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Modification of Crum's Theorem for 'Discrete' Quantum Mechanics

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Crum's theorem in one-dimensional quantum mechanics asserts the existence of an associated Hamiltonian system for any given Hamiltonian with the complete set of eigenvalues and eigenfunctions. The associated system is iso-spectral to the original one except for the lowest energy state, which is deleted. A modification due to Krein-Adler provides algebraic construction of a new complete Hamiltonian system by deleting a finite number of energy levels. Here we present a discrete version of the modification based on Crum's theorem for the 'discrete' quantum mechanics developed by two of the present authors.

Subject Index: 010, 012, 064

§1. Introduction

Crum's seminal paper of 1955[1)] has played an essential role in elucidating the structure of one-dimensional quantum mechanical systems in general and exactly solvable ones, in particular. Throughout this paper, we mean 'exact solvability' in the Schrödinger picture, namely a quantum system is exactly solved when the complete set of eigenvalues and eigenfunctions are known. Many exactly solvable quantum mechanical Hamiltonians were constructed and investigated by combining shape invariance[2)] and Crum's theorem,[1),3)] or the factorisation method[4)] or the method of the so-called supersymmetric quantum mechanics.[5)] It is interesting to note that most of these shape invariant systems are also solvable in the Heisenberg picture.[6)] Exactly solvable quantum mechanical systems of one and many degrees of freedom are not only important in their own right but also have fundamental applications in various disciplines of physics/mathematics, e.g. the Fokker-Planck equations[7)] and their discretised version, birth and death processes,[8)] to name a few.

Shape invariance is a sufficient condition for exactly solvable quantum mechanical systems. The number of shape invariant systems, however, was quite limited; only about a dozen until the recent discovery[9)–11)] of the several types of infinitely many shape invariant Hamiltonians[12)–16)] which led to the infinitely many exceptional Laguerre, Jacobi, Wilson and Askey-Wilson polynomials. Many methods were proposed to derive exactly solvable but non-shape invariant quantum mechanical systems from known shape invariant ones.[17)–21)] (We apologise to those whose work we have missed.) Among them Krein-Adler's modification[18)] of Crum's theorem is the most comprehensive way to generate infinitely many variants of exactly solvable Hamiltonians and their eigenfunctions, starting from an exactly solvable one. The derived system is *iso-spectral* with the original one except that a finite number of energy levels are deleted. If the original system has polynomial eigenfunctions, as is usually the case, the derived systems have also polynomial eigenfunctions. By construction, these polynomials constitute a complete set of orthogonal functions. But they do not qualify to be called *exceptional orthogonal polynomials*[12)–16)] since some members of certain degrees are *missing* due to the *deletion*.

The discrete quantum mechanics is a deformation of the ordinary quantum mechanics in the sense that the Schrödinger equation is a second order *difference* equation instead of differential. In the formulation of Odake and Sasaki,[22)–25)] the algebraic and analytical structure of quantum mechanics as well as shape invariance and exact solvability are retained in the discrete version. The eigenfunctions of the exactly solvable one-dimensional discrete quantum mechanics are the Askey-scheme of hypergeometric orthogonal polynomials and their q-versions,[26)–29)] e.g. the continuous Hahn, the Wilson and the Askey-Wilson polynomials. These examples are all shape invariant and they are also solvable in the Heisenberg picture.[6),22)] The dynamical symmetry algebra of these systems are the Askey-Wilson algebras[30),31)] and degenerations, which contain the q-oscillator algebra.[32)] The discrete version of Crum's theorem was also established recently.[33),34)]

In this paper we present the discrete quantum mechanics version of Adler's modification[18)] of Crum's theorem. It allows to generate an infinite variety of exactly solvable discrete quantum Hamiltonian systems. The insight obtained from Crum's theorems and their modification, in the ordinary and the discrete quantum mechanics, is essential for the recent derivation of the infinite numbers of shape invariant systems and the new exceptional orthogonal polynomials.[12),13),15)] We will discuss the main results, the specialisation to the cases of polynomial eigenfunctions and simplest example for various exactly solvable cases; first for the ordinary quantum mechanics and then for the discrete versions. The reason is two-fold; firstly to introduce appropriate notion and notation in the familiar cases of the ordinary quantum mechanics. Secondly we choose to reveal the underlying logical processes which are not easy to fathom in Adler's paper[18)] or in Crum's original article.[1)] As seen in the subsequent sections, the logical structures of the associated Hamiltonian systems and their modification by *deletion* of energy levels are shared by the ordinary and the discrete quantum mechanics.

This paper is organised as follows. In §2, Adler's modification of Crum's theorem is recapitulated in appropriate notation for our purposes. The specialisation to the cases of polynomial eigenfunctions is discussed in some detail. Section 3 provides the discrete quantum mechanics version of the modification of Crum's theorem. Again the specialisation to the cases of polynomial eigenfunctions is mentioned. Appendix gives the simplest examples of the modified Hamiltonian systems obtained by deleting the lowest lying ℓ excited states for various exactly solvable Hamiltonians. Appendix A provides three examples from the ordinary quantum mechanics, the harmonic oscillator, the radial oscillator, the Darboux-Pöschl-Teller potential. Appendix B is for the four examples from the discrete quantum mechanics, the Hamiltonians of the Meixner-Pollaczek, the continuous Hahn, the Wilson and the Askey-Wilson polynomials,[22),25)] which are known to reduce to the Hermite, the Laguerre and the Jacobi polynomials in certain limits, respectively.

§2. Ordinary quantum mechanics

2.1. *Adler's modification of Crum's theorem*

Let us start with a generic one-dimensional quantum mechanical (QM) system having discrete semi-infinite energy levels only:

$$0 = \mathcal{E}_0 < \mathcal{E}_1 < \mathcal{E}_2 < \cdots . \tag{2·1}$$

Here we have chosen the constant part of the Hamiltonian so that the groundstate energy is zero. Then the Hamiltonian is *positive semi-definite* and can be factorised,

$$\mathcal{H} = p^2 + U(x) = p^2 + \Bigl(\frac{d\mathcal{W}(x)}{dx}\Bigr)^2 + \frac{d^2\mathcal{W}(x)}{dx^2}, \qquad p = -i\frac{d}{dx}, \tag{2·2}$$

$$= \mathcal{A}^\dagger\mathcal{A}, \qquad \mathcal{A} \stackrel{\text{def}}{=} \frac{d}{dx} - \frac{d\mathcal{W}(x)}{dx}, \quad \mathcal{A}^\dagger = -\frac{d}{dx} - \frac{d\mathcal{W}(x)}{dx}. \tag{2·3}$$

Here a real and smooth function $\mathcal{W}(x) \in \mathbb{C}^\infty$ is called a *prepotential* and it parametrises the groundstate wavefunction $\phi_0(x)$, which has *no node* and can be chosen real and positive:

$$\phi_0(x) = e^{\mathcal{W}(x)}. \tag{2·4}$$

It is trivial to verify

$$\mathcal{A}\phi_0(x) = 0 \ \Rightarrow \ \mathcal{H}\phi_0(x) = 0. \tag{2·5}$$

In one dimension all the energy levels are non-degenerate. By construction all the eigenfunctions are square-integrable and orthogonal with each other and form a complete basis of the Hilbert space:

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x), \qquad n \in \mathbb{Z}_+, \tag{2·6}$$

$$\int_{x_1}^{x_2} \phi_n(x)^*\phi_m(x)dx = h_n\delta_{nm}, \quad 0 < h_n < \infty, \quad n, m \in \mathbb{Z}_+, \tag{2·7}$$

where $\mathbb{Z}_+$ is the set of non-negative integers $\{0, 1, 2, \ldots\}$. It is well-known that the n-th excited wavefunction $\phi_n(x)$ has n nodes in the interior. For simplicity we choose all the eigenfunctions to be real. A few exactly solvable examples are given in Appendix.

Let us choose a set of ℓ distinct non-negative integers $\mathcal{D} \stackrel{\text{def}}{=} \{d_1, d_2, \ldots, d_\ell\} \subset \mathbb{Z}_+$, satisfying the condition

$$\prod_{j=1}^{\ell}(m - d_j) \geq 0, \quad \forall m \in \mathbb{Z}_+. \tag{2·8}$$

This condition means that the set $\mathcal{D}$ consists of several clusters, each containing an *even number* of *contiguous* integers

$$d_{k_1}, d_{k_1}+1, \cdots, d_{k_2} \ ; \ d_{k_3}, d_{k_3}+1, \cdots, d_{k_4} \ ; \ d_{k_5}, d_{k_5}+1, \cdots, d_{k_6}, \ ; \ \cdots , \tag{2·9}$$

where $d_{k_2}+1<d_{k_3}$, $d_{k_4}+1<d_{k_5}$, $\cdots$. If $d_{k_1}=0$ for the lowest lying cluster, it could contain an even or odd number of contiguous integers. The set $\mathcal{D}$ specifies the energy levels to be *deleted*. This simply reflects the fact that no singularity arises when two neighbouring levels are deleted. In the ordinary QM, the zeros of the two neighbouring eigenfunctions ϕ_j and ϕ_{j+1} interlace with each other. This fact is essential for the non-singularity of the potential after deletion. See Adler's paper[18] for a proof. The situation is essentially the same in the discrete QM. However, in the dQM to be discussed in the subsequent section, due to the lack of general theorem, the interlacing of the zeros of the two neighbouring eigenfunctions ϕ_j and ϕ_{j+1} must be verified for each specific Hamiltonian. Deleting an arbitrary number of contiguous energy levels starting from the groundstate ($d_{k_1}=0$) is achieved by the original Crum's theorem.[1]

Next we will construct Hamiltonian systems corresponding to the successive deletions $\mathcal{H}_{d_1\ldots}$ (and $\mathcal{A}_{d_1\ldots}$, $\mathcal{A}^\dagger_{d_1\ldots}$, etc.) step by step, algebraically. It should be noted that some quantities in the intermediate steps could be singular. For given d_1 the first Hamiltonian $\mathcal{H}$ can be expressed in two different ways:

$$\mathcal{H}=\mathcal{A}^\dagger\mathcal{A}=\mathcal{A}^\dagger_{d_1}\mathcal{A}_{d_1}+\mathcal{E}_{d_1},\quad \mathcal{A}_{d_1}\phi_{d_1}=0, \tag{2·10}$$

$$\mathcal{A}_{d_1}\stackrel{\text{def}}{=}\frac{d}{dx}-\frac{d\mathcal{W}_{d_1}(x)}{dx},\quad \mathcal{A}^\dagger_{d_1}\stackrel{\text{def}}{=}-\frac{d}{dx}-\frac{d\mathcal{W}_{d_1}(x)}{dx},\quad \mathcal{W}_{d_1}(x)\stackrel{\text{def}}{=}\log\phi_{d_1}(x), \tag{2·11}$$

$$U(x)=\Bigl(\frac{d\mathcal{W}(x)}{dx}\Bigr)^2+\frac{d^2\mathcal{W}(x)}{dx^2}=\Bigl(\frac{d\mathcal{W}_{d_1}(x)}{dx}\Bigr)^2+\frac{d^2\mathcal{W}_{d_1}(x)}{dx^2}+\mathcal{E}_{d_1}. \tag{2·12}$$

Unless $d_1=0$, $\mathcal{W}_{d_1}(x)$ is singular due to the zeros of $\phi_{d_1}(x)$. It is very important to note that $\mathcal{A}^\dagger_{d_1}$ in (2·11) is a 'formal adjoint' of $\mathcal{A}_{d_1}$. We stick to this notation, since the algebraic structure of various expressions appearing in the deletion processes, from (2·10) to (2·34), are best described by using the 'formal adjoint'. These define a new Hamiltonian system

$$\mathcal{H}_{d_1}\stackrel{\text{def}}{=}\mathcal{A}_{d_1}\mathcal{A}^\dagger_{d_1}+\mathcal{E}_{d_1}=p^2+U_{d_1}(x), \tag{2·13}$$

$$U_{d_1}(x)\stackrel{\text{def}}{=}\Bigl(\frac{d\mathcal{W}_{d_1}(x)}{dx}\Bigr)^2-\frac{d^2\mathcal{W}_{d_1}(x)}{dx^2}+\mathcal{E}_{d_1}, \tag{2·14}$$

with the 'eigenfunctions'

$$\mathcal{H}_{d_1}\phi_{d_1\,n}(x)=\mathcal{E}_n\phi_{d_1\,n}(x),\quad \phi_{d_1\,n}(x)\stackrel{\text{def}}{=}\mathcal{A}_{d_1}\phi_n(x),\quad n\in\mathbb{Z}_+\backslash\{d_1\}. \tag{2·15}$$

Note that the energy level d_1 is now deleted, $\phi_{d_1\,d_1}(x)\equiv 0$, from the set of 'eigenfunctions' $\{\phi_{d_1\,n}(x)\}$ of the new Hamiltonian $\mathcal{H}_{d_1}$.

