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Cohomotopy sets of projective planes

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Abstract

We set $\mathbf{F}=\mathbf{R}$ (*real*), $\mathbf{C}$ (*complex*), $\mathbf{H}$ (*quaternion*), $\mathbf{O}$ (*octonian*) and $d=\dim_{\mathbf{R}}\mathbf{F}$. We denote by $\mathbf{F}P^2$ the $\mathbf{F}$-projective plane. The purpose of this note is to determine the cohomotopy set

$$\pi^n(\mathbf{F}P^2)=[\mathbf{F}P^2, S^n].$$

Let $h=h(\mathbf{F}): S^{2d-1}\to S^d$ be the Hopf map. Then we have a cell structure $\mathbf{F}P^2=S^d\cup_h e^{2d}$ and a cofiber sequence:

$$S^{2d-1}\xrightarrow{h} S^d\xrightarrow{i}\mathbf{F}P^2\xrightarrow{p} S^{2d}\xrightarrow{\Sigma h} S^{d+1}\longrightarrow\cdots, \tag{1}$$

where i is the inclusion map, $p=p(\mathbf{F})$ is a map pinching S^d to one point and Σh is the reduced suspension of h. Our result is given by the table on page 7. Its essence is stated as follows.

Theorem $h^*: \pi_d(S^n)\to\pi_{2d-1}(S^n)$ *is a monomorphism and there exists a bijection*

$$\pi^n(\mathbf{F}P^2)=p^*\pi_{2d}(S^n)\cong\pi_{2d}(S^n)/(\Sigma h)^*\pi_{d+1}(S^n).$$

Our method is to use the cofiber sequence (1) and our calculation is based on the results of homotopy groups of spheres given by Toda (1962 [3]).

1 The real, complex and quaternionic planes

Throughout this note we use the following exact sequence induced from the cofiber sequence (1):

$$\pi_{2d-1}(S^n)\xleftarrow{h^*}\pi_d(S^n)\xleftarrow{i^*}\pi^n(\mathbf{F}P^2)\xleftarrow{p^*}\pi_{2d}(S^n)\xleftarrow{(\Sigma h)^*}\pi_{d+1}(S^n). \tag{2}$$

In general no group structure exists on the set $\pi^n(\mathbf{F}P^2)$ except for the case $n\geq d$

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$+1$. To express our result simply, we give a virtual group structure in this set so that $p^*: \pi_{2d}(S^n) \longrightarrow \pi^n(\mathrm{FP}^2)$ is a homomorphism. This group structure is realized for $n \geq d+1$ since $\pi^n(\mathrm{FP}^2)$ is stable. We also note that the Hopf multiplication of S^n for $n=3$ or 7 induces the homomorphism $p^*: \pi_{2d}(S^n) \longrightarrow \pi^n(\mathrm{FP}^2)$.

As is easily seen, we obtain the following: $\pi^1(\mathbf{F}\mathrm{P}^2)=0$, $\pi^{2d}(\mathbf{F}\mathrm{P}^2)=\mathbf{Z}\{p(\mathbf{F})\}$ except for $\mathbf{F}=\mathbf{R}$, $\pi^n(\mathbf{F}\mathrm{P}^2)=0$ for $2d<n$ and $\pi^n(\mathbf{F}\mathrm{P}^2) \cong \pi_{2d}(S^n) \cong \pi^S_{2d-n}(S^0)$ for $d+1<n$, where $\pi^S_k(S^0)$ stands for the k-stem stable homotopy group of a sphere. So it suffices to work in the case $2 \leq n \leq d+1$.

