Some homotopy groups of the mod 4 Moore space

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Abstract

Let $M^n$ be a Moore space of type $(\mathbb{Z}_4, n-1)$. We calculate the homotopy groups $\pi_{2n-k}(M^n)$ in the range $k=3, 4$ and $n \leq 24$. The methods are based on Toda's composition methods and we use Gray's cellular structure of the homotopy fiber of the pinching map from $M^n$ to an $n$-sphere $S^n$ and also use James' exact sequence including a relative homotopy group $\pi_\ast(M^n, S^{n-1})$.

1 Introduction and summary

We denote by $\iota_n \in \pi_n(S^n)$ the homotopy class of the identity map of $S^n$ and by $M^n_\sigma = S^{n-1} \cup \sigma \cdot e^n$ a Moore space of type $(\mathbb{Z}_q, n-1)$. In particular we set $M^n_\sigma = M^n_4$. The purpose of the present note is to determine the stable group $\pi_{2n-4}(M^n)$ or $\pi_{2n-2}(M^n)$ and the metastable group $\pi_{2n-3}(M^n)$ for $n \leq 24$. For example, the notation $(\mathbb{Z}_4)^r \oplus (\mathbb{Z}_2)^s$ or $4^r + 2^s$ means the abelian group

\[
\frac{\mathbb{Z}_4 \oplus \cdots \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}{r \oplus s}
\]

Our result is stated as follows.

Theorem 1.1 $\pi_{2n-2}(M^2) \cong \pi_{2n-4}(M^n) \cong (\mathbb{Z}_4)^r \oplus (\mathbb{Z}_2)^s$, where $r$ and $s$ are given in the following table.

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*The second author is a student of the first.
Theorem 1.2 (i) \( \pi_{n-3}(M^n) \) for \( n \leq 10 \) is isomorphic to the group \( G \) in the following table.

\[
\begin{array}{c|cccccccccc}
  n & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  \hline 
  G & 4 & 8 & 2^2 & 8+2^2 & 4+2 & 4 & 2 & 8+2^2 & 4+2^3 \\
\end{array}
\]

(ii) \( \pi_{n-3}(M^n) \cong (Z_2)^r \oplus (Z_3)^s \) for \( n \geq 11 \), where \( r \) and \( s \) are given in the following table.

\[
\begin{array}{cccccccccc}
  r & 1 & 0 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 3 & 2 & 2 & 0 \\
  s & 5 & 5 & 1 & 1 & 0 & 1 & 3 & 4 & 6 & 6 & 2 & 2 & 2 & 5 \\
\end{array}
\]

We determine \( \pi_{n-3}(M^n) \) for \( 3 \leq n \leq 7 \).

Proposition 1.3 \( \pi_2(M^3) \cong (Z_2)^2 \), \( \pi_4(M^4) \cong (Z_2) \oplus (Z_3)^2 \), \( \pi_5(M^5) \cong (Z_4) \oplus (Z_3)^2 \), \( \pi_6(M^6) \cong (Z_2) \oplus (Z_3)^2 \) and \( \pi_7(M^7) \cong (Z_2)^2 \).

Our result overlaps with that of Baues–Buth [4]. The result of Theorem 1.2 for \( n = 5 \) and 9 corrects the corresponding result of Shinpo [13].

Our method is to use the composition methods developed by Toda [15]. We use the second stage in the cellular decomposition of the homotopy fiber of the collapsing map \( p : M^n \to S^n \) obtained by Gray [6]. We also use the James exact sequence (James [7]) including the relative homotopy group \( \pi_n(M^n, S^{n-1}) \). The key step determining the group extension of \( \pi_n(M^n) \) is Lemma 2.5 which ensures that elements of \( \pi_n(M^n) \) induced from those of \( \pi_n(M^n, S^{n-1}) \) are of order 2. We use the notations and the results of [15], [9] and [8] freely.

2 Fundamental facts

For a pair of spaces \( (X, A) \), let \( i_{A, X} : A \to X \) be the inclusion and \( p_{X, A} : X \to X/A \) be the map pinching \( A \) to one point. In particular we set \( i_n = i_{A, X} \) and \( p_n = p_{X, A} \) for \( (X, A) = (M^n, S^{n-1}) \). Let \( \iota_n \) be the identity class of \( M^n \). Let \( \eta_n \in \pi_S(S^3) \) be the Hopf map and \( \eta_n = \sum_{i=0}^{n-2} \eta_i \) for \( n \geq 2 \). For integers \( a \) and \( b \), we denote by \( (a, b) \) the greatest common divisor of \( a \) and \( b \). We set \( \eta = \sum a_i \eta_i, \iota' = \sum a_i \iota_i, \iota = \sum a_i \iota_i \) and \( p = \sum a_i p_i \). Then, as is well known ([2]),

\[
\{M^2, M^3\} = Z_q(\iota') \oplus Z_{4q, 3}(\iota p) \quad (q \not= 2 \text{ mod } 4)
\]

and

\[
\{M^2, M^4\} = Z_{2q}(\iota') \quad (q = 2 \text{ mod } 4)
\]

Although we know the group structure of \( [M^2, M^3] \) by Corollary III.D.15 of [3] or by Proposition 11 of [1], we show the following.
Lemma 2.1  (i) \([M^n_3, M^n_3]\cong \mathbb{Z}\{a_3\} \oplus \mathbb{Z}\{ia_3p_3\}\) for \(q \neq 2 \text{ mod } 4\).

(ii) \([M^n_3, M^n_3]\cong \mathbb{Z}\{a_3\} \oplus \mathbb{Z}\{aia_3p_3-2a_3\}\) for \(q = 2 \text{ mod } 4\), where \(a\) is an odd integer.

Proof. First we note that \([M^n_3, M^n_3] \cong [M^n_3, \Omega M^n_3]\) is abelian by making use of Theorem X.3.10 of [16].