Suppose we have determined $\mathcal{H}_{d_1\ldots d_s}$ and $\phi_{d_1\ldots d_s\,n}(x)$ with s deletions. They have the following properties:

$$\mathcal{H}_{d_1\ldots d_s}\stackrel{\text{def}}{=}\mathcal{A}_{d_1\ldots d_s}\mathcal{A}^\dagger_{d_1\ldots d_s}+\mathcal{E}_{d_s}\stackrel{\text{def}}{=}p^2+U_{d_1\ldots d_s}(x), \tag{2·16}$$

$$\mathcal{W}_{d_1\ldots d_s}(x)\stackrel{\text{def}}{=}\log\phi_{d_1\ldots d_s}(x), \tag{2·17}$$

$$\mathcal{A}_{d_1\ldots d_s}\stackrel{\text{def}}{=}\frac{d}{dx}-\frac{d\mathcal{W}_{d_1\ldots d_s}(x)}{dx},\quad \mathcal{A}^\dagger_{d_1\ldots d_s}\stackrel{\text{def}}{=}-\frac{d}{dx}-\frac{d\mathcal{W}_{d_1\ldots d_s}(x)}{dx}, \tag{2·18}$$

$$U_{d_1\,\ldots\,d_s}(x) \stackrel{\text{def}}{=} \Bigl(\frac{d\mathcal{W}_{d_1\,\ldots\,d_s}(x)}{dx}\Bigr)^2 - \frac{d^2\mathcal{W}_{d_1\,\ldots\,d_s}(x)}{dx^2} + \mathcal{E}_{d_s}, \tag{2·19}$$

$$\phi_{d_1\,\ldots\,d_s\,n}(x) \stackrel{\text{def}}{=} \mathcal{A}_{d_1\,\ldots\,d_s}\phi_{d_1\,\ldots\,d_{s-1}\,n}(x) \quad (n\in\mathbb{Z}_+\backslash\{d_1,\ldots,d_s\}), \tag{2·20}$$

$$\mathcal{H}_{d_1\,\ldots\,d_s}\phi_{d_1\,\ldots\,d_s\,n}(x) = \mathcal{E}_n\phi_{d_1\,\ldots\,d_s\,n}(x) \quad (n\in\mathbb{Z}_+\backslash\{d_1,\ldots,d_s\}). \tag{2·21}$$

We have also

$$\phi_{d_1\,\ldots\,d_{s-1}\,n}(x) = \frac{\mathcal{A}^{\dagger}_{d_1\,\ldots\,d_s}}{\mathcal{E}_n-\mathcal{E}_{d_s}}\phi_{d_1\,\ldots\,d_s\,n}(x) \quad (n\in\mathbb{Z}_+\backslash\{d_1,\ldots,d_s\}). \tag{2·22}$$

Next we will define a new Hamiltonian system with one more deletion of the level d_{s+1}. We can show

$$\mathcal{H}_{d_1\,\ldots\,d_s} = \mathcal{A}^{\dagger}_{d_1\,\ldots\,d_s\,d_{s+1}}\mathcal{A}_{d_1\,\ldots\,d_s\,d_{s+1}} + \mathcal{E}_{d_{s+1}}, \quad \mathcal{A}_{d_1\,\ldots\,d_s\,d_{s+1}}\phi_{d_1\,\ldots\,d_s\,d_{s+1}} = 0, \tag{2·23}$$

$$\mathcal{A}_{d_1\,\ldots\,d_s\,d_{s+1}} \stackrel{\text{def}}{=} \frac{d}{dx} - \frac{d\mathcal{W}_{d_1\,\ldots\,d_s\,d_{s+1}}(x)}{dx}, \quad \mathcal{A}^{\dagger}_{d_1\,\ldots\,d_s\,d_{s+1}} \stackrel{\text{def}}{=} -\frac{d}{dx} - \frac{d\mathcal{W}_{d_1\,\ldots\,d_s\,d_{s+1}}(x)}{dx}, \tag{2·24}$$

$$\mathcal{W}_{d_1\,\ldots\,d_s\,d_{s+1}}(x) \stackrel{\text{def}}{=} \log\phi_{d_1\,\ldots\,d_s\,d_{s+1}}(x), \tag{2·25}$$

$$U_{d_1\,\ldots\,d_s}(x) = \Bigl(\frac{d\mathcal{W}_{d_1\,\ldots\,d_s\,d_{s+1}}(x)}{dx}\Bigr)^2 + \frac{d^2\mathcal{W}_{d_1\,\ldots\,d_s\,d_{s+1}}(x)}{dx^2} + \mathcal{E}_{d_{s+1}}. \tag{2·26}$$

These determine a new Hamiltonian system with $s+1$ deletions:

$$\mathcal{H}_{d_1\,\ldots\,d_{s+1}} \stackrel{\text{def}}{=} \mathcal{A}_{d_1\,\ldots\,d_{s+1}}\mathcal{A}^{\dagger}_{d_1\,\ldots\,d_{s+1}} + \mathcal{E}_{d_{s+1}} \stackrel{\text{def}}{=} p^2 + U_{d_1\,\ldots\,d_{s+1}}(x), \tag{2·27}$$

$$U_{d_1\,\ldots\,d_{s+1}}(x) \stackrel{\text{def}}{=} \Bigl(\frac{d\mathcal{W}_{d_1\,\ldots\,d_{s+1}}(x)}{dx}\Bigr)^2 - \frac{d^2\mathcal{W}_{d_1\,\ldots\,d_{s+1}}(x)}{dx^2} + \mathcal{E}_{d_{s+1}}, \tag{2·28}$$

$$\phi_{d_1\,\ldots\,d_{s+1}\,n}(x) \stackrel{\text{def}}{=} \mathcal{A}_{d_1\,\ldots\,d_{s+1}}\phi_{d_1\,\ldots\,d_s\,n}(x) \quad (n\in\mathbb{Z}_+\backslash\{d_1,\ldots,d_{s+1}\}), \tag{2·29}$$

$$\mathcal{H}_{d_1\,\ldots\,d_{s+1}}\phi_{d_1\,\ldots\,d_{s+1}\,n}(x) = \mathcal{E}_n\phi_{d_1\,\ldots\,d_{s+1}\,n}(x) \quad (n\in\mathbb{Z}_+\backslash\{d_1,\ldots,d_{s+1}\}). \tag{2·30}$$

After deleting all the $\mathcal{D}=\{d_1,\cdots,d_\ell\}$ energy levels, the resulting Hamiltonian system $\mathcal{H}_{\mathcal{D}}\equiv\mathcal{H}_{d_1\,\ldots\,d_\ell}$, $\mathcal{A}_{\mathcal{D}}\equiv\mathcal{A}_{d_1\,\ldots\,d_\ell}$, etc has the following form:

$$\mathcal{H}_{\mathcal{D}} \stackrel{\text{def}}{=} \mathcal{A}_{\mathcal{D}}\mathcal{A}^{\dagger}_{\mathcal{D}} + \mathcal{E}_{d_\ell} \stackrel{\text{def}}{=} p^2 + U_{\mathcal{D}}(x), \tag{2·31}$$

$$U_{\mathcal{D}}(x) \stackrel{\text{def}}{=} \Bigl(\frac{d\mathcal{W}_{\mathcal{D}}(x)}{dx}\Bigr)^2 - \frac{d^2\mathcal{W}_{\mathcal{D}}(x)}{dx^2} + \mathcal{E}_{d_\ell}, \quad \mathcal{W}_{\mathcal{D}}(x) \stackrel{\text{def}}{=} \log\phi_{d_1\,\cdots\,d_\ell}(x), \tag{2·32}$$

$$\phi_{\mathcal{D}\,n}(x) \stackrel{\text{def}}{=} \mathcal{A}_{\mathcal{D}}\phi_{d_1\,\cdots\,d_{\ell-1}\,n}(x) \quad (n\in\mathbb{Z}_+\backslash\mathcal{D}), \tag{2·33}$$

$$\mathcal{H}_{\mathcal{D}}\phi_{\mathcal{D}\,n}(x) = \mathcal{E}_n\phi_{\mathcal{D}\,n}(x) \quad (n\in\mathbb{Z}_+\backslash\mathcal{D}). \tag{2·34}$$

Now that $\mathcal{H}_{\mathcal{D}}$ has the lowest energy level μ:

$$\mu \stackrel{\text{def}}{=} \min\{n\,|\,n\in\mathbb{Z}_+\backslash\mathcal{D}\}, \tag{2·35}$$

with the groundstate wavefunction $\bar{\phi}_\mu(x)$

$$\bar{\phi}_\mu(x) \stackrel{\text{def}}{=} \phi_{\mathcal{D}\,\mu}(x) \equiv \phi_{d_1\,\cdots\,d_\ell\,\mu}(x). \tag{2·36}$$

As usual the Hamiltonian system can be expressed simply in terms of the groundstate wavefunction $\bar{\phi}_\mu(x)$, which we will denote by new symbols $\bar{\mathcal{H}}$, $\bar{\mathcal{A}}$, etc:

$$\bar{\mathcal{H}} \equiv \mathcal{H}_{\mathcal{D}} \stackrel{\text{def}}{=} \bar{\mathcal{A}}^\dagger \bar{\mathcal{A}} + \mathcal{E}_\mu \stackrel{\text{def}}{=} p^2 + \bar{U}(x), \tag{2·37}$$

$$\bar{\mathcal{A}} \equiv \mathcal{A}_{\mathcal{D}\,\mu} \stackrel{\text{def}}{=} \frac{d}{dx} - \frac{d\bar{\mathcal{W}}(x)}{dx}, \quad \bar{\mathcal{A}}^\dagger \equiv \mathcal{A}_{\mathcal{D}\,\mu}^\dagger \stackrel{\text{def}}{=} -\frac{d}{dx} - \frac{d\bar{\mathcal{W}}(x)}{dx}, \tag{2·38}$$

$$\bar{U}(x) \equiv U_{\mathcal{D}\,\mu}(x) \stackrel{\text{def}}{=} \Bigl(\frac{d\bar{\mathcal{W}}(x)}{dx}\Bigr)^2 + \frac{d^2\bar{\mathcal{W}}(x)}{dx^2} + \mathcal{E}_\mu, \quad \bar{\mathcal{W}}(x) \equiv \mathcal{W}_{\mathcal{D}\,\mu}(x) \stackrel{\text{def}}{=} \log \bar{\phi}_\mu(x), \tag{2·39}$$

$$\bar{\mathcal{H}}\bar{\phi}_n(x) = \mathcal{E}_n\bar{\phi}_n(x), \quad \bar{\phi}_n(x) \equiv \phi_{\mathcal{D}\,n}(x) \qquad (n \in \mathbb{Z}_+\backslash\mathcal{D}). \tag{2·40}$$

As shown by Krein-Adler,[18] the results can be expressed succinctly:

$$\bar{\phi}_n(x) = \frac{\mathrm{W}[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_\ell}, \phi_n](x)}{\mathrm{W}[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_\ell}](x)} \quad (n \in \mathbb{Z}_+\backslash\mathcal{D}), \tag{2·41}$$

$$\bar{U}(x) \equiv U_{d_1,\ldots,d_\ell}(x) = U(x) - 2\frac{d^2}{dx^2}\Bigl(\log \mathrm{W}\,[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_\ell}](x)\Bigr) \quad (\ell \geq 0), \tag{2·42}$$

in which the Wronskian determinant is defined by

$$\mathrm{W}\,[f_1, \ldots, f_n](x) \stackrel{\text{def}}{=} \det\Bigl(\frac{d^{j-1}f_k(x)}{dx^{j-1}}\Bigr)_{1\leq j,k\leq n}. \tag{2·43}$$

For $n = 0$, we set $\mathrm{W}\,[\cdot](x) = 1$. In deriving the determinant formulas (2·41) and (2·42) use is made of the properties of the Wronskian

$$\mathrm{W}[gf_1, gf_2, \ldots, gf_n](x) = g(x)^n \mathrm{W}[f_1, f_2, \ldots, f_n](x), \tag{2·44}$$

$$\begin{aligned}&\mathrm{W}\bigl[\mathrm{W}[f_1, f_2, \ldots, f_n, g], \mathrm{W}[f_1, f_2, \ldots, f_n, h]\,\bigr](x)\\ &= \mathrm{W}[f_1, f_2, \ldots, f_n](x)\,\mathrm{W}[f_1, f_2, \ldots, f_n, g, h](x) \qquad (n \geq 0).\end{aligned} \tag{2·45}$$

Let us note that $U_{d_1\,\ldots\,d_\ell}(x)$ and $\phi_{d_1,\ldots\,d_\ell\,n}(x)$ are symmetric with respect to $d_1, \ldots, d_\ell$, and thus $\bar{\mathcal{H}} \equiv \mathcal{H}_{d_1\,\ldots\,d_\ell}$ is independent of the order of $\{d_j\}$.

Let us state Adler's theorem again; *If the set of deleted energy levels* $d_1, \ldots, d_\ell$ *satisfy the condition* (2·8)*, the Hamiltonian* $\bar{\mathcal{H}} \equiv \mathcal{H}_{d_1\,\ldots\,d_\ell} = p^2 + \bar{U}(x)$ *with* (2·42) *is well-defined and hermitian, and its complete set of eigenfunctions* (2·40) *are given by* (2·41). Crum's theorem corresponds to the choice $\{d_1, \ldots, d_\ell\} = \{0, 1, \ldots, \ell - 1\}$, and the resulting lowest energy level is $\mu = \ell$ and there is no deleted energy levels above the new groundstate.

2.2. *Polynomial eigenfunctions*

In this subsection we consider the typical case of shape invariant systems in which the eigenfunctions consist of the orthogonal polynomials $\{P_n\}$:

$$\phi_n(x) = \phi_0(x)P_n(\eta(x)), \quad \phi_0(x) = e^{\mathcal{W}(x)}, \tag{2·46}$$

in which $\eta(x)$ is called the *sinusoidal coordinate*. As shown in detail in the examples in Appendix A, $\eta(x) = x$ for the harmonic oscillator (the Hermite polynomials),

$\eta(x) = x^2$ for the radial oscillator (the Laguerre polynomials) and $\eta(x) = \cos 2x$ for the Darboux-Pöschl-Teller potential (the Jacobi polynomials). The groundstate wavefunction $\phi_0(x)$ provides the orthogonality weight function

$$\int_{x_1}^{x_2} \phi_0(x)^2 \, P_n(\eta(x)) P_m(\eta(x)) dx = h_n \delta_{nm}, \quad n, m \in \mathbb{Z}_+. \tag{2·47}$$

In this case, the modification of Crum's theorem produces the eigenfunctions $\{\bar{\phi}_n(x)\}$ which again consist of polynomials in $\eta(x)$. By using (2·44) and

$$\check{f}_j(x) \stackrel{\text{def}}{=} f_j(\eta(x)), \quad \mathrm{W}[\check{f}_1, \check{f}_2, \ldots, \check{f}_n](x) = \bigl(\tfrac{d\eta(x)}{dx}\bigr)^{\frac{1}{2}n(n-1)} \mathrm{W}[f_1, f_2, \ldots, f_n](\eta(x)), \tag{2·48}$$

we obtain a simple expression of the eigenfunctions

$$\bar{\phi}_n(x) = \phi_0(x) \bigl(\tfrac{d\eta(x)}{dx}\bigr)^{\ell} \frac{\mathrm{W}[P_{d_1}, P_{d_2}, \ldots, P_{d_\ell}, P_n](\eta(x))}{\mathrm{W}[P_{d_1}, P_{d_2}, \ldots, P_{d_\ell}](\eta(x))}. \tag{2·49}$$

This simply means that the resulting eigenfunctions are again polynomials in $\eta(x)$:

$$\bar{\phi}_n(x) = \bar{\psi}(x) \mathcal{P}_n(\eta(x)), \tag{2·50}$$

$$\bar{\psi}(x) \stackrel{\text{def}}{=} \frac{\phi_0(x)\bigl(\frac{d\eta(x)}{dx}\bigr)^{\ell}}{\mathrm{W}[P_{d_1}, P_{d_2}, \ldots, P_{d_\ell}](\eta(x))}, \quad \mathcal{P}_n(\eta) \stackrel{\text{def}}{=} \mathrm{W}[P_{d_1}, P_{d_2}, \ldots, P_{d_\ell}, P_n](\eta), \tag{2·51}$$

satisfying the orthogonality relation

$$\int_{x_1}^{x_2} \bar{\psi}(x)^2 \, \mathcal{P}_n(\eta(x)) \mathcal{P}_m(\eta(x)) dx = \bar{h}_n \delta_{nm}, \quad n, m \in \mathbb{Z}_+ \backslash \mathcal{D}. \tag{2·52}$$

Let us emphasise that n is not the *degree* in η and by construction, ℓ members are missing: $\mathcal{P}_{d_1} = \mathcal{P}_{d_2} = \cdots = \mathcal{P}_{d_\ell} \equiv 0$. Therefore these polynomials cannot be called *exceptional orthogonal polynomials*.[9),12),15)]

§3. 'Discrete' quantum mechanics

Let us begin with a few general remarks on the one-dimensional discrete QM with pure imaginary shifts. See 25) for the general introduction to the discrete quantum mechanics with pure imaginary shifts and 34) for Crum's theorem in the discrete QM. In the discrete QM, the dynamical variables are, as in the ordinary QM, the coordinate x, which takes value in an infinite or a semi-infinite or a finite range of the real axis and the canonical momentum p, which is realised as a differential operator $p = -i\partial_x$. Since the momentum operator appears in exponentiated forms $e^{\pm\gamma p}$, $\gamma \in \mathbb{R}$, in a Hamiltonian, it causes finite pure imaginary shifts in the wavefunction $e^{\pm\gamma p}\psi(x) = \psi(x \mp i\gamma)$. This requires the wavefunction as well as other functions appearing in the Hamiltonian to be *analytic* in x within a certain complex domain including the physical region of the coordinate. Let us introduce the $*$-operation on an analytic function, $* : f \mapsto f^*$. If $f(x) = \sum_n a_n x^n$, $a_n \in \mathbb{C}$, then $f^*(x) \stackrel{\text{def}}{=} \sum_n a_n^* x^n$, in which a_n^* is the complex conjugation of a_n. Obviously $f^{**}(x) = f(x)$ and $f(x)^* = f^*(x^*)$. If a function satisfies $f^* = f$, we call it a 'real' function, for it takes real values on the real line.