By abuse of notation, we often use the same letter to denote a map and its homotopy class. First we recall the following results of homotopy groups of spheres (Toda 1962 [3]):

$$\pi_n(S^n)=\mathbf{Z}\{\iota_n\}(n \geq 1);\ \pi_3(S^2)=\mathbf{Z}\{\eta_2\};$$

$$\pi_{n+1}(S^n)=\mathbf{Z}_2\{\eta_n\}(n \geq 3);\ \pi_{n+2}(S^n)=\mathbf{Z}_2\{\eta_n^2\}(n \geq 2),$$

where $\eta_2=h(\mathbf{C})$, $\eta_n=\sum^{n-2}\eta_2$ and $\eta_n^2=\eta_n \circ \eta_{n+1}$ for $n \geq 2$.

Obviously we have the following:

$$\pi^2(\mathbf{R}\mathrm{P}^2)=\mathbf{Z}_2\{p(\mathbf{R})\};\ \pi^2(\mathbf{C}\mathrm{P}^2)=0.$$

We show

Lemma 1 $\eta_{2*}: \pi^3(\mathbf{F}\mathrm{P}^2) \rightarrow \pi^2(\mathbf{F}\mathrm{P}^2)$ *is bijective if* $\mathbf{F}=\mathbf{H}$ *or* $\mathbf{O}$.

Proof. By making use of the Hopf fibration $\eta_2: S^3 \rightarrow S^2$, we have an exact sequence (Mimura-Toda 1991 [1])

$$0=[\mathbf{F}\mathrm{P}^2, S^1] \longrightarrow [\mathbf{F}\mathrm{P}^2, S^3] \xrightarrow{h_*} [\mathbf{F}\mathrm{P}^2, S^2] \xrightarrow{i'_*} [\mathbf{F}\mathrm{P}^2, BS^1],$$

where $BS^1=K(\mathbf{Z}, 2)$ is the classifying space of S^1 and $i': S^2 \hookrightarrow BS^1$ is the inclusion map. $[\mathbf{F}\mathrm{P}^2, BS^1]$ is isomorphic to the cohomology group $\tilde{H}^2(\mathbf{F}\mathrm{P}^2 : \mathbf{Z})$ which is trivial in our case. This completes the proof. □

We recall the following (Toda 1962 [3]):
$\pi_6(S^3)=\mathbf{Z}_4\{\nu'\} \oplus \mathbf{Z}_3\{\alpha_1(3)\}$; $\pi_7(S^4)=\mathbf{Z}\{\nu_4\} \oplus \mathbf{Z}_4\{\sum\nu'\} \oplus \mathbf{Z}_3\{\alpha_1(4)\}$; $\pi_{n+3}(S^n)=\mathbf{Z}_8\{\nu_n\} \oplus \mathbf{Z}_3\{\alpha_1(n)\}$ ($\nu_n=\sum^{n-4}\nu_4$ for $n \geq 4$ and $\alpha_1(n)=\sum^{n-3}\alpha_1(3)$ for $n \geq 3$); $2\nu_n=\sum^{n-3}\nu'$ for $n \geq 5$. We can take $h=h(\mathbf{H})=\nu_4+\alpha_1(4)$, and so $\sum h=\nu_5+\alpha_1(5)$. We also recall the following:

$$\pi_7(S^3)=\mathbf{Z}_2\{\nu' \circ \eta_6\};\quad \nu' \circ \eta_6=\eta_3 \circ \nu_4;\ \eta_3 \circ \textstyle\sum\nu'=0;$$
$$\pi_8(S^4)=\mathbf{Z}_2\{\nu_4 \circ \eta_7\} \oplus \mathbf{Z}_2\{\textstyle\sum\nu' \circ \eta_7\};\ \pi_9(S^5)=\mathbf{Z}_2\{\nu_5 \circ \eta_8\};$$
$$\pi_8(S^3)=\mathbf{Z}_2\{\nu' \circ \eta_6^2\};\quad \pi_9(S^4)=\mathbf{Z}_2\{\nu_4 \circ \eta_7^2\} \oplus \mathbf{Z}_2\{\textstyle\sum\nu' \circ \eta_7^2\};$$
$$\pi_4^S(S^0)=\pi_5^S(S^0)=0;\quad \pi_6^S(S^0) \cong \mathbf{Z}_2.$$

