KNO scaling function of modified negative binomial distribution

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Abstract

We investigate the KNO scaling function of the modified negative binomial distribution (MNBD), because this MNBD can explain the oscillating behaviors of the cumulant moment observed in $e^+e^-$ annihilations and in hadronic collisions. By using a straightforward method and the Poisson transform we derive the KNO scaling function from the MNBD. The KNO form of experimental data in $e^+e^-$ collisions and hadronic collisions are analyzed by the KNO scaling function of the MNBD and that of the negative binomial distribution (NBD). The KNO scaling function of the MNBD describes the data as well as that of the NBD.

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1 Introduction

Recently it has been found that cumulant moments of the multiplicity distributions both in $e^+e^-$ annihilations and hadronic collisions show prominent oscillatory behaviors when plotted as a function of their order $q$ [1]. In [1] this behavior was attributed to the QCD-type of branching processes apparently taking place in those reactions. However, in [2, 3] we have shown that the same behavior of the moments emerges essentially from the modified negative binomial distribution (MNBD) (which actually describes the data much better than the negative binomial distribution (NBD) [4]). This distribution can be derived from the pure birth process with the initial condition given by the binomial distribution [4].

In this paper we shall derive the KNO scaling function of the MNBD both by the straightforward method (i.e., proceeding to the limit of large multiplicities $n$ and large average multiplicities $\langle n \rangle$ while keeping the scaling variable $z = n / \langle n \rangle$ finite and

\footnote{It is interesting to mention here that it comes also from the concept of purely bosonic sources as presented recently in [6].}
fixed and by using the Poisson transform. Using this KNO scaling function we shall analyze the observed multiplicity distributions in $e^+e^-$ annihilations and in hadronic collisions.

2 KNO scaling function

Let us remind that the MNBD is given by the following function [2, 3, 4, 5]

\[
P(0) = \left[ \frac{1 + r_1}{1 + r_2} \right]^N; \]

\[
P(n) = \frac{1}{n!} \left( \frac{r_1}{r_2} \right)^n \left( \frac{r_3}{1 + r_2} \right) \sum_{j=1}^N \frac{C_j}{\Gamma(j)} \left( \frac{r_2 - r_1}{r_1} \right)^j \frac{1}{(1 + r_2)^j},
\]

where $N$ (the number of excited hadrons) is an integer, $r_1$ is real and $r_2 > 0$

\[
r_1 = \frac{1}{2} \left( C_2 - 1 - \frac{1}{N} \right) \langle n \rangle - \frac{1}{2},
\]

\[
r_2 = \frac{1}{2} \left( C_2 - 1 - \frac{1}{N} \right) \langle n \rangle - \frac{1}{2}.
\]

Moreover we have the generating function of the MNBD

\[
\Pi(u) = \sum_{n=0}^\infty P(n) u^n = [1 - r_1 (u-1)]^N [1 - r_2 (u-1)]^{-N}. (3)
\]

On the other hand, the NBD has the following form

\[
P(n) = \frac{\Gamma(n+k)}{\Gamma(n+1) \Gamma(k)} \left( \frac{k}{\langle n \rangle} \right)^k \left( 1 + \frac{k}{\langle n \rangle} \right)^{-(n+k)},
\]

where $k > 0$. Its corresponding KNO scaling function is the gamma distribution

\[
\Psi(z) = \frac{k^k}{\Gamma(k)} e^{-kz} z^{k-1}.
\]

2.1 The straightforward method

Traditionally the KNO scaling function is derived from the multiplicity distribution $P(n)$ multiplied by the corresponding mean multiplicity $\langle n \rangle$ by going to the large multiplicity $n$ and large mean multiplicity $\langle n \rangle$ limit while keeping their ratio, $z = \lim_{n,\langle n \rangle \to \infty} n/\langle n \rangle$ fixed. In our case, starting from Eq. (1) we arrive at the following function

\[
\Psi(z) = \lim_{n,\langle n \rangle \to \infty} \langle n \rangle P(n)
\]

\[
= \left( \frac{r_1}{r_2} \right)^N e^{-\frac{\langle n \rangle}{r_2} - \sum_{j=1}^N \frac{C_j}{\Gamma(j)} \left( \frac{r_2 - r_1}{r_1} \right)^j \left( \frac{\langle n \rangle}{r_2} \right)^j} z^{-1}.
\]

The parameters $r_1'$ and $r_2'$ in Eq. (5) are given by

\[
r_1' = \frac{1}{2} \left( C_2 - 1 - \frac{1}{N} \right) \langle n \rangle
\]

\[
r_2' = \frac{1}{2} \left( C_2 - 1 + \frac{1}{N} \right) \langle n \rangle.
\]
which are slightly different from \( r_1, r_2 \) given by Eq. (1), because \( \langle n \rangle > 1 \). It should be noticed that the normalization of Eq. (5) differs now from the unity,

\[
\int_0^\infty \psi(z)\,dz = 1 - \left( \frac{r_1}{r_2} \right)^N, \tag{7}
\]

where the second term corresponds to the term \( \langle n \rangle P(0) \) in Eq. (1).

