On a theorem of MacCluer and Shapiro

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Abstract

Let $u$ be a holomorphic function in the unit ball $B$ of $C^n$ and $\varphi$ be a univalent holomorphic self-map of $B$. We give some sufficient conditions for $u$ and $\varphi$ that the weighted composition operator $uC_{\varphi}$ is bounded or compact on the Hardy spaces $H^p(B)$ and the weighted Bergman spaces $A^p(\nu_\alpha)$ $(0 < p < \infty, -1 < \alpha < \infty)$. This our result is a generalization of a theorem of B. D. MacCluer and J. H. Shapiro[9] concerning the composition operator $C_{\varphi}$. And we also give similar sufficient conditions for such operator to be metrically bounded or metrically compact on the Privalov spaces $N^p(B)$ $(1 < p < \infty)$ and the weighted Bergman–Privalov spaces $(AN)^p(\nu_\alpha)$ $(1 \leq p < \infty, -1 < \alpha < \infty)$.

1 Introduction

Let $n > 1$ be a fixed integer. Let $B = B_n$ and $S = \partial B$ denote the unit ball and the unit sphere of the complex $n$-dimensional Euclidean space $C^n$, respectively. Let $\nu$ and $\sigma$ denote the normalized Lebesgue measure on $B$ and on $S$, respectively. For each $\alpha \in (-1, \infty)$, we set $c_\alpha = \Gamma(n + \alpha + 1)/\Gamma(n + 1)\Gamma(\alpha + 1)$ and $d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^{\alpha}d\nu(z)$ $(z \in B)$. Note that $\nu_\alpha(B) = 1$. Let $H(B)$ denote the space of all holomorphic functions in $B$. For each $p \in (0, \infty)$ and $\alpha \in (-1, \infty)$, the Hardy space $H^p(B)$ and the weighted Bergman space $A^p(\nu_\alpha)$ are as usual defined by

$$H^p(B) = \left\{ f \in H(B) : \| f \|^p_{H^p(B)} = \sup_{0 < r < 1} \int_S |f_r|^p d\sigma < \infty \right\},$$

$$A^p(\nu_\alpha) = \left\{ f \in H(B) : \| f \|^p_{A^p(\nu_\alpha)} = \int_B |f|^p d\nu_\alpha < \infty \right\},$$

where $f_r(z) = f(rz)$ for $r \in (0, 1), z \in C^n$ with $rz \in B$. As in [17], the Privalov space $N^p(B)$ $(1 < p < \infty)$ is defined by

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The Nevanlinna space $N(B)$ is as usual defined:

$$N(B) = \left\{ f \in H(B) : \| f \|_{N(B)} = \sup_{0 < r < 1} \int_{S} (\log(1 + |f_r|))^p d\sigma < \infty \right\}.$$ 

For the sake of convenience, the symbol $N^{1}(B)$ as well as $N(B)$ is used to denote the Nevanlinna space. For each $p \in [1, \infty)$ and $\alpha \in (-1, \infty)$, we define the weighted Bergman–Privalov space $(AN)^p(\nu_\alpha)$ by

$$(AN)^p(\nu_\alpha) = \left\{ f \in H(B) : \| f \|_{(AN)^p(\nu_\alpha)} = \int_{B} (\log(1 + |f|))^p d\nu_\alpha < \infty \right\}.$$ 

Let $ST^{2}(\mathbb{R})$ denote the class of those nondecreasing convex functions $\chi : \mathbb{R} \to [0, \infty)$ which are twice differentiable. Moreover, we define $ST^{2}(\mathbb{R}) = \{ \chi \in ST^{2}(\mathbb{R}) : \lim_{t \to \infty} \frac{\chi(t)}{t} = \infty \}$. For $\alpha \in (-1, \infty)$ and $\chi \in ST^{2}(\mathbb{R})$, we define $\| \cdot \|_{x, \alpha}$ as follows:

$$\| f \|_{x, \alpha} = \begin{cases} \sup_{0 < r < 1} \int_{S} \chi(\log|f_r|) d\sigma & \text{if } \alpha = -1, \\ \int_{B} \chi(\log|f|) d\nu_\alpha & \text{if } \alpha > -1, \end{cases}$$

for $f \in H(B)$. If $\chi(t) = t^p$ ($t \in \mathbb{R}$, $0 < p < \infty$), then $\| f \|_{x, \alpha} = \| f \|_{N^p(\nu_\alpha)}$ for $\alpha \in (-1, \infty)$ and $\| f \|_{x, -1} = \| f \|_{N^p(\nu_\alpha)}$. If $\chi(t) = (\log(1 + e^t))^p$ ($t \in \mathbb{R}$, $1 \leq p < \infty$), then $\| f \|_{x, \alpha} = \| f \|_{(AN)^p(\nu_\alpha)}$ for $\alpha \in (-1, \infty)$ and $\| f \|_{x, -1} = \| f \|_{N^p(\nu_\alpha)}$. For the sake of convenience, we define $A^p(\nu_{-1}) = H^p(B)$ ($0 < p < \infty$) and $(AN)^p(\nu_{-1}) = N^p(\nu_{1})$ ($1 \leq p < \infty$).

If $u \in H(B)$ and $\varphi$ is a holomorphic self-map of $B$, then $u$ and $\varphi$ induce a linear operator $uC_\varphi$ on $H(B)$ by means of the equation $uC_\varphi f = u \cdot (f \circ \varphi)$. This $uC_\varphi$ is called the weighted composition operator induced by $u$ and $\varphi$. In the case $u \equiv 1$ in $B$, $uC_\varphi$ is the composition operator $C_\varphi$. In the present paper we study the operator $uC_\varphi$ on the above function spaces.

In 1986, B. D. MacCluer and J. H. Shapiro got the following result about the boundedness and the compactness of $C_\varphi$ on $H^p(B)$ and on $A^p(\nu_\alpha)$:

**Theorem 1.1** ([9], B. D. MacCluer–J. H. Shapiro). Suppose that $\varphi : B \to B$ is a univalent holomorphic map, and that the Fréchet derivative of $\varphi^{-1}$ is bounded on $\varphi(B)$. Then

(a) For each $p \in (0, \infty)$ and $\alpha \in (-1, \infty)$, $C_\varphi$ is bounded on $H^p(B)$ and on $A^p(\nu_\alpha)$.

(b) For $p \in (0, \infty)$ and $\alpha \in (-1, \infty)$, $C_\varphi$ is compact on $H^p(B)$ and on $A^p(\nu_\alpha)$ if and only if

$$\lim_{|z| \to 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$$

Recently, $C_\varphi$ on $N^p(B_1)$ have been studied by J. S. Choa and H. O. Kim [2, 3, 4]. And $C_\varphi$ on $(AN)^n(\nu)$ (in the case $n=1$) have been studied by J. Jarchow and J. Xiao [7, 18]. Following [4], we say that a linear operator $T$ is metrically bounded on $(AN)^p(\nu_\alpha)$ if there exists a positive constant $L$ such that $\| Tf \|_{(AN)^p(\nu_\alpha)} \leq L \| f \|_{(AN)^p(\nu_\alpha)}$ for all $f \in$
(AN)\rho(\nu_a). And we say that T is metrically compact on (AN)\rho(\nu_a) if T maps every closed ball \(B_R = \{f \in (AN)\rho(\nu_a) : \|f\|_{(AN)\rho(\nu_a)} \leq R\} (0 < R < \infty)\) into a relatively compact set in (AN)\rho(\nu_a).

This paper is organized as follows. In Section 2, we enumerate 13 lemmas that will be used afterward. In Section 3, we give some sufficient conditions for u and \phi that \(uC_\phi\) is bounded or compact on \(H^p(B)\) and on \(A^p(\nu_a)\). This result is a generalization of Theorem 1.1. Finally, in Section 4 we also give similar sufficient conditions for \(uC_\phi\) to be metrically bounded or metrically compact on \(N^p(B)\) and on \((AN)\rho(\nu_a)\). As a corollary of this, we obtain an analogous version of Theorem 1.1 with respect to \(N^p(B)\) and \((AN)\rho(\nu_a)\).

2 Preliminaries

For each \(a \in [-1, \infty)\), we define the nonnegative decreasing function \(K_a\) by

\[
K_a(t) = \begin{cases} 
2nc_a \int_t^1 \rho^{n-1}(1-\rho^2)^a \log \frac{\rho}{t} d\rho & \text{if } a > -1, \\
\log \frac{1}{t} & \text{if } a = -1,
\end{cases}
\]

for all \(t \in (0, 1]\). It is obvious that \(K_a(1) = 0\) and \(K_a(t) > 0\) for all \(t \in (0, 1)\). The following lemma is easily verified ([1, Proposition 2.3]).

Lemma 2.1. (a) \(K_a(t) < 1 - t^2\) if \(\frac{1}{2} < t < 1\).

(b) For each \(a \in (-1, \infty)\),

\[
\lim_{t \to 1} K_a(t) = \frac{2nc_a}{(1-t^2)^{a+2}}.
\]

Lemma 2.2. Let \(-1 \leq a < \infty, 0 < r < 1\) and \(z \in rB\setminus\{0\}\). Then it holds that

\[
K_a(|z|) \leq \log \frac{1}{r} + K_a(\frac{|z|}{r}).
\]

Proof. See [1, pages 48-49].

Lemma 2.3. Suppose \(-1 \leq a < \infty, \chi \in ST^\infty(\mathbb{R}), 0 < r \leq 1\) and \(f \in H(B)\setminus\{0\}\). Then

\[
\|f_r\|_{L_a} = \chi(\log |f(0)|) + \frac{1}{2n} \int_{\mathbb{R}} \chi''(\log |f(z)|) \frac{|Rf(z)|^2}{|f(z)|^2 |z|^2 |z| - 2(a-1)K_a(\frac{|z|}{r})} dv(z),
\]

where \((Rf)(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)\) is the radial derivative of \(f\).

Proof. In the case \(r = 1\), this lemma is just [10, Lemma 3.10]. If \(0 < r < 1\) and \(a = -1\), it is just [10, Lemma 3.7]. If \(0 < r < 1\) and \(-1 < a < \infty\), it follows from [10, Lemma 3.10] with a simple change of variables.

Lemma 2.4. (a) Let \(1 \leq p < \infty\) and \(-1 < a < \infty\). Every \(f \in H(B)\setminus\{0\}\) satisfies the following inequalities:
\[
\frac{a_n \Gamma(n+\alpha+1)}{2^{\alpha+\alpha}(n+\alpha+1)\Gamma(\alpha+2)} \int_B \Delta((\log(1+|f|))(z)(1-|z|^2)\nu(z) \\
+ \{\log(1+|f(0)|)\}^p \\
\leq \|f\|_{AN^p(\nu_\alpha)}^{p}
\leq \frac{b_n 2^{\alpha+\alpha} \Gamma(n+\alpha+1)}{(n+\alpha+1)\Gamma(\alpha+1)} \int_B \Delta((\log(1+|f|))(z)(1-|z|^2)\nu(z) \\
+ \int_s \{\log(1+|f(0)|)\}^p d\sigma,
\]

where \(\Delta\) is the Laplacian with respect to the Bergman metric on \(B\), \(a_n = \frac{n+1}{2^{\alpha+\alpha}(n+\alpha+1)\Gamma(\alpha+2)}\), \(b_n = \frac{2^{\alpha+\alpha}(n+\alpha+1)}{(n+\alpha+1)\Gamma(\alpha+1)}\) and \(\alpha^* = \max\{0, \alpha\}\).

(b) Let \(1 < p < \infty\). Every \(f \in H(B)\setminus\{0\}\) satisfies the following inequalities:
\[
\frac{2a_n \Gamma(n)}{n} \int_B \Delta((\log(1+|f|))(z) \frac{\nu(z)}{1-|z|^2} + \{\log(1+|f(0)|)\}^p \\
\leq \|f\|_{AN^p(\nu_\alpha)}^{p}
\leq \frac{b_n \Gamma(n)}{2n} \int_B \Delta((\log(1+|f|))(z) \frac{\nu(z)}{1-|z|^2} + \int_s \{\log(1+|f(0)|)\}^p d\sigma.
\]

Proof. (a) is just [11, Theorem 1(a)]. By letting \(\alpha \downarrow -1\) in (a), we obtain (b) (cf. [11, the proof of Theorem 2]).