By Proposition 7.1 of [12], the order of \(a_3\) is \(q\) or \(2q\) according as \(q \neq 2 \text{ mod } 4\) or \(q = 2 \text{ mod } 4\). We have \(\pi_5(M^n_3) = \mathbb{Z}\{i_3\}\). We consider the following exact sequence induced from the cofibration starting with \(q_5\):

\[0 \to \pi_4(M^n_3) \xrightarrow{\partial_*} \pi_4([M^n_3, M^n_3]) \xrightarrow{\alpha_*} \pi_5(M^n_3) \xrightarrow{\xi_*} \pi_5(M^n_3)\]

By [14], \(\pi_5(M^n_3) = \mathbb{Z}_{(2q,q^2)}\{i_3p_3\}\). So the assertion (i) is obtained.

Next assume that \(q = 2 \text{ mod } 4\). Then, in the above exact sequence, we have \(qa_3 = xia_3p_3\) for an integer \(x\). By stabilizing the relation, we have \(qa_3 = xia_3p_3\) and so \(x\) becomes odd. Since 0 = 2qa_3 = 2xia_3p_3 and \(ia_3p_3\) is of order \(q\), we can set \(2x = -aq\) for an odd integer \(a\). So we have the relation \(qa_3 = a \cdot \frac{2}{q} ia_3p_3\), and hence the element \(aia_3p_3 - 2a_3\) is of order \(\frac{q}{2}\). This leads to (ii), completing the proof. \(\square\)

Let \(F\) be the homotopy fiber of the map \(p_n : M^n_3 \to S^n\). According to [6], \(F\) has a homotopy type of a CW-complex \(S^{n-1} \cup e^{2(n-1)} \cup e^{3(n-1)} \cup \cdots\) and the subcomplex \(Y = S^{n-1} \cup e^{2(n-1)}\) of the second stage has the following cell structure.

Lemma 2.2

\[Y = S^{n-1} \cup [i_{(n-1)-y}] e^{2n-2}\]

where \([\ , \ ]\) is the Whitehead product.

The following sequence is exact for \(k \leq 2n - 5 (i = iv, m)\):

\[\pi_{n+k+1}(S^n) \xrightarrow{\Delta} \pi_{n+k}(Y) \xrightarrow{\partial_*} \pi_{n+k}(M^n_3) \xrightarrow{\alpha_*} \pi_{n+k}(S^n).\] (1)

We have a formula

\[\Delta(a \circ \sum \beta) = \Delta(a) \circ \beta (a \in \pi_n(S^n), \beta \in \pi_{n+k}(S^{m-1})).\] (2)

We set \(i' = i_{n-1, Y}\).

Lemma 2.3 \(\Delta_{i_n} = q i'\) for \(n \geq 3\).

Proof. We consider the exact sequence (1) for \(k = -1\):
\[ \pi_n(S^n) \xrightarrow{\Delta} \pi_{n-1}(Y) \xrightarrow{\iota} \pi_{n-1}(M^n_q) \rightarrow 0. \]

Since \( \pi_{n-1}(Y) \cong \mathbb{Z}\{i^n\} \) and \( \pi_{n-1}(M^n_q) \cong \mathbb{Z}\{i_n\} \), we have the assertion. \( \Box \)

Next we show the following result overlapping with that of [14].

**Lemma 2.4** \( \pi_n(M^n_q) = \{i_n \eta_{n-1}\} \cong \begin{cases} \mathbb{Z}_{(2n, q^n)} & (n = 3) \\ \mathbb{Z}_{(q, q)} & (n \geq 4). \end{cases} \)

**Proof.** We consider the exact sequence (1) for \( k = 0 \):

\[ \pi_{n+1}(S^n) \xrightarrow{\Delta} \pi_n(Y) \xrightarrow{\iota} \pi_n(M^n_q) \xrightarrow{p_n} \pi_n(S^n). \]

Since \( \pi_n(M^n_q) = i_n \pi_n(S^n-1) \), \( p_n \), is trivial and \( i_n \) is an epimorphism. Note that \( q \iota_2 \circ \eta_2 = q^2 \eta_n \). So, by (2) and Lemma 2.3, we have

\[ \Delta \eta_n = \Delta q^2 \circ \eta_n = q^2 \eta_n. \]

Since \( \pi_n(Y) = \mathbb{Z}_{2q}\{i^n \eta_2\} \), we have \( \pi_n(M^n_q) \cong \pi_n(Y)/(\text{Im} \Delta) \cong \mathbb{Z}_{(2q, q^n)}. \)

For \( n \geq 4 \), we have \( \pi_n(Y) = \mathbb{Z}_{2q}\{i^n \eta_{n-1}\} \) and

\[ \Delta \eta_n = \Delta q^2 \circ \eta_{n-1} = q^2 \eta_{n-1}. \]

This leads to the assertion, completing the proof. \( \Box \)

Assume that \( q \) is even and set \( q' = \frac{q}{2} \). Then we consider the following commutative diagram between the cofiber sequences:

\[
\begin{array}{ccc}
S^{n-1} & \xrightarrow{2\iota_{n-1}} & S^{n-1} \\
\downarrow = & & \downarrow = \\
S^{n-1} & \xrightarrow{q^2 \iota_{n-1}} & M^n_q \\
\downarrow c_n & & \downarrow p_n \\
S^{n-1} & \xrightarrow{\eta_{n-1} \iota} & M^n_q \\
\downarrow & & \downarrow \\
S^n & \xrightarrow{p} & S^n.
\end{array}
\]

Here \( p : M^n_q \rightarrow S^n \) is the collapsing map, \( i : S^{n-1} \rightarrow M^n_q \) is the inclusion and \( c_n : M^n_q \rightarrow M^n_q \) is the natural map.

When there exists an element \( \beta \in \pi_n(M^n_q) \) satisfying \( p_n \beta = a \) for a given element \( \alpha \in \pi_n(S^n) \), \( \beta \) is called a lift of \( \alpha \). A lift \( \beta \) is written as \([\alpha]\). Now we show a sharper result than that of Lemma 3.3 of [13].
Lemma 2.5 Let \( q \equiv 0 \mod 4 \), \( q' = \frac{q}{2} \) and \( n \geq 3 \). Suppose that \( a \in \pi_n(S^n) \) has a lift \([a] \in \pi_n(M^n)\). Then \( c_n[a] \in \pi_n(M^n) \) is a lift of \( a \). Moreover, if \( 2[a] = 0 \) or \( 2[a] = i_{n-1}a \), then \( c_n[a] \in \pi_n(M^n) \) is of order 2.