The starting point is again a generic one dimensional discrete quantum mechanical Hamiltonian with discrete semi-infinite energy levels only (2·1). Again we assume that the groundstate energy is chosen to be zero $\mathcal{E}_0 = 0$, so that the Hamiltonian is positive semi-definite. The generic factorised Hamiltonian reads[22),25)]

$$\mathcal{H} = \mathcal{A}^\dagger\mathcal{A} = \sqrt{V(x)}\,e^{\gamma p}\sqrt{V^*(x)} + \sqrt{V^*(x)}\,e^{-\gamma p}\sqrt{V(x)} - V(x) - V^*(x), \tag{3·1}$$

$$\mathcal{A} \stackrel{\text{def}}{=} i\bigl(e^{\frac{\gamma}{2}p}\sqrt{V^*(x)} - e^{-\frac{\gamma}{2}p}\sqrt{V(x)}\,\bigr), \quad \mathcal{A}^\dagger \stackrel{\text{def}}{=} -i\bigl(\sqrt{V(x)}\,e^{\frac{\gamma}{2}p} - \sqrt{V^*(x)}\,e^{-\frac{\gamma}{2}p}\bigr). \tag{3·2}$$

Since the $*$-operation for $\mathcal{A}f$, $\mathcal{A}^\dagger f$ and $\mathcal{H}f$ satisfies

$$(\mathcal{A}f)^*(x) = \mathcal{A}f^*(x), \quad (\mathcal{A}^\dagger f)^*(x) = \mathcal{A}^\dagger f^*(x), \quad (\mathcal{H}f)^*(x) = \mathcal{H}f^*(x), \tag{3·3}$$

they map a 'real' function to a 'real' function

$$f^* = f \quad \Rightarrow \quad (\mathcal{A}f)^* = \mathcal{A}f, \;\; (\mathcal{A}^\dagger f)^* = \mathcal{A}^\dagger f, \;\; (\mathcal{H}f)^* = \mathcal{H}f. \tag{3·4}$$

By specifying the function $V(x)$, various explicit examples are obtained.[6),22),25)] A few exactly solvable examples are given in Appendix. The corresponding Schrödinger equation $\mathcal{H}\psi(x) = \mathcal{E}\psi(x)$ is a *difference equation*

$$\begin{aligned}\sqrt{V(x)V^*(x-i\gamma)}\,\psi(x-i\gamma) + \sqrt{V^*(x)V(x+i\gamma)}\,\psi(x+i\gamma)&\\ - \bigl(V(x)+V^*(x)\bigr)\psi(x) = \mathcal{E}\psi(x),&\end{aligned} \tag{3·5}$$

instead of differential in the ordinary QM. Although this equation looks rather complicated, the equation for the polynomial eigenfunctions (3·47) has a familiar form of difference equations. Again the groundstate wavefunction $\phi_0(x)$ is determined as a zero mode of $\mathcal{A}$, $\mathcal{A}\phi_0(x) = 0$ ($\Rightarrow \mathcal{H}\phi_0(x) = 0$), namely,

$$\sqrt{V^*(x - i\tfrac{\gamma}{2})}\phi_0(x - i\tfrac{\gamma}{2}) - \sqrt{V(x + i\tfrac{\gamma}{2})}\phi_0(x + i\tfrac{\gamma}{2}) = 0. \tag{3·6}$$

This dictates how the 'phase' of the potential function V is related to that of the groundstate wavefunction ϕ_0. Here we also assume that the groundstate wavefunction $\phi_0(x)$ has no node and chosen to be real and positive for real x.

Due to the lack of generic theorems in the theory of difference equations, let us assume that all the energy levels are non-degenerate and that all the eigenfunctions are square-integrable and orthogonal with each other and form a complete basis of the Hilbert space:

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x), \qquad n \in \mathbb{Z}_+, \tag{3·7}$$

$$\int_{x_1}^{x_2}\phi_n(x)^*\phi_m(x)dx = h_n\delta_{nm}, \quad 0 < h_n < \infty, \quad n, m \in \mathbb{Z}_+. \tag{3·8}$$

In most explicit examples these statements can be verified straightforwardly. For simplicity we choose all the eigenfunctions to be real on the real axis $\phi_n^* = \phi_n$, which is made possible by (3·4).

3.1. *Modification of Crum's theorem*

The formulation of the modified Crum's theorem in the discrete quantum mechanics goes almost parallel to that in the ordinary quantum mechanics. Again the presentation is purely algebraic. Let us note that various quantities in intermediate steps might have singularities and Hamiltonians might not be hermitian. We choose a set of distinct non-negative integers $\mathcal{D} \stackrel{\text{def}}{=} \{d_1, d_2, \ldots, d_\ell\} \subset \mathbb{Z}_+$, satisfying the condition (2·8) as before. First let us note that the Hamiltonian $\mathcal{H}$ can be rewritten by incorporating the level d_1 as:

$$\mathcal{H} = \mathcal{A}_{d_1}^\dagger \mathcal{A}_{d_1} + \mathcal{E}_{d_1}, \quad \mathcal{A}_{d_1}\phi_{d_1} = 0, \tag{3·9}$$

$$\mathcal{A}_{d_1} \stackrel{\text{def}}{=} i\Big(e^{\frac{\gamma}{2}p}\sqrt{V_{d_1}^*(x)} - e^{-\frac{\gamma}{2}p}\sqrt{V_{d_1}(x)}\,\Big),\ \mathcal{A}_{d_1}^\dagger \stackrel{\text{def}}{=} -i\Big(\sqrt{V_{d_1}(x)}\,e^{\frac{\gamma}{2}p} - \sqrt{V_{d_1}^*(x)}\,e^{-\frac{\gamma}{2}p}\Big), \tag{3·10}$$

$$V_{d_1}(x) \stackrel{\text{def}}{=} \sqrt{V(x)V^*(x-i\gamma)}\,\frac{\phi_{d_1}(x-i\gamma)}{\phi_{d_1}(x)}. \tag{3·11}$$

These define a new Hamiltonian system

$$\mathcal{H}_{d_1} \stackrel{\text{def}}{=} \mathcal{A}_{d_1}\mathcal{A}_{d_1}^\dagger + \mathcal{E}_{d_1}, \tag{3·12}$$

$$\mathcal{H}_{d_1}\phi_{d_1\,n}(x) = \mathcal{E}_n\phi_{d_1\,n}(x), \quad \phi_{d_1\,n}(x) \stackrel{\text{def}}{=} \mathcal{A}_{d_1}\phi_n(x), \quad n \in \mathbb{Z}_+\backslash\{d_1\}. \tag{3·13}$$

Note that the energy level d_1 is now deleted, $\phi_{d_1\,d_1}(x) \equiv 0$, from the set of 'eigenfunctions' $\{\phi_{d_1\,n}(x)\}$ of the new Hamiltonian $\mathcal{H}_{d_1}$.

Suppose we have determined $\mathcal{H}_{d_1\ldots d_s}$ and $\phi_{d_1\ldots d_s\,n}(x)$ with s deletions. They have the following properties

$$\mathcal{H}_{d_1\ldots d_s} \stackrel{\text{def}}{=} \mathcal{A}_{d_1\ldots d_s}\mathcal{A}_{d_1\ldots d_s}^\dagger + \mathcal{E}_{d_s}, \tag{3·14}$$

$$\mathcal{A}_{d_1\ldots d_s} \stackrel{\text{def}}{=} i\Big(e^{\frac{\gamma}{2}p}\sqrt{V_{d_1\ldots d_s}^*(x)} - e^{-\frac{\gamma}{2}p}\sqrt{V_{d_1\ldots d_s}(x)}\,\Big),$$

$$\mathcal{A}_{d_1\ldots d_s}^\dagger \stackrel{\text{def}}{=} -i\Big(\sqrt{V_{d_1\ldots d_s}(x)}\,e^{\frac{\gamma}{2}p} - \sqrt{V_{d_1\ldots d_s}^*(x)}\,e^{-\frac{\gamma}{2}p}\Big), \tag{3·15}$$

$$V_{d_1\ldots d_s}(x) \stackrel{\text{def}}{=} \begin{cases} \sqrt{V_{d_1\ldots d_{s-1}}(x-i\frac{\gamma}{2})V_{d_1\ldots d_{s-1}}^*(x-i\frac{\gamma}{2})}\,\dfrac{\phi_{d_1\ldots d_s}(x-i\gamma)}{\phi_{d_1\ldots d_s}(x)} & (s\geq 2),\\[2ex] \sqrt{V(x)V^*(x-i\gamma)}\,\dfrac{\phi_{d_1}(x-i\gamma)}{\phi_{d_1}(x)} & (s=1), \end{cases} \tag{3·16}$$

$$\phi_{d_1\ldots d_s\,n}(x) \stackrel{\text{def}}{=} \mathcal{A}_{d_1\ldots d_s}\phi_{d_1\ldots d_{s-1}\,n}(x), \quad \phi_{d_1\ldots d_s\,n}(x) = \phi_{d_1\ldots d_s\,n}^*(x), \tag{3·17}$$

$$\mathcal{H}_{d_1\ldots d_s}\phi_{d_1\ldots d_s\,n}(x) = \mathcal{E}_n\phi_{d_1\ldots d_s\,n}(x), \tag{3·18}$$

where $n \in \mathbb{Z}_+\backslash\{d_1,\ldots,d_s\}$. We have also

$$\phi_{d_1\ldots d_{s-1}\,n}(x) = \frac{\mathcal{A}_{d_1\ldots d_s}^\dagger}{\mathcal{E}_n - \mathcal{E}_{d_s}}\phi_{d_1\ldots d_s\,n}(x) \quad (n \in \mathbb{Z}_+\backslash\{d_1,\ldots,d_s\}). \tag{3·19}$$

Next we will define a new Hamiltonian system with one more deletion of the level d_{s+1}. We can show the following:

$$\mathcal{H}_{d_1\ldots d_s}=\mathcal{A}^\dagger_{d_1\ldots d_s\,d_{s+1}}\mathcal{A}_{d_1\ldots d_s\,d_{s+1}}+\mathcal{E}_{d_{s+1}},\quad \mathcal{A}_{d_1\ldots d_s\,d_{s+1}}\phi_{d_1\ldots d_s\,d_{s+1}}(x)=0, \tag{3·20}$$

$$\mathcal{A}_{d_1\ldots d_s\,d_{s+1}}\stackrel{\text{def}}{=}i\Big(e^{\frac{\gamma}{2}p}\sqrt{V^*_{d_1\ldots d_s\,d_{s+1}}(x)}-e^{-\frac{\gamma}{2}p}\sqrt{V_{d_1\ldots d_s\,d_{s+1}}(x)}\,\Big),$$

$$\mathcal{A}^\dagger_{d_1\ldots d_s\,d_{s+1}}\stackrel{\text{def}}{=}-i\Big(\sqrt{V_{d_1\ldots d_s\,d_{s+1}}(x)}\,e^{\frac{\gamma}{2}p}-\sqrt{V^*_{d_1\ldots d_s\,d_{s+1}}(x)}\,e^{-\frac{\gamma}{2}p}\Big), \tag{3·21}$$

$$V_{d_1\ldots d_s,d_{s+1}}(x)\stackrel{\text{def}}{=}\sqrt{V_{d_1\ldots d_s}(x-i\tfrac{\gamma}{2})V^*_{d_1\ldots d_s}(x-i\tfrac{\gamma}{2})}\,\frac{\phi_{d_1\ldots d_s\,d_{s+1}}(x-i\gamma)}{\phi_{d_1\ldots d_s\,d_{s+1}}(x)}. \tag{3·22}$$

These determine a new Hamiltonian system with $s+1$ deletions:

$$\mathcal{H}_{d_1\ldots d_{s+1}}\stackrel{\text{def}}{=}\mathcal{A}_{d_1\ldots d_{s+1}}\mathcal{A}^\dagger_{d_1\ldots d_{s+1}}+\mathcal{E}_{d_{s+1}}, \tag{3·23}$$

$$\phi_{d_1\ldots d_{s+1}\,n}(x)\stackrel{\text{def}}{=}\mathcal{A}_{d_1\ldots d_{s+1}}\phi_{d_1\ldots d_s\,n}(x),\quad \phi_{d_1\ldots d_{s+1}\,n}(x)=\phi^*_{d_1\ldots d_{s+1}\,n}(x), \tag{3·24}$$

$$\mathcal{H}_{d_1\ldots d_{s+1}}\phi_{d_1\ldots d_{s+1}\,n}(x)=\mathcal{E}_n\phi_{d_1\ldots d_{s+1}\,n}(x), \tag{3·25}$$

where $n\in\mathbb{Z}_+\backslash\{d_1,\ldots,d_{s+1}\}$.