Considering the weighted Bergman spaces \(A^p(\nu_\alpha)\) instead of the weighted Bergman–Privalov spaces \((AN)^p(\nu_\alpha)\), we have the following lemma of which proof is essentially the same as that of Lemma 2.4.

**Lemma 2.5.** Let \(0 < p < \infty\) and \(-1 \leq \alpha < \infty\). Every \(f \in H(B)\setminus\{0\}\) satisfies the following inequalities:
\[
a_{n,\alpha} \int_B \Delta((|f|)^p)(z)(1-|z|^2)\nu(z) + |f(0)|^p \\
\leq \|f\|_{AN^p(\nu_\alpha)}^{p}
\leq b_{n,\alpha} \int_B \Delta((|f|)^p)(z)(1-|z|^2)\nu(z) + \int_s |f(0)|^p d\sigma,
\]

where
\[
a_{n,\alpha} = \begin{cases} 
\frac{a_n \Gamma(n+\alpha+1)}{2^{\alpha+\alpha}(n+\alpha+1)\Gamma(\alpha+2)} & \text{if } \alpha > -1, \\
\frac{2a_n \Gamma(n)}{n} & \text{if } \alpha = -1,
\end{cases}
\]
\[
b_{n,\alpha} = \begin{cases} 
\frac{b_n 2^{\alpha+\alpha} \Gamma(n+\alpha+1)}{(n+\alpha+1)\Gamma(\alpha+1)} & \text{if } \alpha > -1, \\
\frac{b_n \Gamma(n)}{2n} & \text{if } \alpha = -1.
\end{cases}
\]

Let \(\nabla\) denote the gradient with respect to the Bergman metric on \(B\) ([16, p.27]). Then as in [16, p.30], for \(f \in H(B)\) and \(z \in B\),
\[
|\mathbf{\check{f}}(z)|^2 = \frac{2}{n+1} (1-|z|^2) \left[ |(\nabla f)(z)|^2 - |(R f)(z)|^2 \right] \\
\leq \frac{2}{n+1} (1-|z|^2) |(\nabla f)(z)|^2
\]
where \(|(\nabla f)(z)|^2 = \sum_{j=1}^{n} \left| \frac{\partial f}{\partial z_j} (z) \right|^2\). We note that
\[
|(R f)(z)| \leq |z| |(\nabla f)(z)|
\]
and
\[
|\mathbf{\check{f}}(z)|^2 \geq \frac{2}{n+1} (1-|z|^2)^2 |(\nabla f)(z)|^2.
\]
Moreover, we have by a simple computation
\[
\tilde{\lambda}(x(\log|f|))(z) = \frac{1}{2} x''(\log|f|)(z) \frac{|\mathbf{\check{f}}(z)|^2}{|f(z)|^2}
\]
for \(f \in H(B)\) and \(z \in B \setminus Z(f)\), where \(Z(f) = \{w \in B : f(w) = 0\}\).

**Lemma 2.6.** Let \(1 \leq p < \infty\) and \(-1 \leq \alpha < \infty\). Suppose \(f \in H(B)\) and \(z \in B\). Then
\[
\log(1+|f(z)|) \leq \left( \frac{1+|z|}{1-|z|} \right)^{\alpha + \frac{1}{p}} \|f\|_{(AN)^p(\nu_\alpha)}.
\]

**Proof.** See [12, Lemma 1], [15, Proposition 3.3] and [17, p.233]. \(\square\)

By a simple computation with some change of variables in the case \(-1 < \alpha < \infty\), or with Lemma 2.6 in the case \(\alpha = -1\), we can easily prove the following lemma.

**Lemma 2.7.** Suppose \(\alpha \in [-1, \infty), p \in [1, \infty)\) and \(\varphi\) is a biholomorphic map of \(B\) onto \(B\). Then the composition operator \(C_\varphi\) induced by \(\varphi\) is metrically bounded on \((AN)^p(\nu_\alpha)\):
\[
\|C_\varphi f\|_{(AN)^p(\nu_\alpha)} \leq \left( \frac{1+|\varphi(0)|}{1-|\varphi(0)|} \right)^{\alpha + \frac{1}{p}} \|f\|_{(AN)^p(\nu_\alpha)}
\]
for \(f \in (AN)^p(\nu_\alpha)\).

Let \(\varphi : B \rightarrow B\) be a univalent holomorphic map. For \(z \in B\), We define
\[
\Omega_\varphi(z) = \frac{|\varphi'(z)|}{|J_\varphi(z)|^2}
\]
where \(\varphi'(z)\) is the derivative of \(\varphi\) at \(z\), \(\|\varphi'(z)\|\) denotes its norm as a linear transformation on \(C^n\) and \(J_\varphi(z)\) is the complex Jacobian of \(\varphi\) at \(z\). We can easily see the next lemma (see [5], p.171).

**Lemma 2.8.** Suppose that a univalent holomorphic map \(\varphi : B \rightarrow B\) satisfies
\[
\sup_{w \in \varphi(B)} \| (\varphi^{-1})'(w) \| < \infty.
\]
Then \(\Omega_\varphi\) is bounded in \(B\).

**Lemma 2.9.** Let \(1 < p < \infty\). Suppose that \(\varphi : B \rightarrow B\) is a univalent holomorphic map and \(\Omega_\varphi\) is bounded in \(B\). Then \(C_\varphi\) is metrically bounded on \(N^p(B)\). More precisely, there exists a positive constant \(L\) depending only on \(n\) and \(\varphi\) such that for all \(f \in H(B)\)
\[ \|C_{\varphi}f\|_{\mathcal{K}(B)} \leq L\|f\|_{\mathcal{K}(B)}. \]

**Proof.** Set \( a = \varphi(0) \) and \( \psi = \varphi \circ \varphi \), where \( \varphi \) is the involution described in [14, p.25]. Then \( \psi \) is a univalent holomorphic self-map of \( B \) and \( \psi(0) = 0 \). First, we show that \( C_{\psi} \) is metrically bounded on \( N^p(B) \). Take \( f \in H(B) \setminus \{0\} \). Define \( \chi(t) = (\log(1 + e^{t}))^p (t \in \mathbb{R}) \). Since \( \Omega_{\psi}(z) \leq \Omega_{\varphi}(\varphi(z)) \Omega_{\varphi}(z) \) for any \( z \in B \) and \( (\Omega_{\varphi} \circ \varphi) \cdot \Omega_{\varphi} \) is bounded in \( B \), we have
\[
M = \sup_{z \in B} \Omega_{\varphi}(z) < \infty.
\]

Note that \( M \) is a positive constant depending only on \( \varphi \). It follows from the chain rule that for any \( z \in B \)
\[
|\nabla(f \circ \psi)(z)|^2 \leq (|\nabla f \circ \psi(\psi(z))|^2|^\psi'(z)|^2 \leq M(|\nabla f \circ \psi)(z)|^2|J_{\varphi}(z)|^2. \tag{2.4}
\]

Since \( \psi(0) = 0 \), Schwarz's lemma gives
\[
|\psi(z)| \leq |z| \tag{2.5}
\]
for any \( z \in B \). Using (2.1), Lemma 2.1(a), (2.4), (2.5), a change of variables, (2.2), (2.3) and Lemma 2.4(b) one after another, we have
\[
\int_{B \setminus B_{\frac{1}{2}}} \chi''(\log(|f \circ \psi(z)|)) \frac{|R(f \circ \psi)(\psi(z))|^2}{|f \circ \psi(z)|^2} |z|^{-2(n-1)} \log \frac{1}{|z|} \, d\nu(z)
\begin{align*}
\leq & \ 2^{2(n-1)} \int_{B \setminus B_{\frac{1}{2}}} \chi''(\log(|f \circ \psi(z)|)) \frac{|\nabla f \circ \psi(\psi(z))|^2}{|f \circ \psi(z)|^2} (1 - |\psi(z)|^2) |J_{\varphi}(z)|^2 \, d\nu(z) \\
\leq & \ 2^{2(n-1)} M \\
\times & \ \int_{B \setminus B_{\frac{1}{2}}} \chi''(\log(|f(z)|)) \frac{|\nabla f(\psi(z))|^2}{|f(\psi(z))|^2} (1 - |\psi(z)|^2) |J_{\varphi}(z)|^2 \, d\nu(z) \\
= & \ 2^{2(n-1)} M \int_{B \setminus B_{\frac{1}{2}}} \chi''(\log(|f(z)|)) \frac{|\nabla f(\psi(z))|^2}{|f(z)|^2} (1 - |\psi(z)|^2) \, d\nu(z) \\
\leq & \ 2^{2(n-1)} M \int_{B \setminus B_{\frac{1}{2}}} \chi''(\log(|f(z)|)) \frac{|\nabla f(z)|^2}{|f(z)|^2} (1 - |z|^2) \, d\nu(z) \\
\leq & \ 2^{2(n-1)} M \int_{B \setminus B_{\frac{1}{2}}} \chi''(\log(|f(z)|)) \frac{|\nabla f(z)|^2}{|f(z)|^2} (1 - |z|^2) \, d\nu(z) \\
\leq & \ 2^{2(n-1)} M \left( \frac{2}{\log 2} \right) |f\|_{\mathcal{K}(B)}. \tag{2.6}
\end{align*}
\]

On the other hand, since \( \frac{\log 3 - \log 2}{\log 2} \log \frac{1}{t} \leq \frac{3}{4t} \) for all \( t \in (0, \frac{1}{2}] \), it follows from Lemma 2.3 that
\[
\int_{B \setminus B_{\frac{1}{2}}} \chi''(\log(|f \circ \psi(z)|)) \frac{|R(f \circ \psi)(\psi(z))|^2}{|f \circ \psi(z)|^2} |z|^{-2(n-1)} \log \frac{1}{|z|} \, d\nu(z)
\leq \frac{\log 2}{\log 3 - \log 2} \\
\times \int_{B \setminus B_{\frac{1}{2}}} \chi''(\log(|f \circ \psi(z)|)) \frac{|R(f \circ \psi)(\psi(z))|^2}{|f \circ \psi(z)|^2} |z|^{-2(n-1)} \log \frac{3}{4|z|} \, d\nu(z)
\leq \frac{2n \log 2}{\log 3 - \log 2} |f\|_{\mathcal{K}(B)}. \tag{2.7}
\]
By Lemma 2.6, we have

\[ \|(f \circ \psi)_{\overline{a}}\|^p_{\mathcal{D}} = \int_3 \log(1 + |(f \circ \psi)_{\overline{a}}|^2) \, d\sigma \leq \max_{w \in \varphi(\mathcal{D})} \log(1 + |f(w)|)^p \]
\[ \leq 2^n \max_{w \in \varphi(\mathcal{D})} (1 - |w|)^{-n} \|f\|^p_{\mathcal{D}}. \]  

(2.8)

(2.7) and (2.8) show that

\[ \int_{\frac{1}{2}B} \chi''(\log |(f \circ \psi)(z)|) \frac{|R(f \circ \psi)(z)|^2}{|(f \circ \psi)(z)|^2} |z|^{-2(n-1)\log \frac{1}{|z|}} \nu(z) \]
\[ \leq \frac{2^{n+1}n \log 2}{\log 3 - \log 2} \max_{w \in \varphi(\mathcal{D})} (1 - |w|)^{-n} \|f\|^p_{\mathcal{D}}. \]  

(2.9)

Since \( \psi(0) = 0 \), it follows from Lemma 2.6 that

\[ \{(\log(1 + |(f \circ \psi)(0)|)^p \leq \|f\|^p_{\mathcal{D}}. \]  

(2.10)

By (2.6), (2.9), (2.10) and Lemma 2.3, we have

\[ \|f \circ \psi\|^p_{\mathcal{D}} \leq L_1 \|f\|^p_{\mathcal{D}} \]

where

\[ L_1 = \frac{2^{n+1}(n+1)M}{4a_0 \Gamma(n)} + \frac{2^n \log 2}{\log 3 - \log 2} \max_{w \in \varphi(\mathcal{D})} (1 - |w|)^{-n} + 1. \]

Hence \( C_\psi \) is metrically bounded on \( N^p(B) \). Since \( \varphi \) is a biholomorphic map of \( B \) onto \( B \), Lemma 2.7 implies that \( C_{\varphi} \) is also metrically bounded on \( N^p(B) \). It holds that \( C_\varphi = C_{\varphi \circ C_\psi} \) because \( \varphi = \varphi \circ \psi \). Thus we conclude that \( C_\psi \) is metrically bounded on \( N^p(B) \):

\[ \|C_\psi f\|^p_{\mathcal{D}} \leq L_1 \|f\|^p_{\mathcal{D}} \]

for all \( f \in H(B) \) where

\[ L = L_1 \left( \frac{1 + |\varphi(a)(0)|}{1 - |\varphi(a)(0)|} \right)^n = L_1 \left( \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^n. \]

Note that \( L \) is a positive constant depending only on \( n \) and \( \varphi \). \( \square \)

For each \( \alpha \in (-1, \infty) \) and \( \chi \in ST^\alpha(R) \), we define the weighted Bergman-Orlicz space \( A_\chi(\nu_\alpha) \) by

\[ A_\chi(\nu_\alpha) = \{ f \in H(B) : \|f\|_{L, \alpha} < \infty \}. \]

**Lemma 2.10.** It holds that

\[ (AN)^{(\nu_\alpha)} = \bigcup_{\chi \in ST^\alpha(R)} A_\chi(\nu_\alpha). \]

for any \( \alpha \in (-1, \infty) \).