Proof. By the diagram (3), we have \( p_n \circ c_n \circ [a] = p[a] = a \). If \( 2[a] = 0 \), then \( 2c_n[a] = 0 \). Suppose that \( 2[a] = i_{n-1}a \). Then we have \( 2c_n[a] = c_n \circ i \circ \eta_{n-1} \circ \alpha = i_n \circ q' \circ \eta_{n-1} \circ \alpha = 0 \) for \( n \geq 4 \). Note that \( 2c_n \circ \alpha = 0 \) if \( a \) is lifted to \( M^n \) ([10]). For \( n = 3 \), we have

\[
2c_n[a] = c_n \circ i \circ \eta_{n-1} \circ \alpha \\
= i_n \circ q' \circ \eta_{n-1} \circ \alpha \\
= i_n \circ \eta_{n-1} \circ (q')^2 \circ \alpha \\
= 0.
\]

This completes the proof. \( \square \)

The following elements are known to be lifted to \( M^n \) ([10]): \( \eta_n \) (\( n \geq 3 \)), \( \nu_2 \) (\( n \geq 5 \)), \( \varepsilon_n \) (\( n \geq 4 \)), \( \mu_n \) (\( n \geq 4 \)), \( s_n \) (\( n \geq 11 \)), \( \sigma_n \) (\( n \geq 17 \)), \( \omega_n \) (\( n \geq 15 \)), \( \bar{y}_n \) (\( n \geq 4 \)), \( \sigma_n \) (\( n \geq 8 \)). The problem whether \( \varepsilon_n, \mu_n, \bar{y}_n \) can be lifted to \( M^n \) and \( \sigma_l \) can be lifted to \( M^n \) is open ([10]).

3 Proof of the theorems

A lift of an element \( \alpha_n \in \pi_n(S^n) \) is also denoted as \( \bar{\alpha}_{n-1} \in \pi_n(M^n) \). We recall that \( \pi_d(M^3) \) is isomorphic to \( \mathbb{Z} \) and it is generated by a lift \( \bar{\eta}_3 \) of \( \eta_3 \) and \( 2\bar{\eta}_2 = i\bar{\eta}_2 \). We show the following ([4]).

Example 3.1 \( \pi_d(M^3) = \mathbb{Z}[c_3 \bar{\eta}_3] \oplus \mathbb{Z} [i \bar{\eta}_2] \) for \( q \equiv 0 \mod 4 \).

Proof. We consider the exact sequence (1) for \( n = 3 \) and \( k = 1 \):

\[\pi_3(S^3) \xrightarrow{\Delta} \pi_3(Y) \xrightarrow{i_n} \pi_3(M^3) \xrightarrow{p_{*}} \pi_3(S^3) .\]

By (2) and the proof of Lemma 2.4, we have

\[\Delta(\bar{\eta}_3) = \Delta \eta_3 \ast \eta_3 = q^2 \eta_2 \ast \eta_3 = 0.\]

Obviously \( \pi_d(Y) = \mathbb{Z}[i \bar{\eta}_2] \). By Lemma 2.5 for \( \bar{\eta}_3, c_3 \bar{\eta}_3 \) is of order 2. This completes the proof. \( \square \)

Hereafter we shall work in the 2-primary components of homotopy groups, unless otherwise stated.

The stable group \( \pi_d(M^3) \) has the exponent 4 because \( 4c' = 0 \in (M^3, M^3) \). By making use of the exact sequence induced from the cofibration starting with \( 4\bar{\iota}_1 \), we can
determine the stable group $\pi_k(M^3)$ for $k \leq 23$. For example, we determine the group in the case of $k=3, 4, 8$ and $17$. The rest of Theorem 1.1 is obtained by the similar argument.

**Example 3.2**

(i) $\pi_0(M^3) = \mathbb{Z}_2[i\eta] \oplus \mathbb{Z}_2[i\eta^2]$ and $
\pi_3(M^3) = \mathbb{Z}_2[\tilde{\eta}] \oplus \mathbb{Z}_2[i\nu].$

(ii) $\pi_2(M^3) = \mathbb{Z}_2[8\rho] \oplus \mathbb{Z}_2[\eta\kappa] \oplus \mathbb{Z}_2[i\eta^4] \oplus \mathbb{Z}_2[i\eta\rho],$ where $8\rho$ is a coextension of $8\rho.$

**Proof.** The first half of (i) is easily obtained.

In the exact sequence

$$\cdots \rightarrow \pi_4(S^1) \rightarrow \pi_4(M^3) \rightarrow \pi_4(S^2) \rightarrow 0,$$

we have $p_*(\tilde{\eta}) = \eta^2$ and the order of $i\nu$ is 4. This leads to the second half of (i).

In the exact sequence

$$\cdots \rightarrow \pi_4(S^1) \rightarrow \pi_4(M^3) \rightarrow \pi_4(S^2) \rightarrow 0,$$

we know $\pi_4(S^3) = \mathbb{Z}_2[\nu^2]$ and $\pi_4(S^1) = \mathbb{Z}_4[\sigma].$ The order of $i\sigma$ is 4. The order of $\nu^2 = \Sigma^2(c_0\nu^2)$ is 2 because $\nu^2 \in \pi_1(M^3)$ is of order 2 ([10]). This leads to (ii).