After deleting all the $\mathcal{D}=\{d_1,\cdots,d_\ell\}$ energy levels, the resulting Hamiltonian system $\mathcal{H}_\mathcal{D}\equiv\mathcal{H}_{d_1\ldots d_\ell}$, $\mathcal{A}_\mathcal{D}\equiv\mathcal{A}_{d_1\ldots d_\ell}$, etc. has the following form:

$$\mathcal{H}_\mathcal{D}\stackrel{\text{def}}{=}\mathcal{A}_\mathcal{D}\mathcal{A}^\dagger_\mathcal{D}+\mathcal{E}_{d_\ell}, \tag{3·26}$$

$$\mathcal{A}_\mathcal{D}\stackrel{\text{def}}{=}i\Big(e^{\frac{\gamma}{2}p}\sqrt{V^*_\mathcal{D}(x)}-e^{-\frac{\gamma}{2}p}\sqrt{V_\mathcal{D}(x)}\Big),\quad \mathcal{A}^\dagger_\mathcal{D}\stackrel{\text{def}}{=}-i\Big(\sqrt{V_\mathcal{D}(x)}\,e^{\frac{\gamma}{2}p}-\sqrt{V^*_\mathcal{D}(x)}\,e^{-\frac{\gamma}{2}p}\Big), \tag{3·27}$$

$$V_\mathcal{D}(x)\stackrel{\text{def}}{=}\sqrt{V_{d_1\ldots d_{\ell-1}}(x-i\tfrac{\gamma}{2})V^*_{d_1\ldots d_{\ell-1}}(x-i\tfrac{\gamma}{2})}\,\frac{\phi_\mathcal{D}(x-i\gamma)}{\phi_\mathcal{D}(x)}, \tag{3·28}$$

$$\phi_{\mathcal{D}\,n}(x)\stackrel{\text{def}}{=}\mathcal{A}_\mathcal{D}\phi_{d_1\ldots d_{\ell-1}\,n}(x),\quad \phi_{\mathcal{D}\,n}(x)=\phi^*_{\mathcal{D}\,n}(x),\quad (n\in\mathbb{Z}_+\backslash\mathcal{D}), \tag{3·29}$$

$$\mathcal{H}_\mathcal{D}\phi_{\mathcal{D}\,n}(x)=\mathcal{E}_n\phi_{\mathcal{D}\,n}(x)\quad (n\in\mathbb{Z}_+\backslash\mathcal{D}). \tag{3·30}$$

Now that $\mathcal{H}_\mathcal{D}$ has the lowest energy level μ:

$$\mu\stackrel{\text{def}}{=}\min\{n\,|\,n\in\mathbb{Z}_+\backslash\mathcal{D}\}, \tag{3·31}$$

with the groundstate wavefunction $\bar{\phi}_\mu(x)$

$$\bar{\phi}_\mu(x)\stackrel{\text{def}}{=}\phi_{\mathcal{D}\,\mu}(x)\equiv\phi_{d_1\cdots d_\ell\,\mu}(x). \tag{3·32}$$

Then the Hamiltonian system can be expressed simply in terms of the groundstate wavefunction $\bar{\phi}_\mu(x)$, which we will denote by new symbols $\bar{\mathcal{H}}$, $\bar{\mathcal{A}}$, etc.:

$$\bar{\mathcal{H}}\equiv\mathcal{H}_\mathcal{D}\stackrel{\text{def}}{=}\bar{\mathcal{A}}^\dagger\bar{\mathcal{A}}+\mathcal{E}_\mu,\qquad \bar{\mathcal{A}}\bar{\phi}_\mu(x)=0, \tag{3·33}$$

$$\bar{\mathcal{A}}\equiv\mathcal{A}_{\mathcal{D}\,\mu}\stackrel{\text{def}}{=}i\Big(e^{\frac{\gamma}{2}p}\sqrt{\bar{V}^*(x)}-e^{-\frac{\gamma}{2}p}\sqrt{\bar{V}(x)}\Big),$$

$$\bar{\mathcal{A}}^\dagger \equiv \mathcal{A}^\dagger_{\mathcal{D}\,\mu} \stackrel{\text{def}}{=} -i\Big(\sqrt{\bar{V}(x)}\, e^{\frac{\gamma}{2}p} - \sqrt{\bar{V}^*(x)}\, e^{-\frac{\gamma}{2}p}\Big), \tag{3·34}$$

$$\bar{V}(x) \equiv V_{\mathcal{D}\,\mu}(x) \stackrel{\text{def}}{=} \sqrt{V_{\mathcal{D}}(x - i\tfrac{\gamma}{2})V^*_{\mathcal{D}}(x - i\tfrac{\gamma}{2})}\, \frac{\bar{\phi}_\mu(x - i\gamma)}{\bar{\phi}_\mu(x)}, \tag{3·35}$$

$$\bar{\mathcal{H}}\bar{\phi}_n(x) = \mathcal{E}_n\bar{\phi}_n(x), \quad \bar{\phi}_n(x) \equiv \phi_{\mathcal{D}\,n}(x) \qquad (n \in \mathbb{Z}_+\backslash\mathcal{D}). \tag{3·36}$$

The discrete counterpart of the determinant formulas (2·41)–(2·42) requires a deformation of the Wronskian, the Casorati determinant, which has a good limiting property:

$$\mathrm{W}_\gamma[f_1, \ldots, f_n](x) \stackrel{\text{def}}{=} i^{\frac{1}{2}n(n-1)} \det\Big(f_k(x + i\tfrac{n+1-2j}{2}\gamma)\Big)_{1\le j,k\le n}, \tag{3·37}$$

$$\lim_{\gamma\to 0} \gamma^{-\frac{1}{2}n(n-1)}\mathrm{W}_\gamma[f_1, f_2, \ldots, f_n](x) = \mathrm{W}\,[f_1, f_2, \ldots, f_n](x), \tag{3·38}$$

(for $n = 0$, we set $\mathrm{W}_\gamma\,[\cdot](x) = 1$.). It satisfies

$$\mathrm{W}_\gamma[f_1, \ldots, f_n]^*(x) = \mathrm{W}_\gamma[f_1^*, \ldots, f_n^*](x), \tag{3·39}$$

$$\mathrm{W}_\gamma[gf_1, gf_2, \ldots, gf_n] = \prod_{j=1}^{n} g(x + i\tfrac{n+1-2j}{2}\gamma) \cdot \mathrm{W}_\gamma[f_1, f_2, \ldots, f_n](x), \tag{3·40}$$

$$\begin{aligned}&\mathrm{W}_\gamma\big[\mathrm{W}_\gamma[f_1, f_2, \ldots, f_n, g], \mathrm{W}_\gamma[f_1, f_2, \ldots, f_n, h]\,\big](x)\\ &= \mathrm{W}_\gamma[f_1, f_2, \ldots, f_n](x)\,\mathrm{W}_\gamma[f_1, f_2, \ldots, f_n, g, h](x) \quad (n \ge 0).\end{aligned} \tag{3·41}$$

By using the Casorati determinant we obtain ($\ell \ge 0$)

$$\bar{\phi}_n(x) \equiv \phi_{d_1\ldots d_\ell\, n}(x) = \sqrt{\prod_{j=1}^{\ell} V_{d_1\ldots d_j}(x + i\tfrac{\ell+1-j}{2}\gamma)}\; \frac{\mathrm{W}_\gamma\,[\phi_{d_1}, \ldots, \phi_{d_\ell}, \phi_n](x)}{\mathrm{W}_\gamma\,[\phi_{d_1}, \ldots, \phi_{d_\ell}](x - i\frac{\gamma}{2})}, \tag{3·42}$$

$$\begin{aligned}\bar{V}(x) \equiv V_{d_1\ldots d_\ell\,\mu}(x) &= \sqrt{V(x - i\tfrac{\ell}{2}\gamma)V^*(x - i\tfrac{\ell+2}{2}\gamma)}\\ &\times \frac{\mathrm{W}_\gamma\,[\phi_{d_1}, \ldots, \phi_{d_\ell}](x + i\frac{\gamma}{2})}{\mathrm{W}_\gamma\,[\phi_{d_1}, \ldots, \phi_{d_\ell}](x - i\frac{\gamma}{2})}\, \frac{\mathrm{W}_\gamma\,[\phi_{d_1}, \ldots, \phi_{d_\ell}, \phi_\mu](x - i\gamma)}{\mathrm{W}_\gamma\,[\phi_{d_1}, \ldots, \phi_{d_\ell}, \phi_\mu](x)}.\end{aligned} \tag{3·43}$$

We also have ($\ell \ge 0$)

$$\begin{aligned}&\prod_{j=1}^{\ell} V_{d_1\ldots d_j}(x + i\tfrac{\ell+1-j}{2}\gamma)\\ &= \sqrt{\prod_{j=0}^{\ell-1} V(x + i\tfrac{\ell-2j}{2}\gamma)V^*(x - i\tfrac{\ell-2j}{2}\gamma)}\; \frac{\mathrm{W}_\gamma\,[\phi_{d_1}, \ldots, \phi_{d_\ell}](x - i\frac{\gamma}{2})}{\mathrm{W}_\gamma\,[\phi_{d_1}, \ldots, \phi_{d_\ell}](x + i\frac{\gamma}{2})}.\end{aligned} \tag{3·44}$$

Therefore $V_{d_1\ldots d_\ell}(x)$ and $\phi_{d_1,\ldots d_\ell\, n}(x)$ are symmetric with respect to $d_1, \ldots, d_\ell$, and $\mathcal{H}_{d_1\ldots d_\ell}$ is independent of the order of $\{d_j\}$.

Let us state the discrete QM analogue of Adler's theorem; *If the set of deleted energy levels* $\mathcal{D} = \{d_1, \ldots, d_\ell\}$ *satisfy the condition* (2·8), *the modified Hamiltonian* *is given by* $\bar{\mathcal{H}} = \mathcal{H}_{d_1\,\ldots\,d_\ell} = \bar{\mathcal{A}}^\dagger\bar{\mathcal{A}} + \mathcal{E}_\mu$ *with the potential function given by* (3·43) *and its eigenfunctions are given by* (3·42). The discrete QM version of Crum's theorem[34)] corresponds to the choice $\{d_1,\ldots,d_\ell\} = \{0,1,\ldots,\ell-1\}$ and the new groundstate is at the level $\mu = \ell$ and there is no vacant energy level above that. Due to the lack of generic theorems in the theory of difference equations, the hermiticity of the resulting Hamiltonian $\bar{\mathcal{H}}$ and the non-singularity of the eigenfunctions $\bar{\phi}_n(x)$ cannot be proved categorically for the discrete QM, even when the condition (2·8) is satisfied by the deleted levels. See Appendix A of 25) for a detailed discussion of the self-adjointness of the Hamiltonians in discrete QM. It should be stressed that in most practical cases, in particular, in the cases of polynomial eigenfunctions, the hermiticity of the Hamiltonian $\bar{\mathcal{H}}$ and non-singularity of the eigenfunctions $\{\bar{\phi}_n(x)\}$ are satisfied.

3.2. *Polynomial eigenfunctions*

In this subsection we consider the typical case of shape invariant systems in which the eigenfunctions consist of the orthogonal polynomials $\{P_n\}$:

$$\phi_n(x) = \phi_0(x)P_n(\eta(x)), \quad \mathcal{A}\phi_0(x) = 0, \tag{3·45}$$

in which $\eta(x)$ is called the *sinusoidal coordinate* [see (B·9)]. The groundstate wavefunction $\phi_0(x)$ provides the orthogonality weight function

$$\int_{x_1}^{x_2}\phi_0(x)^2\,P_n(\eta(x))P_m(\eta(x))dx = h_n\delta_{nm}, \quad n,\,m\in\mathbb{Z}_+. \tag{3·46}$$

The difference equation for $\{P_n\}$ looks much simpler than the Schrödinger equation (3·5):

$$V(x)\big(P_n(\eta(x-i\gamma)) - P_n(\eta(x))\big) + V^*(x)\big(P_n(\eta(x+i\gamma)) - P_n(\eta(x))\big) = \mathcal{E}_nP_n(\eta(x)). \tag{3·47}$$

For the explicit forms of $V(x)$ [see for example (B·10)], these are the equations that determine the hypergeometric orthogonal polynomials, e.g. the Meixner-Pollaczek (MP), the continuous Hahn (cH), the Wilson (W) and the Askey-Wilson (AW) polynomials. In fact, the above form of the *difference equation* (3·47) is independent of the fact that P_n is a polynomial or not. It is obtained simply by the similarity transformation of the Hamiltonian (3·1) in terms of the groundstate wavefunction $\phi_0(x)$:

$$\widetilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0(x)^{-1}\circ\mathcal{H}\circ\phi_0(x) = V(x)\,e^{\gamma p} + V^*(x)\,e^{-\gamma p} - V(x) - V^*(x). \tag{3·48}$$

In the case of polynomial eigenfunctions, the modification of Crum's theorem produces the eigenfunctions $\{\bar{\phi}_n(x)\}$ which again consist of polynomials in $\eta(x)$. By using the property (3·40) we have

$$\mathrm{W}_\gamma[\phi_1,\ldots,\phi_\ell](x) = \prod_{j=1}^{\ell}\phi_0\big(x + i\tfrac{\ell+1-2j}{2}\gamma\big)\cdot\mathrm{W}_\gamma[\check{P}_1,\ldots,\check{P}_\ell](x), \tag{3·49}$$

$$\mathrm{W}_\gamma[\phi_1,\ldots,\phi_\ell,\phi_n](x) = \prod_{j=1}^{\ell+1}\phi_0\big(x + i\tfrac{\ell+2-2j}{2}\gamma\big)\cdot\mathrm{W}_\gamma[\check{P}_1,\ldots,\check{P}_\ell,\check{P}_n](x), \tag{3·50}$$

where $\check{P}_n(x) \stackrel{\text{def}}{=} P_n(\eta(x))$. Corresponding to the formula (2·48) in the ordinary QM, we have

$$\check{f}_j(x) \stackrel{\text{def}}{=} f_j(\eta(x)), \quad f_j(\eta)\text{: polynomial in } \eta,$$
$$\mathrm{W}_\gamma[\check{f}_1, \check{f}_2, \ldots, \check{f}_n](x) = \varphi_n(x) \times \big(\text{polynomial in } \eta(x)\big), \tag{3·51}$$

in which $\varphi_n(x)$ is defined in (B·39) (and $\varphi(x)$ is defined in (B·20)).