**Proof.** It is easily seen that
Take \( f \in (AN)^n(\nu_a) \). By the subharmonicity of the function \( \log(1+|f|) \) in \( B \), we have
\[
\lim_{r \downarrow 1} \int_B \log(1+|f_r|) \, d\nu_a = \sup_{0<r<1} \int_B \log(1+|f_r|) \, d\nu_a \leq \int_B \log(1+|f|) \, d\nu_a < \infty.
\]  
(2.11)
On the other hand, Fatou's lemma gives
\[
\int_B \log(1+|f|) \, d\nu_a = \int_B \liminf_{r \downarrow 1} \log(1+|f_r|) \, d\nu_a.
\]  
(2.12)
By (2.11) and (2.12), we obtain
\[
\lim_{r \downarrow 1} \int_B \log(1+|f_r|) \, d\nu_a = \int_B \log(1+|f|) \, d\nu_a < \infty.
\]
It follows from [6, Chap. V, Lemma 1.4] and [6, Chap. V, Theorem 1.3] that the family \( \{\log(1+|f_r|)\}_{0<r<1} \) is uniformly integrable with respect to the measure \( \nu_a \). The de la Vallée Poussin's theorem ([13, Theorem 3.10]) therefore implies that
\[
\int_B \chi(\log(1+|f|)) \, d\nu_a = \sup_{0<r<1} \int_B \chi(\log(1+|f_r|)) \, d\nu_a < \infty
\]
for some \( \chi \in ST^2(\mathbb{R}) \). Thus \( f \in A_\chi(\nu_a) \). This completes the proof. \( \square \)

**Lemma 2.11.** Let \( 1 \leq p < \infty \) and \(-1 < a < \infty\). Suppose that \( \varphi : B \to B \) is a univalent holomorphic map and \( \Omega_\varphi \) is bounded in \( B \). Then \( C_\varphi \) is metrically bounded on \( (AN)^p(\nu_a) \). More precisely, there exists a positive constant \( L_a \) depending only on \( n, a \) and \( \varphi \) such that
\[
\| C_\varphi f \|^p_{(AN)^p(\nu_a)} \leq L_a \| f \|^p_{(AN)^p(\nu_a)}
\]
for all \( f \in (AN)^p(\nu_a) \).

**Proof.** First we consider the case \( 1 < p < \infty \). Define \( \chi(t) = (\log(1+e^t))^p(t \in \mathbb{R}) \). Take \( f \in H(B) \setminus \{0\} \). As in the proof of Lemma 2.9, we set \( \alpha = \varphi(0) \) and \( \psi = \varphi_{\alpha} \circ \varphi \). Like (2.4) and (2.5), we have
\[
|\nabla (f \circ \psi)(z)|^p \leq M |(\nabla f) \circ \psi(z)|^p |\psi(z)|^p, \quad |\psi(z)| \leq |z|
\]  
(2.13)
for all \( z \in B \), where \( M = \sup_{z \in \partial \Omega_\varphi} \varphi(z) < \infty \). By Lemma 2.1(b), there are a positive constant \( d_{a,n} \) depending only \( a \) and \( n \), and \( r_0 \in (\frac{1}{2}, 1) \) such that
\[
K_a(|z|) \leq d_{a,n} (1 - |z|^2)^{a+2}
\]  
(2.14)
for all \( z \in B \setminus r_0 B \). By using (2.1)~(2.3), (2.13), (2.14) and Lemma 2.4(a), we have
\[
\int_{B \setminus r_0 B} \chi'(\log(|f \circ \psi(z)|)) \frac{|R(f \circ \psi)(z)|^2}{|R_{f \circ \psi}(z)|^2} |z|^{-2(n-1)} K_a(|z|) \, d\nu(z)
\]
\begin{align*}
\leq & 2^{2(n-1)} d_{a,n} M \int_{B \setminus r_0 B} \chi''(\log(1/\psi(z))) \left\{ \frac{|\nabla f(\psi(z))|^2}{\psi(z)} \right\} (1-|\psi(z)|^2)^\alpha \nu(z) \\
= & 2^{2(n-1)} d_{a,n} M \int_{B \setminus r_0 B} \chi''(\log|f(w)|) \left\{ \frac{|\nabla f(w)|^2}{|f(w)|^2} \right\} (1-|w|^2)^\alpha \nu(w) \\
\leq & \frac{2^{2(n-1)} d_{a,n} (n+1)M}{2} \int_{B} \chi''(\log|f(w)|) \left\{ \frac{|\nabla f(w)|^2}{|f(w)|^2} \right\} (1-|w|^2)^\alpha \nu(w) \\
= & 2^{2(n-1)} d_{a,n} (n+1)M \int_{B} \tilde{A}(\{\log(1+|f|)\})^\alpha |w|(1-|w|^2) \nu(w) \\
\leq & 2^{2(n-1)} d_{a,n} (n+1)M \|(f)\|_{(AN)^p(\nu_a)}.
\end{align*}

where \( A_{n,a} = \{a_n \Gamma(n+a+1)/(2^{a+a}(n+a+1)\Gamma(a+2)) \}. \)

On the other hand, by Lemma 2.2 we can easily see that for all \( z \in r_0 B \setminus \{0\} \)

\[ K_a(z) \leq \left[ (\log \frac{1}{r_0}) \left\{ K_a(\frac{r_0}{r_1}) \right\} \right]^{-1} K_a(\frac{|z|}{r_1}) \]

\[ = C_a K_a(\frac{|z|}{r_1}), \]

where \( r_1 = \frac{1+r_0}{2} \) and \( C_a = (\log \frac{1}{r_0}) \{K_a(\frac{r_0}{r_1})\}^{-1} + 1. \) It follows from (2.16) and Lemma 2.3 that

\[ \int_{r_0 B} \chi''(\log f(\psi(z))) \left\{ \frac{|R(f \circ \psi(z))|^2}{|f \circ \psi(z)|^2} \right\} |z|^{-2(n-1)} K_a(z) \nu(z) \]

\[ \leq C_a \int_{r_0 B} \chi''(\log f(\psi(z))) \left\{ \frac{|R(f \circ \psi(z))|^2}{|f \circ \psi(z)|^2} \right\} |z|^{-2(n-1)} K_a(\frac{|z|}{r_1}) \nu(z) \]

\[ \leq 2nC_a\|(C_Af)_{r_1}\|_{(AN)^p(\nu_a)}. \]

Moreover, by Lemma 2.6, we have

\[ \|(C_Af)_{r_1}\|_{(AN)^p(\nu_a)} \leq \max_{w \in \psi(\tau, B)} \left\{ \frac{1+|w|}{1-|w|} \right\}^{n+a+1} \|(f)\|_{(AN)^p(\nu_a)}, \]

\[ \{\log(1+|f(0)|)\}^p \leq \|(f)\|_{(AN)^p(\nu_a)}. \]

It follows from (2.15), (2.17)~(2.19) and Lemma 2.3 that \( C_\psi \) is metrically bounded on \( (AN)^p(\nu_a) : \)

\[ \|C_\psi f\|_{(AN)^p(\nu_a)} \leq D_a \|(f)\|_{(AN)^p(\nu_a)} \]

for any \( p \in (1, \infty) \) and any \( f \in H(B) \), where

\[ D_a = \frac{2^{2(n-1)} d_{a,n} (n+1)M}{2n A_{n,a}} + C_a \max_{w \in \psi(\tau, B)} \left\{ \frac{1+|w|}{1-|w|} \right\}^{n+a+1} + 1. \]

Since \( C_\psi = C_\psi \circ C_{\psi_a} \), it follows from Lemma 2.7 that

\[ \|C_\psi f\|_{(AN)^p(\nu_a)} \leq L_a \|(f)\|_{(AN)^p(\nu_a)} \]

(2.20)

for any \( p \in (1, \infty) \) and any \( f \in H(B) \), where
\[ L_a = D_a \left( \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^p. \]

Note that \( L_a \) is a positive constant depending only on \( n, \alpha \) and \( \varphi \).

Now we consider the case \( p = 1 \). Take \( f \in (AN)^p(\nu_a) \). For any \( r \in (0, 1) \) and any \( p \in (1, \infty) \), we have by (2.20)
\[
\int_B \{ \log(1 + |f(r\varphi(z))|) \}^p d\nu_a(z) \leq L_a \int_B \{ \log(1 + |f(yz)|) \}^p d\nu_a(z).
\]
By taking the limit as \( p \to 1 \) above, it holds that
\[
\int_B \log(1 + |f(r\varphi(z))|) d\nu_a(z) \leq L_a \int_B \log(1 + |f(yz)|) d\nu_a(z). \tag{2.21}
\]
By Lemma 2.10, the family \( \{ \log(1 + |f(z)|) \}_{0 < r < 1} \) is uniformly integrable with respect to the measure \( \nu_a \). Hence
\[
\lim_{r \to 1} \int_B \log(1 + |f(z)|) d\nu_a(z) = \int_B \log(1 + |f(z)|) d\nu_a(z). \tag{2.22}
\]
On the other hand, Fatou's lemma gives
\[
\int_B \log(1 + |f(\varphi(z))|) d\nu_a(z) \leq \liminf_{r \to 1} \int_B \log(1 + |f(r\varphi(z))|) d\nu_a(z). \tag{2.23}
\]
(2.21)~(2.23) show that
\[
\| f \|_{(AN)^p(\nu_a)} \leq L_a \| f \|_{(AN)^p(\nu_a)}. \]
This completes the proof.

The next lemma is a characterization of the compactness of \( UC_\varphi \) on \( H^p(B) \) and on \( A^p(\nu_a) \) in terms of sequential convergence. Its proof, which we omit, is based on the fact that bounded subsets of \( H^p(B) \) (respectively \( A^p(\nu_a) \)) are normal families. (cf. [5, Lemma 3.11])

**Lemma 2.12.** Let \( 0 < p < \infty \) and \( -1 \leq a < \infty \). Suppose that \( u \in H(B) \) and a holomorphic self-map \( \varphi \) of \( B \) satisfy \( UC_\varphi \)(\( A^p(\nu_a) \)) \( \subset A^p(\nu_a) \). Then \( UC_\varphi \) is compact on \( A^p(\nu_a) \) if and only if for every bounded sequence \( \{ f_j \} \) in \( A^p(\nu_a) \) which converges to \( 0 \) uniformly on compact subsets of \( B \), we have \( \lim_{j \to \infty} \| uC\varphi f_j \|_{A^p(\nu_a)} = 0 \).

