begin{aligned}
\text{XJ2} : & (1 - \eta^2) \partial_\eta^2 P_{\ell,n}(\eta; \boldsymbol{\lambda}) \\
& + \left(h - g - (g + h + 2\ell + 1)\eta - 2 \frac{(1 - \eta^2) \partial_\eta \xi_\ell(\eta; \boldsymbol{\lambda})}{\xi_\ell(\eta; \boldsymbol{\lambda})} \right) \partial_\eta P_{\ell,n}(\eta; \boldsymbol{\lambda}) \\
& + \left(\frac{2(g + \frac{1}{2})(1 + \eta) \partial_\eta \xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\xi_\ell(\eta; \boldsymbol{\lambda})} \right. \\
& \quad \left. + \ell(\ell + h - g - 1) + n(n + g + h + 2\ell) \right) P_{\ell,n}(\eta; \boldsymbol{\lambda}) = 0.
\end{aligned} \tag{4.96}$$

The zeros of the deforming polynomial $\xi_\ell(\eta_j; \boldsymbol{\lambda}) = 0$ ($j = 1, \dots, \ell$) are the extra regular singularities on top of those for the original Laguerre and Jacobi polynomials. In all the four cases, the characteristic exponents at these extra ℓ zeros are the same:

$$\rho = 0, 3. \tag{4.97}$$

The new polynomial solutions have $\rho = 0$ at all the extra singularities. The XJ polynomials provide an infinite family ($n = 0, 1, 2, \dots$) of global solutions of the Fuchsian differential equations (4.95)-(4.96) with $3 + \ell$ regular singularities. To the best of our knowledge, global solutions of a Fuchsian differential equation with more than four singularities had been utterly unknown, although in this case, the locations of the singularities are very special.

4.5 Intertwining the Original and the Deformed Systems

Here we present the transformation intertwining the original and the deformed Hamiltonian systems. It provides a simple derivation of the bi-linear expression of the new orthogonal polynomials in the original and the deforming polynomials. It also yields another proof of the shape invariance of the deformed systems. This type of transformations are sometimes called Darboux-Crum transformations.

The general setting of the transformation theory is common to oQM, idQM and rdQM. Let us define a pair of Hamiltonians $\hat{\mathcal{H}}_\ell^{(\pm)}(\boldsymbol{\lambda})$ in terms of the operators $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger$,

$$\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}), \quad \hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger, \tag{4.98}$$

and consider their Schrödinger equations:

$$\hat{\mathcal{H}}_\ell^{(\pm)}(\boldsymbol{\lambda})\hat{\phi}_{\ell,n}^{(\pm)}(x; \boldsymbol{\lambda}) = \hat{\mathcal{E}}_{\ell,n}^{(\pm)}(\boldsymbol{\lambda})\hat{\phi}_{\ell,n}^{(\pm)}(x; \boldsymbol{\lambda}) \quad (n = 0, 1, 2, \dots). \quad (4.99)$$

By definition, all the eigenfunctions must be square integrable (square summable). The pair of Hamiltonians are intertwined:

$$\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda})\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger = \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger = \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda}), \quad (4.100)$$

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda}) = \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) = \hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda})\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}). \quad (4.101)$$

In all the cases to be discussed here, we have $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\hat{\phi}_{\ell,n}^{(+)}(x; \boldsymbol{\lambda}) \neq 0$ and $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\phi}_{\ell,n}^{(-)}(x; \boldsymbol{\lambda}) \neq 0$. This situation is called the ‘broken susy’ case in the supersymmetric quantum mechanics. This means that the two systems are exactly iso-spectral and there is one-to-one correspondence between the eigenfunctions:

$$\hat{\mathcal{E}}_{\ell,n}^{(+)}(\boldsymbol{\lambda}) = \hat{\mathcal{E}}_{\ell,n}^{(-)}(\boldsymbol{\lambda}), \quad (4.102)$$

$$\hat{\phi}_{\ell,n}^{(-)}(x; \boldsymbol{\lambda}) \propto \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\hat{\phi}_{\ell,n}^{(+)}(x; \boldsymbol{\lambda}), \quad \hat{\phi}_{\ell,n}^{(+)}(x; \boldsymbol{\lambda}) \propto \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\phi}_{\ell,n}^{(-)}(x; \boldsymbol{\lambda}). \quad (4.103)$$

The strategy is to find the operators $\hat{\mathcal{A}}_\ell$ and $\hat{\mathcal{A}}_\ell^\dagger$ in such a way that $\hat{\mathcal{H}}_\ell^{(+)}$ is the original Hamiltonian (up to an additive constant) shown in §2.1 and $\hat{\mathcal{H}}_\ell^{(-)}$ becomes the deformed Hamiltonian (up to an additive constant) demonstrated in §4.3. Then the above formula (4.103) gives the expression of the new orthogonal polynomials in terms of the original ones.

