

Symmetry Reduction, Gauge Transformation and Orbifold

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We study a mechanism of symmetry reduction in a higher-dimensional field theory upon orbifold compactification. Split multiplets appear unless all components in a multiplet of a symmetry group have a common parity on an orbifold. A gauge transformation property is also examined.

§ 1. Introduction

Recently, a new possibility¹⁾ has been proposed to reconcile the coupling unification scenario with the triplet-doublet mass splitting based on a 5-dimensional (5D) supersymmetric (SUSY) model with $SU(5)$ gauge symmetry. The minimal supersymmetric standard model (MSSM) is derived on a 4D wall through compactification on $S^1/(Z_2 \times Z_2)$.^{*)} The excellent characteristics of this model have been studied.^{1),12)**)} The key features are as follows.

- Unless components in a multiplet have a common $Z_2 \times Z_2$ parity on the orbifold, the lowest modes in 4D fields do not form full multiplets of $SU(5)$. It realizes a triplet-doublet splitting and an SM- X, Y gauge multiplets splitting with a suitable assignment of $Z_2 \times Z_2$ parity.
- A specific type of $SU(5)$ gauge symmetry exists on one of 4D walls (a visible wall) as well as in the bulk. It leads to a coupling unification at the zero-th order approximation.
- 5D bulk fields and 4D fields on the visible wall belong to some representations of $SU(5)$. It guarantees the quantization of charge.

We expect that similar features hold in a class of higher-dimensional grand unified theory (GUT) as suggested in Ref. 12). Concretely,

1. Unless all components in a multiplet of some unified gauge group G_U have a

^{*)}Recently, Barbieri, Hall and Nomura have constructed a constrained standard model upon a compactification of a 5D SUSY model on the orbifold $S^1/(Z_2 \times Z_2)$.²⁾ They used $Z_2 \times Z_2$ parity to reduce SUSY. There are also several works on model building through a reduction of SUSY³⁾⁻⁷⁾ by the use of a discrete symmetry and a reduction of gauge symmetry⁸⁾ by the use of Z_2 parity. Attempts to construct unified models have been made through dimensional reduction over coset space.⁹⁾ The study of higher-dimensional SUSY grand unified theories traces back to the work by Fayet.¹⁰⁾

^{**)}There are several works on the other type of 5D unified models with 1D orbifold, i.e., 5D $SU(5)$ model with S^1/Z_2 ⁸⁾, 5D $SU(5)$ model with $S^1/(Z_2 \times Z_2)$ ¹³⁾ and 5D SUSY $SU(5)$ model with S^1/Z_2 ¹⁴⁾.

common parity on an orbifold, split multiplets appear after the integration of the extra space because the lowest modes, in general, do not form full multiplets of G_U .

2. The higher-dimensional gauge symmetry is realized as an invariance under the gauge transformation whose gauge functions have a definite parity on an orbifold, and hence the gauge symmetry at some points on the orbifold turns out to be a reduced one whose generators are commutable to a parity operator.

In this paper, we study the above features in GUTs on an orbifold, which would be important for a construction of a realistic model and an exploration of the origin of symmetries in the SM.

This paper is organized as follows. In the next section, we study a mechanism of symmetry reduction due to an intrinsic parity on an orbifold. We discuss the reduction of gauge symmetry, a gauge transformation property and its phenomenological implications in §3. Section 4 is devoted to conclusions and discussion.

§ 2. Splitting from Z_N parity

The space-time is assumed to be factorized into a product of 4D Minkowski space-time M^4 and the $2n$ -dimensional ($2n$ -D) orbifold $O^{2n} \equiv T^{2n}/\prod_N Z_N$, whose coordinates are denoted by x^μ ($\mu=0,1,2,3$) and $y^{\hat{\mu}}$ ($\hat{\mu}=1,2,\dots,2n$), respectively. The notation x^M ($M=0,1,2,3,5,\dots,2n+4$) is also used for coordinates. The orbifold O^{2n} is obtained by dividing a $2n$ -D torus T^{2n} with Z_N rotations which are automorphisms of T^{2n} .³⁾ The Z_N rotation is diagonalizable under a suitable complex basis (z^i, \bar{z}^i) ($i=1,2,\dots,n$) for the extra space and is given by the transformation $z^i \rightarrow z'^i = \theta_j^i z^j$. Here θ_j^i is an element of Z_N transformation written by