For the metrically compactness of \( UC_\varphi \) on \( H^p(B) \) (respectively on \( (AN)^p(\nu_a) \)), an analogous result of Lemma 2.12 holds:

**Lemma 2.13.** Let \( 1 \leq p < \infty \) and \( -1 \leq a < \infty \). Suppose that \( u \in H(B) \) and a holomorphic self-map \( \varphi \) of \( B \) satisfy \( (UC_\varphi)(\( (AN)^p(\nu_a) \)) \( \subset (AN)^p(\nu_a) \). Then \( UC_\varphi \) is metrically compact on \( (AN)^p(\nu_a) \) if and only if for every bounded sequence \( \{ f_j \} \) in \( (AN)^p(\nu_a) \) which converges to \( 0 \) uniformly on compact subsets of \( B \), we have \( \lim_{j \to \infty} \| uC\varphi f_j \|_{(AN)^p(\nu_a)} = 0 \).

### 3 \( UC_\varphi \) on \( H^p(B) \) and \( A^p(\nu_a) \)

**Theorem 3.1.** Let \( 0 < p < \infty, u \in H(B) \setminus \{0\} \) and let \( \varphi : B \to B \) be a univalent holomorphic map such that \( \Omega_\varphi \) is bounded in \( B \).
(a) Suppose \( u \) and \( \varphi \) satisfy the following conditions:
\[
\limsup_{|z| \to 1} |u(z)|^{p-2} |(\nabla u)(z)|^2 (1-|z|^2)^2 < \infty,
\]
(3.1)
\[
\limsup_{|z| \to 1} \frac{|u(z)|^p (1-|z|^2)}{1-|\varphi(z)|^2} < \infty.
\]
(3.2)

Then \( uC_\varphi \) is bounded on \( H^p(B) \).

(b) Suppose \( u \) and \( \varphi \) satisfy the following conditions:
\[
\lim_{|z| \to 1} |u(z)|^{p-2} |(\nabla u)(z)|^2 (1-|z|^2) = 0,
\]
(3.3)
\[
\lim_{|z| \to 1} \frac{|u(z)|^p (1-|z|^2)}{1-|\varphi(z)|^2} = 0.
\]
(3.4)

Then \( uC_\varphi \) is compact on \( H^p(B) \).

Proof. Put \( Tf = uC_\varphi f \) for \( f \in H(B) \). If we show that \( Tf \in H^p(B) \) whenever \( f \in H^p(B) \), the closed graph theorem will give that \( T \) is bounded on \( H^p(B) \). Since \( \varphi \) is a univalent holomorphic self-map of \( B \) and \( \Omega_\varphi \) is bounded in \( B \), by Theorem 1.1(a), \( C_\varphi \) is bounded on \( H^p(B) \) (see [5, Theorem 3.41]). Thus \( \|C_\varphi f\|_{H^p(B)} \leq \|C_\varphi\| \|f\|_{H^p(B)} \) for all \( f \in H^p(B) \). And so,
\[
\int_B |C_\varphi f|^p d\nu \leq \|C_\varphi\|^p \|f\|^p_{H^p(B)}. \tag{3.5}
\]

It follows from the chain rule that for any \( z \in B \)
\[
|\nabla (Tf)(z)|^2 \leq 2|(|\nabla u|(z) (C_\varphi f)(z)|^2
+ 2|u(z)|^2 |(\nabla f)(\varphi(z))|^2 |f'(z)|^2
\leq 2|(|\nabla u|(z) f(\varphi(z))|^2
+ 2M|u(z)|^2 |(\nabla f)(\varphi(z))|^2 |J_\varphi(z)|^2, \tag{3.6}
\]
where \( M \equiv \sup_{z \in \Omega_\varphi} |\varphi(z)| < \infty \). By (3.1) and (3.2), there are two positive constants \( \epsilon_1, \epsilon_2 \) and \( r_0 \in (\frac{1}{2}, 1) \) such that
\[
|u(z)|^{p-2} |(\nabla u)(z)|^2 (1-|z|^2) < \epsilon_1, \tag{3.7}
\]
\[
\frac{|u(z)|^p (1-|z|^2)}{1-|\varphi(z)|^2} < \epsilon_2, \tag{3.8}
\]
for any \( z \in B \setminus r_0 B \). Take \( f \in H^p(B) \setminus \{0\} \). By (2.1)\(\sim\) (2.2), (3.5)\(\sim\) (3.8), Lemma 2.1(a) and a change of variables, we have
\[
\int_{B \setminus r_0 B} \frac{|R(Tf)(z)|^p}{|z|^2} |(Tf)(z)|^{p-2} |z|^{-2(n-1)} \log \frac{1}{|z|} d\nu(z)
\leq 2^{n-1} \left[ \int_{B \setminus r_0 B} |(\nabla u)(z)|^2 |u(z)|^{p-2} |f(\varphi(z))|^2 (1-|z|^2) d\nu(z)
+ M \int_{B \setminus r_0 B} |u(z)|^2 |(\nabla f)(\varphi(z))|^2 |f(\varphi(z))|^2 |(1-|z|^2)| J_\varphi(z)^2 d\nu(z) \right]
\]
\[
\begin{align*}
&\leq 2^{2p-1}\left[ e_1 \int_{B_{n/2}} |(C_\psi f)(z)|^p d\nu(z) \\
&\quad + Me_2 \int_{B_{n/2}} |(\nabla f)(\varphi(z))|^p |f(\varphi(z))|^p |1-|\varphi(z)|^2|^p |\varphi(z)|^2 d\nu(z) \right] \\
&\leq 2^{2p-1}\left[ e_1 \int_{B_{n/2}} |(C_\psi f)(z)|^p d\nu(z) \\
&\quad + Me_2 \int_{B_{n/2}} |(\nabla f)(w)|^p |f(w)|^p |1-|w|^2|^p d\nu(w) \right] \\
&\leq 2^{2p-1} e_1 \|C_\psi\|^p \|f\|^p_{H^p(B)} \\
&\quad + 2^{2p-2}(n+1) Me_2 \int_{B_{n/2}} |(\nabla f)(w)|^2 |f(w)|^p |1-|w|^2|^p \frac{d\nu(w)}{1-|w|^2}. \quad (3.9)
\end{align*}
\]

By (2.3) and Lemma 2.5, we have
\[
\int_{B_{n/2}} |(\nabla f)(w)|^p |f(w)|^p |1-|w|^2|^p \frac{d\nu(w)}{1-|w|^2} \leq \frac{2}{a_{n-1} p^2} \|f\|^p_{H^p(B)}. \quad (3.10)
\]

It follows from (3.9) and (3.10) that
\[
\int_{B_{n/2}} \left| \frac{|(T f)(z)|^p}{|z|^2} \right|^p \left| (T f)(z) \right|^{p-2} |z|^{-2(n-1) \log \frac{1}{|z|}} \frac{d\nu(z)}{|z|} \\
\leq 2^{2p-1}\left[ e_1 \|C_\psi\|^p + \frac{(n+1) Me_2}{a_{n-1} p^2} \right] \|f\|^p_{H^p(B)} < \infty. \quad (3.11)
\]

On the other hand, we set \( \eta = \frac{1}{2} + \eta_0 \) and \( \delta = 1 - \log \frac{\eta_1}{\log \eta_0} \). We can easily see that \( 0 < \delta < 1 \) and for all \( t \in (0, \eta_0] \)
\[
\delta \log \frac{1}{t} \leq \log \frac{\eta_1}{t}. \quad (3.12)
\]

By using (3.12) and Lemma 2.3, we obtain
\[
\int_{B_{n/2}} \left| \frac{|(T f)(z)|^2}{|z|^2} \right|^p \left| (T f)(z) \right|^{p-2} |z|^{-2(n-1) \log \frac{1}{|z|}} \frac{d\nu(z)}{|z|} \\
\leq \frac{1}{\delta} \int_{B_{n/2}} \left| \frac{|(T f)(z)|^2}{|z|^2} \right|^p \left| (T f)(z) \right|^{p-2} |z|^{-2(n-1) \log \frac{\eta_1}{|z|}} \frac{d\nu(z)}{|z|} \\
\leq \frac{2n}{\delta p^2} \|B \|_{H^p(B)} \\
\leq \frac{2n}{\delta p^2} \max \left| u(z) \right|^p \max_{x \in \Omega(\eta_1)} \left| f(z) \right|^p < \infty. \quad (3.13)
\]

(3.11), (3.13) and Lemma 2.3 show \( \|T f\|^p_{H^p(B)} < \infty \). Hence \( uC_\psi \in H^p(B) \). This completes the proof of (a).

To prove (b), suppose that \( \{f_j\} \) is a sequence in \( H^p(B) \) which converges to zero uniformly on compact subsets of \( B \) and \( \|f_j\|^p_{H^p(B)} \leq L < \infty \) for all \( j \in \mathbb{N} \). Let \( \varepsilon > 0 \) be given. By (3.3) and (3.4), we can choose \( \eta_0 \in \left( \frac{1}{2}, 1 \right) \) such that
\[
|u(z)|^{p-2} |(\nabla u)(z)|^2 (1-|z|^2) < \varepsilon, \quad (14)
\]
\[
\left| \frac{u(z)}{1-|\varphi(z)|^2} \right|^p (1-|\varphi(z)|^2) < \varepsilon. \quad (15)
\]
for any \( z \in B \setminus r_0 \bar{B} \). By (a), it holds that \( T(H^p(B)) \subseteq H^p(B) \). If we show that \( \lim_{j \to \infty} \| T_{f_j} \|_{H^p(B)} = 0 \), then Lemma 2.12 will give that \( T = uC_\varphi \) is compact on \( H^p(B) \). In the same way as in the proof of (a), by using (3.14), (3.15) and Lemma 2.5, we have

\[
\int_{B \setminus r_0 \bar{B}} \frac{|R(T_{f_j}(z))|^p}{|z|^2} |(T_{f_j}(z))^{p-2}| |z|^{-2(n-1)\log \frac{1}{|z|}} dv(z)
\leq 2^{2n-1} L \left[\|C_\varphi\|_p + \frac{(n+1)M}{a_{n-1}p^p} \right] \epsilon \tag{3.16}
\]

for all \( j \in \mathbb{N} \).

On the other hand, as in (3.13), we obtain for all \( j \in \mathbb{N} \)

\[
\int_{r_0} \frac{|R(T_{f_j}(z))|^p}{|z|^2} |(T_{f_j}(z))^{p-2}| |z|^{-2(n-1)\log \frac{1}{|z|}} dv(z)
\leq \frac{2n}{\partial^p} \max_{z \in \varphi(r_1)} |u(z)|^p \max_{z \in \varphi(r_1)} |f_j(z)|^p \tag{3.17}
\]

where \( r_1 = \frac{1+\log r_0}{2} \) and \( \delta = 1 - \frac{\log r_1}{\log r_0} \). Since \( \{f_j\} \) converges to zero uniformly on compact subsets of \( B \),

\[
\lim_{j \to \infty} \max_{z \in \varphi(r_1)} |f_j(z)|^p = 0 \tag{3.18}
\]

(3.17) and (3.18) show that

\[
\lim_{j \to \infty} \int_{r_0} \frac{|R(T_{f_j}(z))|^p}{|z|^2} |(T_{f_j}(z))^{p-2}| |z|^{-2(n-1)\log \frac{1}{|z|}} dv(z) = 0. \tag{3.19}
\]

(3.16), (3.19) and Lemma 2.3 imply that \( \lim_{j \to \infty} \| T_{f_j} \|_{H^p(B)} = 0 \). This completes the proof of (b). \( \square \)

**Theorem 3.2.** Let \( 0 < p < \infty \) and \( -1 < \alpha < \infty \). Let \( u \in H(B) \) and \( \varphi : B \to B \) be a univalent holomorphic map such that \( \Omega_\varphi \) is bounded in \( B \).

(a) Suppose \( u \) and \( \varphi \) satisfy the following conditions:

\[
\limsup_{|z| \to 1} |u(z)|^{p-2} |(\nabla u)(z)|^2 (1 - |z|^2)^2 < \infty, \tag{3.20}
\]

\[
\limsup_{|z| \to 1} \left| u(z) \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha+2} \right| < \infty. \tag{3.21}
\]

Then \( uC_\varphi \) is bounded on \( A^p(\nu_\alpha) \).

(b) Suppose \( u \) and \( \varphi \) satisfy the following conditions:

\[
\lim_{|z| \to 1} |u(z)|^{p-2} |(\nabla u)(z)|^2 (1 - |z|^2)^2 = 0, \tag{3.22}
\]

\[
\lim_{|z| \to 1} \left| u(z) \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha+2} \right| = 0. \tag{3.23}
\]

Then \( uC_\varphi \) is compact on \( A^p(\nu_\alpha) \).