We define operators $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{A}}_\ell^\dagger(\boldsymbol{\lambda})$ as follows:

oQM:

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{d}{dx} - \partial_x \hat{w}_\ell(x; \boldsymbol{\lambda}), \quad \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger = -\frac{d}{dx} - \partial_x \hat{w}_\ell(x; \boldsymbol{\lambda}), \quad (4.104)$$

$$\hat{w}_\ell(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \log \xi_\ell(\eta(x); \boldsymbol{\lambda}) + \begin{cases} \frac{1}{2}x^2 + (g + \ell - 1) \log x & : \text{XL1} \\ w(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta})) & : \text{XL2, XJ1, XJ2} \end{cases} \cdot \quad (4.105)$$

(Note that $\frac{1}{2}x^2 + (g + \ell - 1) \log x = w(ix; \boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}) + \text{const.}$ for XL1.)

idQM:

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} i(e^{\frac{\gamma}{2}p} \sqrt{\hat{V}_\ell^*(x; \boldsymbol{\lambda})} - e^{-\frac{\gamma}{2}p} \sqrt{\hat{V}_\ell(x; \boldsymbol{\lambda})}),$$

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger = -i(\sqrt{\hat{V}_\ell(x; \boldsymbol{\lambda})} e^{\frac{\gamma}{2}p} - \sqrt{\hat{V}_\ell^*(x; \boldsymbol{\lambda})} e^{-\frac{\gamma}{2}p}), \quad (4.106)$$

$$\hat{V}_\ell(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} V(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta})) \frac{\xi_\ell(\eta(x - i\gamma); \boldsymbol{\lambda})}{\xi_\ell(\eta(x); \boldsymbol{\lambda})}, \quad (4.107)$$

$$\hat{V}_\ell^*(x; \boldsymbol{\lambda}) = V^*(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta})) \frac{\xi_\ell(\eta(x + i\gamma); \boldsymbol{\lambda})}{\xi_\ell(\eta(x); \boldsymbol{\lambda})}, \quad (4.108)$$

rdQM:

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \sqrt{\hat{B}_\ell(x; \boldsymbol{\lambda})} - e^\partial \sqrt{\hat{D}_\ell(x; \boldsymbol{\lambda})}, \quad \hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger = \sqrt{\hat{B}_\ell(x; \boldsymbol{\lambda})} - \sqrt{\hat{D}_\ell(x; \boldsymbol{\lambda})} e^{-\partial}, \quad (4.109)$$

$$\hat{B}_\ell(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{\check{\xi}_\ell(x + 1; \boldsymbol{\lambda})}{\check{\xi}_\ell(x; \boldsymbol{\lambda})} \times \begin{cases} -D(-(x + \beta + \ell - 1); \boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}) & : \text{XM} \\ B(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta})) & : \text{XR, XqR} \end{cases}, \quad (4.110)$$

$$\hat{D}_\ell(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{\check{\xi}_\ell(x - 1; \boldsymbol{\lambda})}{\check{\xi}_\ell(x; \boldsymbol{\lambda})} \times \begin{cases} -B(-(x + \beta + \ell - 1); \boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}) & : \text{XM} \\ D(x; \mathbf{t}(\boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta})) & : \text{XR, XqR} \end{cases}. \quad (4.111)$$

Then, by using the two identities of the deforming polynomials (4.3)–(4.4), (4.15)–(4.16) and (4.25)–(4.26), we can show that [36, 38, 37, 11]

$$\hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda}) = \hat{\kappa}_\ell(\boldsymbol{\lambda})(\mathcal{H}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) + \hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda})), \quad (4.112)$$

$$\hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda}) = \hat{\kappa}_\ell(\boldsymbol{\lambda})(\mathcal{H}_\ell(\boldsymbol{\lambda}) + \hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda})), \quad (4.113)$$

where $\hat{\kappa}_\ell(\boldsymbol{\lambda})$ is

$$\hat{\kappa}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} 1 : \text{oQM}, \quad \hat{\kappa}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 1 & : \text{XMP, XW} \\ (a_1 a_2 q^\ell)^{-1} & : \text{XAW} \end{cases}, \quad (4.114)$$

$$\hat{\kappa}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 1 & : \text{XM, XR} \\ (abd^{-1}q^\ell)^{-1} & : \text{XqR} \end{cases}.$$

Therefore the original system with the shifted parameters ($\mathcal{H}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}})$) and the deformed system ($\mathcal{H}_\ell(\boldsymbol{\lambda})$) are exactly isospectral.

Based on the results (4.112)–(4.113), we have

$$\hat{\phi}_{\ell,n}^{(+)}(x; \boldsymbol{\lambda}) = \phi_n(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}), \quad \hat{\phi}_{\ell,n}^{(-)}(x; \boldsymbol{\lambda}) = \phi_{\ell,n}(x; \boldsymbol{\lambda}), \quad (4.115)$$

$$\hat{\mathcal{E}}_{\ell,n}^{(\pm)}(\boldsymbol{\lambda}) = \hat{\kappa}_\ell(\boldsymbol{\lambda})(\mathcal{E}_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) + \hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda})) = \hat{\kappa}_\ell(\boldsymbol{\lambda})(\mathcal{E}_{\ell,n}(\boldsymbol{\lambda}) + \hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda})). \quad (4.116)$$