In the exact sequence

$$0 \rightarrow \pi_6(S^1) \rightarrow \pi_6(M^3) \rightarrow \pi_6(S^2) \rightarrow \cdots,$$

we know $\pi_6(S^3) = \mathbb{Z}_2[\rho] \oplus \mathbb{Z}_2[\eta\kappa]$ and $\pi_6(S^1) = \mathbb{Z}_4[\eta^4] \oplus \mathbb{Z}_4[\eta\rho].$ Since $p_8\rho = 8\rho,$ the order of $8\rho$ is 4. This leads to (iii), completing the proof. □

We show

**Lemma 3.3** $\pi_7(M^5) = \mathbb{Z}_2[6\nu] \oplus \mathbb{Z}_2[6\nu] \oplus \mathbb{Z}_2[\eta\nu].$

**Proof.** In the exact sequence (1) for $n=5, k=2$:

$$\pi_5(S^5) \rightarrow \pi_5(Y) \rightarrow \pi_5(M^5) \rightarrow \pi_5(S^5),$$

we have $p_*(\eta\nu) = \eta^2$ and $\pi_5(Y) = \mathbb{Z}_2[i\nu] \oplus \mathbb{Z}_2[i\Sigma\nu']$ because $4[\nu, \nu] = 8\nu.$ Here, by abuse of notation, $\Sigma\nu'$ stands for a generator of the direct summand $Z_{12}$ of $\pi_5(S^9).$

By Theorem XI.8.9 of [16], we have $4[\nu, \nu] = 4\nu + 6[\nu, \nu] = 16\nu - 6\Sigma\nu'.$ So we have $\Delta\nu = \Delta\nu - 4\nu - 6i\nu = -6i\nu$. This leads to the group $\pi_7(M^5),$ completing the proof. □

Let $A$ be a connected CW-complex and $X = A \cup e^n$ be the complex formed by attaching an $n$-cell. Let $CY$ be the reduced cone of a pointed space $Y.$ For an element
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Let $\alpha \in \pi_{k-1}(Y)$, we denote by $\alpha' \in \pi_k(CY, Y)$ an element satisfying $\partial \alpha' = \alpha$, where $\partial : \pi_k(CY, Y) \to \pi_{k-1}(Y)$ is the connecting isomorphism. Let $\omega \in \pi_k(X, A)$ be the characteristic map of the $n$-cell $e^n$ of $X$. For $\alpha \in \pi_{k-1}(Y)$, we set $\overline{\alpha} = \omega \alpha' \in \pi_k(X, A)$. Let $\omega \in \pi_n(M^n, S^{n-1})$ and $\gamma \in \pi_n(M^n, S^{n-1})$ be the characteristic maps of the $n$-cells of $M^n$ and $M^n$ respectively. Let $[\omega, \epsilon_{n-1}] \in \pi_{2n-2}(M^n, S^{n-1})$ be the relative Whitehead product ([5]). Then, by (3) and [5], we can take $\omega = c_n \gamma$ and we have

$$c_n[\gamma, \epsilon_{n-1}] = 2[\omega, \epsilon_{n-1}]. \quad (4)$$

Let $j : (M^n, *) \to (M^n, S^{n-1})$ be the inclusion. We show

**Lemma 3.4** $\pi_0(M^n) = \mathbb{Z}_4[\epsilon_0] \oplus \mathbb{Z}_2[\epsilon_0] \oplus \mathbb{Z}_2[\epsilon_0]$. 

**Proof.** We consider the homotopy exact sequence of pair $(M^n, S^n)$:

$$\pi_0(M^n, S^n) \to \pi_0(S^n) \to \pi_0(M^n, S^n).$$

Since $\pi_0(M^n, S^n) = \mathbb{Z}_4[\epsilon_0]$ and $j_* \epsilon_0 = \epsilon_0$, $j_*$ is a split epimorphism. By Theorem 2.1 of [7], $\pi_0(M^n, S^n)$ is generated by elements $\epsilon_0$ and $[\omega, \epsilon_0]$. By Theorem XI.8.9 of [16], $\partial \eta_0 = 4 \epsilon_0 \circ \eta_0 = 4 \epsilon_0 + 6[\epsilon_0, \eta_0] = 16 \epsilon_0 - 6 \Sigma \sigma'$. By [5], $\partial[\omega, \epsilon_0] = -4[\epsilon_0, \eta_0] = -8 \epsilon_0 + 4 \Sigma \sigma'$. So we have

$$\partial(2 \epsilon_0 + 3[\omega, \epsilon_0]) = 2 \Sigma \sigma'$$

and

$$\partial(2 \epsilon_0 + 3[\omega, \epsilon_0]) = 8 \epsilon_0.$$

This determines the group $\pi_0(M^n)$, completing the proof. \(\square\)

We recall that $\pi_0(S^n) = \mathbb{Z}_4[\epsilon_0]$, $2 \epsilon_0 = \Sigma \sigma''$ and $H(\sigma') = \eta_0 (\{15\})$. By the Hilton formula ([16]), $2 \epsilon_0 \circ \sigma'' = 2 \Sigma \sigma'' + [\epsilon_0, \epsilon_0] \circ H(\sigma') = \Sigma \sigma''$, because $[\epsilon_0, \epsilon_0] \circ \eta_0 = [\eta_0, \eta_0] = \eta_0 \circ [\epsilon_0, \epsilon_0] = 0$. So we have the relation $4 \epsilon_0 \circ \sigma'' = 2 \Sigma \sigma'' = 0$ and a coextension $\sigma' \in \pi_0(M^n)$ of $\sigma''$ is taken as a representative of a Toda bracket $(i_\ast, 4 \epsilon_0, \sigma')$. $\sigma'$ is a lift of $\Sigma \sigma''$. We recall that $\pi_4(S^n) = \mathbb{Z}_4[\epsilon_0] \oplus \mathbb{Z}_2[\epsilon_0]$ ([15]). Then we show

**Lemma 3.5** By a suitable choice of a coextension $\sigma'$, the order of $\sigma'$ is 4.

**Proof.** We have $4 \sigma' \in \{i_\ast, 4 \epsilon_0, \sigma''\} \circ 4 \epsilon_0 = \frac{3}{4} \{4 \epsilon_0, \sigma'', 4 \epsilon_0\}$. By Corollary 3.7 of [15], we have $\{2 \epsilon_0, \Sigma \sigma'', 2 \epsilon_0\} \equiv (\Sigma \sigma'') \eta_3 = 0 \mod 2 \pi_4(S^n) = \{2 \epsilon_0\}$. So we have $\{2 \epsilon_0, \Sigma \sigma'', 2 \epsilon_0\} \equiv 0 \mod 2 \epsilon_0 \circ \pi_4(S^n) + 2 \pi_4(S^n) = \{2 \epsilon_0\}$. Here we have used the relations $2 \epsilon_0 \circ \epsilon_0 = 0$ and $2 \epsilon_0 \circ \epsilon_0 = 4 \epsilon_0$.