$$\begin{aligned} \theta_j^i &= \text{diag}\left(\exp\frac{2\pi i m_1}{N}, \exp\frac{2\pi i m_2}{N}, \dots, \exp\frac{2\pi i m_n}{N}\right) \\ &\equiv \text{diag}(\theta_1, \theta_2, \dots, \theta_n) \end{aligned} \quad (2\cdot 1)$$

where m_i are integers. The T^{2n} is regarded as a $2n$ -D lattice that the point z^i is identified with $z^i + n^l e_i^l$ where n^l are integers and e_i^l are shift vectors on the lattice. There are points fixed by the discrete transformation. They are called fixed points,^{*)} which are denoted by z_{fp}^i and satisfy the relation $z_{fp}^i = \theta_j^i z_{fp}^j + n^l e_i^l$.

Here we study a field theory on 2D Z_3 orbifold as an example. The Z_3 orbifold is obtained by dividing the $SU(3)$ root lattice $\Gamma_{SU(3)}$ with a Z_3 rotation whose element is $\theta = \exp\frac{2\pi i}{3}$. The shift vectors on $\Gamma_{SU(3)}$ are given by 1 and $\omega \equiv \exp\frac{2\pi i}{3}$.^{**)} Hence the

^{*)}Since fixed points are singular points on the space, orbifolds are not manifolds. We assume that this singularity does not cause any trouble in an underlying theory.

^{**)}As we take a normalization where a size of extra space equals that of $\Gamma_{SU(3)}$, we should consider that the compact space has a physical size $2\pi R$ on the estimation of a magnitude of physical quantities.

following identification holds on the orbifold,

$$z \sim z+1 \sim z + \omega \sim \omega z. \quad (2.2)$$

There are three kinds of fixed points,

$$z_{fp} = 0, \frac{2+\omega}{3}, \frac{1+2\omega}{3}. \quad (2.3)$$

An intrinsic Z_3 parity of the bulk field $\phi(x^\mu, z, \bar{z})$ is defined by the transformation

$$\phi(x^\mu, z, \bar{z}) \rightarrow \phi(x^\mu, \omega z, \omega^2 \bar{z}) = P\phi(x^\mu, z, \bar{z}) \quad (2.4)$$

By definition, P possesses only the eigenvalues 1, ω or ω^2 . We denote the fields that are eigenfunctions of P as ϕ_{ω^0} , ϕ_{ω^1} , ϕ_{ω^2} where the subscript corresponds to the eigenvalue of P . The 6D fields ϕ_{ω^l} ($l=0,1,2$) are Fourier expanded as

$$\phi_{\omega^l}(x^\mu, z, \bar{z}) = \sum_{n,m} \phi_{\omega^l}^{(nm)}(x^\mu) f_{nm}^{\omega^l}(z, \bar{z}) \quad (2.5)$$

where n and m are integers, and $f_{nm}^{\omega^l}(z, \bar{z})$ are eigenfunctions of P whose eigenvalues are ω^l . The $f_{nm}^{\omega^l}(z, \bar{z})$ are written as

$$f_{nm}^{\omega^0}(z, \bar{z}) = f_{nm}(z, \bar{z}) + f_{nm}(\omega z, \omega^2 \bar{z}) + f_{nm}(\omega^2 z, \omega \bar{z}), \quad (2.6)$$

$$f_{nm}^{\omega^1}(z, \bar{z}) = f_{nm}(z, \bar{z}) + \omega^2 f_{nm}(\omega z, \omega^2 \bar{z}) + \omega f_{nm}(\omega^2 z, \omega \bar{z}), \quad (2.7)$$

$$f_{nm}^{\omega^2}(z, \bar{z}) = f_{nm}(z, \bar{z}) + \omega f_{nm}(\omega z, \omega^2 \bar{z}) + \omega^2 f_{nm}(\omega^2 z, \omega \bar{z}), \quad (2.8)$$

by the use of a function $f_{nm}(z, \bar{z})$ which satisfies periodic boundary conditions

$$f_{nm}(z, \bar{z}) = f_{nm}(z+1, \bar{z}+1) = f_{nm}(z+\omega, \bar{z}+\omega^2). \quad (2.9)$$

The explicit form of $f_{nm}(z, \bar{z})$ is given by

$$f_{nm}(z, \bar{z}) = \exp\left(\pi i \left(\left(n - \frac{n+2m}{\sqrt{3}} i \right) z + \left(n + \frac{n+2m}{\sqrt{3}} i \right) \bar{z} \right) \right). \quad (2.10)$$

From the expressions (2.5)-(2.10), we find the following features of eigenfunctions.