The correspondence of the pair of eigenfunctions $\hat{\phi}_{\ell,n}^{(\pm)}(x)$ with their own normalisation specified in the preceding sections is related by

$$\hat{\phi}_{\ell,n}^{(-)}(x; \boldsymbol{\lambda}) = \frac{\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\hat{\phi}_{\ell,n}^{(+)}(x; \boldsymbol{\lambda})}{\sqrt{\hat{\kappa}_\ell(\boldsymbol{\lambda})\hat{f}_{\ell,n}(\boldsymbol{\lambda})}} \times \begin{cases} 1 & : \text{oQM, idQM} \\ \sqrt{\check{\xi}_\ell(1; \boldsymbol{\lambda})s_\ell(\boldsymbol{\lambda})} & : \text{rdQM} \end{cases}, \quad (4.117)$$

$$\hat{\phi}_{\ell,n}^{(+)}(x; \boldsymbol{\lambda}) = \frac{\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger\hat{\phi}_{\ell,n}^{(-)}(x; \boldsymbol{\lambda})}{\sqrt{\hat{\kappa}_\ell(\boldsymbol{\lambda})\hat{b}_{\ell,n}(\boldsymbol{\lambda})}} \times \begin{cases} 1 & : \text{oQM, idQM} \\ \frac{1}{\sqrt{\check{\xi}_\ell(1; \boldsymbol{\lambda})s_\ell(\boldsymbol{\lambda})}} & : \text{rdQM} \end{cases}. \quad (4.118)$$

By removing the effects of the orthogonality weight functions, we define the operators $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda})$

$$\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \psi_\ell(x; \boldsymbol{\lambda})^{-1} \circ \frac{\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})}{\sqrt{\hat{\kappa}_\ell(\boldsymbol{\lambda})}} \circ \phi_0(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \sqrt{\check{\xi}_\ell(1; \boldsymbol{\lambda})_{s_\ell(\boldsymbol{\lambda})}} & : \text{rdQM} \end{cases}, \quad (4.119)$$

$$\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}})^{-1} \circ \frac{\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})^\dagger}{\sqrt{\hat{\kappa}_\ell(\boldsymbol{\lambda})}} \circ \psi_\ell(x; \boldsymbol{\lambda}) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \frac{1}{\sqrt{\check{\xi}_\ell(1; \boldsymbol{\lambda})_{s_\ell(\boldsymbol{\lambda})}}} & : \text{rdQM} \end{cases}. \quad (4.120)$$

Their explicit forms are the following:

oQM:

$$\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda}) = 2 \left(d_2(\eta) \xi_\ell(\eta; \boldsymbol{\lambda}) \frac{d}{d\eta} - d_1(\boldsymbol{\lambda}) \xi_\ell(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta}) \right) \times \begin{cases} \pm 1 & : \text{XL1/XL2} \\ \pm 1 & : \text{XJ1/XJ2} \end{cases}, \quad (4.121)$$

$$\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda}) = \frac{-2}{\xi_\ell(\eta; \boldsymbol{\lambda})} \left(\frac{c_2(\eta)}{d_2(\eta)} \frac{d}{d\eta} + d_3(\boldsymbol{\lambda}, \ell) \right) \times \begin{cases} \pm 1 & : \text{XL1/XL2} \\ \pm 1 & : \text{XJ1/XJ2} \end{cases}, \quad (4.122)$$

idQM:

$$\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda}) = \frac{-i}{\varphi(x)} \left(v_1(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \xi_\ell(\eta(x + i\frac{\gamma}{2}); \boldsymbol{\lambda}) e^{\frac{\gamma}{2}p} - v_1^*(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \xi_\ell(\eta(x - i\frac{\gamma}{2}); \boldsymbol{\lambda}) e^{-\frac{\gamma}{2}p} \right), \quad (4.123)$$

$$\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda}) = \frac{1}{\xi_\ell(\eta(x); \boldsymbol{\lambda})} \frac{-i}{\varphi(x)} \left(v_2(x; \boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}) e^{\frac{\gamma}{2}p} - v_2^*(x; \boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}) e^{-\frac{\gamma}{2}p} \right), \quad (4.124)$$

rdQM:

$$\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda}) = \frac{1}{\varphi(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}})} \left(v_1^B(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \check{\xi}_\ell(x; \boldsymbol{\lambda}) e^\partial - v_1^D(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta}) \check{\xi}_\ell(x + 1; \boldsymbol{\lambda}) \right), \quad (4.125)$$

$$\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda}) = \frac{1}{\check{\xi}_\ell(x; \boldsymbol{\lambda})} \frac{1}{\varphi(x; \boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\lambda} + \tilde{\boldsymbol{\delta}})} \left(v_2^B(x; \boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}) - v_2^D(x; \boldsymbol{\lambda} + (\ell - 1)\boldsymbol{\delta}) e^{-\partial} \right). \quad (4.126)$$

The operators $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda})$ act as the forward and backward shift operators connecting the original polynomials P_n and the exceptional polynomials $P_{\ell,n}$:

$$\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda}) \check{P}_n(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) = \hat{f}_{\ell,n}(\boldsymbol{\lambda}) \check{P}_{\ell,n}(x; \boldsymbol{\lambda}), \quad (4.127)$$

$$\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda}) \check{P}_{\ell,n}(x; \boldsymbol{\lambda}) = \hat{b}_{\ell,n}(\boldsymbol{\lambda}) \check{P}_n(x; \boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}). \quad (4.128)$$

The former relation (4.127) with the explicit forms of $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ (4.121), (4.123) and (4.125) provides the explicit expressions (4.41), (4.48) and (4.52) of the exceptional orthogonal polynomials. In terms of $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ and $\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda})$, the relations (4.112)–(4.113) become

$$\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda}) \hat{\mathcal{F}}_\ell(\boldsymbol{\lambda}) = \tilde{\mathcal{H}}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) + \hat{f}_{\ell,0}(\boldsymbol{\lambda}) \hat{b}_{\ell,0}(\boldsymbol{\lambda}), \quad (4.129)$$

$$\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})\hat{\mathcal{B}}_\ell(\boldsymbol{\lambda}) = \tilde{\mathcal{H}}_\ell(\boldsymbol{\lambda}) + \hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda}). \quad (4.130)$$