### 2.2 The Poisson transform

The KNO scaling function \( \psi(z, t) \) can be also obtained by using the inverse Poisson transform \([7]\) in which it is related to the distribution function \( P(n, t) \) as

\[
P(n, t) \overset{\text{inverse Poisson trans.}}{\longrightarrow} \psi(z, t)
\]

As a result we obtain in this approach that

\[
P(n, t) = \int_0^\infty \frac{(\omega)^n}{n!} e^{-\omega} \psi \left( \frac{\omega}{\langle n \rangle / \alpha}; t \right) \frac{d\omega}{\langle n \rangle / \alpha}, \tag{8}
\]

\[
\psi \left( \frac{\omega}{\langle n \rangle / \alpha}; t \right) = \frac{1}{2\pi} e^{\omega} \left( \begin{array}{c} \langle n \rangle \\ \alpha \end{array} \right) \int_{-\infty}^{0} e^{-\omega \nu} \sum_{n=0}^{\infty} \left( \frac{\nu}{\alpha} \right)^n P(n, t)\,d\nu.
\]

where \( \Pi(u, t) \) is the generating function of \( P(n, t) \). These equations hold also in the stationary function \((t = 0)\).

Using now the generating function Eq. (3), \( \psi(z) \) is given by the following inverse Laplace transform

\[
\psi(z) = \frac{1}{2\pi i} \int_{s-i\omega}^{s+i\omega} e^{\alpha s} \left( \begin{array}{c} r_1 \\ r_2 \end{array} \right)^N \sum_{j=0}^{N} C_j \left( \begin{array}{c} r_2-r_1 \langle n \rangle \\ r_2 \end{array} \right)^j \left( \begin{array}{c} s + \langle n \rangle \\ \alpha \end{array} \right)^{-j} \,ds. \tag{10}
\]

Then we arrive at the KNO scaling function for the MNBD\(^2\)

\[
\psi(z) = \left( \frac{r_1}{r_2} \right)^N \delta(z-e) + \left( \frac{r_1}{r_2} \right)^N e^{\frac{iz}{r_1}} \sum_{j=1}^{N} C_j \frac{1}{\Gamma(j)} \left( \frac{r_2-r_1^*}{r_2} \right)^j \left( \begin{array}{c} s + \langle n \rangle \\ \alpha \end{array} \right)^{j-1} e^{-\frac{iz}{r_2}} L_N^{(1)} \left( -\frac{r_2-r_1^*}{r_2} \frac{\langle n \rangle}{z} \right), \tag{11}
\]

where \( L_N^{(a)}(x) \) is the Laguerre's polynomial. In Eq. (11) the first term corresponds to the constant term in Eq. (10) (or \( \langle n \rangle P(0) \)). Because the normalization is now just the unity, in what follows we shall use this function as the KNO scaling function of the MNBD in the analysis of data.

\(^2\)In the actual analysis the delta function term occurring here will be approximated by the following expression

\[
\delta(z-e) \approx \lim_{a \rightarrow \infty} \sqrt{\frac{a}{2\pi}} \exp \left[ -a(z-e)^2 / 2 \right].
\]
3 Analysis of experimental data

We shall investigate now the applicability of the MNBD as presented in its KNO form (i.e., using Eq. (11)) to the description of the observed multiplicity distributions in $e^+e^-$ annihilations[8]-[13] and in hadronic collisions[14,15]. Table I shows obtained parameters $C_2$, $N$ and the corresponding values of the $x_{min}^2$. Moreover it shows other parameters $r_1'$ and $r_2'$ calculated from $C_2$ and $N$ according to Eq. (6).

The corresponding KNO scaling functions for our MNBD are compared with the KNO form of the above mentioned multiplicity distributions in Fig. 1(a) - 1(f) and Fig. 2(a) - 2(g).