- The 4D fields $\phi_{\omega^l}^{(nm)}(x^\mu)$ acquire mass $(n^2 + \frac{(n+2m)^2}{3})^{1/2}/R$ upon compactification.
- The 4D fields with $n=m=0$ (4D zero modes) appear from 6D fields whose Z_3 parity is 1, i.e., the $\phi_{\omega^0}(x^\mu, z, \bar{z})$ has 4D zero mode.
- The 6D fields whose Z_3 parity is ω or ω^2 vanish on the fixed points, i.e., $\phi_{\omega^l}(x^\mu, z_{fp}, \bar{z}_{fp}) = \phi_{\omega^2}(x^\mu, z_{fp}, \bar{z}_{fp}) = 0$.

Let us study the case in which a field $\Phi(x^\mu, z, \bar{z})$ is an N_f -plet under some symmetry group. The components of Φ are denoted by $\Phi = (\phi_1, \phi_2, \dots, \phi_{N_f})^T$. The Z_3 transformation of Φ is given by the same form as (2.4), but in this case P is an $N_f \times N_f$ matrix which satisfies $P^3 = I$, where I is the unit matrix. The Z_3 invariance of the Lagrangian density does not necessarily require that P be proportional to I . Unless all components of Φ have a common Z_3 parity, the splitting in a multiplet occurs upon compactification

because of the lack of zero modes in components with Z_3 parity other than one.

The generalization on a model with a generic orbifold is straightforward. Hence, we expect that *in a class of higher-dimensional GUT on an orbifold, unless all components in a multiplet of some unified gauge group G_U have a common parity on an orbifold, split multiplets appear after the integration of the extra space because zero modes, in general, do not form full multiplets of G_U .*

§ 3. Gauge transformation property

We apply the mechanism of symmetry reduction discussed in the previous section to GUTs on $M^4 \times O^{2n}$. Here we consider a non-SUSY model for simplicity. The SUSY extension is straightforward. We take two basic assumptions. One is that the gauge boson $A_M(x^\mu, z^i, \bar{z}^i) = A_M^\alpha(x^\mu, z^i, \bar{z}^i) T^\alpha$ and a scalar field $\Phi(x^\mu, z^i, \bar{z}^i)$ exist in the bulk. Here the T^α are gauge generators and the $\Phi(x^\mu, z^i, \bar{z}^i)$ belongs to a vector representation of a unified group G_U . The other is that our visible world is one of 4D walls at a certain point on the orbifold and matter fields are located on the wall.

The action integral is given by

$$S = \int \mathcal{L}_{bulk} d^{4+2n}x + \sum_p \int \mathcal{L}_{fp}^{(p)} d^{4+2n}x, \quad (3.1)$$

$$\mathcal{L}_{bulk} \equiv -\frac{1}{2} \text{tr} F_{MN} F^{MN} + |D_M \Phi|^2 - V(|\Phi|^2) \quad (3.2)$$

where $D_M \equiv \partial_M - ig_U A_M(x^M)$, g_U is a $(4+2n)$ -D gauge coupling constant and $\mathcal{L}_{fp}^{(p)}$ is a contribution from the p -th 4D wall. The above Lagrangian density \mathcal{L}_{bulk} is invariant under a Z_N transformation and a gauge transformation defined as follows. The Z_N transformation for A_M and Φ is given by