The other simple consequences of these relations are

$$\hat{\mathcal{E}}_{\ell,n}^{(\pm)}(\boldsymbol{\lambda}) = \hat{\kappa}_\ell(\boldsymbol{\lambda})\hat{f}_{\ell,n}(\boldsymbol{\lambda})\hat{b}_{\ell,n}(\boldsymbol{\lambda}), \quad \mathcal{E}_n(\boldsymbol{\lambda} + \ell\boldsymbol{\delta}) = \hat{f}_{\ell,n}(\boldsymbol{\lambda})\hat{b}_{\ell,n}(\boldsymbol{\lambda}) - \hat{f}_{\ell,0}(\boldsymbol{\lambda})\hat{b}_{\ell,0}(\boldsymbol{\lambda}). \quad (4.131)$$

Orthogonality relations (4.43)–(4.44), (4.50)–(4.51) and (4.55)–(4.56) can be shown by using these intertwining relations [38, 37, 11].

It is interesting to note that the operator $\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})$ intertwines those of the original and deformed systems $\mathcal{A}(\boldsymbol{\lambda})$ and $\mathcal{A}_\ell(\boldsymbol{\lambda})$:

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta})\mathcal{A}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) = \mathcal{A}_\ell(\boldsymbol{\lambda})\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda}), \quad (4.132)$$

$$\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}})^\dagger = \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta}). \quad (4.133)$$

In terms of the definitions of the forward shift operators $\mathcal{F}(\boldsymbol{\lambda})$ (2.128), $\mathcal{F}_\ell(\boldsymbol{\lambda})$ (4.77), $\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})$ (4.119), and $\mathcal{B}(\boldsymbol{\lambda})$ (2.130), $\mathcal{B}_\ell(\boldsymbol{\lambda})$ (4.78), the above relations are rewritten as:

$$\hat{s}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta})\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta})\mathcal{F}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) = \hat{s}_\ell(\boldsymbol{\lambda})\mathcal{F}_\ell(\boldsymbol{\lambda})\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda}), \quad (4.134)$$

$$\hat{s}_\ell(\boldsymbol{\lambda})\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda})\mathcal{B}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) = \hat{s}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta})\mathcal{B}_\ell(\boldsymbol{\lambda})\hat{\mathcal{F}}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (4.135)$$

where $\hat{s}_\ell(\boldsymbol{\lambda})$ is

$$\hat{s}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} 1 \quad : \text{oQM}, \quad \hat{s}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \sqrt{\hat{\kappa}_\ell(\boldsymbol{\lambda})} \quad : \text{idQM}, \quad \hat{s}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \hat{\kappa}_\ell(\boldsymbol{\lambda}) \times \begin{cases} \beta + \ell - 1 & : \text{XM} \\ c + \ell - 1 & : \text{XR} \\ 1 - cq^{\ell-1} & : \text{XqR} \end{cases}. \quad (4.136)$$

These relations can be proven by explicit calculation with the help of the two identities of the deforming polynomial (4.3)–(4.4), (4.15)–(4.16) and (4.25)–(4.26), and provide a proof of (4.73)–(4.74) and (4.79)–(4.80) without recourse to the shape invariance of the deformed system, see [38, 37, 11].

The intertwining relation offers another proof of the shape invariance of the ℓ -th new orthogonal polynomials through the established shape invariance of the original polynomials

as depicted in the following commutative diagram:

$$\begin{array}{ccc}
\boxed{\begin{array}{c} \hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda} + \boldsymbol{\delta}) \\ \propto \mathcal{H}(\boldsymbol{\lambda} + (\ell + 1)\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) + c(\boldsymbol{\lambda} + \boldsymbol{\delta}) \end{array}} & \xrightarrow{\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta})} & \boxed{\begin{array}{c} \hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda} + \boldsymbol{\delta}) \\ \propto \mathcal{H}_\ell(\boldsymbol{\lambda} + \boldsymbol{\delta}) + c(\boldsymbol{\lambda} + \boldsymbol{\delta}) \end{array}} \\
\begin{array}{c} \text{established} \\ \text{shape invariance} \end{array} \uparrow & & \uparrow \begin{array}{c} \text{shape} \\ \text{invariance} \end{array} \\
\boxed{\begin{array}{c} \hat{\mathcal{H}}_\ell^{(+)}(\boldsymbol{\lambda}) \\ \propto \mathcal{H}(\boldsymbol{\lambda} + \ell\boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) + c(\boldsymbol{\lambda}) \end{array}} & \xrightarrow{\hat{\mathcal{A}}_\ell(\boldsymbol{\lambda})} & \boxed{\begin{array}{c} \hat{\mathcal{H}}_\ell^{(-)}(\boldsymbol{\lambda}) \\ \propto \mathcal{H}_\ell(\boldsymbol{\lambda}) + c(\boldsymbol{\lambda}) \end{array}} \\
\text{original polynomial} & & \text{new polynomial}
\end{array}$$

Two ways of proving shape invariance of the new orthogonal polynomials system.