<table>
<thead>
<tr>
<th>Eq. (11)</th>
<th>$\sqrt{s}$ [GeV]</th>
<th>$N$</th>
<th>$C_2$</th>
<th>$r_1'$</th>
<th>$r_2'$</th>
<th>$x_{min}^2$/NDF</th>
</tr>
</thead>
<tbody>
<tr>
<td>TASSO</td>
<td>14</td>
<td>31</td>
<td>$1.101 \pm 0.003$</td>
<td>$0.320 \pm 0.020$</td>
<td>$0.620 \pm 0.031$</td>
<td>10.8/11</td>
</tr>
<tr>
<td>TASSO</td>
<td>22</td>
<td>90</td>
<td>$1.099 \pm 0.003$</td>
<td>$0.497 \pm 0.027$</td>
<td>$0.622 \pm 0.031$</td>
<td>3.2/12</td>
</tr>
<tr>
<td>HRS</td>
<td>29</td>
<td>90</td>
<td>$1.081 \pm 0.002$</td>
<td>$0.450 \pm 0.012$</td>
<td>$0.593 \pm 0.019$</td>
<td>11.1/12</td>
</tr>
<tr>
<td>TASSO</td>
<td>34.8</td>
<td>35</td>
<td>$1.094 \pm 0.001$</td>
<td>$0.445 \pm 0.017$</td>
<td>$0.833 \pm 0.029$</td>
<td>8.7/16</td>
</tr>
<tr>
<td>TASSO</td>
<td>43.6</td>
<td>90</td>
<td>$1.091 \pm 0.002$</td>
<td>$0.602 \pm 0.024$</td>
<td>$0.770 \pm 0.028$</td>
<td>11.3/17</td>
</tr>
<tr>
<td>AMY</td>
<td>50</td>
<td>20</td>
<td>$1.083 \pm 0.004$</td>
<td>$0.268 \pm 0.033$</td>
<td>$1.080 \pm 0.042$</td>
<td>2.5/17</td>
</tr>
<tr>
<td>AMY</td>
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<td>$1.082 \pm 0.002$</td>
<td>$0.582 \pm 0.024$</td>
<td>$0.828 \pm 0.029$</td>
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</tr>
<tr>
<td>AMY</td>
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<td>48</td>
<td>$1.087 \pm 0.004$</td>
<td>$0.591 \pm 0.037$</td>
<td>$0.902 \pm 0.038$</td>
<td>3.8/18</td>
</tr>
<tr>
<td>SLD</td>
<td>91.2</td>
<td>19</td>
<td>$1.091 \pm 0.001$</td>
<td>$0.401 \pm 0.011$</td>
<td>$1.499 \pm 0.014$</td>
<td>34.7/22</td>
</tr>
<tr>
<td>L3</td>
<td>91.2</td>
<td>11</td>
<td>$1.095 \pm 0.003$</td>
<td>$0.043 \pm 0.031$</td>
<td>$1.933 \pm 0.058$</td>
<td>9.6/21</td>
</tr>
<tr>
<td>DELPHI</td>
<td>91.2</td>
<td>12</td>
<td>$1.091 \pm 0.001$</td>
<td>$0.081 \pm 0.011$</td>
<td>$1.853 \pm 0.025$</td>
<td>17.0/23</td>
</tr>
<tr>
<td>OPAL</td>
<td>91.2</td>
<td>11</td>
<td>$1.092 \pm 0.002$</td>
<td>$0.012 \pm 0.021$</td>
<td>$1.957 \pm 0.045$</td>
<td>4.5/25</td>
</tr>
<tr>
<td>OPAL</td>
<td>133</td>
<td>11</td>
<td>$1.093 \pm 0.004$</td>
<td>$0.024 \pm 0.047$</td>
<td>$2.152 \pm 0.076$</td>
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<table>
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<tr>
<th>Eq. (4)</th>
<th>$\sqrt{s}$ [GeV]</th>
<th>$k$</th>
<th>$x_{min}^2$/NDF</th>
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<td>TASSO</td>
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<td>18.7/12</td>
</tr>
<tr>
<td>TASSO</td>
<td>22</td>
<td>10.26 \pm 0.11</td>
<td>15.2/13</td>
</tr>
<tr>
<td>HRS</td>
<td>29</td>
<td>12.34 \pm 0.30</td>
<td>36.9/13</td>
</tr>
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<td>TASSO</td>
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<td>10.81 \pm 0.15</td>
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</tr>
<tr>
<td>TASSO</td>
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<td>11.40 \pm 0.22</td>
<td>41.3/18</td>
</tr>
<tr>
<td>AMY</td>
<td>50</td>
<td>11.75 \pm 0.74</td>
<td>4.4/18</td>
</tr>
<tr>
<td>AMY</td>
<td>57</td>
<td>12.12 \pm 0.40</td>
<td>17.9/19</td>
</tr>
<tr>
<td>AMY</td>
<td>60</td>
<td>11.26 \pm 0.60</td>
<td>8.1/19</td>
</tr>
<tr>
<td>SLD</td>
<td>91.2</td>
<td>11.02 \pm 0.07</td>
<td>199.6/23</td>
</tr>
<tr>
<td>L3</td>
<td>91.2</td>
<td>10.57 \pm 0.18</td>
<td>9.8/22</td>
</tr>
<tr>
<td>DELPHI</td>
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<td>11.00 \pm 0.10</td>
<td>21.0/24</td>
</tr>
<tr>
<td>OPAL</td>
<td>91.2</td>
<td>10.86 \pm 0.18</td>
<td>4.4/26</td>
</tr>
<tr>
<td>OPAL</td>
<td>133</td>
<td>10.30 \pm 0.51</td>
<td>6.7/23</td>
</tr>
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Table 1 (a)
Table I(b) The parameters of the KNO scaling function of the MNBD used in the analysis compared with those of the NBD for the charged multiplicity in: (a) e⁺e⁻ collisions and (b) hadronic collisions.

