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Vector Analysis on Sobolev Spaces, II

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Abstract In our previous paper ([1]), vector analysis on a Sobolev space $W^k(X)$ was investigated and the possibility to give a geometric example of Kerner's higher gauge theory ([6],[7]) was discussed. In this paper, we give a simple example of geometric space on which the exterior derivation is not nilpotent but its n-th power vanishes ($n \geqq 3$), by using suitable subspace of $(\infty-p)$-forms. This provides a geometric example of Kerner's higher gauge theory. To discuss its algebraic counter part, we also treat Clifford algebra on $W^k(X)$ with infinite degree spinors.

Introduction. Let X be an n-dimensional compact (spin) manifold, D a fixed first order non-degenrate selfadjoint elliptic (pseudo) differential operator on X (acting on smooth sections of some hermotian vectro bundle E over X). Fixing a Riemannian metric on X, the Hilbert space of sections of E is denoted by $L^2(X)$. Its inner product (determined by the metric) is denoted by $(\ ,\ )$.

On $L^2(X)$ (the closed extension of) D allowes spectral decomposition

$$D=\sum_\lambda(\ ,e_\lambda)e_\lambda.$$

Since D is non-degenerate, the k-th Sobolev metric on X is determined by the inner product $(\ ,\ )_k$ determined by

$$(e_\lambda, e_\mu)=sgn\lambda|\lambda|^k\delta_{\lambda\mu}$$

([8]). This metric is same to the metric defined by $\|f\|_k=\|D^kf\|$, where $\|f\|$ is the norm of f in $L^2(X)$. The Sobolev duality between $W^{-k}(X)$ and $W^k(X)$ is given by

$$\langle u,f\rangle=(G^ku,D^kf),\quad G \textit{ is the Green operator of } D.$$

Since a p-form on $W^k(X)$ is an element of $\Lambda^pW^{-k}(X)$, the Sobolev $-k$ completion of the space of alternative functions (sections) of $\overbrace{X\times\dots\times X}^{p}$, and an $(\infty-p)$-form should be the dual of p-forms, we have defined an $(\infty-p)$-form on U, an open set of $W^k(X)$ to be a Frechet differentiable map from U to $\Lambda^pW^k(X)$ ([1],[2]).

∞-forms on U are difined to be scalar functions multiplied by $(det\ D)^k$. Here det

D is defined by using spectral zeta and eta functions $\zeta_{|D|}(s)$ and $\eta_D(s)$ of $|D|$ and D. Since we are working in local, there are no essential difference between ∞-forms and scalar forms (0-forms). But in the global study, this leads a geometric definition of the determinant bundle ([2]). We say $\zeta_{|D|}(0)=\nu$ to be the virtual dimension of $W^k(X)$. ν is used several calculations includng $(\infty-p)$-forms and need to be an integer ([1]). Later we will show introducing Clifford argument, *mod*. 4 class of ν has meanings.

In [1], Grassmann calculations and differential and integral calculuses of $(\infty-p)$-forms are investigated. One of bigg difference between finite forms and $(\infty-p)$-forms is the following fact ([1]).

Theorem. *An exterior differentiable $(\infty-p)$-form is exact.*

Consequently, the exterior derivation operator d is not nilpotent on the space of $(\infty-p)$-forms. Here we give some illustrated examples

Example 1. As an $(\infty-p)$-form, $|D|^{-s}$ is written as

$$|D|^{-s}=\sum|\lambda_n|^{-s}x_n\Lambda^{\infty-\{n\}}dx, \quad d|D|^{-s}=\zeta_{|D|}(s)\Lambda^{\infty}dx,$$
$$|D|^{-s}=d\left(\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}|\lambda_n|^{-s}\right)x_nx_{n+1}\Lambda^{\infty-\{n,n+1\}}dx\right).$$

Example 2. For the volume form $\Lambda^{\infty}dx$, we have

$$\begin{aligned}\Lambda^{\infty}dx &= d(x_1\Lambda^{\infty-\{1\}}dx)=d^2\left(\sum_{n=1}^{\infty}x_nx_{n+1}\Lambda^{\infty-\{n,n+1\}}dx\right)\\ &= d^3\left(\sum_{n=2}^{\infty}x_1x_nx_{n+1}\Lambda^{\infty-\{1,n,n+1\}}dx\right)\\ &= d^4\left(\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}x_nx_{n+1}x_mx_{m+1}\Lambda^{\infty-\{n,n+1,m,m+1\}}dx\right).\end{aligned}$$

Non vanishing of the power of d comes from the fact the equation

$$dg=f, \quad f \text{ an } (\infty-p)\text{-form},$$

is a system of infinite linear equations which is formally subdeterminant. So to get good theory of Poincare lemma and so on, we must impose some boundary conditin to the system $dg=f$, that is to restrict both g and f some appropriate class of $(\infty-p)$-forms.