Historically the first members of the X_1 Laguerre polynomials were discussed in the framework of ‘conditionally exactly solvable problems’ [77] in 1997. About ten years later the first members of the X_1 Laguerre and Jacobi polynomials were constructed by Gómez-Ullate et al [67] in 2008 in the framework of the Sturm-Liouville theory. They were rederived as the main part of the eigenfunctions of shape invariant quantum mechanical Hamiltonians by Quesne and collaborators [68]. In 2009 the present authors derived the infinitely many X_ℓ Laguerre and Jacobi polynomials by deforming the Hamiltonian systems of the radial oscillator and the Pöschl-Teller potential in terms of the eigenpolynomials of degree ℓ [7]. The main idea was very simple. The X_1 Jacobi Hamiltonian of Quesne [68] in our notation read

$$\begin{aligned}
\mathcal{H}_1(g, h) = & -\frac{d^2}{dx^2} + \frac{g(g+1)}{\sin^2 x} + \frac{h(h+1)}{\cos^2 x} - (2+g+h)^2 \\
& + \frac{8(g+h+1)}{1+g+h+(g-h)\cos 2x} - \frac{8(2g+1)(2h+1)}{(1+g+h+(g-h)\cos 2x)^2}.
\end{aligned} \tag{4.137}$$

Its groundstate wavefunction was

$$\begin{aligned}
\phi_{1,0}(x; g, h) &= (\sin x)^{g+1}(\cos x)^{h+1} \frac{3+g+h+(g-h)\cos 2x}{1+g+h+(g-h)\cos 2x} \\
&= (\sin x)^{g+1}(\cos x)^{h+1} \frac{P_1^{(g+2-\frac{3}{2}, -h-2-\frac{1}{2})}(\cos 2x)}{P_1^{(g+1-\frac{3}{2}, -h-1-\frac{1}{2})}(\cos 2x)}.
\end{aligned} \tag{4.138}$$

From this it did not take long to guess the general ℓ form:

$$\phi_{\ell,0}(x; g, h) = (\sin x)^{g+\ell}(\cos x)^{h+\ell} \frac{P_\ell^{(g+\ell+1-\frac{3}{2}, -h-\ell-1-\frac{1}{2})}(\cos 2x)}{P_\ell^{(g+\ell-\frac{3}{2}, -h-\ell-\frac{1}{2})}(\cos 2x)}, \tag{4.139}$$

which was $e^{w_\ell(x,\lambda)}$ in (4.63) for XJ1. After the discovery of the second lowest members ($\ell = 2$) of the XL2 family [78], the entire XJ2 and XL2 families were constructed in [8]. Then the construction of the exceptional Wilson and Askey-Wilson polynomials [10] and the exceptional Racah and q -Racah polynomials [11] followed by taking shape invariance as a guiding principle. The Fuchsian properties and many other aspects of XL and XJ polynomials were demonstrated in [9]. The intertwining relations were developed in [36, 37, 11]. The general knowledge of the solution spaces of exactly solvable (discrete) quantum mechanical systems governed by Crum's theorem [27] and its modifications [29, 24, 25, 26] had been very helpful for the discovery of various exceptional orthogonal polynomials. Slightly different formulation of the Darboux-Crum transformation for the XL polynomials were reported in [79, 80].

Naively one expects that most exactly solvable QM Hamiltonian systems would admit similar shape invariant deformations. Since the 'twisting' of the parameters is essential, the systems corresponding to the Hermite and the Gegenbauer ($g = h$ case of the Jacobi) polynomials do not admit such a deformation. So far, in addition to the examples presented in this review, the exceptional polynomials for the continuous Hahn [37], dual (q)-Hahn and little q -Jacobi [11] are explicitly known. Various exceptional orthogonal polynomials can be obtained from XAW and XqR polynomials by taking appropriate limits. It is a big challenge to construct various new orthogonal polynomials, for example, those corresponding to the Morse and Scarf potentials and the (q)-Hahn polynomials, and to study them in detail.

5 Summary and Comments

Discrete quantum mechanics, as formulated and developed by the present authors during the last decade, is briefly reviewed following the logical structure rather than the historical developments. The parallelism and contrast among the ordinary quantum mechanics (oQM) and the discrete quantum mechanics (dQM) are emphasised. As far as possible, one universal formula valid for oQM and dQM with the pure imaginary shifts (idQM) and the real shifts (rdQM) is presented first and various ramifications follow. In one aspect, this is a reformulation of the theory of bi-spectral orthogonal polynomials in the language and logic of quantum mechanics and every quantity is given explicitly without recourse to the moment problem as exemplified in §2. While it offers structural understanding of various specific properties of the bi-spectral polynomials, a new unified theory is established to generate all

the known bi-spectral polynomials from the first principle, as reviewed in §3. As evidenced by the discovery of various kinds of infinite families of new orthogonal polynomials reviewed in §4, this reformulation has been extremely successful.

Here are a few words on how our project of discrete quantum mechanics started. The classical and quantum integrability were very closely related in the multi-particle dynamics of Calogero-Moser systems [43]. Reflecting the quantised eigenvalues etc, the corresponding classical quantities showed Diophantine properties at various levels [81]. In particular, the classical equilibrium points of the Calogero-Moser systems were described by the zeros of the classical polynomials, the Hermite, Laguerre and Jacobi [82, 83, 84], which in turn constituted the eigenpolynomials of single particle oQM. The classical equilibrium point, namely the minimum of the classical potential, could be better rephrased by the *maximum of the groundstate eigenfunction* $\phi_0(x) = e^{w(x)}$, or the maximum of the prepotential $w(x)$. Stieltjes investigated the equilibrium problems of multi-particle systems having ‘logarithmic potential’ which was nothing but the corresponding prepotential. Investigation of the discrete counterparts of the Calogero-Moser systems, that is, the Ruijsenaars-Schneider-van-Diejen [46] systems revealed similar Diophantine properties [85]. Later it was discovered that the classical equilibrium points of the Ruijsenaars-Schneider-van-Diejen systems were described by the zeros of the Wilson and Askey-Wilson polynomials [86, 87]. Then it was quite natural to seek for a single particle quantum mechanical formulation in which the Wilson and Askey-Wilson polynomials were the eigenpolynomials. Thus the first discrete quantum mechanics was born [12]. The factorised Hamiltonian was the guiding principle.