These simply mean that the resulting eigenfunctions $\{\bar{\phi}_n(x)\}$ (3·42) are again polynomials in $\eta(x)$:

$$\bar{\phi}_n(x) = \bar{\psi}(x)\, \mathcal{P}_n(\eta(x)), \tag{3·52}$$

$$\bar{\psi}(x) \stackrel{\text{def}}{=} \sqrt{\prod_{j=1}^{\ell} V_{d_1 \ldots d_j}(x + i\tfrac{\ell+1-j}{2}\gamma)\, \frac{\varphi_{\ell+1}(x)}{\varphi_\ell(x - i\frac{\gamma}{2})}\, \frac{\phi_0(x + i\frac{\ell}{2}\gamma)}{\mathcal{Q}(\eta(x - i\frac{\gamma}{2}))}}$$
$$= \sqrt{\prod_{k=0}^{\ell-1} \varphi(x - i\tfrac{k}{2}\gamma) \sqrt{V(x + i\tfrac{\ell-2k}{2}\gamma)} \cdot \frac{\phi_0(x - i\frac{\ell}{2}\gamma)}{\mathcal{Q}(\eta(x - i\frac{\gamma}{2}))}}$$
$$\times \sqrt{\prod_{k=0}^{\ell-1} \varphi(x + i\tfrac{k}{2}\gamma) \sqrt{V^*(x - i\tfrac{\ell-2k}{2}\gamma)} \cdot \frac{\phi_0(x + i\frac{\ell}{2}\gamma)}{\mathcal{Q}(\eta(x + i\frac{\gamma}{2}))}}\,, \tag{3·53}$$

in which $\mathcal{P}_n(\eta(x))$ and $\mathcal{Q}(\eta(x))$ are certain polynomials in $\eta(x)$ defined by

$$\mathrm{W}_\gamma[\check{P}_{d_1}, \ldots, \check{P}_{d_\ell}, \check{P}_n] = \varphi_{\ell+1}(x) \times \mathcal{P}_n(\eta(x)), \quad \mathrm{W}_\gamma[\check{P}_{d_1}, \ldots, \check{P}_{d_\ell}] = \varphi_\ell(x) \times \mathcal{Q}(\eta(x)). \tag{3·54}$$

The polynomials $\{\mathcal{P}_n\}$ form a complete basis of the Hilbert space and satisfy the orthogonality relations

$$\int_{x_1}^{x_2} \bar{\psi}(x)^2\, \mathcal{P}_n(\eta(x))\mathcal{P}_m(\eta(x))dx = \bar{h}_n \delta_{nm}, \quad n,\, m \in \mathbb{Z}_+ \backslash \mathcal{D}. \tag{3·55}$$

Let us emphasise that n is the level of the original eigenfunction and not the *degree* in η. The degree of $\mathcal{P}_n(\eta)$ depends on the set $\mathcal{D}$, and it can be calculated explicitly from (3·54). By construction ℓ members are missing: $\mathcal{P}_{d_1} = \mathcal{P}_{d_2} = \cdots = \mathcal{P}_{d_\ell} \equiv 0$. Therefore these polynomials cannot be called *exceptional orthogonal polynomials*.[9),12),15)]

§4. Summary and comments

Theory of exactly solvable discrete QM is less developed than that of the ordinary QM. Up to date, the known exactly solvable discrete quantum systems are all shape invariant[24),25)] and in one to one correspondence with the known (q)-hypergeometric orthogonal polynomials.[26)–29)] Small progress was made in this direction[31)] by introducing several new sinusoidal coordinates for the construction of new types of exactly solvable Hamiltonians. Roughly speaking, this approach attempts to create the discrete analogues of various Morse type potentials and the soliton potentials. In this paper we pursue another direction; to construct infinitely many exactly solvable quantum systems by *deforming* the known exactly solvable one. In the ordinary QM, the modification of Crum's theorem[1)] due to Krein-Adler[18)] allows to produce an essentially iso-spectral Hamiltonian by deleting a finite number of energy levels from the original system. The set of deleted levels must satisfy certain condition (2·8), but there are infinitely many possible deletions leading to infinitely many exactly solvable systems starting from a known one. The discrete analogue of Adler's modification is presented in this paper in parallel with the original version, since the algebraic structure is common. We also comment on the practical cases when the eigenfunctions consist of orthogonal polynomials. The eigenfunctions of the resulting system also consist of orthogonal polynomials. But certain members of these polynomials are missing due to the deletion.

Very special and simple examples, in which all the excited states from the first to the ℓ-th are deleted (see Fig. 2), are presented explicitly in Appendix. As will be commented shortly, these examples were instrumental for the discovery of the infinitely many shape invariant systems and the corresponding infinitely many exceptional orthogonal polynomials.[12),13)] In the ordinary QM, the corresponding prepotential has a very simple form (A·18):

$$w_\ell(x;\boldsymbol{\lambda}) = \mathcal{W}(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}) + \log\frac{1}{\xi_\ell(\eta(x);\boldsymbol{\lambda})},$$

to be compared with the prepotential for the Hamiltonian of the ℓ-th exceptional orthogonal polynomial (14) and (28) of 12):

$$w_\ell(x;\boldsymbol{\lambda}) = \mathcal{W}(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}) + \log\frac{\xi_\ell(\eta(x);\boldsymbol{\lambda}+\boldsymbol{\delta})}{\xi_\ell(\eta(x);\boldsymbol{\lambda})}. \tag{4·1}$$

In the discrete QM, the corresponding formula is (B·35)

$$V_\ell(x;\boldsymbol{\lambda}) = \kappa^\ell\frac{\xi_\ell(\eta(x+i\frac{\gamma}{2});\boldsymbol{\lambda})}{\xi_\ell(\eta(x-i\frac{\gamma}{2});\boldsymbol{\lambda})}V(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}),$$

to be compared with the corresponding formula for the Hamiltonian of the ℓ-th exceptional orthogonal polynomial (30) of 13):

$$V_\ell(x;\boldsymbol{\lambda}) = \frac{\xi_\ell(\eta(x-i\gamma);\boldsymbol{\lambda}+\boldsymbol{\delta})}{\xi_\ell(\eta(x);\boldsymbol{\lambda}+\boldsymbol{\delta})}\frac{\xi_\ell(\eta(x+i\frac{\gamma}{2});\boldsymbol{\lambda})}{\xi_\ell(\eta(x-i\frac{\gamma}{2});\boldsymbol{\lambda})}V(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}). \tag{4·2}$$

The addition (multiplication) of the deforming polynomial with the shifted parameters $\xi_\ell(\eta(x);\boldsymbol{\lambda}+\boldsymbol{\delta})$ would achieve the shape invariance. Since the harmonic oscillator has no shiftable parameter, we have $\xi_\ell(\eta(x);\boldsymbol{\lambda}) = \xi_\ell(\eta(x);\boldsymbol{\lambda}+\boldsymbol{\delta})$. This also 'explains' non-existence of exceptional Hermite polynomials. In contrast to the Hermite polynomial, the continuous Hahn polynomial has four real parameters. We can construct the corresponding exceptional continuous Hahn polynomials with three real parameters, which will be reported elsewhere.

The actual function forms of the deforming polynomial ξ_ℓ in Appendix are not the same as those for the exceptional orthogonal polynomials. For the ordinary QM examples, see (A·23) vs (13) and (27) in 12) and for the discrete QM examples, see (B·41) vs (64) and (78) in 13). But they share some interesting features.

Before closing this section, let us remark that the present modification of Crum's theorem is applicable to the Hamiltonian systems of various species of the infinite family of exceptional orthogonal polynomials,[12),13),15)] as well as to those of the classical orthogonal polynomials including the Wilson and the Askey-Wilson polynomials.

Acknowledgements

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Appendix

In Appendix we present very special and simple examples of an application of Adler's theorem, in which the eigenstates $\phi_1, \phi_2, \ldots, \phi_\ell$ are deleted. In other words, $\mathcal{D} = \{d_1, d_2, \ldots, d_\ell\} = \{1, 2, \ldots, \ell\}$, that is, the modified groundstate level is the same as that of the original theory $\mu = 0$. The situation is illustrated in Fig. 2, which should be compared with Fig. 1, depicting the generic case discussed in §§2.1 and 3.1.

The black circles denote the energy levels, whereas the white circles denote *deleted* energy levels. We write $\bar{\mathcal{H}} = \mathcal{H}_{12\ldots\ell}$, $\bar{\phi}_n = \phi_{12\ldots\ell n}$, $\bar{\mathcal{A}} = \mathcal{A}_{12\ldots\ell}$, $\bar{V} = V_{12\ldots\ell}$ etc. as $\mathcal{H}_\ell$, $\phi_{\ell,n}$, $\mathcal{A}_\ell$, V_ℓ etc. This Hamiltonian $\mathcal{H}_\ell = \mathcal{A}_\ell^\dagger \mathcal{A}_\ell$ is non-singular for even ℓ but may be singular for odd ℓ. Since algebraic formulas such as the

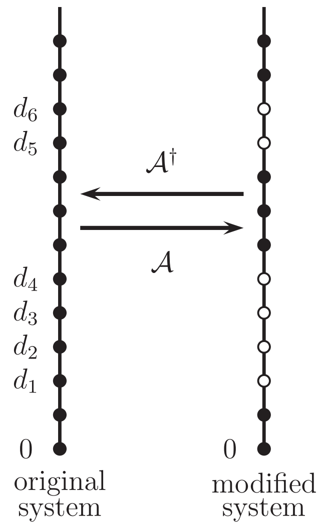

Fig. 1. Generic case.

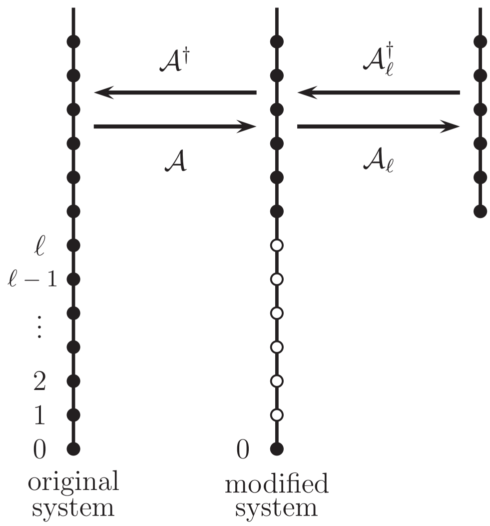

Fig. 2. Special case.

Wronskians and Casoratians are valid for even and odd ℓ, we present various formulas without restricting to the even ℓ. The original systems are shape invariant but the $(\phi_1,\ldots,\phi_\ell)$-deleted systems $\mathcal{H}_\ell$ are not. The rightmost vertical line in Fig. 2 corresponds to the Hamiltonian system $\mathcal{H}'_\ell=\mathcal{A}_\ell\mathcal{A}_\ell^\dagger$, which is shape invariant and it is obtained from $\mathcal{H}_\ell$ by one more step of Crum's method. This study helped us to find the new shape invariant systems and exceptional orthogonal polynomials.[12),13)]

Appendix A
—— *The Ordinary QM* ——

Here we apply Adler's theorem to the harmonic oscillator, the radial oscillator and the Darboux-Pöschl-Teller potential, whose eigenfunctions are described by the classical orthogonal polynomials. That is, the Hermite, Laguerre and Jacobi polynomials, to be abbreviated as H, L and J, respectively. These original systems are shape invariant, meaning a very special form of parameter dependence, (A·8), (A·9). Here we display the parameter dependence explicitly by $\boldsymbol{\lambda}$, which represents the set of the parameters.

A.1. *The original systems*

Here we summarise various properties of the original Hamiltonian systems to be compared with the specially modified systems to be presented in A.2. Let us start with the Hamiltonians, Schrödinger equations and eigenfunctions ($x_1<x<x_2$):

$$\mathcal{H}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\mathcal{A}(\boldsymbol{\lambda})^\dagger\mathcal{A}(\boldsymbol{\lambda}),\ \mathcal{A}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\frac{d}{dx}-\frac{d\mathcal{W}(x;\boldsymbol{\lambda})}{dx},\ \mathcal{A}(\boldsymbol{\lambda})^\dagger=-\frac{d}{dx}-\frac{d\mathcal{W}(x;\boldsymbol{\lambda})}{dx},\quad (\text{A·1})$$

$$\mathcal{H}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda})=\mathcal{E}_n(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda})\quad(n=0,1,2,\ldots),\quad (\text{A·2})$$

$$\phi_n(x;\boldsymbol{\lambda})=\phi_0(x;\boldsymbol{\lambda})P_n(\eta(x);\boldsymbol{\lambda}),\quad \phi_0(x;\boldsymbol{\lambda})=e^{\mathcal{W}(x;\boldsymbol{\lambda})}.\quad (\text{A·3})$$

Here $\eta(x)$ is the sinusoidal coordinate, $\mathcal{W}(x;\boldsymbol{\lambda})$ is the prepotential and $\mathcal{E}_n(\boldsymbol{\lambda})$ is the n-th energy eigenvalue:

$$\eta(x)\stackrel{\text{def}}{=}\begin{cases}x, & x_1=-\infty,\quad x_2=\infty,\quad :\text{H}\\ x^2, & x_1=0,\quad x_2=\infty,\quad :\text{L}\\ \cos 2x, & x_1=0,\quad x_2=\frac{\pi}{2},\quad :\text{J}\end{cases},\quad \boldsymbol{\lambda}\stackrel{\text{def}}{=}\begin{cases}\text{none} & :\text{H}\\ g, \quad g>0 & :\text{L}\\ (g,h),\quad g,h>0 & :\text{J}\end{cases},\quad (\text{A·4})$$

$$\mathcal{W}(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\begin{cases}-\frac{1}{2}x^2 & :\text{H}\\ -\frac{1}{2}x^2+g\log x & :\text{L}\\ g\log\sin x+h\log\cos x & :\text{J}\end{cases},\quad \mathcal{E}_n(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\begin{cases}2n & :\text{H}\\ 4n & :\text{L}\\ 4n(n+g+h) & :\text{J}\end{cases}.\quad (\text{A·5})$$