**Proof.** Take \( f \in A^p(\nu_\alpha) \setminus \{0\} \). As in the proof of Theorem 3.1,
\[
|\nabla (uC_{\varphi}) (z) |^2 \leq 2 |(\nabla u) (z) |^2 | f (\varphi (z)) |^2 \\
+ 2M |u(z)|^p |(\nabla f) (\varphi(z))|^2 |J_f (z)|^2 
\]
(3.24)
for any \( z \in B \), where \( M = \sup_{z \in B} |\varphi (z) | < \infty \). By Theorem 1.1(a), \( C_{\varphi} \) is bounded on \( A^p (\nu_{\varphi}) \).

By (3.20) and (3.21), there exist positive constants \( \varepsilon_1, \varepsilon_2 \) and \( n_0 \geq \frac{3}{2}, 1 \) such that
\[
|u(z)|^{p-2}(\nabla u(z))^2 |1-|z|^2| < \varepsilon_1, 
\]
(3.25)
\[
|u(z)|^p \left( \frac{1-|z|^2}{1-|\varphi(z)|^2} \right) ^{\alpha+2} < \varepsilon_2 
\]
(3.26)
for any \( z \in B \setminus n_B \). Furthermore, we have by Lemma 2.1(b)
\[
K_{n_0} (|z|) \leq d_{n_0} (1-|z|^2)^{n_0} 
\]
(3.27)
for any \( z \in B \setminus n_B \), where \( d_{n_0} \) is a positive constant depending only on \( n \) and \( \alpha \). By (3.24) \sim (3.27), Lemma 2.5 and the same argument as in the proof of Theorem 3.1(a), we have
\[
\int_{\mathbb{D}} \frac{|R(uC_{\varphi}) (z) |^2}{|z|^p} |(uC_{\varphi}) (z) |^{p-2} |z|^{-2(n-1)} K_n (|z|) d\nu (z) \\
\leq \left[ \frac{2^{n-1} d_{n_0} \varepsilon_1 |C_{\varphi}|^p}{C_{\varphi}} \\
+ \frac{2^{n-1}(n+1) d_{n_0} \varepsilon_2 M}{d_{n_0} p^2} \right] \| f \|_{A^p (\nu_{\varphi})} < \infty. 
\]
(3.28)
On the other hand, by (2.16) and Lemma 2.3, we obtain
\[
\int_{\mathbb{D}} \frac{|R(uC_{\varphi}) (z) |^2}{|z|^p} |(uC_{\varphi}) (z) |^{p-2} |z|^{-2(n-1)} K_n (|z|) d\nu (z) \\
\leq C_n \int_{\mathbb{D}} \frac{|R(uC_{\varphi}) (z) |^2}{|z|^p} |(uC_{\varphi}) (z) |^{p-2} |z|^{-2(n-1)} K_n (|z|) d\nu (z) \\
\leq \frac{2 n C_{\varphi}}{p^2} \| (uC_{\varphi}) \|_{A^p (\nu_{\varphi})} < \infty, 
\]
(3.29)
where \( r_1 = \frac{1+r_0}{2} \) and \( C_n = (\log \frac{1}{r_0}) (K_n (\frac{r_0}{r_1}))^{-1} + 1 \). (3.28), (3.29) and Lemma 2.3 give that \( \| uC_{\varphi} \|_{A^p (\nu_{\varphi})} < \infty \), that is, \( uC_{\varphi} \in A^p (\nu_{\varphi}) \). This completes the proof of (a).

In order to prove (b), suppose that \( \{ f_j \} \) is a bounded sequence in \( A^p (\nu_{\varphi}) \) which converges to zero uniformly on compact subsets of \( B \). As in the proof of Theorem 3.1(b), we can show that \( \lim_{j \to \infty} \| uC_{\varphi} f_j \|_{A^p (\nu_{\varphi})} = 0 \). It follows from Lemma 2.12 that \( uC_{\varphi} \) is compact on \( A^p (\nu_{\varphi}) \). The proof is now complete.

4 \( uC_{\varphi} \) on \( \N^p (B) \) and \( (AN)^p (\nu_{\varphi}) \)

Theorem 4.1. Let \( 1 < p < \infty \), \( u \in H(B) \setminus \{ 0 \} \) and let \( \varphi : B \to B \) be a univalent holomorphic map such that \( \Omega_{\varphi} \) is bounded in \( B \).

(a) Suppose \( u \) and \( \varphi \) satisfy the following conditions:

(1) \( u \) is bounded in \( B \).
(2) \( \varphi \) is a univalent function.
(3) \( \Omega_{\varphi} \) is bounded in \( B \).

(b) For any \( \lambda > 0 \), there exists a constant \( C_{\lambda} \) such that \( \| uC_{\varphi} \|_{A^p (\nu_{\varphi})} \leq C_{\lambda} \| f \|_{A^p (\nu_{\varphi})} \).
(i) When $1 < p \leq 2$,
\[
\limsup_{|z| \to 1} \left[ \max \left( |u(z)|^{p-4} \right) |(\nabla u)(z)|^2 (1 - |z|^2) \right] < \infty, \tag{4.1}
\]
\[
\limsup_{|z| \to 1} \left[ \max \left( |u(z)|^{p-3} \right) |(\nabla u)(z)|^2 \right] < \infty. \tag{4.2}
\]

(ii) When $2 < p < \infty$,
\[
\limsup_{|z| \to 1} \left[ \max \left( |u(z)|^{-2} \right) |u(z)|^2 |(\nabla u)(z)|^2 (1 - |z|^2) \right] < \infty, \tag{4.3}
\]
\[
\limsup_{|z| \to 1} \left[ \max \left( |u(z)|^{-1} \right) \left( |u(z)|^p \right) \right] < \infty. \tag{4.4}
\]

Then $uC_\varphi$ is metrically bounded on $N^p(B)$.

(b) Suppose $u$ and $\varphi$ satisfy the following conditions:

(i) When $1 < p \leq 2$,
\[
\lim_{|z| \to 1} \left[ \max \left( |u(z)|^{p-4} \right) |(\nabla u)(z)|^2 (1 - |z|^2) \right] = 0, \tag{4.5}
\]
\[
\limsup_{|z| \to 1} \left[ \max \left( |u(z)|^{p-3} \right) |u(z)|^2 (1 - |z|^2) \right] = 0. \tag{4.6}
\]

(ii) When $2 < p < \infty$,
\[
\lim_{|z| \to 1} \left[ \max \left( |u(z)|^{-2} \right) |u(z)|^2 |(\nabla u)(z)|^2 (1 - |z|^2) \right] = 0, \tag{4.7}
\]
\[
\limsup_{|z| \to 1} \left[ \max \left( |u(z)|^{-1} \right) \left( |u(z)|^p \right) \right] = 0. \tag{4.8}
\]

Then $uC_\varphi$ is metrically compact on $N^p(B)$.

Proof. Take $f \in N^p(B) \setminus \{0\}$. Since $\varphi$ is a univalent holomorphic self-map of $B$ and $\Omega_\varphi$ is bounded in $B$, by Lemma 2.9, $C_\varphi$ is metrically bounded on $N^p(B)$, that is, $\|C_\varphi f\|_{N^p(B)} \leq L \|f\|_{N^p(B)}$ where $L$ is a positive constant depending only on $n$ and $\varphi$. And so,
\[
\int_B \left( \log(1 + |C_\varphi f|)^p \right) d\nu \leq \|C_\varphi f\|_{N^p(B)} \leq L \|f\|_{N^p(B)}. \tag{4.9}
\]

By (4.1)～(4.4), there are positive constants $\epsilon_1$, $\epsilon_2$ and $\gamma_0 \in (\frac{1}{2}, 1)$ such that when $1 < p \leq 2$,
\[
\max \left( |u(z)|^{p-4} \right) |(\nabla u)(z)|^2 (1 - |z|^2) < \epsilon_1, \tag{4.10}
\]
\[
\max \left( |u(z)|^{p-3} \right) |u(z)|^2 (1 - |z|^2) < \epsilon_2, \tag{4.11}
\]
when $2 < p < \infty$,
\[
\max \left( |u(z)|^{-2} \right) |u(z)|^2 |(\nabla u)(z)|^2 (1 - |z|^2) < \epsilon_1, \tag{4.12}
\]
\[
\max \left( |u(z)|^{-1} \right) \left( |u(z)|^p \right) (1 - |z|^2) < \epsilon_2. \tag{4.13}
\]
for any \( z \in B \setminus r_0B \). Define

\[
E_1 = \{ z \in B : |u(z)| \leq 1 \}, \quad E'_1 = E_1 \cap (B \setminus r_0B), \\
E_2 = \{ z \in B : |u(z)| > 1 \}, \quad E'_2 = E_2 \cap (B \setminus r_0B).
\]

It is obvious that the following inequalities hold:

\[
(p-1)|u_C\phi| + \log(1+|u_C\phi|) \leq (p-1)|C\phi| + \log(1+|C\phi|), \tag{4.14}
\]

\[
\frac{1}{(1+|u_C\phi|)^2} \leq \frac{|u|^2}{(1+|C\phi|)^2} \tag{4.15}
\]

in \( E_1 \), and

\[
(p-1) + \frac{\log(1+|u_C\phi|)}{|u_C\phi|} \leq (p-1) + \frac{\log(1+|C\phi|)}{|C\phi|} \tag{4.16}
\]

in \( E_2 \). We can also easily see that if \( 1 \leq p \leq 2 \),

\[
\frac{\log(1+|u_C\phi|)}{|u_C\phi|} \leq \frac{\log(1+|C\phi|)}{|C\phi|} \tag{4.17}
\]

\[
\frac{\log(1+|u_C\phi|)}{(1+|u_C\phi|)^2} \leq \frac{\log(1+|C\phi|)}{(1+|C\phi|)^2} \tag{4.18}
\]

When \( 2 < p < \infty \),

\[
\log(1+|u_C\phi|) \leq \log(1+|C\phi|) \tag{4.19}
\]

\[
\log(1+|u_C\phi|) \leq |u| \log(1+|C\phi|) \tag{4.20}
\]

Put \( \chi(t) = (\log(1+e^t))^p \) (\( t \in \mathbb{R} \)). By a simple computation, we have

\[
\chi''(\log|u_C\phi|) = p \frac{\log(1+|u_C\phi|)}{(1+|u_C\phi|)^2}
\times [(p-1)|u_C\phi| + \log(1+|u_C\phi|)]|u_C\phi| \tag{4.21}
\]

in \( B \). By (2.1), Lemma 2.1(a) and (3.6), we have

\[
\int_{B \setminus r_0B} \chi''(\log(|u_C\phi|(z))) \frac{|R(u_C\phi)(z)|^2}{|u_C\phi(z)|^2} |\nabla u(z)|^2 |(C\phi)(z)|^2 (1-|z|^2) dv(z)
\leq 2^{2n-1} \left[ \int_{B \setminus r_0B} \chi''(\log(|u_C\phi|(z))) \frac{|\nabla u(z)|^2 |C\phi(z)|^2}{|u_C\phi(z)|^2} (1-|z|^2) dv(z)
+ M \int_{B \setminus r_0B} \chi''(\log(|u_C\phi|(z))) \frac{|u(z)|^2 |\nabla \psi(z)|^2}{|u_C\phi(z)|^2} (1-|z|^2) J_\psi(z)^2 dv(z) \right] \tag{4.22}
\]

where \( M = \sup_{r_0 < r} Q_\psi(z) < \infty \). Let \( V_1 \) and \( V_2 \) be defined by

\[
V_1 = \int_{B \setminus r_0B} \chi''(\log(|u_C\phi|(z))) \frac{|\nabla u(z)|^2 |C\phi(z)|^2}{|u_C\phi(z)|^2} (1-|z|^2) dv(z), \tag{4.23}
\]

\[
V_2 = \int_{B \setminus r_0B} \chi''(\log(|u_C\phi|(z))) \frac{|u(z)|^2 |\nabla \psi(z)|^2}{|u_C\phi(z)|^2} (1-|z|^2) J_\psi(z)^2 dv(z). \tag{4.24}
\]