$$\begin{aligned} A_\mu(x^\mu, z^i, \bar{z}^i) &\rightarrow A_\mu(x^\mu, z'^i, \bar{z}'^i) = P A_\mu(x^\mu, z^i, \bar{z}^i) P^{-1}, \\ A_{z^i}(x^\mu, z^i, \bar{z}^i) &\rightarrow A_{z^i}(x^\mu, z'^i, \bar{z}'^i) = \theta_i^{-1} P A_{z^i}(x^\mu, z^i, \bar{z}^i) P^{-1}, \\ A_{\bar{z}^i}(x^\mu, z^i, \bar{z}^i) &\rightarrow A_{\bar{z}^i}(x^\mu, z'^i, \bar{z}'^i) = \theta_i P A_{\bar{z}^i}(x^\mu, z^i, \bar{z}^i) P^{-1}, \\ \Phi(x^\mu, z^i, \bar{z}^i) &\rightarrow \Phi(x^\mu, z'^i, \bar{z}'^i) = P \Phi(x^\mu, z^i, \bar{z}^i) \end{aligned} \quad (3.3)$$

where P is Z_N parity operator, $z'^i = \theta_i z^i$ and $\bar{z}'^i = \bar{\theta}_i \bar{z}^i$. The gauge transformation for A_M and Φ is given by

$$\begin{aligned} A_M(x^\mu, z^i, \bar{z}^i) &\rightarrow A'_M(x^\mu, z^i, \bar{z}^i) = U A_M(x^\mu, z^i, \bar{z}^i) U^{-1} + \frac{i}{g_U} U \partial_M U^{-1}, \\ \Phi(x^\mu, z^i, \bar{z}^i) &\rightarrow \Phi'(x^\mu, z^i, \bar{z}^i) = U \Phi(x^\mu, z^i, \bar{z}^i) \end{aligned} \quad (3.4)$$

where U is a space-time dependent gauge transformation matrix. The Z_N transformation is, in general, not commutable to a gauge transformation with generic gauge functions, unless P is proportional to the unit matrix. But, when there is a relation $P T^\alpha P^{-1} = \theta^{k_\alpha} T^\alpha$ and the group structure constants $f^{a\beta\gamma}$ vanish for $k_\alpha + k_\beta \neq k_\gamma \pmod{N}$, there survives a specific type of unified gauge symmetry, which is compatible with the

Z_N transformation, based on a gauge transformation matrix given by

$$U(x^M) = \exp(i\xi_{\theta^{k_a}}^\alpha(x^M)T^\alpha) \quad (3\cdot5)$$

where gauge functions $\xi_{\theta^{k_a}}^\alpha(x^M)$ are eigenfunctions with eigenvalue θ^{k_a} for Z_N parity. Actually the gauge transformation matrix (3·5) is obtained from the requirement that a Z_N parity of A_M^α and Φ_k equals that of A_M^α and Φ_k or that the Z_N parity assignment of each component in a multiplet is preserved after the gauge transformation, i.e.,

$$PU(x^\mu, z^i, \bar{z}^i) = U(x^\mu, z'^i, \bar{z}'^i)P. \quad (3\cdot6)$$

The reduction of gauge symmetry occurs at a fixed point z_{fp}^i because the $\xi_{\theta^{k_a}}^\alpha(x^M)$ vanish at z_{fp}^i for $k_a \neq 0 \pmod{N}$. The residual gauge group is a subgroup of G_U , whose generators are commutable to Z_N parity operator. The interaction on z_{fp}^i is constrained from the symmetry there. For example, the Lagrangian density on z_{fp}^i should be invariant under both Z_N parity and the residual gauge transformation.

The above feature can be generalized in the case with a generic orbifold as a statement that *in higher-dimensional space-time, there exists a specific type of unified gauge symmetry based on gauge functions with a definite parity on an orbifold. Hence the gauge symmetry is reduced to a smaller one whose generators are commutable to a parity operator at some points on the orbifold because some of gauge functions vanish there.*