We recall $\pi_{10}(S^3)=\mathbf{Z}_3\{\alpha_2(3)\}\oplus\mathbf{Z}_5\{\alpha_1'(3)\}$. We set $\alpha_2(n)=\Sigma^{n-3}\alpha_2(3)$ and $\alpha_1'(n)=\Sigma^{n-3}\alpha_1'(3)$ for $n\geq 3$. Then we know $\pi_{11}(S^4)=\mathbf{Z}_3\{\alpha_2(4)\}\oplus\mathbf{Z}_5\{\alpha_1'(4)\}$; $\pi_{15}(S^8)=\mathbf{Z}\{\sigma_8\}\oplus\mathbf{Z}_8\{\Sigma\sigma'\}\oplus\mathbf{Z}_3\{\alpha_2(8)\}\oplus\mathbf{Z}_5\{\alpha_1'(8)\}$; $\pi_{16}(S^9)=\mathbf{Z}_{16}\{\sigma_9\}\oplus\mathbf{Z}_3\{\alpha_2(9)\}\oplus\mathbf{Z}_5\{\alpha_1'(9)\}$. We can take $h=h(\mathbf{O})=\sigma_8+\alpha_2(8)+\alpha_1'(8)$, and so $\Sigma h=\sigma_9+\alpha_2(9)+\alpha_1'(9)$.

Since $\Sigma h(\mathbf{F})$ is a generator of $\pi_{2d}(S^{d+1})$ except for $\mathbf{F}=\mathbf{R}$, the exact sequence (2) implies $\pi^{d+1}(\mathrm{FP}^2)=0$ except for $\mathbf{F}=\mathbf{R}$. We show

Lemma 2 $\pi^4(\mathrm{HP}^2)=\mathbf{Z}_2\{\nu_4\circ\eta_7\circ p\}$ and $\pi^3(\mathrm{HP}^2)=\mathbf{Z}_2\{\nu'\circ\eta_6^2\circ p\}$.

Proof. We consider the exact sequence (2) for $n=4$:

$$\pi_7(S^4)\xleftarrow{h^*}\pi_4(S^4)\xleftarrow{i^*}\pi^4(\mathrm{HP}^2)\xleftarrow{p^*}\pi_8(S^4)\xleftarrow{(\Sigma h)^*}\pi_5(S^4).$$

h^* is a monomorphism and we have $(\Sigma h)^*(\eta_4)=\eta_4\circ\nu_5+\eta_4\circ\alpha_1(5)=\Sigma\nu'\circ\eta_7$ since $\eta_4\circ\alpha_1(5)=0$. This leads to the first half.

Next we consider the exact sequence (2) for $n=3$:

$$\pi_7(S^3)\xleftarrow{h^*}\pi_4(S^3)\xleftarrow{i^*}\pi^3(\mathrm{HP}^2)\xleftarrow{p^*}\pi_8(S^3)\xleftarrow{(\Sigma h)^*}\pi_5(S^3).$$

Since $h^*(\eta_3)=\eta_3\circ\nu_4=\nu'\circ\eta_6$, h^* is a monomorphism. We have

$$\begin{aligned}(\Sigma h)^*(\eta_3^2)&=\eta_3\circ\eta_4\circ\nu_5\\&=\eta_3\circ\Sigma\nu'\circ\eta_7\\&=0.\end{aligned}$$

This leads to the second half and completes the proof. □

2 The Cayley projective plane

Hereafter we deal with the cohomotopy set $\pi^n(\mathrm{OP}^2)$. We recall the following:

$$\nu'\circ\nu_6=0;\ \eta_6\circ\sigma'=4\bar{\nu}_6;\ \eta_7\circ\Sigma\sigma'=0;\ \eta_6\circ\bar{\nu}_7=\bar{\nu}_6\circ\eta_{14}=\nu_6^3;$$