By (4.21) and (4.23), we have
\[ V_1 = p \int_{\mathbb{R}^n \setminus \mathbb{S}^n} \left[ \frac{\log(1 + |(u_{C_{p0}})(z)|)}{1 + |(u_{C_{p0}})(z)|} \right]^{p-2} \left\{ (p-1) + \frac{\log(1 + |(u_{C_{p0}})(z)|)}{|(u_{C_{p0}})(z)|} \right\} (\nabla u)(z) \right] dv(z) \\
\leq p^2 \int_{\mathbb{R}^n \setminus \mathbb{S}^n} \frac{\log(1 + |(u_{C_{p0}})(z)|)}{1 + |(u_{C_{p0}})(z)|} \frac{(|(u_{C_{p0}})(z)|)^2}{\nabla u(z)} \frac{((C_{p0})(z))^p}{(1 - |z|^p)} dv(z). \] (4.25)

Let \( V_{1,k} (k=1, 2) \) be defined by
\[ V_{1,k} = \int_{E_k} \frac{\log(1 + |(u_{C_{p0}})(z)|)}{1 + |(u_{C_{p0}})(z)|} \frac{|(\nabla u)(z)|^2}{(|(u_{C_{p0}})(z)|^2)} ((C_{p0})(z))^p (1 - |z|^p) dv(z). \] (4.26)

Note that the following elementary inequality holds:
\[ \frac{t}{1+t} \leq \log(1+t) \quad (t \geq 0). \] (4.27)

Let \( 1 < p \leq 2 \). By (4.15), (4.17), (4.26) and (4.27), we have
\[ V_{1,1} \leq \int_{E_1} \frac{|u(z)|^{p-2} \log(1 + |(C_{p0})(z)|)}{1 + |(C_{p0})(z)|} \frac{|(\nabla u)(z)|^2}{(|(u_{C_{p0}})(z)|^2)} ((C_{p0})(z))^p (1 - |z|^p) dv(z) \\
= \int_{E_1} |u(z)|^{p-4} \log(1 + |(C_{p0})(z)|) \frac{(|(C_{p0})(z)|)^3}{1 + |(C_{p0})(z)|} \frac{|(\nabla u)(z)|^2}{(|(u_{C_{p0}})(z)|^2)} ((C_{p0})(z))^p (1 - |z|^p) dv(z) \\
\leq \int_{E_1} |u(z)|^{p-4} \log(1 + |(C_{p0})(z)|) |(\nabla u)(z)|^2 (1 - |z|^p) dv(z). \] (4.28)

And we have by (4.18), (4.26) and (4.27),
\[ V_{1,2} \leq \int_{E_2} \frac{\log(1 + |(C_{p0})(z)|)}{1 + |(C_{p0})(z)|} \frac{|(\nabla u)(z)|^p}{(|(u_{C_{p0}})(z)|^2)} ((C_{p0})(z))^p (1 - |z|^p) dv(z) \\
\leq \int_{E_1} \log(1 + |(C_{p0})(z)|) |(\nabla u)(z)|^p (1 - |z|^p) dv(z). \] (4.29)

It follows from (4.25), (4.26), (4.28) and (4.29) that
\[ V_1 \leq p^2 \int_{E_1} |u(z)|^{p-4} \log(1 + |(C_{p0})(z)|) |(\nabla u)(z)|^p (1 - |z|^p) dv(z) \\
+ p^2 \int_{E_2} \log(1 + |(C_{p0})(z)|) |(\nabla u)(z)|^p (1 - |z|^p) dv(z) \\
= p^2 \int_{\mathbb{R}^n \setminus \mathbb{S}^n} \log(1 + |(C_{p0})(z)|) \max(|u(z)|^{p-4}, 1) |(\nabla u)(z)|^p (1 - |z|^p) dv(z). \] (4.30)

By (4.9), (4.10) and (4.30), we obtain
\[ V_1 \leq \epsilon p^2 \int_{\mathbb{R}^n \setminus \mathbb{S}^n} \log(1 + |C_{p0}(z)|)^p dv \leq \epsilon; p^2 L \|f\|_{L^p(M)}. \] (4.31)

When \( 2 < p < \infty \), by (4.15), (4.19), (4.26) and (4.27), we have
\[ V_{1,1} \leq \int_{E_1} \frac{\log(1 + |(C_{p0})(z)|)^{p-2}}{|(u_{C_{p0}})(z)|^2} |(\nabla u)(z)|^p ((C_{p0})(z))^p (1 - |z|^p) dv(z) \\
\leq \int_{E_1} |u(z)|^{p-2} |(\nabla u)(z)|^p (1 - |z|^p) dv(z). \] (4.32)
And we have by (4.20), (4.26) and (4.27),
\begin{align*}
V_{1,2} & \leq \int_{E_{3}} \frac{|u(z)|^{p-2}[(\log(1+|(C_{\psi}f)(z)|))^{p-2}[(\nabla u)(z)]^{2}|(C_{\psi}f)(z)|^{2}(1-|z|^{2})}d\nu(z) \\
& \leq \int_{E_{3}} [(\log(1+|(C_{\psi}f)(z)|))]^{p}|u(z)|^{p-2}[(\nabla u)(z)]^{2}(1-|z|^{2})d\nu(z).
\end{align*}
(4.33)

By (4.25), (4.26), (4.32) and (4.33),
\begin{align*}
V_{1} & \leq p^{2} \int_{E_{3}} [(\log(1+|(C_{\psi}f)(z)|))]^{p}|u(z)|^{p-2}[(\nabla u)(z)]^{2}(1-|z|^{2})d\nu(z) \\
& \quad + p^{2} \int_{E_{3}} [(\log(1+|(C_{\psi}f)(z)|))]^{p}|u(z)|^{p-2}[(\nabla u)(z)]^{2}(1-|z|^{2})d\nu(z) \\
& \quad = p^{2} \int_{B_{r_{0}}^{(3)}} [(\log(1+|(C_{\psi}f)(z)|))]^{p}\max(|u(z)|^{-2},|u(z)|^{p-2}) \\
& \quad \times [(\nabla u)(z)]^{2}(1-|z|^{2})d\nu(z).
\end{align*}
(4.34)

(4.9), (4.12) and (4.34) give
\begin{align*}
V_{1} & \leq \epsilon_{1}p^{2}L||f||_{L^{p}(B)}.
\end{align*}
(4.35)

On the other hand, by (4.21) and (4.24), we have
\begin{align*}
V_{2} & = p \int_{B_{r_{0}}^{(3)}} \frac{[(\log(1+|(uC_{\psi}f)(z)|))]^{p-2}}{[(1+|(uC_{\psi}f)(z)|)]^{2}} \left\{ (p-1) + \frac{\log(1+|(uC_{\psi}f)(z)|)}{|(uC_{\psi}f)(z)|} \right\} \\
& \quad \times |u(z)|^{2}[(\nabla f)(\psi(z))]^{2}(1-|z|^{2})|J_{\psi}(z)|^{2}d\nu(z).
\end{align*}
(4.36)

Let $V_{2,k}$ ($k=1, 2$) be defined by
\begin{align*}
V_{2,k} & = \int_{E_{3}} \frac{[(\log(1+|(uC_{\psi}f)(z)|))]^{p-2}}{[(1+|(uC_{\psi}f)(z)|)]^{2}} \left\{ (p-1) + \frac{\log(1+|(uC_{\psi}f)(z)|)}{|(uC_{\psi}f)(z)|} \right\} \\
& \quad \times |u(z)|^{2}[(\nabla f)(\psi(z))]^{2}(1-|z|^{2})|J_{\psi}(z)|^{2}d\nu(z).
\end{align*}
(4.37)

Let $1 < p \leq 2$. By (4.14), (4.15), (4.17), (4.21) and (4.37), we have
\begin{align*}
V_{2,1} & \leq \int_{E_{3}} \frac{|u(z)|^{p-2}[(\log(1+|(C_{\psi}f)(z)|))]^{p-2}}{|u(z)|^{2}(1+|(C_{\psi}f)(z)|)^{2}} \\
& \quad \times \left\{ (p-1)|(C_{\psi}f)(z)| + \log(1+|(C_{\psi}f)(z)|) \right\} \\
& \quad \times \frac{|u(z)|}{|(C_{\psi}f)(z)|}[(\nabla f)(\psi(z))]^{2}(1-|z|^{2})|J_{\psi}(z)|^{2}d\nu(z) \\
& = \int_{E_{3}} \frac{[(\log(1+|(C_{\psi}f)(z)|))]^{p-2}}{[(1+|(C_{\psi}f)(z)|)]^{2}} \left\{ (p-1) + \frac{\log(1+|(C_{\psi}f)(z)|)}{|(C_{\psi}f)(z)|} \right\} |(C_{\psi}f)(z)|^{2} \\
& \quad \times \frac{|u(z)|^{p-2}}{|(C_{\psi}f)(z)|^{2}}[(\nabla f)(\psi(z))]^{2}(1-|z|^{2})|J_{\psi}(z)|^{2}d\nu(z) \\
& = \frac{1}{p} \int_{E_{3}} \chi^{\psi}[(\log|f(\psi(z))|)]^{p}[(\nabla f)(\psi(z))]^{2} \frac{|u(z)|^{p-2}(1-|z|^{2})|J_{\psi}(z)|^{2}}{|f(\psi(z))|^{2}}d\nu(z).
\end{align*}
(4.38)

And we have by (4.16), (4.18), (4.21) and (4.37),
\[ V_{2,2} \leq \int_{\mathbb{R}^n} \left[ \frac{\log(1 + |(C_{tgf}(z)|)}{1 + |(C_{tgf}(z))|^2} \right]^{p-2} \left( (p-1) + \frac{\log(1 + |(C_{tgf}(z))|)}{|(C_{tgf}(z))|} \right) \times |u(z)|^2 |(\nabla f)(\varphi(z))|^2 (1 - |z|^2) |J_{\varphi}(z)|^2 \, dv(z) \]
\[ = \frac{1}{p} \int_{\mathbb{R}^n} \chi''(\log(\mathbf{f}(\varphi(z)))) \left| \frac{(\nabla f)(\varphi(z))}{f(\varphi(z))} \right|^2 |u(z)|^2 (1 - |z|^2) |J_{\varphi}(z)|^2 \, dv(z). \]  
(4.39)

(4.11) and (4.36)~(4.39) give
\[ V_2 \leq \int_{\mathbb{R}^n} \chi''(\log(\mathbf{f}(\varphi(z)))) \left| \frac{(\nabla f)(\varphi(z))}{f(\varphi(z))} \right|^2 |u(z)|^{p-3} (1 - |z|^2) |J_{\varphi}(z)|^2 \, dv(z) \]
\[ + \int_{\mathbb{R}^n} \chi''(\log(\mathbf{f}(\varphi(z)))) \left| \frac{(\nabla f)(\varphi(z))}{f(\varphi(z))} \right|^2 |u(z)|^2 (1 - |z|^2) |J_{\varphi}(z)|^2 \, dv(z) \]
\[ = \int_{\mathbb{R}^n} \chi''(\log(\mathbf{f}(\varphi(z)))) \left| \frac{(\nabla f)(\varphi(z))}{f(\varphi(z))} \right|^2 |u(z)|^2 (1 - |z|^2) |J_{\varphi}(z)|^2 \, dv(z) \]
\[ \leq \epsilon_1 \int_{\mathbb{R}^n} \chi''(\log(\mathbf{f}(\varphi(z)))) \left| \frac{(\nabla f)(\varphi(z))}{f(\varphi(z))} \right|^2 (1 - |\varphi(z)|^2) |J_{\varphi}(z)|^2 \, dv(z). \]  
(4.40)

From (2.2), (2.3), (4.40), Lemma 2.4(b) and a change of variables, it follows that
\[ V_2 \leq \epsilon_2 \int_{\mathbb{R}^n} \chi''(\log(\mathbf{f}(\varphi(z)))) \left| \frac{(\nabla f)(\varphi(z))}{f(\varphi(z))} \right|^2 (1 - |z|^2) \, dv(z) \]
\[ \leq \epsilon_2 (n+1) \int_{\mathbb{R}^n} \chi''(\log(\mathbf{f}(\varphi(z)))) \left| \frac{(\nabla f)(\varphi(z))}{f(\varphi(z))} \right|^2 \frac{dv(z)}{1 - |z|^2} \]
\[ = \epsilon_2 (n+1) \int_{\mathbb{R}^n} \Delta(\log(1 + |f|)^2) \left( \frac{dv(z)}{1 - |z|^2} \right) \]
\[ \leq \frac{\epsilon_2 (n+1)}{2a_{2n}} \|f\|_{W^{2n}(\mathbb{R}^n)}. \]  
(4.41)