<table>
<thead>
<tr>
<th>Eq. 10</th>
<th>√5 [GeV]</th>
<th>N</th>
<th>C₁</th>
<th>r₁'</th>
<th>r₂'</th>
<th>χ²_min/NDF</th>
</tr>
</thead>
<tbody>
<tr>
<td>FNAL</td>
<td>23.9</td>
<td>90</td>
<td>1.251 ± 0.007</td>
<td>1.020 ± 0.033</td>
<td>1.114 ± 0.034</td>
<td>26.1 / 12</td>
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<tr>
<td>ISR</td>
<td>30.4</td>
<td>7</td>
<td>1.187 ± 0.005</td>
<td>0.233 ± 0.027</td>
<td>1.738 ± 0.035</td>
<td>15.2 / 15</td>
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<tr>
<td>FNAL</td>
<td>38.8</td>
<td>90</td>
<td>1.265 ± 0.005</td>
<td>1.302 ± 0.032</td>
<td>1.416 ± 0.033</td>
<td>44.2 / 14</td>
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<tr>
<td>ISR</td>
<td>44.5</td>
<td>6</td>
<td>1.198 ± 0.005</td>
<td>0.189 ± 0.024</td>
<td>2.203 ± 0.034</td>
<td>5.2  / 17</td>
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<tr>
<td>ISR</td>
<td>52.6</td>
<td>9</td>
<td>1.205 ± 0.004</td>
<td>0.599 ± 0.026</td>
<td>2.017 ± 0.034</td>
<td>4.8  / 19</td>
</tr>
<tr>
<td>ISR</td>
<td>62.2</td>
<td>90</td>
<td>1.194 ± 0.005</td>
<td>1.246 ± 0.037</td>
<td>1.398 ± 0.038</td>
<td>21.8 / 18</td>
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<tr>
<td>UA5</td>
<td>200</td>
<td>4</td>
<td>1.264 ± 0.011</td>
<td>0.150 ± 0.118</td>
<td>5.500 ± 0.156</td>
<td>7.6  / 29</td>
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<tr>
<td>UA5</td>
<td>546</td>
<td>4</td>
<td>1.275 ± 0.004</td>
<td>0.368 ± 0.060</td>
<td>7.718 ± 0.243</td>
<td>54.2 / 45</td>
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<tr>
<td>UA5</td>
<td>900</td>
<td>4</td>
<td>1.301 ± 0.009</td>
<td>0.908 ± 0.024</td>
<td>9.808 ± 0.257</td>
<td>78.2 / 52</td>
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<tr>
<th>Eq. (5)</th>
<th>√5 [GeV]</th>
<th>k</th>
<th>χ²_min/NDF</th>
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<tbody>
<tr>
<td>FNAL</td>
<td>23.9</td>
<td>4.19 ± 0.10</td>
<td>74.1 / 13</td>
</tr>
<tr>
<td>ISR</td>
<td>30.4</td>
<td>5.27 ± 0.14</td>
<td>18.8 / 16</td>
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<tr>
<td>FNAL</td>
<td>38.8</td>
<td>3.86 ± 0.07</td>
<td>130.1 / 15</td>
</tr>
<tr>
<td>ISR</td>
<td>44.5</td>
<td>4.92 ± 0.15</td>
<td>7.9  / 18</td>
</tr>
<tr>
<td>ISR</td>
<td>52.6</td>
<td>4.72 ± 0.10</td>
<td>38.8 / 20</td>
</tr>
<tr>
<td>ISR</td>
<td>62.2</td>
<td>5.25 ± 0.13</td>
<td>55.5 / 19</td>
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<td>UA5</td>
<td>200</td>
<td>3.79 ± 0.16</td>
<td>7.6  / 30</td>
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<tr>
<td>UA5</td>
<td>546</td>
<td>3.60 ± 0.06</td>
<td>52.7 / 46</td>
</tr>
<tr>
<td>UA5</td>
<td>900</td>
<td>3.26 ± 0.09</td>
<td>62.8 / 53</td>
</tr>
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</table>