The eigenfunction consists of an orthogonal polynomial $P_n(\eta;\boldsymbol{\lambda})$, a polynomial of degree n in η, ($P_n(\eta;\boldsymbol{\lambda})=0$ for $n<0$):

$$P_n(\eta;\boldsymbol{\lambda})\stackrel{\text{def}}{=}c_n(\boldsymbol{\lambda})P_n^{\text{monic}}(\eta;\boldsymbol{\lambda}),\quad (\text{A·6})$$

$$P_n(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} H_n(\eta) & : \mathrm{H} \\ L_n^{(g-\frac{1}{2})}(\eta) & : \mathrm{L} \\ P_n^{(g-\frac{1}{2},h-\frac{1}{2})}(\eta) & : \mathrm{J} \end{cases}, \quad c_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 2^n & : \mathrm{H} \\ \frac{(-1)^n}{n!} & : \mathrm{L} \\ \frac{(n+g+h)_n}{2^n n!} & : \mathrm{J} \end{cases}, \tag{A·7}$$

in which $(a)_n$ is the Pochhammer symbol. Shape invariance means

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger = \mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}), \quad \boldsymbol{\delta} \stackrel{\text{def}}{=} \begin{cases} \text{none} & : \mathrm{H} \\ 1 & : \mathrm{L} \\ (1,1) & : \mathrm{J} \end{cases}, \tag{A·8}$$

or equivalently,

$$\Bigl(\frac{d\mathcal{W}(x;\boldsymbol{\lambda})}{dx}\Bigr)^2 - \frac{d^2\mathcal{W}(x;\boldsymbol{\lambda})}{dx^2} = \Bigl(\frac{d\mathcal{W}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}{dx}\Bigr)^2 + \frac{d^2\mathcal{W}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}{dx^2} + \mathcal{E}_1(\boldsymbol{\lambda}). \tag{A·9}$$

The action of $\mathcal{A}(\boldsymbol{\lambda})$ and $\mathcal{A}(\boldsymbol{\lambda})^\dagger$ on the eigenfunction is:

$$\mathcal{A}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})\phi_{n-1}\bigl(x;\boldsymbol{\lambda}+\boldsymbol{\delta}\bigr), \quad \mathcal{A}(\boldsymbol{\lambda})^\dagger\phi_{n-1}\bigl(x;\boldsymbol{\lambda}+\boldsymbol{\delta}\bigr) = b_{n-1}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}). \tag{A·10}$$

Here the coefficients $f_n(\boldsymbol{\lambda})$ and $b_{n-1}(\boldsymbol{\lambda})$ are the factors of $\mathcal{E}_n(\boldsymbol{\lambda})$:

$$f_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 2n & : \mathrm{H} \\ -2 & : \mathrm{L} \\ -2(n+g+h) & : \mathrm{J} \end{cases}, \quad b_{n-1}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 1 & : \mathrm{H} \\ -2n & : \mathrm{L,J} \end{cases}, \tag{A·11}$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})b_{n-1}(\boldsymbol{\lambda}).$$

The forward and backward shift operators, $\mathcal{F}(\boldsymbol{\lambda})$ and $\mathcal{B}(\boldsymbol{\lambda})$, are defined by:

$$\mathcal{F}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1}\circ\mathcal{A}(\boldsymbol{\lambda})\circ\phi_0(x;\boldsymbol{\lambda}) = \frac{\phi_0(x;\boldsymbol{\lambda})}{\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}\frac{d}{dx}, \tag{A·12}$$

$$\begin{aligned}\mathcal{B}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \phi_0(x;\boldsymbol{\lambda})^{-1}\circ\mathcal{A}(\boldsymbol{\lambda})^\dagger\circ\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) \\ &= -\frac{\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\phi_0(x;\boldsymbol{\lambda})}\Bigl(\frac{d}{dx} + \partial_x\bigl(\mathcal{W}(x;\boldsymbol{\lambda}) + \mathcal{W}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\bigr)\Bigr),\end{aligned} \tag{A·13}$$

and their action on the polynomial is:

$$\mathcal{F}(\boldsymbol{\lambda})P_n(\eta(x);\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})P_{n-1}(\eta(x);\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{A·14}$$

$$\mathcal{B}(\boldsymbol{\lambda})P_{n-1}(\eta(x);\boldsymbol{\lambda}+\boldsymbol{\delta}) = b_{n-1}(\boldsymbol{\lambda})P_n(\eta(x);\boldsymbol{\lambda}). \tag{A·15}$$

Note that $\mathcal{F}(\boldsymbol{\lambda})$ and $\mathcal{B}(\boldsymbol{\lambda})$ can also be expressed in terms of η only.[16] The orthogonality reads

$$\int_{x_1}^{x_2}\phi_0(x;\boldsymbol{\lambda})^2\,P_n(\eta(x);\boldsymbol{\lambda})P_m(\eta(x);\boldsymbol{\lambda})dx = h_n(\boldsymbol{\lambda})\delta_{nm}, \tag{A·16}$$

$$h_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 2^n n!\sqrt{\pi} & : \mathrm{H} \\ \frac{1}{2\,n!}\Gamma(n+g+\frac{1}{2}) & : \mathrm{L} \\ \frac{\Gamma(n+g+\frac{1}{2})\Gamma(n+h+\frac{1}{2})}{2\,n!(2n+g+h)\Gamma(n+g+h)} & : \mathrm{J} \end{cases}. \tag{A·17}$$

A.2. *The* $(\phi_1,\ldots,\phi_\ell)$*-deleted systems*

The prepotential of the modified system is obtained from (2·51) up to an additive constant:

$$w_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \log\phi_{\ell,0}(x) = \mathcal{W}(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}) - \log\xi_\ell(\eta(x);\boldsymbol{\lambda}). \tag{A·18}$$

It is a polynomial ($\xi_\ell(\eta(x);\boldsymbol{\lambda})$) deformation of the shape invariant one $\mathcal{W}(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})$. Note that the normalization of ξ_ℓ does not affect the Hamiltonian. The explicit forms of the deforming polynomial $\xi_\ell(\eta;\boldsymbol{\lambda})$ will be given in (A·23). For even ℓ, the polynomial $\xi_\ell(\eta(x);\boldsymbol{\lambda})$ has no zero in the range of x and the modified Hamiltonian system is hermitian, which reads:

$$\mathcal{A}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{d}{dx} - \frac{dw_\ell(x;\boldsymbol{\lambda})}{dx}, \quad \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger = -\frac{d}{dx} - \frac{dw_\ell(x;\boldsymbol{\lambda})}{dx}, \tag{A·19}$$

$$\mathcal{H}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\mathcal{A}_\ell(\boldsymbol{\lambda}), \tag{A·20}$$

$$\mathcal{H}_\ell(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}) \quad (n=0,\ell+1,\ell+2,\ldots). \tag{A·21}$$

This system is not shape invariant. As mentioned in §4, the above form of the deformed prepotential (A·18) is closely related to that of the exceptional Laguerre and Jacobi polynomials.

A degree ℓ polynomial in η, $\xi_\ell(\eta;\boldsymbol{\lambda})$ is defined by

$$\mathrm{W}[P_1,\ldots,P_\ell](\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \prod_{k=1}^{\ell} k!\,c_k(\boldsymbol{\lambda})\cdot\xi_\ell(\eta;\boldsymbol{\lambda}), \tag{A·22}$$

and the explicit forms are:

$$\xi_\ell(\eta;\boldsymbol{\lambda}) = \begin{cases} \frac{1}{2^\ell\ell!\,i^\ell}\,H_\ell(i\eta) & :\mathrm{H} \\ L_\ell^{(-g-\ell-\frac12)}(-\eta) & :\mathrm{L} \\ \frac{(-2)^\ell}{(g+h+1)_\ell}P_\ell^{(-g-\ell-\frac12,-h-\ell-\frac12)}(\eta) & :\mathrm{J} \end{cases}. \tag{A·23}$$

The eigenfunctions are

$$\phi_{\ell,0}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} e^{w_\ell(x;\boldsymbol{\lambda})} = \frac{\phi_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}{\xi_\ell\bigl(\eta(x);\boldsymbol{\lambda}\bigr)}, \quad \phi_{\ell,n}(x;\boldsymbol{\lambda}) = \phi_{\ell,0}(x;\boldsymbol{\lambda})P_{\ell,n}\bigl(\eta(x);\boldsymbol{\lambda}\bigr), \tag{A·24}$$

$$\mathrm{W}[P_1,\ldots,P_\ell,P_n](\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \prod_{k=1}^{\ell} k!\,c_k(\boldsymbol{\lambda})\cdot(-1)^\ell P_{\ell,n}(\eta;\boldsymbol{\lambda}) \quad \bigl(\Rightarrow P_{\ell,0}(\eta;\boldsymbol{\lambda})=1\bigr). \tag{A·25}$$

Note that $P_{\ell,n}(\eta;\boldsymbol{\lambda})$ is a polynomial of degree n in η and $P_{0,n}(\eta;\boldsymbol{\lambda}) = P_n(\eta;\boldsymbol{\lambda})$ and $P_{\ell,n}(\eta;\boldsymbol{\lambda})= 0$ for $1\le n\le\ell$. We set $P_{\ell,n}(\eta;\boldsymbol{\lambda}) = 0$ for $n<0$. For even ℓ, the eigenpolynomial $P_{\ell,n}(\eta(x);\boldsymbol{\lambda})$ $(n\ge\ell+1)$ has $n-\ell$ zeros in the range of x. The operators $\mathcal{A}_\ell(\boldsymbol{\lambda})$ and $\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger$ connect the modified system $\mathcal{H}_\ell(\boldsymbol{\lambda}) = \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\mathcal{A}_\ell(\boldsymbol{\lambda})$ to the shape invariant system $\mathcal{H}'_\ell(\boldsymbol{\lambda}) = \mathcal{A}_\ell(\boldsymbol{\lambda})\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger$ with the parameters $\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}$, which is denoted by the rightmost vertical line in Fig. 2. The n-th level $(n\ge\ell+1)$ of the modified system $\mathcal{H}_\ell$ is *iso-spectral* with the $n-\ell-1$-th level of the new shape invariant system $\mathcal{H}'_\ell$:

$$\mathcal{A}_\ell(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}) = f_{\ell,n}(\boldsymbol{\lambda})\phi_{n-\ell-1}\bigl(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}\bigr), \tag{A·26}$$

$$\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\phi_{n-\ell-1}\bigl(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}\bigr) = b_{\ell,n-1}(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}). \tag{A·27}$$

Here $f_{\ell,n}(\boldsymbol{\lambda})$ and $b_{\ell,n-1}(\boldsymbol{\lambda})$ are defined by

$$\begin{aligned} f_{\ell,n}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} f_n(\boldsymbol{\lambda})\times A,\\ b_{\ell,n-1}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} b_{n-1}(\boldsymbol{\lambda})\times A^{-1}, \end{aligned}\qquad A = \left\{\begin{array}{ll} (-2)^\ell(n-\ell)_\ell & : \mathrm{H}\\ 1 & : \mathrm{L}\\ (-2)^{-\ell}(n+g+h+1)_\ell & : \mathrm{J} \end{array}\right., \tag{A·28}$$

and they factorise $\mathcal{E}_n(\boldsymbol{\lambda})$, $\mathcal{E}_n(\boldsymbol{\lambda}) = f_{\ell,n}(\boldsymbol{\lambda})b_{\ell,n-1}(\boldsymbol{\lambda})$. The forward and backward shift operators $\mathcal{F}_\ell(\boldsymbol{\lambda})$ and $\mathcal{B}_\ell(\boldsymbol{\lambda})$, which act on the polynomial eigenfunctions, are defined by:

$$\begin{aligned} \mathcal{F}_\ell(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \phi_0\bigl(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}\bigr)^{-1}\circ\mathcal{A}_\ell(\boldsymbol{\lambda})\circ\phi_{\ell,0}(x;\boldsymbol{\lambda})\\ &= \frac{\phi_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}{\phi_0(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta})}\frac{1}{\xi_\ell(\eta(x);\boldsymbol{\lambda})}\frac{d}{dx}, \end{aligned} \tag{A·29}$$

$$\begin{aligned} \mathcal{B}_\ell(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \phi_{\ell,0}(x;\boldsymbol{\lambda})^{-1}\circ\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\circ\phi_0\bigl(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}\bigr)\\ &= -\frac{\phi_0(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta})}{\phi_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}\,\xi_\ell(\eta(x);\boldsymbol{\lambda})\\ &\quad\times\Bigl(\frac{d}{dx}+\partial_x\bigl(\mathcal{W}(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})+\mathcal{W}(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta})\bigr)-\frac{\partial_x\xi_\ell(\eta(x);\boldsymbol{\lambda})}{\xi_\ell(\eta(x);\boldsymbol{\lambda})}\Bigr). \end{aligned} \tag{A·30}$$

Their action on the polynomials is ($n\geq\ell+1$):

$$\mathcal{F}_\ell(\boldsymbol{\lambda})P_{\ell,n}(\eta(x);\boldsymbol{\lambda}) = f_{\ell,n}(\boldsymbol{\lambda})P_{n-\ell-1}\bigl(\eta(x);\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}\bigr), \tag{A·31}$$

$$\mathcal{B}_\ell(\boldsymbol{\lambda})P_{n-\ell-1}\bigl(\eta(x);\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}\bigr) = b_{\ell,n-1}(\boldsymbol{\lambda})P_{\ell,n}(\eta(x);\boldsymbol{\lambda}). \tag{A·32}$$

Note that $\mathcal{F}_\ell(\boldsymbol{\lambda})$ and $\mathcal{B}_\ell(\boldsymbol{\lambda})$ can be expressed in terms of η.[16] For $n\geq\ell+1$, the above relation (A·32) provides a simple formula of the modified eigenpolynomial $P_{\ell,n}(\eta;\boldsymbol{\lambda})$ in terms of $\xi_\ell(\eta;\boldsymbol{\lambda})$ and the original eigenpolynomial $P_n(\eta;\boldsymbol{\lambda})$:

$$\begin{aligned} &b_{\ell,n-1}(\boldsymbol{\lambda})f_{n-\ell}(\boldsymbol{\lambda}+\ell\boldsymbol{\delta})P_{\ell,n}(\eta;\boldsymbol{\lambda})\\ &= \mathcal{E}_{n-\ell}(\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\xi_\ell(\eta;\boldsymbol{\lambda})P_{n-\ell}(\eta;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})+4c_2(\eta)\partial_\eta\xi_\ell(\eta;\boldsymbol{\lambda})\,\partial_\eta P_{n-\ell}(\eta;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}), \end{aligned} \tag{A·33}$$

in which the coefficient $c_2(\eta)$ is given by

$$c_2(\eta) \stackrel{\text{def}}{=} \left\{\begin{array}{ll} \frac14 & : \mathrm{H}\\ \eta & : \mathrm{L}\\ 1-\eta^2 & : \mathrm{J} \end{array}\right.. \tag{A·34}$$

The orthogonality relation for even ℓ is:

$$\int_{x_1}^{x_2}\phi_{\ell,0}(x;\boldsymbol{\lambda})^2\,P_{\ell,n}(\eta(x);\boldsymbol{\lambda})P_{\ell,m}(\eta(x);\boldsymbol{\lambda})dx = h_{\ell,n}(\boldsymbol{\lambda})\delta_{nm}, \tag{A·35}$$

$$h_{\ell,n}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (n-\ell)_\ell\, h_n(\boldsymbol{\lambda}) \times \left\{ \begin{array}{ll} 2^\ell & : \mathrm{H} \\ 1 & : \mathrm{L} \\ 4^{-\ell}(n+g+h+1)_\ell & : \mathrm{J} \end{array} \right. , \quad (n=0,\ n \geq \ell+1). \tag{A·36}$$

A few historical remarks are in order. Dubov et al.[17] derived in 1992 an exactly solvable Hamiltonian system of a deformed harmonic oscillator, which corresponds to the $\ell = 2$ case of this appendix. Their paper, written about two years before Adler's, relied on rather heuristic arguments. Recently Quesne[11] derived exactly solvable and non-shape invariant systems of deformed radial oscillator and deformed DPT potential, both are called type III. These results again correspond to the $\ell = 2$ cases of the radial oscillator and the DPT potential of this appendix.