The discrete quantum mechanics is still in its infancy. Although the single particle dynamics has been understood rather well by now, the multi-particle dynamics is virtually untouched. A lot of interesting problems, for example, the construction of multi-particle version of various results in this review, are waiting to be clarified.

Due to the length limit, many interesting and important topics could not be covered in this review, for example, the limits from dQM to oQM and those of the corresponding polynomials, continuous ℓ version of XL and XJ, etc. We refer to our papers [7]–[38] and [86] and others.

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Appendix: Symbols and Definitions

Here are several symbols and definitions related to the (q) -hypergeometric functions.

◦ shifted factorial (Pochhammer symbol) $(a)_n$:

$$(a)_n \stackrel{\text{def}}{=} \prod_{k=1}^n (a+k-1) = a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}. \quad (\text{A.1})$$

◦ q -shifted factorial (q -Pochhammer symbol) $(a; q)_n$:

$$(a; q)_n \stackrel{\text{def}}{=} \prod_{k=1}^n (1 - aq^{k-1}) = (1-a)(1-aq) \cdots (1-aq^{n-1}). \quad (\text{A.2})$$

◦ Hypergeometric series ${}_rF_s$:

$${}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| z \right) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r)_n z^n}{(b_1, \dots, b_s)_n n!}, \quad (\text{A.3})$$

where $(a_1, \dots, a_r)_n \stackrel{\text{def}}{=} \prod_{j=1}^r (a_j)_n = (a_1)_n \cdots (a_r)_n$.

◦ q -Hypergeometric series (the basic hypergeometric series) ${}_r\phi_s$:

$${}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z \right) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n} (-1)^{(1+s-r)n} q^{(1+s-r)n(n-1)/2} \frac{z^n}{(q; q)_n}, \quad (\text{A.4})$$

where $(a_1, \dots, a_r; q)_n \stackrel{\text{def}}{=} \prod_{j=1}^r (a_j; q)_n = (a_1; q)_n \cdots (a_r; q)_n$.

References

- [1] Grünbaum F A and Haine L 1997 *Bispectral Darboux transformations: an extension of the Krall polynomials*, IMRN No. 8, 359-392
- [2] Nikiforov A F, Suslov S K and Uvarov V B 1991 *Classical Orthogonal Polynomials of a Discrete Variable* Springer-Verlag, Berlin

- [3] Andrews G E, Askey R and Roy R 1999 *Special Functions* Encyclopedia of mathematics and its applications, Cambridge Univ. Press, Cambridge
- [4] Gasper G and Rahman M 2004 *Basic Hypergeometric Series, second edition* Encyclopedia of mathematics and its applications, Cambridge University Press, Cambridge
- [5] Ismail M E H 2005 *Classical and quantum orthogonal polynomials in one variable* Encyclopedia of mathematics and its applications, Cambridge Univ. Press, Cambridge
- [6] Koekoek R and Swarttouw R F 1996 The Askey-scheme of hypergeometric orthogonal polynomials and its q -analogue (arXiv:math.CA/9602214)
- [7] Odake S and Sasaki R 2009 *Phys. Lett.* **B679** 414-417
- [8] Odake S and Sasaki R 2010 *Phys. Lett.* **B684** 173-176 (Remark: J1(J2) in this reference corresponds to J2(J1) in later references.)
- [9] Ho C-L, Odake S and Sasaki R Properties of the exceptional (X_ℓ) Laguerre and Jacobi polynomials (arXiv:0912.5447[math-ph])
- [10] Odake S and Sasaki R 2009 *Phys. Lett.* **B682** 130-136
- [11] Odake S and Sasaki R 2011 *Prog. Theor. Phys.* **125** 851-870
- [12] Odake S and Sasaki R 2005 *J. Nonlinear Math. Phys.* **12** Suppl. 1 507-521
- [13] Odake S and Sasaki R 2005 *J. Math. Phys.* **46** 063513 (10 pp)
- [14] Odake S and Sasaki R 2006 *J. Math. Phys.* **47** 102102 (33 pp)
- [15] Odake S and Sasaki R 2006 *Phys. Lett.* **B641** 112-117
- [16] Odake S and Sasaki R 2007 *J. Math. Phys.* **48** 082106 (12 pp)
- [17] Odake S and Sasaki R 2007 *J. Math. Phys.* **48** 122105 (8 pp)
- [18] Sasaki R 2007 *J. Math. Phys.* **48** 122104 (11 pp)
- [19] Odake S and Sasaki R 2008 *Phys. Lett.* **B663** 141-145