We have
\[
\{ \alpha_0, \sigma', 4 \alpha_1 \} \subset \{ \alpha_0, 2 \alpha_0, \sigma', 4 \alpha_1 \} = \{ \alpha_0, \sigma'', 4 \alpha_1 \}
\]
\[
\supset (2 \alpha_0, \sigma'', 2 \alpha_1) \circ 2 \alpha_1 \\
\equiv 0 \mod (4 \nu_0 + 2 \alpha_0 \circ \pi_4(S^9) = \{ 4 \nu_0 \}).
\]
So we have \( 4 \sigma'' = 4a_i \nu_i \) for \( a \in \mathbb{Z} \). We set \( \sigma'' = \sigma'' - a_i \nu_i \). Then \( p_7 \sigma'' = \sum \sigma'' = 2 \sigma' \) and \( 4 \sigma'' = 0 \). By renaming \( \sigma'' \) as \( \sigma'' \), we have the assertion. This completes the proof. \( \square \)

We set \( 4 \sigma_n = \sum^{n-9} \sigma''(n \geq 9) \). Since \( p_{n+1} 4 \sigma_n = 4 \sigma_{n+1} \), the order of \( 4 \sigma_n \) \( (n \geq 9) \) is 4. We show

**Lemma 3.6** \( \pi_1(M^{10}) = \mathbb{Z}[4 \sigma_0] \oplus \mathbb{Z}[i_{10} \alpha_0, \xi_0] \oplus \mathbb{Z}[i_0 \nu_0] \oplus \mathbb{Z}[i_0 \nu_0] \oplus \mathbb{Z}[i_0 \xi_0] \).

**Proof.** We consider the exact sequence (1) for \( n=10 \) and \( k=7 \):

\[ \pi_0(S^{10}) \overset{\Delta}{\rightarrow} \pi_1(Y) \overset{i}{\rightarrow} \pi_1(M^{10}) \overset{p_7}{\rightarrow} \pi_1(S^{10}). \]

We have \( \text{Im} \ \Delta = 0 \) and \( \pi_1(Y) \cong \pi_1(S^9) = \mathbb{Z}[\nu_0] \oplus \mathbb{Z}[\nu_0] \oplus \mathbb{Z}[\xi_0] \) by [15] and Lemma 2. This completes the proof. \( \square \)

The group \( \pi_{n+3}(M^n) \) for \( 11 \leq n \leq 13 \) is given as follows ([13]).

**Example 3.7** (i) \( \pi_{10}(M^{11}) = \mathbb{Z}[i_{11} \alpha_1, \xi_0] \oplus \mathbb{Z}[i_{11} \nu_0] \oplus \mathbb{Z}[i_{11} \mu_0] \oplus \mathbb{Z}[\eta_{11} \epsilon_1] \oplus \mathbb{Z}[\xi_0] \).

(ii) \( \pi_{10}(M^{12}) = \mathbb{Z}[i_{12} \sigma_{11} \nu_0] \oplus \mathbb{Z}[i_{12} \eta_{11} \mu_0] \oplus \mathbb{Z}[\nu_0 \nu_0] \oplus \mathbb{Z}[\xi_0] \).

(iii) \( \pi_{10}(M^{13}) = \mathbb{Z}[i_{13} \alpha_1, \alpha_1] \oplus \mathbb{Z}[i_{13} \mu_{12}] \oplus \mathbb{Z}[\eta_{11} \xi_0] \).

Although we can get the group \( \pi_{n+3}(M^n) \) quickly ([13]), we take a roundabout way. First we recall that \( \alpha_0 \in \pi_{20}(S^{10}) \) is not lifted to \( M^{10} \) ([10]) and has the property ([15], [9])

\[ 2 \alpha_0 \circ \alpha_1 = 2 \alpha_1 = 0. \]

So we can define a coextension \( \tilde{\kappa}_0 \in \{ i, 2 \alpha_1, \alpha_0 \} \subset \pi_{20}(M^{11}) \) of \( \kappa_{10} \). We know \( \pi_{20}(S^{10}) = \mathbb{Z}[\sum \sigma''] \oplus \mathbb{Z}[\eta_{10} \kappa_1] \oplus \mathbb{Z}[\sigma_0 \nu_1] \) and \( \pi_{20}(S^{11}) = \mathbb{Z}[\sum \sigma''] \oplus \mathbb{Z}[\eta_{11} \kappa_1] \). Then we show

**Lemma 3.8** By a suitable choice of a coextension \( \tilde{\kappa}_0 \),

\[ 2 \tilde{\kappa}_0 = i \eta_{10} \kappa_1 \mod i \sigma_0 \nu_1 \]

and the order of \( \tilde{\kappa}_0 \) is 4.