Appendix B
—— *The Discrete QM* ——

Here we apply Adler's theorem to the shape invariant, therefore solvable, systems whose eigenfunctions are described by the orthogonal polynomials; the Meixner-Pollaczek (we set the parameter $\phi = \frac{\pi}{2}$), continuous Hahn, Wilson and Askey-Wilson polynomials,[29] to be abbreviated as MP, cH, W and AW, respectively. See 22) and 25) for the discrete QM treatment of these polynomials.

B.1. *The original systems*

Here we summarise various properties of the original Hamiltonian systems to be compared with the specially modified systems to be presented in B.2. Let us start with the Hamiltonians, Schrödinger equations and eigenfunctions ($x_1 < x < x_2$):

$$\mathcal{A}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} i\big(e^{\frac{\gamma}{2}p}\sqrt{V^*(x;\boldsymbol{\lambda})} - e^{-\frac{\gamma}{2}p}\sqrt{V(x;\boldsymbol{\lambda})}\,\big),$$
$$\mathcal{A}(\boldsymbol{\lambda})^\dagger \stackrel{\text{def}}{=} -i\big(\sqrt{V(x;\boldsymbol{\lambda})}\,e^{\frac{\gamma}{2}p} - \sqrt{V^*(x;\boldsymbol{\lambda})}\,e^{-\frac{\gamma}{2}p}\big), \tag{B·1}$$
$$\mathcal{H}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}(\boldsymbol{\lambda})^\dagger\mathcal{A}(\boldsymbol{\lambda}), \tag{B·2}$$
$$\mathcal{H}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}) \quad (n=0,1,2,\ldots), \tag{B·3}$$
$$\phi_n(x;\boldsymbol{\lambda}) = \phi_0(x;\boldsymbol{\lambda})P_n(\eta(x);\boldsymbol{\lambda}). \tag{B·4}$$

The set of parameters $\boldsymbol{\lambda}$ are

$$\mathrm{MP}: \quad \boldsymbol{\lambda} \stackrel{\text{def}}{=} a, \quad a>0, \tag{B·5}$$
$$\mathrm{cH}: \quad \boldsymbol{\lambda} \stackrel{\text{def}}{=} (a_1,a_2), \quad \mathrm{Re}\,a_i > 0 \ \ (i=1,2), \tag{B·6}$$
$$\mathrm{W}: \quad \boldsymbol{\lambda} \stackrel{\text{def}}{=} (a_1,a_2,a_3,a_4), \quad \mathrm{Re}\,a_i>0 \ \ (i=1,\ldots,4),$$
$$\{a_1^*,a_2^*,a_3^*,a_4^*\} = \{a_1,a_2,a_3,a_4\} \quad \text{(as a set)}, \tag{B·7}$$
$$\mathrm{AW}: \quad q^{\boldsymbol{\lambda}} \stackrel{\text{def}}{=} (a_1,a_2,a_3,a_4), \quad |a_i|<1, \quad (i=1,\ldots,4), \quad 0<q<1,$$
$$\{a_1^*,a_2^*,a_3^*,a_4^*\} = \{a_1,a_2,a_3,a_4\} \ \ \text{(as a set)}, \tag{B·8}$$

where $q^{(\lambda_1,\lambda_2,\ldots)} \stackrel{\text{def}}{=} (q^{\lambda_1}, q^{\lambda_2}, \ldots)$. The the sinusoidal coordinate $\eta(x)$ is,

$$\eta(x) \stackrel{\text{def}}{=} \begin{cases} x, & x_1=-\infty, \quad x_2=\infty, \quad \gamma=1 \quad : \mathrm{MP}, \mathrm{cH} \\ x^2, & x_1=0, \quad x_2=\infty, \quad \gamma=1 \quad : \mathrm{W} \\ \cos x, & x_1=0, \quad x_2=\pi, \quad \gamma=\log q \quad : \mathrm{AW} \end{cases}. \tag{B·9}$$

The potential function $V(x;\boldsymbol{\lambda})$ and energy eigenvalue $\mathcal{E}_n(\boldsymbol{\lambda})$ are

$$V(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} a+ix & : \mathrm{MP} \\ (a_1+ix)(a_2+ix) & : \mathrm{cH} \\ \bigl(2ix(2ix+1)\bigr)^{-1}\prod_{j=1}^4(a_j+ix) & : \mathrm{W} \\ \bigl((1-e^{2ix})(1-qe^{2ix})\bigr)^{-1}\prod_{j=1}^4(1-a_je^{ix}) & : \mathrm{AW} \end{cases}, \tag{B·10}$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 2n & : \mathrm{MP} \\ n(n+b_1-1), \quad b_1 \stackrel{\text{def}}{=} a_1+a_2+a_1^*+a_2^* & : \mathrm{cH} \\ n(n+b_1-1), \quad b_1 \stackrel{\text{def}}{=} a_1+a_2+a_3+a_4 & : \mathrm{W} \\ (q^{-n}-1)(1-b_4q^{n-1}), \quad b_4 \stackrel{\text{def}}{=} a_1a_2a_3a_4 & : \mathrm{AW} \end{cases}. \tag{B·11}$$

The eigenfunction is described by the orthogonal polynomial $P_n(\eta;\boldsymbol{\lambda})$, a polynomial of degree n in η:

$$P_n(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} c_n(\boldsymbol{\lambda})P_n^{\text{monic}}(\eta;\boldsymbol{\lambda}), \tag{B·12}$$

$$P_n(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} P_n^{(a)}(\eta;\frac{\pi}{2}) & : \mathrm{MP} \\ p_n(\eta;a_1,a_2,a_1^*,a_2^*) & : \mathrm{cH} \\ W_n(\eta;a_1,a_2,a_3,a_4) & : \mathrm{W} \\ p_n(\eta;a_1,a_2,a_3,a_4|q) & : \mathrm{AW} \end{cases}, \quad c_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{n!}\,2^n & : \mathrm{MP} \\ \frac{1}{n!}\,(n+b_1-1)_n & : \mathrm{cH} \\ (-1)^n(n+b_1-1)_n & : \mathrm{W} \\ 2^n(b_4q^{n-1};q)_n & : \mathrm{AW} \end{cases}, \tag{B·13}$$

in which $(a;q)_n$ is the q-Pochhammer symbol. We set $P_n(\eta;\boldsymbol{\lambda})=0$ for $n<0$. The shape invariance relations involve one more parameter κ:

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger = \kappa\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}), \tag{B·14}$$

$$\boldsymbol{\delta} \stackrel{\text{def}}{=} \begin{cases} \frac12 & : \mathrm{MP} \\ (\frac12,\frac12) & : \mathrm{cH} \\ (\frac12,\frac12,\frac12,\frac12) & : \mathrm{W}, \mathrm{AW} \end{cases}, \quad \kappa \stackrel{\text{def}}{=} \begin{cases} 1 & : \mathrm{MP}, \mathrm{cH}, \mathrm{W} \\ q^{-1} & : \mathrm{AW} \end{cases}, \tag{B·15}$$

or equivalently,

$$V(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda})V^*(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda}) = \kappa^2\,V(x;\boldsymbol{\lambda}+\boldsymbol{\delta})V^*(x-i\gamma;\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{B·16}$$

$$V(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda}) + V^*(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda}) = \kappa\bigl(V(x;\boldsymbol{\lambda}+\boldsymbol{\delta})+V^*(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\bigr) - \mathcal{E}_1(\boldsymbol{\lambda}). \tag{B·17}$$

The groundstate wavefunction $\phi_0(x;\boldsymbol{\lambda})$ is determined by

$$\sqrt{V^*(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda})}\,\phi_0(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda}) = \sqrt{V(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})}\,\phi_0(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda}), \tag{B·18}$$

and its explicit forms are:

$$\phi_0(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} \sqrt{\Gamma(a+ix)\Gamma(a-ix)} & : \text{MP} \\ \sqrt{\Gamma(a_1+ix)\Gamma(a_2+ix)\Gamma(a_1^*-ix)\Gamma(a_2^*-ix)} & : \text{cH} \\ \sqrt{(\Gamma(2ix)\Gamma(-2ix))^{-1}\prod_{j=1}^4 \Gamma(a_j+ix)\Gamma(a_j-ix)} & : \text{W} \\ \sqrt{(e^{2ix};q)_\infty(e^{-2ix};q)_\infty \prod_{j=1}^4 (a_je^{ix};q)_\infty^{-1}(a_je^{-ix};q)_\infty^{-1}} & : \text{AW} \end{cases}. \tag{B·19}$$

We introduce an auxiliary function $\varphi(x)$ with the properties:

$$\varphi(x) \stackrel{\text{def}}{=} \begin{cases} 1 & : \text{MP, cH} \\ 2x & : \text{W} \\ 2\sin x & : \text{AW} \end{cases}, \tag{B·20}$$

$$\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) = \varphi(x)\sqrt{V(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})}\,\phi_0(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda}), \tag{B·21}$$

$$V(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) = \kappa^{-1}\frac{\varphi(x-i\gamma)}{\varphi(x)}V(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda}). \tag{B·22}$$

The sinusoidal coordinate $\eta(x)$ has the following properties:

$$\eta(x-i\tfrac{k\gamma}{2}) - \eta(x+i\tfrac{k\gamma}{2}) = -i\varphi(x)\times\begin{cases} k & : \text{MP, cH, W} \\ \sinh\frac{-k\gamma}{2} & : \text{AW} \end{cases}, \tag{B·23}$$

$$\eta(x-i\tfrac{k\gamma}{2}) + \eta(x+i\tfrac{k\gamma}{2}) = \begin{cases} 2\eta(x) & : \text{MP, cH} \\ 2\eta(x)-\frac{1}{2}k^2 & : \text{W} \\ 2\eta(x)\cosh\frac{k\gamma}{2} & : \text{AW} \end{cases}, \tag{B·24}$$

$$\eta(x-i\tfrac{k\gamma}{2})\eta(x+i\tfrac{k\gamma}{2}) = \begin{cases} \eta(x)^2+\frac{1}{4}k^2 & : \text{MP, cH} \\ \bigl(\eta(x)+\frac{1}{4}k^2\bigr)^2 & : \text{W} \\ \eta(x)^2+\sinh^2\frac{k\gamma}{2} & : \text{AW} \end{cases}. \tag{B·25}$$

These mean that for a polynomial $P(\eta)$ in η, $i\varphi(x)^{-1}\bigl(P(\eta(x-i\frac{k\gamma}{2}))-P(\eta(x+i\frac{k\gamma}{2}))\bigr)$ is another polynomial in $\eta(x)$. The action of $\mathcal{A}(\boldsymbol{\lambda})$ and $\mathcal{A}(\boldsymbol{\lambda})^\dagger$ on the eigenfunctions is

$$\mathcal{A}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})\phi_{n-1}\bigl(x;\boldsymbol{\lambda}+\boldsymbol{\delta}\bigr), \quad \mathcal{A}(\boldsymbol{\lambda})^\dagger\phi_{n-1}\bigl(x;\boldsymbol{\lambda}+\boldsymbol{\delta}\bigr) = b_{n-1}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}). \tag{B·26}$$

The factors of the energy eigenvalue, $f_n(\boldsymbol{\lambda})$ and $b_{n-1}(\boldsymbol{\lambda})$, $\mathcal{E}_n(\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})b_{n-1}(\boldsymbol{\lambda})$, are given by

$$f_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 2 & : \text{MP} \\ n+b_1-1 & : \text{cH} \\ -n(n+b_1-1) & : \text{W} \\ q^{\frac{n}{2}}(q^{-n}-1)(1-b_4q^{n-1}) & : \text{AW} \end{cases}, \quad b_{n-1}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} n & : \text{MP, cH} \\ -1 & : \text{W} \\ q^{-\frac{n}{2}} & : \text{AW} \end{cases}. \tag{B·27}$$

The forward and backward shift operators $\mathcal{F}(\boldsymbol{\lambda})$ and $\mathcal{B}(\boldsymbol{\lambda})$ are defined by

$$\mathcal{F}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1}\circ\mathcal{A}(\boldsymbol{\lambda})\circ\phi_0(x;\boldsymbol{\lambda}) = i\varphi(x)^{-1}(e^{\frac{\gamma}{2}p}-e^{-\frac{\gamma}{2}p}), \tag{B·28}$$

$$\mathcal{B}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x;\boldsymbol{\lambda})^{-1} \circ \mathcal{A}(\boldsymbol{\lambda})^\dagger \circ \phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) = -i\big(V(x;\boldsymbol{\lambda})e^{\frac{\gamma}{2}p} - V^*(x;\boldsymbol{\lambda})e^{-\frac{\gamma}{2}p}\big)\varphi(x), \tag{B·29}$$

and their action on the polynomials is

$$\mathcal{F}(\boldsymbol{\lambda})P_n(\eta(x);\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})P_{n-1}(\eta(x);\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{B·30}$$

$$\mathcal{B}(\boldsymbol{\lambda})P_{n-1}(\eta(x);\boldsymbol{\lambda}+\boldsymbol{\delta}) = b_{n-1}(\boldsymbol{\lambda})P_n(\eta(x);\boldsymbol{\lambda}). \tag{B·31}$$