- [20] Odake S and Sasaki R 2008 *J. Math. Phys.* **49** 053503 (43 pp) (The dual q -Meixner polynomial in § 5.2.4 and dual q -Charlier polynomial in § 5.2.8 should be deleted because the hermiticity of the Hamiltonian is lost for these two cases.)
- [21] Odake S and Sasaki R 2008 *Prog. Theor. Phys.* **119** 663-700
- [22] Sasaki R 2008 *J. Nonlinear Math. Phys.* **15** Suppl. 3 373-384
- [23] Odake S and Sasaki R 2010 *J. Math. Phys.* **51** 083502 (24 pp)
- [24] Odake S and Sasaki R 2009 *Prog. Theor. Phys.* **122** 1067-1079
- [25] García-Gutiérrez L, Odake S and Sasaki R 2010 *Prog. Theor. Phys.* **124** 1-26
- [26] Odake S and Sasaki R 2011 *Prog. Theor. Phys.* **126** 1-34
- [27] Crum M M 1955 *Quart. J. Math.* **6** 121-127
- [28] Krein M G 1957 *Doklady Acad. Nauk. CCCP* **113** 970-973
- [29] Adler V É 1994 *Theor. Math. Phys.* **101** 1381-1386
- [30] Gendenshtein L E 1983 *JETP Lett.* **38** 356-359
- [31] Dabrowska J W, Khare A and Sukhatme U P 1988 *J. Phys. A: Math. Gen.* **21** L195-L200
- [32] Leonard D 1982 *SIAM J. Math. Anal.* **13** 656-663
- [33] Bannai E and Ito T 1984 *Algebraic Combinatorics I: Association Schemes* Benjamin/Cummings, Menlo Park, CA; Terwilliger P 2004 Two linear transformations each tridiagonal with respect to an eigenbasis of the other; an algebraic approach to the Askey scheme of orthogonal polynomials *Lecture notes for the summer school on orthogonal polynomials and special functions, Universidad Carlos III de Madrid, Leganes, Spain, July 8–July 18, 2004* (arXiv:math.QA/0408390)
- [34] Zhedanov A S 1992 *Theor. Math. Phys.* **89** 1146-1157
- [35] Odake S and Sasaki R 2010 *J. Math. Phys.* **51** 053513 (9 pp)
- [36] Sasaki R, Tsujimoto S and Zhedanov A 2010 *J. Phys. A: Math. Theor.* **43** 315204 (20 pp)

- [37] Odake S and Sasaki R 2010 *J. Phys. A: Math. Theor.* **43** 335201 (18 pp)
- [38] Odake S and Sasaki R 2011 *J. Phys. A: Math. Theor.* **44** 195203 (14 pp)
- [39] Sasaki R 2011 *Phil. Trans. R. Soc.* **A369** 1301-1318
- [40] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21-68
- [41] See for example: Miller Jr W 1969 *J. Math. Anal. Appl.* **28** 383-399; Spiridonov V, Vinet L and Zhedanov A 1997 *J. Phys. A: Math. Gen.* **30** 7621-7637; Bangerezako G 1999 *J. Comp. Appl. Math.* **107** 219-232; Bangerezako G and Hounkonnou M N 2001 *J. Phys. A: Math. Gen.* **34** 5653-5666
- [42] Spiridonov V, Vinet L and Zhedanov A 1993 *Lett. Math. Phys.* **29** 63-73
- [43] Calogero F 1971 *J. Math. Phys.* **12** 419-436; Sutherland B 1972 *Phys. Rev.* **A5** 1372-1376; Moser J 1975 *Adv. Math.* **16** 197-220
- [44] Bordner A J, Manton N S and Sasaki R 2000 *Prog. Theor. Phys.* **103** 463-487
- [45] Khastgir S P, Pocklington A J and Sasaki R 2000 *J. Phys. A: Math. Gen.* **A33** 9033-9064
- [46] Ruijsenaars S N M and Schneider H 1986 *Ann. Phys.* **170** 370-405; Ruijsenaars S N M 1987 *Comm. Math. Phys.* **110** 191-213; van Diejen J F 1994 *J. Math. Phys.* **35** 2983-3004; *J. Phys. A: Math. Gen.* **A28** (1995) L369-L374; *J. Math. Phys.* **36** (1995) 1299-1323; *Proceedings of the IV Wigner Symposium* (Guadalajara, 1995), 470-477, World Sci. Publ., River Edge, NJ (1996)
- [47] Degasperis A and Ruijsenaars S N M 2001 *Ann. Phys.* **293** 92-109
- [48] Darboux G 1882 *C. R. Acad. Paris* **94** 1456-1459
- [49] Gaillard P and Matveev V B 2009 *J. Phys. A: Math. Theor.* **A42** 404009 (16 pp)
- [50] Witten E 1981 *Nucl. Phys.* **B188** 513-554
- [51] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267-385
- [52] Andrianov A A, Ioffe M V and Spiridonov V P 1993 *Phys. Lett.* **A174** 273-279
- [53] Klishevich S M and Plyushchay M S 2001 *Nucl. Phys.* **B616** 403-418