**Proof.** By Corollary 3.7 of [15], we have
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\[ \{2\epsilon_{11}, \kappa_{11}, 2\epsilon_{25}\} \ni \kappa_{11} \eta_{25} = \eta_{11} \kappa_{12} \mod 2\pi_9(S^{11}) = (2\Sigma^3 \rho'). \]

Since \( \Sigma \{2\epsilon_{10}, \kappa_{10}, 2\epsilon_{24}\} \subset \{2\epsilon_{11}, \kappa_{11}, 2\epsilon_{25}\} \), we have

\[ \{2\epsilon_{10}, \kappa_{10}, 2\epsilon_{24}\} \ni \eta_{10} \kappa_{11} \mod \{ \sigma_{10} \nu_{17} \} + 2\pi_{25}(S^{10}) = (2\Sigma \rho', \sigma_{10} \nu_{17}). \]

So we have

\[ 2\kappa_{10} \in \{ i, 2\epsilon_{10}, \kappa_{10} \} \circ 2\epsilon_{25} \]
\[ = - i(2\epsilon_{10}, \kappa_{10}, 2\epsilon_{24}) \]
\[ \ni 2\eta_{10} \kappa_{11} \mod \{ 2i \Sigma \rho', \sigma_{10} \nu_{17} \}. \]

So, by a suitable choice of a coextension \( \kappa_{10} \), we have the relation. In the stable range, we have \( i\eta_9 \neq 0 \) in \( \pi_9(M^2) \). Hence the order of \( \kappa_{10} \) is 4. This completes the proof. \( \square \)

Hereafter we set \( \kappa_n = \Sigma^{n-10} \kappa_{10} \) for \( n \geq 10 \) for the coextension \( \kappa_{10} \) in Lemma 3.8. Since \( \sigma_1 \nu_{18} = 0 \), we have \( \kappa_n = i\eta_{9n} \kappa_{11} \) for \( n \geq 11 \).

We recall that \( \sigma_6^2 \) is not lifted to \( M^2 \) ([10]). The following is a byproduct of our roundabout way.

**Lemma 3.9** \([\epsilon_{10}, \epsilon_{10}] \in \{2\epsilon_{10}, \sigma_6^2, 2\epsilon_{25}\} \mod 2\pi_9(S^{16})\).

**Proof.** Since \( 2\sigma_6^2 = [\epsilon_{10}, \epsilon_{10}] \), a Toda bracket \( \{2\epsilon_{10}, \sigma_6^2, 4\epsilon_{25}\} \) is well defined. By Proposition 2.6 of [15], we have

\[ H\{2\epsilon_{10}, \sigma_6^2, 4\epsilon_{25}\} = - \Delta^{-1}(2\sigma_6^2) \circ 4\epsilon_{25} = - 4\epsilon_{25} = \pm 2H([\epsilon_{10}, \epsilon_{10}]). \]

The indeterminacy of \( \{2\epsilon_{10}, \sigma_6^2, 4\epsilon_{25}\} \) is \( 2\epsilon_{10} + \Sigma \pi_9(S^{15}) + 4\pi_9(S^{16}) = \{4[\epsilon_{10}, \epsilon_{10}], 2\rho_{16}\} \). So \( \{2\epsilon_{10}, \sigma_6^2, 4\epsilon_{25}\} \) contains \( 2[\epsilon_{10}, \epsilon_{10}] \) modulo elements of \( \Sigma \pi_9(S^{15}) = \{4[\epsilon_{10}, \epsilon_{10}], \eta_{10} \kappa_{17}\} \). In the stable case, \( \eta_{10} \neq 0 \) and \( \langle 2i, \sigma^2, 4\epsilon \rangle \neq 0 \mod 2\pi_9(S^9) = (2\rho) \). Hence we have

\[ 2[\epsilon_{10}, \epsilon_{10}] \in \{2\epsilon_{10}, \sigma_6^2, 4\epsilon_{25}\} \mod \{4[\epsilon_{10}, \epsilon_{10}], 2\rho_{16}\}. \]

We have

\[ \{2\epsilon_{10}, \sigma_6^2, 4\epsilon_{25}\} \subset \{2\epsilon_{10}, \sigma_6^2, 4\epsilon_{25}\} \supset 2\{2\epsilon_{10}, \sigma_6^2, 2\epsilon_{25}\} \]
\[ \mod 2\epsilon_{10} + \pi_9(S^{15}) + 4\pi_9(S^{16}) = \{4[\epsilon_{10}, \epsilon_{10}], 2\rho_{16}\} \]

So, for any element \( \alpha \in \{2\epsilon_{10}, \sigma_6^2, 4\epsilon_{25}\} \), we have

\[ 2\alpha = 2[\epsilon_{10}, \epsilon_{10}] \mod \{4[\epsilon_{10}, \epsilon_{10}], 2\rho_{16}\}. \]

This implies the relation \( \alpha = [\epsilon_{10}, \epsilon_{10}] \mod \{2[\epsilon_{10}, \epsilon_{10}], \rho_{16}, \eta_{10} \kappa_{17}\} \). By the same argument as the above in the stable range, we have
\[
\alpha = [\ell_{16}, \ell_{16}] \mod (2[\ell_{16}, \ell_{16}], 2\rho_{16}) = 2\pi_3(S^{16}).
\]

The indeterminacy of \([2\ell_{16}, \sigma_{16}, 2\omega_6]\) is \(2\ell_{16} \circ \pi_3(S^{16}) + 2\pi_3(S^{16}) = 2\pi_3(S^{16}).\) Hence we have \([\ell_{16}, \ell_{16}] \in (2\ell_{16}, \sigma_{16}^2, 2\omega_6) \mod 2\pi_3(S^{16}).\) This completes the proof. \(\Box\)

Let \(\tilde{\sigma}_{16} \in \pi_3(M_{17}^2)\) be a coextension of \(\sigma_{16}.\) Then we show

**Lemma 3.10** \(\pi_3(M_{17}^2) = \mathbb{Z}_4(i\sigma_{16}) \oplus \mathbb{Z}_4(i\sigma_{16}^2) \oplus \mathbb{Z}_2(i\rho_{16})\), where \(2\eta_{16} = i\eta_{16} \pi_{17}\) and \(2\sigma_{16} = i[\ell_{16}, \ell_{16}].\) For a suitable choice of a coextension \(i_{16}.\)

**Proof.** In the exact sequence

\[
\begin{align*}
\text{ng}^2(\ell_{16}, S^{16}) &\rightarrow \pi_3(S^{16}) \rightarrow \pi_3(M_{17}^2) \rightarrow \pi_3(S^{16}),
\end{align*}
\]

we know \(\pi_3(M_{17}^2, S^{16}) = \pi_3(S^{16}) = \mathbb{Z}_4(\sigma_{16}^2) \oplus \mathbb{Z}_2(\pi_{16}).\) By Theorem 2.1 of [7], we have \(\eta_{16} = \pi_{16} \mathbb{Z}_2(\pi_{16} \sigma_{16}) \oplus \mathbb{Z}_2(\pi_{16} \pi_{16}).\) We have \(\partial(\rho_{16}) = 2\rho_{16},\) \(\partial(\pi_{16}) = 0\) and \(\partial([\eta_{16}, \pi_{16}]) = -2[\ell_{16}, \ell_{16}].\) We know \(2\eta_{16} = i\eta_{16} \pi_{17}.