When \(2 < p < \infty\), by (4.14), (4.15), (4.19), (4.21) and (4.37) we have
\[ V_{2,1} \leq \int_{\mathbb{R}^n} \left[ \frac{\log(1 + |(C_{tgf}(z))|)}{u(z)}(1 + |(C_{tgf}(z))|^2) \right]^{p-2} \times \left( (p-1) |(C_{tgf}(z))| + \log(1 + |(C_{tgf}(z))|) \right) \]
\[ \times \frac{|u(z)|}{|(C_{tgf}(z))|} |(\nabla f)(\varphi(z))|^2 (1 - |z|^2) |J_{\varphi}(z)|^2 \, dv(z) \]
\[ = \frac{1}{p} \int_{\mathbb{R}^n} \chi''(\log(\mathbf{f}(\varphi(z)))) \left| \frac{(\nabla f)(\varphi(z))}{f(\varphi(z))} \right|^2 |u(z)|^{-1} (1 - |z|^2) |J_{\varphi}(z)|^2 \, dv(z). \]  
(4.42)

And we have by (4.16), (4.20), (4.21) and (4.37),
\[ V_{2,2} \leq \int_{\mathbb{R}^n} \left[ \frac{u(z)^{p-2}(\log(1 + |(C_{tgf}(z))|))^{p-2}}{1 + |(C_{tgf}(z))|^2} \right] \left( (p-1) + \frac{\log(1 + |(C_{tgf}(z))|)}{|(C_{tgf}(z))|} \right) \]
\[ \times |u(z)|^2 |(\nabla f)(\varphi(z))|^2 (1 - |z|^2) |J_{\varphi}(z)|^2 \, dv(z) \]
\[ = \frac{1}{p} \int_{\mathbb{R}^n} \chi''(\log(\mathbf{f}(\varphi(z)))) \left| \frac{(\nabla f)(\varphi(z))}{f(\varphi(z))} \right|^2 |u(z)|^p (1 - |z|^2) |J_{\varphi}(z)|^2 \, dv(z). \]  
(4.43)

By (4.13), (4.36), (4.37), (4.42) and (4.43), we obtain
\[ V_2 \leq \int B \chi''(\log[f(\varphi(z))] \frac{(|\nabla f(\varphi(z))|^2}{f(\varphi(z))})^p |u(z)|^{-1} (1 - |z|^2) J_\varphi(z)^2 dv(z) \\
+ \int B \chi''(\log[f(\varphi(z))] \frac{(|\nabla f(\varphi(z))|^2}{f(\varphi(z))})^p |u(z)|^{-1} (1 - |z|^2) J_\varphi(z)^2 dv(z) \\
= \int B \chi''(\log[f(\varphi(z))] \frac{(|\nabla f(\varphi(z))|^2}{f(\varphi(z))})^p \\
\times \max(|u(z)|^{-1}, |u(z)|^2) (1 - |z|^2)^n J_\varphi(z)^2 dv(z) \\
\leq \varepsilon_1 \int B \chi''(\log[f(\varphi(z))] \frac{(|\nabla f(\varphi(z))|^2}{f(\varphi(z))})^p (1 - |\varphi(z)|^2) J_\varphi(z)^2 dv(z). \tag{4.44} \]

As in (4.41), we have
\[ V_2 \leq \varepsilon_1 n \frac{n+1}{2 \alpha_n \Gamma(n)} \|f\|_{\mathcal{B}_p (B)}. \tag{4.45} \]

By (4.22)~(4.24), (4.31), (4.35), (4.41) and (4.45), we obtain
\[ \int B \chi'' \left( \log \left( \frac{|(UC_{\varphi})f(z)|^2}{|UC_{\varphi}|(z)^2} \right) |z|^{-2n-1} \log \frac{1}{|z|} \right) dv(z) \leq 2^{2n-1} \left[ \varepsilon_1 p^2 L + \varepsilon_1 n \frac{n+1}{2 \alpha_n \Gamma(n)} \right] \|f\|_{\mathcal{B}_p (B)}. \tag{4.46} \]

By (3.12), Lemma 2.3 and Lemma 2.6, we have
\[ \int B \chi'' \left( \log \left( \frac{|(UC_{\varphi})f(z)|^2}{|UC_{\varphi}|(z)^2} \right) |z|^{-2n-1} \log \frac{1}{|z|} \right) dv(z) \leq \frac{1}{\delta} \int B \chi'' \left( \log \left( \frac{|(UC_{\varphi})f(z)|^2}{|UC_{\varphi}|(z)^2} \right) |z|^{-2n-1} \log \frac{1}{|z|} \right) dv(z) \leq \frac{2n}{\delta} \| (UC_{\varphi}) \|_{\mathcal{B}_p (B)} \]
\[ \leq 2n \int \left[ \max \{1, |u(z)| \} \right]^p \left( \log(1 + |(C_{\varphi}) f(z)|) \right)^p d\sigma \]
\[ \leq \frac{2n}{\delta} \max \{1, |u(z)| \}^p \max \left\{ \log(1 + |f(z)|) \right\}^p \]
\[ \leq \frac{2n}{\delta} \max \{1, |u(z)| \}^p \max \left\{ \frac{1 + |z|}{1 - |z|} \right\}^n \|f\|_{\mathcal{B}_p (B)}. \tag{4.47} \]

where \( r_1 = \frac{1 + r_0}{2} \) and \( \delta = 1 - \frac{\log r_1}{\log r_0} \). Moreover, we have by Lemma 2.6
\[ \| (1 + |u(0) f(\varphi(0))|)^p \| = \left[ \max \{1, |u(0)| \} \right]^p \left\{ \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right\}^n \|f\|_{\mathcal{B}_p (B)}. \tag{4.48} \]

By (4.46)~(4.48) and Lemma 2.3, we obtain
\[ \| UC_{\varphi} \|_{\mathcal{B}_p (B)} \leq \left[ \frac{2n}{\delta} \left( \varepsilon_1 p^2 L + \varepsilon_1 n \frac{n+1}{2 \alpha_n \Gamma(n)} \right) \right] \]
\[ + \frac{1}{\delta} \max \{1, |u(z)| \}^p \max \left\{ \log(1 + |f(z)|) \right\}^p \]
\[ \left[ \max \{1, |u(0)| \} \right]^p \left\{ \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right\}^n \|f\|_{\mathcal{B}_p (B)}. \tag{4.49} \]

Hence \( UC_{\varphi} \) is metrically bounded on \( N^p(B) \). This completes the proof of (a).
To prove (b), suppose that \((f_j)\) is a sequence in \(N^p(B)\) which converges to zero uniformly on compact subsets of \(B\) and \(\|f_j\|_{N^p(B)} \leq \gamma < \infty\) for all \(j \in \mathbb{N}\). Let \(\varepsilon > 0\) be given. By (4.5)~(4.8), we can choose \(r_0 \in (\frac{1}{2}, 1)\) such that if \(1 < p \leq 2\),
\[
\max \{|u(z)|^{p-1}|(\nabla u)(z)|^2(1-|z|^2) < \varepsilon, \tag{4.50}
\]
\[
\max \{|u(z)|^{p-1}|\varphi(z)|^2(1-|z|^2) < \varepsilon, \tag{4.51}
\]
if \(2 < p < \infty\),
\[
\max \{|u(z)|^{-2}|u(z)|^{p-2}|(\nabla u)(z)|^2(1-|z|^2) < \varepsilon, \tag{4.52}
\]
\[
\max \{|u(z)|^{-2}|\varphi(z)|^2(1-|z|^2) < \varepsilon \tag{4.53}
\]
for any \(z \in B \setminus r_0 B\). By (a), it holds that \((uC_\varepsilon)(N^p(B)) \subseteq N^p(B)\). By the same argument as in the proof of (a), we have instead of (4.46)
\[
\int_{B \setminus r_0 B} 
\chi''(\log \|uC_\varepsilon f_j(z)\|) \left| \frac{R(uC_\varepsilon f_j(z))^2}{|uC_\varepsilon f_j(z)|^2} \right|^2 |z|^{-2}(n-1) \log \frac{1}{|z|} \, dv(z)
\leq 2^{2n-1} \left[ p^2 L + \frac{n(n+1)M}{2 a_n \Gamma(n)} \right] \varepsilon \tag{4.54}
\]
for all \(j \in \mathbb{N}\), by virtue of (4.50)~(4.53).

On the other hand, as in (4.47) we obtain for all \(j \in \mathbb{N}\)
\[
\int_{r_0 B} \chi''(\log \|uC_\varepsilon f_j(z)\|) \left| \frac{R(uC_\varepsilon f_j(z))^2}{|uC_\varepsilon f_j(z)|^2} \right|^2 |z|^{-2}(n-1) \log \frac{1}{|z|} \, dv(z)
\leq \frac{2n}{\delta} \max_{z \in \partial B} \{1 + |u(z)|^p \max_{z \in \Omega(\tau_1)} \log(1 + |f_j(z)|)^p \} \tag{4.55}
\]
where \(\tau_1 = \frac{1 + r_0}{2}\) and \(\delta = 1 - \log \frac{r_1}{\log r_0}\). Since \((f_j)\) converges to zero uniformly on compact subsets of \(B\),
\[
\lim_{j \to \infty} \max_{z \in \Omega(\tau_1)} \log(1 + |f_j(z)|)^p = 0. \tag{4.56}
\]
By (4.55) and (4.56), we have
\[
\lim_{j \to \infty} \int_{r_0 B} \chi''(\log \|uC_\varepsilon f_j(z)\|) \left| \frac{R(uC_\varepsilon f_j(z))^2}{|uC_\varepsilon f_j(z)|^2} \right|^2 |z|^{-2}(n-1) \log \frac{1}{|z|} \, dv(z) = 0. \tag{4.57}
\]
By (4.54), (4.57) and Lemma 2.3, we can conclude that \(\lim_{j \to \infty} \|uC_\varepsilon f_j\|_{N^p(B)} = 0\). Thus Lemma 2.13 implies that \(uC_\varepsilon\) is metrically compact on \(N^p(B)\). This completes the proof.

**Corollary 1.** Let \(1 < p < \infty\). Suppose that \(\varphi\) is a univalent holomorphic self-map of \(B\) such that \(\Omega_\varphi\) is bounded in \(B\). Then \(C_\varphi\) is metrically compact on \(N^p(B)\) if and only if
\[
\lim_{|z| \to 1} \frac{1-|z|^p}{1-|\varphi(z)|^p} = 0. \tag{4.56}
\]

**Proof.** Sufficiency is the case \(u \equiv 1\) of Theorem 4.1(b). Suppose that \(C_\varphi\) is metrically
compact on $N^p(B)$. Since $\varphi$ is a holomorphic self-map of $B$, $\varphi$ has a radial limit $\varphi^*(\zeta) = \lim_{r \to 1} \varphi(r\zeta) \in \overline{B}$ for almost every $\zeta \in \Sigma$. First we show that the pull-back measure $\mu_\varphi = \sigma \circ \varphi^{-1}$ satisfies the following condition:

$$\mu_\varphi(S(\zeta, h)) = o(h^p) \quad \text{as } h \to 0$$

(4.58)

uniformly in $\zeta \in \Sigma$, where $S(\zeta, h) = \{z \in \overline{B} : |1 - \langle z, \zeta \rangle| < h\}$. We assume that it does not hold that $\mu_\varphi(S(\zeta, h)) = o(h^p)$ uniformly in $\zeta \in \Sigma$. Then there are $\{\zeta_i\} \subset \Sigma$, $\{(h_j) \in (0, 1)$ with $h_j \downarrow 0$ ($j \to \infty$) and $\delta_0 > 0$ such that

$$\mu_\varphi(S(\zeta_j, h_j)) \geq \delta_0 h_j^p$$

(4.59)

for all $j \in \mathbb{N}$. Put $a_j = (1 - h_j)^{1/j} (j \in \mathbb{N})$. Define

$$f_j(z) = (1 - |a_j|)^{1/j} \exp \left( \frac{1 - |a_j|^2}{(1 - \langle z, a_j \rangle^2)^{1/j}} \right)^p$$

(4.60)

for $z \in B$ and $j \in \mathbb{N}$. We can easily see that $f_j$ is in the ball algebra $A(B)$ and

$$\|f_j\|_{A(B)} \leq 2^{p - 1} ((\log 2)^p + 1)$$

for all $j \in \mathbb{N}$. Moreover, by (4.60) we see that $(f_j)$ converges to 0 uniformly on compact subsets of $B$. Since $C_\varphi$ is metrically compact, we have

$$\lim_{j \to \infty} \|C_\varphi f_j\|_{N^p(B)} = 0,$$

(4.61)

by Lemma 2.13.