- [54] Aoyama H, Sato M and Tanaka T 2001 *Phys. Lett.* **B503** 423-429
- [55] Yermolayeva O and Zhedanov A 1999 *Meth. Appl. Anal.* **6** 1-20
- [56] Risken H 1996 *The Fokker-Planck Equation* (2nd. ed.) Springer-Verlag, Berlin
- [57] Parthasarathy P R and Lenin R B 2004 *Birth and death processes (BDP) models with applications* American Sciences Press, Inc. Columbus, Ohio (2004)
- [58] Sasaki R 2009 *J. Math. Phys.* **50** 103509 (18 pp)
- [59] Chihara T S 1978 *An Introduction to orthogonal polynomials*, Gordon and Breach, New York (1978)
- [60] Nieto M M and Simmons Jr L M 1978 *Phys. Rev. Lett.* **41** 207-210; *Phys. Rev.* **D20** (1979) 1321-1331; *Phys. Rev.* **D20** (1979) 1332-1341; *Phys. Rev.* **D20** (1979) 1342-1350
- [61] Vinet L and Zhedanov A 2008 *SIGMA* **4** 015 (22 pp)
- [62] Fukui T and Aizawa N 1993 *Phys. Lett.* **A180** 308-313; Gazeau J P and Klauder J R 1999 *J. Phys. A: Math. Gen.* **A32** 123-132; Antoine J-P, Gazeau J-P, Monceau P, Klauder J R and Penson K A 2001 *J. Math. Phys.* **42** 2349-2387; Kinani A H El and Daoud M 2001 *Phys. Lett.* **A283** 291-299; *J. Math. Phys.* **43** (2002) 714-733; Alexio A N F and Balantekin A B 2004 *J. Phys. A: Math. Gen.* **A37** 8513-8528
- [63] Atakishiyev N M and Suslov S K 1991 *Theor. Math. Phys.* **85** 1055-1062; Askey R, Atakishiyev N M and Suslov S K 1993 *Symmetries in Science, VI*, edited by Gruber B, Plenum Press, New York, 57-63 (1993); Floreanini R, LeTourneux J and Vinet L 1995 *J. Phys. A: Math. Gen.* **A28** L287-L294; Atakishiyev M N and Klimyk A U 2007 *J. Phys. A: Math. Theor.* **A40** 9311-9317; Borzov V V and Damaskinsky E V 2008 *Theor. Math. Phys.* **155** 536-543
- [64] Granovskii Ya I, Lutzenko I M and Zhedanov A 1992 *Ann. Phys.* **217** 1-20
- [65] Routh E 1884 *Proc. London Math. Soc.* **16** 245-261
- [66] Bochner S 1929 *Math. Zeit.* **29** 730-736

- [67] Gómez-Ullate D, Kamran N and Milson R 2010 *J. Approx Theory* **162** 987-1006; *J. Math. Anal. Appl.* **359** (2009) 352-367
- [68] Quesne C 2008 *J. Phys. A: Math. Gen.* **A41** 392001 (6 pp); Bagchi B, Quesne C and Roychoudhury R 2009 *Pramana J. Phys.* **73** 337-347
- [69] Grünbaum F A and Haine L 1996 *J. Com. Appl. Math.* **68** (1996) 103-114
- [70] Vinet L and Zhedanov A 2008 *J. Comput. Appl. Math.* **211** 45-56
- [71] Morozov A Y, Perelomov A M, Rosly A A, Shifman M A and Turbiner A V 1990 *Int. J. Mod. Phys.* **A5** 803-832
- [72] Turbiner A V 1988 *Comm. Math. Phys.* **118** 467-474
- [73] Ushveridze A G 1988 *Sov. Phys.-Lebedev Inst. Rep.* **2** 50, 54-58
- [74] Sasaki R and Takasaki K 2001 *J. Phys. A: Math. Gen.* **A34** 9533-9553, Corrigendum *J. Phys. A: Math. Gen.* **A34** (2001) 10335
- [75] Gómez-Ullate D, Kamran N and Milson R 2011 On orthogonal polynomials spanning a non-standard flag (arXiv:1101.5584[math-ph])
- [76] Ho C.-L. and Sasaki R 2011 Zeros of the exceptional Laguerre and Jacobi polynomials (arXiv:1102.5669[math-ph])
- [77] Junker G and Roy P 1997 *Phys. Lett.* **A232** 155-161; *Ann. Phys.* **270** (1998) 155-177
- [78] Quesne C 2009 *SIGMA* **5** 084 (24 pp)
- [79] Dutta D and Roy P 2010 *J. Math. Phys.* **51** 042101 (9 pp)
- [80] Gómez-Ullate D, Kamran N and Milson R 2010 *J. Phys. A: Math. Theor.* **A43** 434016 (16 pp)
- [81] Corrigan E and Sasaki R 2002 *J. Phys. A: Math. Gen.* **A35** 7017-7061
- [82] Stieltjes T 1885 *Compt. Rend.* **100** 439-440, 620-622; See §6.7 of Szegő G *Orthogonal polynomials* Amer. Math. Soc. New York (1939), for a readable explanation of the equilibrium problem.

- [83] Calogero F 1977 *Lett. Nuovo Cim.* **19** 505-507; *Lett. Nuovo Cim.* **22** (1977) 251-253
- [84] Calogero F and Perelomov A M 1978 *Commun. Math. Phys.* **59** 109-116
- [85] Ragnisco O and Sasaki R 2004 *J. Phys. A: Math. Gen.* **A37** 469-479
- [86] Odake S and Sasaki R 2004 *J. Phys. A: Math. Gen.* **A37** 11841-11876; *J. Math. Phys.* **46** (2005) 063513 (10 pp); *Prog. Theor. Phys.* **114** (2005) 1245-1260; *Rokko Lectures in Mathematics* (Kobe University) **18** (2005) 85-110
- [87] van Diejen J F 2005 *J. Nonlinear Math. Phys.* **12** Suppl. 1 689-696