Finally we discuss 4D particle spectrum of a model with $G_U = SU(5)$ on Z_N orbifold. When we take $P = \text{diag}(\theta^k, \theta^k, \theta^k, 1, 1)$ for $k \neq 0 \pmod{N}$, the gauge symmetry is reduced to that of the Standard Model, $G_{SM} \equiv SU(3) \times SU(2) \times U(1)$, in 4D theory.*) This is because some of the gauge generators T^a ($a=1, 2, \dots, 24$) are not commutable with P ,

$$PT^aP^{-1} = T^a, \quad PT^{\hat{a}+}P^{-1} = \theta^k T^{\hat{a}+}, \quad PT^{\hat{a}-}P^{-1} = \theta^{-k} T^{\hat{a}-} \quad (3\cdot7)$$

where the T^a are gauge generators of G_{SM} and the $T^{\hat{a}\pm}$ are other gauge generators. The Z_N parity assignment of 4D fields is given in Table I. The scalar field is divided into two pieces: Φ is divided into the colored triplet piece, ϕ_c , and the $SU(2)$ doublet piece, ϕ_w . In the second column, we give the $SU(3) \times SU(2)$ quantum numbers of the 4D fields. In the third column, Z_N parity of 4D fields is given. We find that the 4D massless fields include SM gauge bosons $A_\mu^{\alpha(00)}$ and a weak Higgs doublet $\phi_w^{(00)}$ and that the triplet-doublet mass splitting of the Higgs multiplets is realized by projecting out zero modes of the colored components. Whether or not extra massless particles appear depends on an assignment of Z_N parity. Let us take 6D $SU(5)$ GUTs as an example. In the case with Z_3 orbifold, the $SU(5)$ is reduced to G_{SM} in 4D theory with $P = \text{diag}(\omega, \omega, \omega, 1, 1)$

*)Our symmetry reduction mechanism is regarded as a field-theoretic version of orbifold breaking mechanism in string theory^{3),15)}. There are works¹⁶⁾ that study the relationship between our symmetry reduction mechanism and Hosotani mechanism¹⁷⁾.

Table I 4D fields and Z_N Parity.

4D fields	Quantum numbers	Z_N parity
$A_\mu^{a(\bar{n}\bar{m})}$	$(8,1)+(1,3)+(1,1)$	1
$A_\mu^{\bar{a}+(\bar{n}\bar{m})}$	$(3,2)$	θ^k
$A_\mu^{\bar{a}-(\bar{n}\bar{m})}$	$(\bar{3},2)$	θ^{-k}
$A_{z_i}^{a(\bar{n}\bar{m})}$	$(8,1)+(1,3)+(1,1)$	θ_i^{-1}
$A_{z_i}^{\bar{a}+(\bar{n}\bar{m})}$	$(3,2)$	$\theta^k \theta_i^{-1}$
$A_{z_i}^{\bar{a}-(\bar{n}\bar{m})}$	$(\bar{3},2)$	$\theta^{-k} \theta_i^{-1}$
$A_{z_i}^{a(\bar{n}\bar{m})}$	$(8,1)+(1,3)+(1,1)$	θ_i
$A_{z_i}^{\bar{a}+(\bar{n}\bar{m})}$	$(3,2)$	$\theta^k \theta_i$
$A_{z_i}^{\bar{a}-(\bar{n}\bar{m})}$	$(\bar{3},2)$	$\theta^{-k} \theta_i$
$\phi_C^{j(\bar{n}\bar{m})}$	$(3,1)$	θ^k
$\phi_W^{j(\bar{n}\bar{m})}$	$(1,2)$	1

and the 4D massless fields consist of SM gauge bosons $A_\mu^{a(00)}$, SM weak Higgs doublet $\phi_W^{(00)}$ and extra 4D scalar fields ($A_z^{\bar{a}+(00)}$, $A_z^{\bar{a}-(00)}$). In the case with Z_4 orbifold, the $SU(5)$ is reduced to G_{SM} in 4D theory with $P = \text{diag}(i^k, i^k, i^k, 1, 1)$ for $k \neq 0(\text{mod } 4)$. If we take $P = \text{diag}(i, i, i, 1, 1)$, extra 4D scalar fields appear. But if we take $P = \text{diag}(-1, -1, -1, 1, 1)$, the 4D massless fields consist of SM gauge bosons $A_\mu^{a(00)}$ and SM weak Higgs doublet $\phi_W^{(00)}$. No extra 4D scalar fields appear.

§ 4. Conclusions and discussion

We have studied a mechanism of symmetry reduction due to an intrinsic parity on an orbifold. In a class of higher-dimensional GUT, unless all components in a multiplet of some unified gauge group G_U have a common parity on an orbifold, split multiplets appear after the integration of the extra space because zero modes do not form full multiplets of G_U . We have discussed the reduction of unified gauge symmetry, gauge transformation property and its phenomenological implications. The higher-dimensional gauge symmetry is realized as an invariance under the gauge transformation whose gauge functions have a definite parity on an orbifold, and hence the gauge symmetry at some points in the compact space turns out to be a reduced one whose generators are commutable to a parity operator.