On the other hand, by using the continuity of the function $F(v) = \text{Re}(1 + v)^{-p}$ ($v \in \mathbb{C}$) at the origin in $\mathbb{C}$, we can choose $j_0 \in \mathbb{N}$ such that

$$\text{Re} \left( 1 + \frac{|a_j| (1 - \langle z, \zeta \rangle)}{1 - |a_j|} \right)^{-p} > \frac{1}{2}$$

(4.62)

for any $j \in \mathbb{N}$ with $j \geq j_0$ and $z \in S(\zeta, h_j)$. Moreover, we have by (4.62)

$$\text{Re} \left( \frac{1 - |a_j|^2}{(1 - \langle z, a_j \rangle)^2} \right)^{1/j} > \frac{1}{2h_j^{1/j}}$$

for any $j \in \mathbb{N}$ with $j \geq j_0$ and $z \in S(\zeta, h_j)$. Thus, for any $j \in \mathbb{N}$ with $j \geq j_0$ and $z \in S(\zeta, h_j)$, we have

$$\log^+ |f_j(z)| = \log \left[ (1 - |a_j|)^{\text{Re} \left( \frac{1 - |a_j|^2}{(1 - \langle z, a_j \rangle)^2} \right)^{1/p}} \right]$$

$$\geq \log \left[ (1 - |a_j|)^{\text{Re} \left( \frac{1}{2h_j^{1/j}} \right)} \right]$$

$$= \log \left[ h_j \exp \left( \frac{1}{2h_j^{1/j}} \right) \right].$$

(4.63)

Hence, by Fatou's lemma and (4.63) we obtain for any $j \in \mathbb{N}$ with $j \geq j_0$
\[
\left\{ \log^+(h, \exp\left(\frac{1}{2h^2}\right)) \right\}^p \mu_\varphi(S(\xi, h)) \\
\leq \int_{\varphi(S(\xi, h))} \{ \log^+ |f_\varphi| \}^p d\mu_\varphi \\
\leq \int_S \{ \log(1 + |f_\varphi \circ \varphi^{-1}|) \}^p d\sigma \\
= \int_S \lim_{\tau \to 1} \{ \log(1 + |(f_\varphi \circ \varphi^{-1})|) \}^p d\sigma \\
\leq \liminf_{\tau \to 1} \int_S \{ \log(1 + |(f_\varphi \circ \varphi^{-1})|) \}^p d\sigma \\
= \|C_{\varphi}|_{H^p(B)} \cdot (4.64)
\]

It follows from (4.61) and (4.64) that
\[
\lim_{j \to \infty} \left\{ \log^+(h, \exp\left(\frac{1}{2h^2}\right)) \right\}^p \mu_\varphi(S(\xi, h)) = 0.
\]
(4.65)

Since
\[
\lim_{j \to \infty} h_j^p \left\{ \log^+(h, \exp\left(\frac{1}{2h^2}\right)) \right\}^p \to \frac{1}{2^p},
\]
(4.65) implies that
\[
\lim_{j \to \infty} \frac{\mu_\varphi(S(\xi, h))}{h_j^p} = 0.
\]

This contradicts (4.59). Thus we establish that (4.58) holds. By MacCluer's Carleson-measure criterion([8]), \( C_\varphi \) is compact on \( H^p(B) \). It follows from Theorem 1.1(b) that
\[
\lim_{|z| \to 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.
\]

This completes the proof. \( \square \)

**Theorem 4.2.** Let \( 1 \leq p < \infty \) and \( -1 < a < \infty \). Let \( u \in H(B) \setminus \{0\} \) and \( \varphi : B \to B \) be a univalent holomorphic map such that \( \Omega_\varphi \) is bounded in \( B \). 

(a) Suppose \( u \) and \( \varphi \) satisfy the following conditions:

(i) When \( 1 \leq p \leq 2 \),
\[
\limsup_{|z| \to 1} \max\left\{ |u(z)|^{p-1}, 1 \right\} |(\nabla u)(z)|^p (1 - |z|^2)^2 < \infty,
\]
(4.66)

(ii) When \( 2 < p < \infty \),
\[
\limsup_{|z| \to 1} \max\left\{ |u(z)|^{p-2}, |u(z)|^2 \right\} \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{p+1/2} < \infty.
\]
(4.67)

Then \( uC_\varphi \) is metrically bounded on \( (AN)^p(\nu_a) \).

(b) Suppose \( u \) and \( \varphi \) satisfy the following conditions:

(i) When \( 1 \leq p \leq 2 \),
\[
\limsup_{|z| \to 1} \max\left\{ |u(z)|^{p-1}, 1 \right\} |(\nabla u)(z)|^p (1 - |z|^2)^2 < \infty,
\]
(4.68)

(ii) When \( 2 < p < \infty \),
\[
\limsup_{|z| \to 1} \max\left\{ |u(z)|^{p-2}, |u(z)|^2 \right\} \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{p+1/2} < \infty.
\]
(4.69)
(i) When $1 \leq p \leq 2$,
\[ \lim_{|z| \to 1^-} \max\{|u(z)|^{p-1}, |\nabla u(z)|^2(1 - |z|^2)^2\} = 0, \]
\[ \lim_{|z| \to 1^-} \max\{|u(z)|^{p-3}, |u(z)|^2 \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{a+2} \} = 0. \]

(ii) When $2 < p < \infty$,
\[ \lim_{|z| \to 1^-} \max\{|u(z)|^{-2}, |u(z)|^{p-2} \}
\left( |\nabla u(z)|^2(1 - |z|^2)^2\right) = 0, \]
\[ \lim_{|z| \to 1^-} \max\{|u(z)|^{-1}, |u(z)|^p \}
\left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{a+2} = 0. \]

Then $uC_{\varphi}$ is metrically compact on $(AN)^p(\nu_0)$.

**Proof.** Take $f \in (AN)^p(\nu_0) \setminus \{0\}$. By the hypothesis of the present theorem and Lemma 2.11, $C_{\varphi}$ is metrically bounded on $(AN)^p(\nu_0)$, that is, $\|C_{\varphi}\|_{(AN)^p(\nu_0)} \leq L\|f\|_{(AN)^p(\nu_0)}$ where $L$ is a positive constant depending only on $n$, $a$ and $\varphi$. By (4.66)~(4.69), there are positive constants $\epsilon_1$, $\epsilon_2$ and $r_0 \in \left(\frac{1}{2}, 1\right)$ such that when $1 < p \leq 2$,
\[ \max\{|u(z)|^{p-1}, |\nabla u(z)|^2(1 - |z|^2)^2\} < \epsilon_1, \]
\[ \max\{|u(z)|^{p-3}, |u(z)|^2 \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{a+2} \} < \epsilon_2, \]
when $2 < p < \infty$,
\[ \max\{|u(z)|^{-2}, |u(z)|^{p-2} \} |\nabla u(z)|^2(1 - |z|^2)^2 < \epsilon_1, \]
\[ \max\{|u(z)|^{-1}, |u(z)|^p \} \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{a+2} < \epsilon_2 \]
for any $z \in B \setminus r_0B$. By (3.27), (4.74)~(4.78), Lemma 2.4(a) and the same argument as in the proof of Theorem 4.1(a), we obtain
\[ \int_{B \setminus r_0B} \chi''(\log(|uC_{\varphi}(z)|)) \left[ \frac{R(uC_{\varphi}(z))^p}{|uC_{\varphi}(z)|^2 |z|^{2a+2} |\varphi(z)|^{a+1} K_a(|z|)} \right] d\nu(z) \]
\[ \leq 2^{2n-1} d_{n,a} \left[ \frac{\epsilon_1 n^2 L}{C_a} + \epsilon_2 (n+1)M \right] \times \frac{2^{a+2}(n+a+1)}{a\Gamma(n+a+1)} \int_{(AN)^p(\nu_0)} d\nu(z) \]
\[ (4.78) \]
where $M = \sup_{z \in \partial B} Q_{\varphi}(z) < \infty$. (cf. (4.46))

On the other hand, by (2.16), Lemma 2.3 and Lemma 2.6, we obtain
\[ \int_{\partial B} \chi''(\log(|uC_{\varphi}(z)|)) \left[ \frac{R(uC_{\varphi}(z))^p}{|uC_{\varphi}(z)|^2 |z|^{2a+2} |\varphi(z)|^{a+1} K_a(|z|)} \right] d\nu(z) \]
\[ \leq C_a \int_{\partial B} \chi''(\log(|uC_{\varphi}(z)|)) \left[ \frac{R(uC_{\varphi}(z))^p}{|uC_{\varphi}(z)|^2 |z|^{2a+2} |\varphi(z)|^{a+1} K_a(|z|)} \right] d\nu(z) \]
\[ \leq 2nC_a \|uC_{\varphi}\|_{r_1(AN)^p(\nu_0)} \]
\[
\leq 2nC_a \max_{x \in r_1 B} (1 + |u(z)|)^p \max_{x \in r_1 B} \{ \log(1 + |f(z)|) \}^p \\
\leq 2nC_a \max_{x \in r_1 B} (1 + |u(z)|)^p \max_{x \in r_1 B} \left( \frac{1 + |z|}{1 - |z|^2} \right)^{n+1+a} \|f\|_{L^p(V_{\mu_a})}.
\]  
(4.79)

where \( r_1 = \frac{1 + r_0}{2} \) and \( C_a = (\log \frac{1}{r_0} (K_a(\frac{r_0}{r_1}))^{-1} + 1 \). (cf. (4.47)) Moreover, we have by Lemma 2.6

\[
\{ \log(1 + |(uC_\varphi)f(0)|) \}^p \leq \max\{1, |u(0)|\} \left[ 1 + \frac{|\varphi(0)|}{(1 - |\varphi(0)|)} \right]^{n+1+a} \|f\|_{L^p(V_{\mu_a})}.
\]  
(4.80)

(4.78) \sim (4.80) and Lemma 2.3 show that \( uC_\varphi \) is metrically bounded on \( (AN)^p(\nu_a) \). This completes the proof of (a).

To prove (b), suppose that \( \{f_j\} \) is a bounded sequence in \( (AN)^p(\nu_a) \) which converges to zero uniformly on compact subsets of \( B \). Then we can show that \( \lim_{j \to \infty} \|uC_\varphi f_j\|_{L^p(V_{\mu_a})} = 0 \), by the same way as that in the proof of Theorem 4.1(b).

Hence, by Lemma 2.13, we conclude that \( uC_\varphi \) is metrically compact on \( (AN)^p(\nu_a) \). The proof is complete.

**Corollary 2.** Let \( 1 \leq p < \infty \) and \( -1 < a < \infty \). Suppose that \( \varphi \) is a univalent holomorphic self-map of \( B \) such that \( \Omega_\varphi \) is bounded in \( B \). Then \( C_\varphi \) is metrically compact on \( (AN)^p(\nu_a) \) if and only if

\[
\lim_{|z| \to 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0
\]

**Proof.** The proof is entirely similar to that of Corollary 1 except that we choose functions

\[
f_j(z) = (1 - |a_j|) \exp \left( \frac{1 - |a_j|^2}{1 - \langle z, a_j \rangle^2} \right)^{\frac{n+1+a}{p}} \quad (z \in B, j \in \mathbb{N})
\]

instead of (4.60).

\[ \square \]

**References**