By Lemma 3.9, we have

\[
2c_{16} \equiv i[\ell_{16}, \ell_{16}] \mod 2\pi_3(S^{16}).
\]

So, by a suitable choice of the coextension \(\tilde{\sigma}_{16},\) we have \(2\tilde{\sigma}_{16} = i[\ell_{16}, \ell_{16}].\) This completes the proof. \(\Box\)

We set

\[
\tilde{\sigma}_{16} = \sum_{n=16}^{n-1} \tilde{\sigma}_{16}^n
\]

for \(n \geq 16.\) Now we show

**Lemma 3.11** \(\pi_3(M_{17}^2) = \mathbb{Z}_4(i\ell_{16} \ell_{16}) \oplus \mathbb{Z}_4(i\ell_{16} \rho_{16}) \oplus \mathbb{Z}_2(i\ell_{16} \pi_{16}) \oplus \mathbb{Z}_2(\ell_{17} \pi_{16}).\)

**Proof.** We consider the exact sequence (1) for \(k=14\) and \(n=17:\)

\[
\pi_3(S^{17}) \rightarrow \pi_3(Y) \rightarrow \pi_3(M_{17}^2) \rightarrow \pi_3(S^{17}).
\]

We know \(\pi_3(S^{17}) = \mathbb{Z}_2(\pi_{17}) \oplus \mathbb{Z}_2(\pi_{17})\) and \(\pi_3(S^{17}) = \mathbb{Z}_2(\rho_{17}) \oplus \mathbb{Z}_2(\pi_{17} \pi_{18}).\) We have \(\pi_3(Y) = \mathbb{Z}_2(i\rho_{16}) \oplus \mathbb{Z}_2(i\ell_{16} \ell_{16}) \oplus \mathbb{Z}_2(i\ell_{16} \pi_{16}).\) We have \(\Delta(\rho_{17}) = 4i\rho_{16}\) and \(\Delta(\pi_{17} \pi_{18}) = 0.\) By Lemmas 2.5 and 3.10, the order of \(c_{17} \pi_{16}\) is 2. By Lemmas 2.5, 3.10 and by (3), we have

\[
2c_{17} \pi_{16} = c_{17} \circ i[\ell_{16}, \ell_{16}] = i_{17} \circ 2\ell_{16} \circ [\ell_{16}, \ell_{16}] = 4i_{17} \circ [\ell_{16}, \ell_{16}].
\]

This completes the proof. \(\Box\)
The following result is easily obtained.

**Lemma 3.12 (i)** \( \pi_4(\mathbb{M}_2^2) = \mathbb{Z}_4[4c_{22}, \delta_{22}] \oplus \mathbb{Z}_4[2c_{22}, \delta_{22}] \oplus \mathbb{Z}_2[2c_{22}, \delta_{22}] \oplus \mathbb{Z}_2[2c_{22}, \delta_{22}], \) where \( 2c_{22} \in \{i_{22}, 4c_{22}, 2\delta_{22}\}. \)

**Lemma 3.12 (ii)** \( \pi_5(\mathbb{M}_2^3) = \mathbb{Z}_4[c_{23}, \delta_{23}] \oplus \mathbb{Z}_4[2c_{23}, \delta_{23}] \oplus \mathbb{Z}_2[c_{23}, \delta_{23}] \oplus \mathbb{Z}_2[2c_{23}, \delta_{23}], \) where \( 2c_{23} \in \{i_{23}, 4c_{23}, 2\delta_{23}\}. \)

**Lemma 3.12 (iii)** \( \pi_6(\mathbb{M}_2^4) = \mathbb{Z}_2[c_{24}, \delta_{24}] \oplus \mathbb{Z}_2[2c_{24}, \delta_{24}] \oplus \mathbb{Z}_2[2c_{24}, \delta_{24}] \oplus \mathbb{Z}_2[2c_{24}, \delta_{24}] \oplus \mathbb{Z}_2[i_{24}, \delta_{24}, \delta_{24}]. \)

The rest of Theorem 1.2 is obtained by the similar argument ([13]).

### 4 Some unstable homotopy groups of \( M^n \)

In this section, we shall prove Proposition 1.3. We recall that \( \pi_6(\mathbb{M}_2^3) = \mathbb{Z}_4[4c_3, \delta_3] \oplus \mathbb{Z}_2[2c_3, \delta_3], \) where \( j_4 = [\gamma, \alpha_3] \) and \( \Sigma \delta = 2i_4 \) ([11]). We show

**Lemma 4.1** There exists an element \( \theta \in \pi_6(M^4) \) satisfying \( j_4 \theta = [\omega, \alpha_3], \) \( 2\theta = c_4 \delta \) and \( \pi_6(M^4) = \mathbb{Z}_4[\theta] \oplus \mathbb{Z}_2[i_4 \nu + 2\theta] \oplus \mathbb{Z}_2[4c_4, \delta_4]. \)

**Proof.** By Theorem 2.1 of [7], we have \( \pi_6(\mathbb{M}_4, S^3) = \mathbb{Z}_4[\eta_3^2] \oplus \mathbb{Z}_4[\omega, \alpha_3] \) and \( \pi_6(\mathbb{M}_4, S^3) = \mathbb{Z}_4[\delta_4] \oplus \mathbb{Z}_2[\omega, \eta_3]. \) In the homotopy exact sequence of a pair \( (\mathbb{M}_4, S^3): \)

\[
\pi_6(\mathbb{M}_4, S^3) \to \pi_6(S^3) \to \pi_6(\mathbb{M}_4, S^3) \to \pi_6(S^3),
\]

we have

\[
\partial \eta_3^2 = 4\iota_4 \eta_3^2 = 0, \quad \partial(\omega, \alpha_3) = -4\iota_4 \omega = 0,
\]

\[
\partial \nu_4 = 4\iota_4 \nu_4 = 4\nu_4 = 0 \quad \text{and} \quad \partial(\omega, \eta_3) = -4\iota_4 \omega = 0.
\]