4 Summary and Discussion

The KNO scaling function of the MNBD is obtained and is applied to the analysis of the observed multiplicity distribution in e⁺e⁻ annihilations and hadronic collisions. The data are also analysed by the gamma distribution which is the KNO scaling function of the NBD. As is seen from Table I, the χ²_min values for the MNBD fit are almost equivalent to those for the NBD fit in both reactions. The result shows to be similar to the case of the analysis of the cumulant moments in hadronic collisions[3]. On the other hand it should be noticed that the MNBD described the data of the cumulant moments much better than the NBD in e⁺e⁻ annihilations[2].

In order to know the stochastic structure of the MNBD in detail, we discuss the following point: The solution obtained from the branching equation of the pure birth process with the immigration under the initial condition of the binomial distribution
Fig. 1(a)  

Fig. 1(b)  

Fig. 1(c)
The KNO form of the charged multiplicity in $e^+e^-$ collisions. The full circles are obtained from data of the following Collaborations: HRS[8], TASSO[9], AMY[10], DELPHI[11] and OPAL[12,13], respectively, in Fig. 1(a)-1(f). The broken line is the KNO scaling function obtained from the MNBD.
Fig. 2(a)

Fig. 2(b)

Fig. 2(c)
KNO scaling function of modified negative binomial distribution

Fig. 2(d) ISR 62.2 GeV

Fig. 2(e) UA5 200 GeV

Fig. 2(f) UA5 546 GeV

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The KNO form of the charged multiplicity in hadronic collisions. The full circles are obtained from data of the following Collaborations: ISR[14] and UA5[15] collaborations, respectively, in Fig. 2(a)-(b). The broken line is the KNO scaling function obtained from the MNBD.

\[ \Pi(u) = \sum_{n=0} P(n) u^n = [1 - r_1(u-1)]^N [1 - r_2(u-1)]^{-k-N}. \]  

where \( k \) is the immigration rate, and

\[ r_1 = \frac{\langle n \rangle}{k} \left( 1 - \sqrt{\frac{k+N}{N} \left[ -k \left( C_2 - 1 - \frac{1}{\langle n \rangle} \right) + 1 \right]} \right) \]
\[ r_2 = \frac{N r_1 + \langle n \rangle}{k + N}. \]  

The MNBD is obtained by neglecting the power \( k \) in \( \Pi(u) \). The generating function of the NBD is given by neglecting the power \( N \) in Eq. (12). The physical meaning of the immigration term \( k \) may be interpreted as a possible contribution from gluons.

Using Eq. (9), we have directly the KNO scaling function for Eq. (12)

\[ \Psi(z) = \left( \frac{r_1}{r_2} \right)^N e^{-z r_1} z^{k-1} \sum_{j=0}^N j! \frac{r_1^{j+k}}{r_2^j} (\langle n \rangle)^j \]
\[ = \frac{\Gamma(N+1)}{\Gamma(N+k)} \left( \frac{r_1}{r_2} \right)^N \left( \frac{\langle n \rangle}{r_2} \right)^k z^{k-1} e^{-z} z^{-1} \sum_{j=0}^N j! \frac{r_1^{j+k}}{r_2^j} (\langle n \rangle)^j \]  

This function becomes the KNO scaling function of the MNBD (Eq. (11)) when \( k \to 0 \), and reduces to the gamma distribution (Eq. (4)) if \( N = 0 \).

In concrete application, we have confirmed that the discrete distribution of Eq. (14) cannot explain the oscillating behaviors of the cumulant moment observed in \( e^+e^- \) annihilations and in hadronic collisions much better than the MNBD (Eq. (1)). However, we are expecting at present that Eq. (14) will become to be useful in analyses of data of
some reactions at higher energies, since it has stochastic characteristics of the MNBD and the NBD.

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References