The orthogonality relation is

$$\int_{x_1}^{x_2} \phi_0(x;\boldsymbol{\lambda})^2\, P_n(\eta(x);\boldsymbol{\lambda})P_m(\eta(x);\boldsymbol{\lambda})dx = h_n(\boldsymbol{\lambda})\delta_{nm}, \tag{B·32}$$

$$h_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 2\pi\, \Gamma(n+2a)\big(n!\, 2^{2a}\big)^{-1} & : \text{MP} \\ 2\pi \prod_{i,j=1}^2 \Gamma(n+a_i+a_j^*)\cdot\big(n!(2n+b_1-1)\Gamma(n+b_1-1)\big)^{-1} & : \text{cH} \\ 2\pi n!\,(n+b_1-1)_n \prod_{1\le i<j\le 4}\Gamma(n+a_i+a_j)\cdot\Gamma(2n+b_1)^{-1} & : \text{W} \\ 2\pi(b_4q^{n-1};q)_n(b_4q^{2n};q)_\infty(q^{n+1};q)_\infty^{-1}\prod_{1\le i<j\le 4}(a_ia_jq^n;q)_\infty^{-1} & : \text{AW} \end{cases}. \tag{B·33}$$

B.2. *The* $(\phi_1,\ldots,\phi_\ell)$*-deleted systems*

The potential function, the Hamiltonian and the Schrödinger equation of the modified system are:

$$V_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{\varphi(x-i\frac{\ell+1}{2}\gamma)\varphi(x-i\frac{\ell}{2}\gamma)}{\varphi(x)\varphi(x-i\frac{\gamma}{2})}\,\frac{\xi_\ell(\eta(x+i\frac{\gamma}{2});\boldsymbol{\lambda})}{\xi_\ell(\eta(x-i\frac{\gamma}{2});\boldsymbol{\lambda})}V(x-i\tfrac{\ell}{2}\gamma;\boldsymbol{\lambda}) \tag{B·34}$$

$$= \kappa^\ell\,\frac{\xi_\ell(\eta(x+i\frac{\gamma}{2});\boldsymbol{\lambda})}{\xi_\ell(\eta(x-i\frac{\gamma}{2});\boldsymbol{\lambda})}V(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}), \tag{B·35}$$

$$\mathcal{A}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} i\big(e^{\frac{\gamma}{2}p}\sqrt{V_\ell^*(x;\boldsymbol{\lambda})} - e^{-\frac{\gamma}{2}p}\sqrt{V_\ell(x;\boldsymbol{\lambda})}\,\big),$$

$$\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger \stackrel{\text{def}}{=} -i\big(\sqrt{V_\ell(x;\boldsymbol{\lambda})}\,e^{\frac{\gamma}{2}p} - \sqrt{V_\ell^*(x;\boldsymbol{\lambda})}\,e^{-\frac{\gamma}{2}p}\big), \tag{B·36}$$

$$\mathcal{H}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\mathcal{A}_\ell(\boldsymbol{\lambda}), \tag{B·37}$$

$$\mathcal{H}_\ell(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}) \quad (n=0,\ell+1,\ell+2,\ldots). \tag{B·38}$$

The explicit forms of the deforming polynomial $\xi_\ell(\eta;\boldsymbol{\lambda})$ will be given in (B·41). For even ℓ, $\xi_\ell(\eta(x);\boldsymbol{\lambda})$ has no zero in the rectangular domain in the complex x plane, $x_1 \le \text{Re}\,x \le x_2$, $|\text{Im}\,x| \le \frac{1}{2}|\gamma|$. Note that the normalization of ξ_ℓ does not affect $\mathcal{H}_\ell$. This system is not shape invariant. The second line of the expression for $V_\ell(x;\boldsymbol{\lambda})$, (B·35), is obtained from the first line (B·34) by repeated use of the formula (B·22) of the auxiliary function φ. As mentioned in §4, this form of the deformed potential function (B·35) is closely related to that of the exceptional Wilson and Askey-Wilson polynomials.

It is convenient to introduce an auxiliary function $\varphi_\ell(x)$:

$$\varphi_\ell(x) \stackrel{\text{def}}{=} \varphi(x)^{[\frac{\ell}{2}]}\prod_{k=1}^{\ell-2}\big(\varphi(x-i\tfrac{k}{2}\gamma)\varphi(x+i\tfrac{k}{2}\gamma)\big)^{[\frac{\ell-k}{2}]}, \tag{B·39}$$

where $[x]$ denotes the greatest integer not exceeding x. Note that $[\frac{\ell}{2}]+2\sum_{k=1}^{\ell-2}[\frac{\ell-k}{2}]=\frac{1}{2}\ell(\ell-1)$. The deforming polynomial $\xi_\ell(\eta;\boldsymbol{\lambda})$ is defined by

$$\begin{aligned}
&\mathrm{W}_\gamma[\check{P}_1,\ldots,\check{P}_\ell](x;\boldsymbol{\lambda}) \qquad \bigl(\check{P}_n(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}P_n(\eta(x);\boldsymbol{\lambda})\bigr)\\
&\stackrel{\text{def}}{=}\prod_{k=1}^{\ell}c_k(\boldsymbol{\lambda})\cdot\varphi_\ell(x)\xi_\ell(\eta(x);\boldsymbol{\lambda})\times\left\{\begin{array}{ll}\prod_{k=1}^{\ell}k! & :\text{MP, cH, W}\\ \prod_{k=1}^{\ell}\prod_{j=1}^{k}\sinh\frac{-j\gamma}{2} & :\text{AW}\end{array}\right..
\end{aligned}\tag{B·40}$$

As in the ordinary QM cases (A·23), it is expressed in terms of the polynomial P_ℓ of the original system with shifted parameters:

$$\xi_\ell(\eta;\boldsymbol{\lambda})=\frac{P_\ell(\eta;-\boldsymbol{\lambda}^*-(\ell-1)\boldsymbol{\delta})}{c_\ell(-\boldsymbol{\lambda}^*-(\ell-1)\boldsymbol{\delta})}\times\left\{\begin{array}{ll}(\ell!)^{-1} & :\text{MP, cH, W}\\ (\prod_{j=1}^{\ell}\sinh\frac{-j\gamma}{2})^{-1} & :\text{AW}\end{array}\right..\tag{B·41}$$

Note that $P_n(\eta;\boldsymbol{\lambda}^*)=P_n(\eta;\boldsymbol{\lambda})$ and $c_n(\boldsymbol{\lambda}^*)=c_n(\boldsymbol{\lambda})$ for the MP, W and AW cases. The eigenfunctions are

$$\phi_{\ell,0}(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\frac{(-1)^\ell\kappa^{\frac{1}{4}\ell(\ell-1)}\phi_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}{\sqrt{\xi_\ell(\eta(x-i\frac{\gamma}{2});\boldsymbol{\lambda})\xi_\ell(\eta(x+i\frac{\gamma}{2});\boldsymbol{\lambda})}},\quad \phi_{\ell,n}(x;\boldsymbol{\lambda})=\phi_{\ell,0}(x;\boldsymbol{\lambda})P_{\ell,n}\bigl(\eta(x);\boldsymbol{\lambda}\bigr),\tag{B·42}$$

$$\begin{aligned}
\mathrm{W}_\gamma[\check{P}_1,\ldots,\check{P}_\ell,\check{P}_n](x;\boldsymbol{\lambda})&\stackrel{\text{def}}{=}\prod_{k=1}^{\ell}c_k(\boldsymbol{\lambda})\cdot\varphi_{\ell+1}(x)(-1)^\ell P_{\ell,n}(\eta(x);\boldsymbol{\lambda})\\
&\quad\times\left\{\begin{array}{ll}\prod_{k=1}^{\ell}k! & :\text{MP, cH, W}\\ \prod_{k=1}^{\ell}\prod_{j=1}^{k}\sinh\frac{-j\gamma}{2} & :\text{AW}\end{array}\right.\quad\bigl(\Rightarrow P_{\ell,0}(\eta;\boldsymbol{\lambda})=1\bigr).
\end{aligned}\tag{B·43}$$

For even ℓ, $P_{\ell,n}(\eta(x);\boldsymbol{\lambda})$ $(n\geq\ell+1)$ has $n-\ell$ zeros in the range of x. Note that $P_{\ell,n}(\eta;\boldsymbol{\lambda})$ is a polynomial of degree n in η and $P_{0,n}(\eta;\boldsymbol{\lambda})=P_n(\eta;\boldsymbol{\lambda})$ and $P_{\ell,n}(\eta;\boldsymbol{\lambda})=0$ for $1\leq n\leq\ell$. We set $P_{\ell,n}(\eta;\boldsymbol{\lambda})=0$ for $n<0$. The operators $\mathcal{A}_\ell(\boldsymbol{\lambda})$ and $\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger$ connect the modified system $\mathcal{H}_\ell(\boldsymbol{\lambda})=\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\mathcal{A}_\ell(\boldsymbol{\lambda})$ to the shape invariant system $\mathcal{H}'_\ell(\boldsymbol{\lambda})=\mathcal{A}_\ell(\boldsymbol{\lambda})\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger$ with the parameters $\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}$, which is denoted by the rightmost vertical line in Fig. 2. The n-th level $(n\geq\ell+1)$ of the modified system $\mathcal{H}_\ell$ is *iso-spectral* with the $n-\ell-1$-th level of the new shape invariant system $\mathcal{H}'_\ell$:

$$\mathcal{A}_\ell(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda})=f_{\ell,n}(\boldsymbol{\lambda})\phi_{n-\ell-1}\bigl(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}\bigr),\tag{B·44}$$

$$\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\phi_{n-\ell-1}\bigl(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}\bigr)=b_{\ell,n-1}(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}).\tag{B·45}$$

Here, $f_{\ell,n}(\boldsymbol{\lambda})$ and $b_{\ell,n-1}(\boldsymbol{\lambda})$ are the factors of the energy eigenvalue, $\mathcal{E}_n(\boldsymbol{\lambda})=f_{\ell,n}(\boldsymbol{\lambda})\times b_{\ell,n-1}(\boldsymbol{\lambda})$, and are defined by

$$\begin{aligned}
f_{\ell,n}(\boldsymbol{\lambda})&\stackrel{\text{def}}{=}f_n(\boldsymbol{\lambda})\times A,\\
b_{\ell,n-1}(\boldsymbol{\lambda})&\stackrel{\text{def}}{=}b_{n-1}(\boldsymbol{\lambda})\times A^{-1},
\end{aligned}\qquad A=\left\{\begin{array}{ll}2^\ell & :\text{MP}\\ (b_1+n)_\ell & :\text{cH}\\ (-1)^\ell(n-\ell)_\ell(b_1+n)_\ell & :\text{W}\\ q^{-\frac{1}{2}\ell n}(q^{n-\ell};q)_\ell(b_4q^n;q)_\ell & :\text{AW}\end{array}\right..\tag{B·46}$$

The forward and backward shift operators $\mathcal{F}_\ell(\boldsymbol{\lambda})$ and $\mathcal{B}_\ell(\boldsymbol{\lambda})$ which act on the polynomial eigenfunctions, are defined by:

$$\mathcal{F}_\ell(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0\bigl(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}\bigr)^{-1}\circ\mathcal{A}_\ell(\boldsymbol{\lambda})\circ\phi_{\ell,0}(x;\boldsymbol{\lambda}) = \frac{(-1)^\ell\kappa^{\frac14\ell(\ell+1)}}{\varphi(x)\xi_\ell(\eta(x);\boldsymbol{\lambda})}\,i\bigl(e^{\frac{\gamma}{2}p}-e^{-\frac{\gamma}{2}p}\bigr), \tag{B·47}$$

$$\begin{aligned}\mathcal{B}_\ell(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \phi_{\ell,0}(x;\boldsymbol{\lambda})^{-1}\circ\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\circ\phi_0\bigl(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}\bigr)\\ &= (-1)^\ell\kappa^{-\frac14\ell(\ell-3)}(-i)\Bigl(V(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\xi_\ell(\eta(x+i\tfrac{\gamma}{2});\boldsymbol{\lambda})e^{\frac{\gamma}{2}p}\\ &\qquad\qquad -V^*(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\xi_\ell(\eta(x-i\tfrac{\gamma}{2});\boldsymbol{\lambda})e^{-\frac{\gamma}{2}p}\Bigr)\varphi(x),\end{aligned} \tag{B·48}$$

and their action on the polynomials is

$$\mathcal{F}_\ell(\boldsymbol{\lambda})P_{\ell,n}(\eta(x);\boldsymbol{\lambda}) = f_{\ell,n}(\boldsymbol{\lambda})P_{n-\ell-1}\bigl(\eta(x);\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}\bigr), \tag{B·49}$$

$$\mathcal{B}_\ell(\boldsymbol{\lambda})P_{n-\ell-1}\bigl(\eta(x);\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}\bigr) = b_{\ell,n-1}(\boldsymbol{\lambda})P_{\ell,n}(\eta(x);\boldsymbol{\lambda}). \tag{B·50}$$

For $n\geq\ell+1$, the above formula (B·50) provides a simple formula of the modified eigenpolynomial $P_{\ell,n}(\eta;\boldsymbol{\lambda})$ in terms of $\xi_\ell(\eta;\boldsymbol{\lambda})$ and the original eigenpolynomial $P_n(\eta;\boldsymbol{\lambda})$:

$$\begin{aligned}&(-1)^\ell\kappa^{\frac14\ell(\ell-3)}b_{\ell,n-1}(\boldsymbol{\lambda})P_{\ell,n}(\eta;\boldsymbol{\lambda})\\ &= -i\Bigl(V(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\xi_\ell(\eta(x+i\tfrac{\gamma}{2});\boldsymbol{\lambda})\varphi(x-i\tfrac{\gamma}{2})P_{n-\ell-1}(\eta(x-i\tfrac{\gamma}{2});\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta})\\ &\quad -V^*(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\xi_\ell(\eta(x-i\tfrac{\gamma}{2});\boldsymbol{\lambda})\varphi(x+i\tfrac{\gamma}{2})P_{n-\ell-1}(\eta(x+i\tfrac{\gamma}{2});\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta})\Bigr).\end{aligned} \tag{B·51}$$

The orthogonality relation for even ℓ is:

$$\int_{x_1}^{x_2}\phi_{\ell,0}(x;\boldsymbol{\lambda})^2\,P_{\ell,n}(\eta(x);\boldsymbol{\lambda})P_{\ell,m}(\eta(x);\boldsymbol{\lambda})dx = h_{\ell,n}(\boldsymbol{\lambda})\delta_{nm}, \tag{B·52}$$

$$h_{\ell,n}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} h_n(\boldsymbol{\lambda})\times\left\{\begin{array}{ll}(n-\ell)_\ell 2^\ell & :\text{MP}\\ (n-\ell)_\ell(b_1+n)_\ell & :\text{cH,W}\\ q^{-\ell n}(q^{n-\ell};q)_\ell(b_4q^n;q)_\ell & :\text{AW}\end{array}\right.,\quad (n=0,\ n\geq\ell+1). \tag{B·53}$$

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