So there exists an element \( \theta \in \pi_6(M^4) \) satisfying \( j_4 \theta = [\omega, \alpha_3] \) and we have a short exact sequence:

\[
0 \to \pi_6(S^3) \overset{i_*}{\to} \pi_6(M^4) \overset{j_*}{\to} \pi_6(M^4, S^3) \to 0.
\]

By (4), we have

\[
\quad j_4(c_4, \delta) = c_4 j_4 \delta = c_4[\gamma, \alpha_3] = 2[\omega, \alpha_3] = 2j_4 \theta.
\]

So we have the relation \( 2\theta = c_4 \delta + a_i \nu \) for an integer \( a. \) Note that we take \( c_6 = \Sigma c_4 \) in the diagram (3). Then, by Lemma 3.3, we have
Therefore we have $2\Theta = a_i \Sigma \nu'$. Since $i \Sigma \nu'$ is not divisible by 2 by Lemma 3.3, $a$ becomes even. So we have the relation $2\Theta = c_i \delta \mod 2i\nu'$. By the diagram (3), we have $4\Theta = 2c_i \delta = c_i \circ i\nu' = i_i \circ 2c_i \circ \nu' = 2i\nu' \circ 0$. Hence the order of $\Theta$ is 8 and $2i\nu' = 4\Theta = 2c_i \delta$.

Thus we have $2\Theta = \pm c_i \delta$ and we get the group $\pi_0(M^i)$. This completes the proof. 

Let $\tilde{\nu}' \in \{i_i, 4i_i, \nu'\} \subset \pi(M^i)$ be a coextension of $\nu'$. Then we show

**Lemma 4.2** $\pi_0(M^i) = \mathbb{Z}_4[\Sigma \tilde{\nu}'] \oplus \mathbb{Z}_2[i_i \nu_i \eta_i] \oplus \mathbb{Z}_2[i_i (\Sigma \nu') \eta_i].$

**Proof.** We consider the exact sequence

$$\pi_0(M^i, S^i) \xrightarrow{\iota} \pi_0(S^i) \xrightarrow{\iota^*} \pi_0(M^i) \xrightarrow{\iota_!} \pi_0(S^i).$$

By Theorem 2.1 of [7], we have $\pi_0(M^i, S^i) = \mathbb{Z}[[\omega, \omega]] \oplus \mathbb{Z}_4[\Sigma \tilde{\nu}']$ and $\pi_0(M^i, S^i) = \mathbb{Z}_2[i_i \nu_i \eta_i] \oplus \mathbb{Z}_2[[\omega, \omega]]$. Here $\Sigma : \pi_0(M^i, S^i) \rightarrow \pi_0(M^i, S^i)$ is the relative suspension. We have $\partial[\omega, \omega] = -8\nu_i$, $j_* \Sigma \tilde{\nu}' = \Sigma \tilde{\nu}'$, $\partial \tilde{\nu}' \eta_i = 4\nu_i \circ \nu_i \circ \eta_i = (16\nu_i - 6\Sigma \nu') \circ \eta_i = 0$ and $\partial[\omega, \nu_i] = 0$. So the following short exact sequence splits:

$$0 \rightarrow \pi_0(S^i) \xrightarrow{\iota^*} \pi_0(M^i) \xrightarrow{\iota_!} \mathbb{Z}_4[\Sigma \tilde{\nu}'] \rightarrow 0.$$

This completes the proof. 

We set $\tilde{\nu}_n = \Sigma^{n-2} \tilde{\nu}'$ for $n \geq 5$. By use of the exact sequence (1) for $n = 6$ and $k = 3$, we have

**Example 4.3** $\pi_0(M^6) = \mathbb{Z}_4[\tilde{\nu}_6] \oplus \mathbb{Z}_2[i_i \nu_i \eta_i].$

By Theorem 1.2 of [7], we have $\pi_0(M^6, S^6) = \mathbb{Z}[[\omega, \omega]]$. Let $\beta \in \pi_0(M^6)$ be an element satisfying $j_* \beta = [\omega, \omega]$. Then we show

**Lemma 4.4** $\pi_0(M^6) = \mathbb{Z}_4[\beta] \oplus \mathbb{Z}_2[i_i \nu_i \eta_i].$

**Proof.** In the exact sequence
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\[ \pi_1(M^6, S^5) \xrightarrow{\partial} \pi_0(S^5) \xrightarrow{i_*} \pi_0(M^6) \xrightarrow{\iota} \pi_0(M^6, S^5) \rightarrow 0, \]

we have \( \pi_1(M^6, S^5)=\mathbb{Z}_2[\{\omega, \eta_b\}] \) and \( \partial[\omega, \eta_b]=0 \). So it suffices to show \( 8\beta=0 \). By the parallel argument to the proof of Lemma 4.1, we have a relation

\[ 2\beta = c_6\lambda + b\iota_6 \nu_6 \eta_b^2 \quad (b=0, 1), \]

where \( \lambda \) is a generator of \( \pi_0(M^6) \approx \mathbb{Z}_2 \) satisfying \( j_*\lambda = [\gamma, \iota_6] \) and \( 4\lambda = \iota_6 \nu_6 \eta_b^2 \) ([11]). By the diagram (3), we have

\[ 8\beta = c_6 \circ \iota_6 \nu_6 \eta_b^2 = \iota_6 \circ 2 \iota_6 \circ \nu_6 \eta_b^2 = 0. \]

This completes the proof. \( \square \)

Finally the following is easily obtained.

**Example 4.5** \( \pi_2(M^7) = \mathbb{Z}_2[\iota_7 \nu_6 \eta_b^2] \).

**References**