$$\pi_{15}(S^7)=\mathbf{Z}_2\{\sigma'\circ\eta_{14}\}\oplus\mathbf{Z}_2\{\bar{\nu}_7\}\oplus\mathbf{Z}_2\{\varepsilon_7\};\ \eta_7\circ\sigma_8=\sigma'\circ\eta_{14}+\bar{\nu}_7+\varepsilon_7;$$

$$\pi_{16}(S^7)=\mathbf{Z}_2\{\sigma'\circ\eta_{14}^2\}\oplus\mathbf{Z}_2\{\nu_7^3\}\oplus\mathbf{Z}_2\{\eta_7\circ\varepsilon_8\}\oplus\mathbf{Z}_2\{\mu_7\};$$

$$\pi_{16}(S^8)=\mathbf{Z}_2\{\sigma_8\circ\eta_{15}\}\oplus\mathbf{Z}_2\{\Sigma\sigma'\circ\eta_{15}\}\oplus\mathbf{Z}_2\{\bar{\nu}_8\}\oplus\mathbf{Z}_2\{\varepsilon_8\};$$

$$\pi_{15}(S^5)=\mathbf{Z}_8\{\nu_5\circ\sigma_8\}\oplus\mathbf{Z}_2\{\eta_6\circ\mu_7\}\oplus\mathbf{Z}_9\{\beta_1(5)\};$$

$$2(\nu_5\circ\sigma_8)=\nu_5\circ\Sigma\sigma';\ 3\beta_1(5)=-\alpha_1(5)\circ\alpha_2(8);$$

$$\pi_{15}(S^6)=\mathbf{Z}_2\{\nu_6^3\}\oplus\mathbf{Z}_2\{\eta_6\circ\varepsilon_7\}\oplus\mathbf{Z}_2\{\mu_6\};$$

$$\pi_{16}(S^6)=\mathbf{Z}_8\{\nu_6\circ\sigma_9\}\oplus\mathbf{Z}_2\{\eta_6\circ\mu_7\}\oplus\mathbf{Z}_9\{\beta_1(6)\}\ (\beta_1(6)=\Sigma\beta_1(5)).$$

We show

Lemma 3 (i) $\pi^8(\mathrm{OP}^2)=\mathbf{Z}_2\{\sigma_8\circ\eta_{15}\circ p\}\oplus\mathbf{Z}_2\{\bar{\nu}_8\circ p\}\oplus\mathbf{Z}_2\{\varepsilon_8\circ p\}$.
(ii) $\pi^7(\mathrm{OP}^2)=\mathbf{Z}_2\{\sigma'\circ\eta_{14}^2\circ p\}\oplus\mathbf{Z}_2\{\eta_7\circ\varepsilon_8\circ p\}\oplus\mathbf{Z}_2\{\mu_7\circ p\}$.

(iii) $\pi^6(\mathrm{OP}^2)=\mathbf{Z}_2\{\eta_6\circ\mu_7\circ p\}\oplus\mathbf{Z}_3\{\beta_1(6)\circ p\}$.

Proof. We consider the exact sequence (2) for $n=8$:

$$\pi_{15}(S^8)\xleftarrow{h^*}\pi_8(S^8)\xleftarrow{i^*}\pi^8(\mathrm{OP}^2)\xleftarrow{p^*}\pi_{16}(S^8)\xleftarrow{(\Sigma h)^*}\pi_9(S^8).$$

h^* is a monomorphism. We have

$$\begin{aligned}(\Sigma h)^*(\eta_8)&=\eta_8\circ\sigma_9\\&=\Sigma(\eta_7\circ\sigma_8)\\&=\Sigma\sigma'\circ\eta_{15}+\bar\nu_8+\varepsilon_8.\end{aligned}$$

So we have (i). Next we consider the exact sequence (2) for $n=7$:

$$\pi_{15}(S^7)\xleftarrow{h^*}\pi_8(S^7)\xleftarrow{i^*}\pi^7(\mathrm{OP}^2)\xleftarrow{p^*}\pi_{16}(S^7)\xleftarrow{(\Sigma h)^*}\pi_9(S^7).$$