The origin of a specific parity assignment is unknown, and we believe that it will be explained in terms of some yet to be constructed underlying theory. The merit of this type of symmetry reduction is that there might be no sizable contribution to the vacuum energy upon compactification because there exists no field with a non-vanishing VEV of $O(M_c)$ in our model.* Here M_c is a compactification scale, which is related

*In the framework of supergravity theory, a large amount of (negative) vacuum energy can be generated on the breakdown of a unified gauge symmetry by Higgs mechanism through the non-vanishing VEV of the superpotential.

to a unification scale M_U .

To construct a more realistic model, it is reasonable to require the following conditions on a 4D theory.

- The coupling unification holds at the zero-th order approximation.
- The quantization of charge is derived.
- The weak scale is stable against radiative corrections.

It is desirable that our 4D world is a specific point on an extra space where a unified gauge symmetry survives from the first and second requirements. The stability of the weak scale can be guaranteed by a SUSY extension of a model. However, in a higher-dimensional SUSY GUT, Higgs multiplet appears as a hypermultiplet and it is difficult to project out all zero modes of colored Higgs multiplets by the use of a single parity. Hence it would be quite interesting to study SUSY GUTs on a more complex orbifold constructed by dividing a torus with several discrete symmetries.

References

- 1) Y. Kawamura, Prog. Theor. Phys. **105** (2001), 999.
- 2) R. Barbieri, L. J. Hall and Y. Nomura, Phys. Rev. **D63** (2001), 105007.
- 3) L. Dixon, J. Harvey, C. Vafa and E. Witten, Nucl. Phys. **B261** (1985), 651 ; **B274** (1986), 285.
- 4) L. Antoniadis, Phys. Lett. **B246** (1990), 377.
L. Antoniadis, C. Muñoz and M. Quirós, Nucl. Phys. **B397** (1993), 515.
L. Antoniadis and K. Benakli, Phys. Lett. **B326** (1994), 69.
- 5) P. Horáva and E. Witten, Nucl. Phys. **B460** (1996), 506 ; **B475** (1996), 94.
- 6) E. A. Mirabelli and M. Peskin, Phys. Rev. **D58** (1998), 065002.
- 7) A. Pomarol and M. Quirós, Phys. Lett. **B438** (1998), 255.
- 8) Y. Kawamura, Prog. Theor. Phys. **103** (2000), 613.
- 9) D. Kapetanakis and G. Zoupanos, Phys. Rep. **219** (1992), 1 and references therein.
- 10) P. Fayet, Nucl. Phys. **B246** (1984), 89 ; Phys. Lett. **146B** (1984), 41.
- 11) G. Altarelli and F. Feruglio, Phys. Lett. **B511** (2001), 257.
- 12) L. J. Hall and Y. Nomura, Phys. Rev. **D64** (2001), 055003.
- 13) Y. Kawamura, Prog. Theor. Phys. **105** (2001), 691.
- 14) A. B. Kobakhidze, Phys. Lett. **B514** (2001), 131.
- 15) L. E. Ibáñez, H. P. Nilles and F. Quevedo, Phys. Lett. **B187** (1987), 25 ;
L. E. Ibáñez, J. E. Kim, H. P. Nilles and F. Quevedo, Phys. Lett. **B191** (1987), 282 ;
L. E. Ibáñez, J. Mass, H. P. Nilles and F. Quevedo, Nucl. Phys. **B301** (1988), 157 ;
Y. Katsuki, Y. Kawamura, T. Kobayashi, N. Ohtsubo, Y. Ono and K. Tanioka, Nucl. Phys. **B341** (1990), 611.
- 16) A. Hebecker and J. March-Russell, Nucl. Phys. **B625** (2001), 128 ;
L. J. Hall, H. Murayama and Y. Nomura, hep-th/0107245.
- 17) Y. Hosotani, Phys. Lett. **B126** (1983), 309 ; Ann. of Phys. **190** (1989), 233.