Since $h^*(\eta_7)=\eta_7\circ\sigma_8=\sigma'\circ\eta_{14}+\bar\nu_7+\varepsilon_7$, h^* is a monomorphism. We have

$$\begin{aligned}(\Sigma h)^*(\eta_7^2)&=\eta_7\circ\eta_8\circ\sigma_9\\&=\eta_7\circ(\Sigma\sigma'\circ\eta_{15}+\bar\nu_8+\varepsilon_8)\\&=\eta_7\circ\Sigma\sigma'\circ\eta_{15}+\eta_7\circ\bar\nu_8+\eta_7\circ\varepsilon_8\\&=\nu_7^3+\eta_7\circ\varepsilon_8.\end{aligned}$$

This leads to (ii).

We consider the exact sequence (2) for $n=6$:

$$\pi_{15}(S^6)\xleftarrow{h^*}\pi_8(S^6)\xleftarrow{i^*}\pi^6(\mathrm{OP}^2)\xleftarrow{p^*}\pi_{16}(S^6)\xleftarrow{(\Sigma h)^*}\pi_9(S^6).$$

We have

$$\begin{aligned}h^*(\eta_6^2)&=\eta_6\circ\eta_7\circ\sigma_8\\&=\eta_6\circ(\sigma'\circ\eta_{14}+\bar\nu_7+\varepsilon_7)\\&=\eta_6\circ\sigma'\circ\eta_{14}+\eta_6\circ\bar\nu_7+\eta_6\circ\varepsilon_7\\&=4\bar\nu_6\circ\eta_{14}+\nu_6^3+\eta_6\circ\varepsilon_7\\&=\nu_6^3+\eta_6\circ\varepsilon_7.\end{aligned}$$

So h^* is a monomorphism. We have $(\Sigma h)^*(\nu_6)=\nu_6\circ\sigma_9$ and $(\Sigma h)^*(\alpha_1(6))=\alpha_1(6)\circ(\alpha_2(9)+\alpha_1'(9))=-3\beta_1(6)$ since $\alpha_1(6)\circ\alpha_1'(9)=0$. This leads to (iii) and completes the proof. □

We recall the following:

$$\nu'\circ\bar\nu_6=\varepsilon_3\circ\nu_{11};\ \pi_{16}(S^5)=\mathbf{Z}_{504}\{\zeta_5\}\oplus\mathbf{Z}_2\{\nu_5\circ\bar\nu_8\}\oplus\mathbf{Z}_2\{\nu_5\circ\varepsilon_8\};$$
$$\pi_{15}(S^4)=\mathbf{Z}_2\{\nu_4\circ\sigma'\circ\eta_{14}\}\oplus\mathbf{Z}_2\{\nu_4\circ\bar\nu_7\}\oplus\mathbf{Z}_2\{\nu_4\circ\varepsilon_7\}\oplus\mathbf{Z}_{84}\{\Sigma\mu'\}\oplus\mathbf{Z}_2\{\varepsilon_4\circ\nu_{12}\}\oplus\mathbf{Z}_2\{\Sigma\nu'\circ\varepsilon_7\}.$$

Here the generators ζ_5 and $\Sigma\mu'$ of the 2-primary components are used to represent $\mathbf{Z}_{504}$ and $\mathbf{Z}_{84}$, respectively. We also recall the following:

$$\pi_{16}(S^4)=\mathbf{Z}_2\{\nu_4\circ\sigma'\circ\eta_{14}^2\}\oplus\mathbf{Z}_2\{\nu_4^4\}\oplus\mathbf{Z}_2\{\nu_4\circ\mu_7\}\oplus\mathbf{Z}_2\{\nu_4\circ\eta_7\circ\varepsilon_8\}$$
$$\oplus\,\mathbf{Z}_2\{\sum\nu'\circ\mu_7\}\oplus\mathbf{Z}_2\{\sum\nu'\circ\eta_7\circ\varepsilon_8\};$$
$$\pi_{15}(S^3)=\mathbf{Z}_2\{\nu'\circ\mu_6\}\oplus\mathbf{Z}_2\{\nu'\circ\eta_6\circ\varepsilon_7\};$$
$$\pi_{16}(S^3)=\mathbf{Z}_2\{\nu'\circ\eta_6\circ\mu_7\}\oplus\mathbf{Z}_3\{\alpha_1(3)\circ\beta_1(6)\};\ \pi_9(S^3)=\mathbf{Z}_3\{\alpha_1(3)\circ\alpha_1(6)\}.$$

We show

Lemma 4 (1) $\pi^5(\mathrm{OP}^2)=\mathbf{Z}_{504}\{\zeta_5\circ p\}\oplus\mathbf{Z}_2\{\nu_5\circ\varepsilon_8\circ p\}$.
(2) $\pi^4(\mathrm{OP}^2)=\mathbf{Z}_2\{\nu_4\circ\sigma'\circ\eta_{14}^2\circ p\}\oplus\mathbf{Z}_2\{\nu_4^4\circ p\}\oplus\mathbf{Z}_2\{\nu_4\circ\mu_7\circ p\}\oplus\mathbf{Z}_2\{\sum\nu'\circ\mu_7\circ p\}$.
(3) $\pi^3(\mathrm{OP}^2)=\mathbf{Z}_2\{\nu'\circ\eta_6\circ\mu_7\circ p\}\oplus\mathbf{Z}_3\{\alpha_1(3)\circ\beta_1(6)\circ p\}$.

Proof. In the exact sequence

$$\pi_{15}(S^5)\xleftarrow{h^*}\pi_8(S^5)\xleftarrow{i^*}\pi^5(\mathrm{OP}^2)\xleftarrow{p^*}\pi_{16}(S^5)\xleftarrow{(\Sigma h)^*}\pi_9(S^5),$$

we have $h^*(\nu_5)=\nu_5\circ\sigma_8$ and $h^*(\alpha_1(5))=\alpha_1(5)\circ\alpha_2(8)=-3\beta_1(5)$. So h^* is a monomorphism. We have

$$\begin{aligned}(\sum h)^*(\nu_5\circ\eta_8)&=\nu_5\circ\eta_8\circ\sigma_9\\&=\nu_5\circ(\sum\sigma'\circ\eta_{15}+\bar{\nu}_8+\varepsilon_8)\\&=\nu_5\circ\sum\sigma'\circ\eta_{15}+\nu_5\circ\bar{\nu}_8+\nu_5\circ\varepsilon_8\\&=\nu_5\circ\bar{\nu}_8+\nu_5\circ\varepsilon_8.\end{aligned}$$

This leads to (i).

We consider the exact sequence

$$\pi_{15}(S^4)\xleftarrow{h^*}\pi_8(S^4)\xleftarrow{i^*}\pi^4(\mathrm{OP}^2)\xleftarrow{p^*}\pi_{16}(S^4)\xleftarrow{(\Sigma h)^*}\pi_9(S^4).$$

By Proposition 2.2.(1) of Ôguchi (1964 [2]), we know $\sum\nu'\circ\sigma'=2\sum\varepsilon'$. We have

$$\begin{aligned}h^*(\sum\nu'\circ\eta_7)&=\sum\nu'\circ\eta_7\circ\sigma_8\\&=\sum\nu'\circ(\sigma'\circ\eta_{14}+\bar{\nu}_7+\varepsilon_7)\\&=\sum\nu'\circ\sigma'\circ\eta_{14}+\sum\nu'\circ\bar{\nu}_7+\sum\nu'\circ\varepsilon_7\\&=\varepsilon_7\circ\nu_{14}+\sum\nu'\circ\varepsilon_7\end{aligned}$$

and

$$\begin{aligned}h^*(\nu_4\circ\eta_7)&=\nu_4\circ\eta_7\circ\sigma_8\\&=\nu_4\circ(\sigma'\circ\eta_{14}+\bar{\nu}_7+\varepsilon_7)\\&=\nu_4\circ\sigma'\circ\eta_{14}+\nu_4\circ\bar{\nu}_7+\nu_4\circ\varepsilon_7.\end{aligned}$$

So h^* is a monomorphism.

$$\begin{aligned}(\Sigma h)^*(\nu_4\circ\eta_7^2)&=\nu_4\circ\eta_7\circ\eta_8\circ\sigma_9\\&=\nu_4\circ\eta_7\circ(\Sigma\sigma'\circ\eta_{15}+\bar{\nu}_8+\varepsilon_8)\\&=\nu_4^4+\nu_4\circ\eta_7\circ\varepsilon_8\end{aligned}$$

and

$$\begin{aligned}(\Sigma h)^*(\Sigma\nu'\circ\eta_7^2)&=\Sigma\nu'\circ\eta_7\circ\eta_8\circ\sigma_9\\&=\Sigma\nu'\circ\eta_7\circ(\Sigma\sigma'\circ\eta_{15}+\bar{\nu}_8+\varepsilon_8)\\&=\Sigma\nu'\circ\eta_7\circ\Sigma\sigma'\circ\eta_{15}+\Sigma\nu'\circ\nu_7^3+\Sigma\nu'\circ\eta_7\circ\varepsilon_8\\&=\Sigma\nu'\circ\eta_7\circ\varepsilon_8.\end{aligned}$$

This leads to (ii).

Next, in the exact sequence

$$\pi_{15}(S^3)\xleftarrow{h^*}\pi_8(S^3)\xleftarrow{i^*}\pi^3(\mathrm{OP}^2)\xleftarrow{p^*}\pi_{16}(S^3)\xleftarrow{(\Sigma h)^*}\pi_9(S^3),$$

we have

$$\begin{aligned}h^*(\nu'\circ\eta_6^2)&=\nu'\circ\eta_6\circ\eta_7\circ\sigma_8\\&=\nu'\circ\eta_6\circ\sigma'\circ\eta_{14}+\nu'\circ\eta_6\circ\bar{\nu}_7+\nu'\circ\eta_6\circ\varepsilon_7\\&=\nu'\circ\eta_6\circ\varepsilon_7.\end{aligned}$$

So h^* is a monomorphism. Finally we have

$$\begin{aligned}(\Sigma h)^*(\alpha_1(3)\circ\alpha_1(6))&=\alpha_1(3)\circ\alpha_1(6)\circ\alpha_2(9)\\&=\alpha_1(3)\circ-3\beta_1(6)\\&=-3\alpha_1(3)\circ\beta_1(6)\\&=0.\end{aligned}$$

This leads to (iii) and completes the proof. □

Thus we have completed the proof of our theorem.

References

[1] MIMURA, M. and TODA, H.: *Topology of Lie groups, I and II,* AMS Translations of Mathematical Monographs **91**, 1991.

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[3] TODA, H.: *Composition methods in homotopy groups of spheres.* Ann. of Math. Studies, **49**, Princeton, 1962.

Table of the result

Our result is summarized in the following table. Here $m+r^k$ means the direct sum of the $(k+1)$ factors $\mathbf{Z}_m \oplus \mathbf{Z}_r \oplus \cdots \oplus \mathbf{Z}_r$ and ∞ means $\mathbf{Z}$.

n	$[\mathbf{R}P^2, S^n]$	$[\mathrm{C}P^2, S^n]$	$[\mathrm{H}P^2, S^n]$	$[\mathrm{O}P^2, S^n]$
1	0	0	0	0
2	2	0	2	6
3	0	0	2	6
4	⋮	∞	2	2^4
5		0	0	$504+2$
6		⋮	2	6
7			2	2^3
8			∞	2^3
9			0	0
10			⋮	2
11				0
12				0
13				24
14				2
15				2
16				∞
17				0
18				⋮