In this paper, we discuss the most simple boundary condition, namely, the finite condition on f. Let f be an $(\infty-p)$-form such that

$$f=\sum_{i_1<\cdots<i_p}f_{i_1\cdots i_p}\Lambda^{\infty-\{i_1,\cdots,i_p\}}dx$$

Then we say f is finite type if f and df are both expressed as finite sums. df can be regarded as linear operator valued function with $(p-1)$-parameters. But finite sum condition of df is not equivalent to take df the vlues in the ideal of finite rank operators on $W^k(X)$, and we can show this second condition follows from the first condition. Since the ortho-normal basis of $\Lambda^p W^k(X)$ is a countable discrete set, finite condition corresponds to the compact support condition. We set $C_f^p(U)$ the space of finite type $(\infty-p)$-forms and

$$C_f(U) = \sum_p C_f^p(U), \quad C_f^o(U) = C^\infty(U), \textit{ the space of } (\infty\text{-}p)\textit{-forms on } U.$$

Then on $C_f(U)$, d becomes nilpotent. Starting from $C_f(U)$, we set

$$C_m(U) = \{f \mid f \textit{ is an } (\infty\text{-}p)\textit{-form and } d^{m-2} f \in C_f(U)\}, \quad m \geqq 3.$$

Then $d^{m-1} \neq 0$ but $d^m = 0$ on $C_m(U)$. Hence $C_m(U)$ gives a geometric example of Kerner's higher gauge theory ([6],[7]). Same space is also defined on the mapping space $Map(X, M)$, where M is a smooth manifold.

Kerner also considered corresponding extended Clifford algebra ([6]). Such model may be constructed when ν is not integral but fractional. But before to do so, we must discuss Clifford algebra corresponding to the Grassmann algebra with $(\infty-p)$-forms. To do so, we need to consider Clifford algebra with the infinite degree spinor. We denote the Clifford algebra over $W^{-k}(X)$ by $C(W^{-k}(X))$, and the infinite degree spinor by e^∞. Then e^∞ must satisfy

$$\begin{aligned} &e^\infty \vee e^\infty = (-1)^{\nu(\nu+1)/2+\nu_-} \qquad (det|D|)^{2k}, \\ &e_\lambda \vee e^\infty = e^\infty \vee e_\lambda, \ \nu \equiv 1, \ mod.\ 2, \qquad e_\lambda \vee e^\infty = -e^\infty \vee e_\lambda, \ \nu \equiv 0, \ mod.\ 2. \end{aligned}$$

Here ν_- is $(\nu - \eta_D(0))/2$, e_λ's are the generators of $C(W^{-k}(X))$. The resulting algebra is denoted by $C(W^{-k}(X))[e^\infty]$. Contrary to the finite degree case, to define Grassmann product of $(\infty-p)$-forms and p-forms by using the Grassmann map and the Clifford product of $C(W^{-k}(X))[e^\infty]$, we need the metric of $W^{-k}(X)$. That is, Grassmann algebra on $W^{-k}(X)$ with $(\infty-p)$-forms, depends on the metric structure. Acording to [9], we can define super Poisson structure on the space of Grassmann algebra on $W^{-k}(X)$ with $(\infty-p)$-forms. Whose precise meanings will be discussed later.

Most part of this paper is restricted to the local study. The half infinite forms (and semi infinite forms) ([11]) are not investigated. But by using proper spinors corresponding to the positive and negative proper values of D, when D is the Dirac operator, we can define half infinite forms and semi infinite forms on $W^{-k}(X)$. In that study we need the integrities of ν and ν_-. So we introduce two parameters m and n and replace the original D by

$$D + mI + n\varepsilon,$$

where ε is the polarization operator $lim_{a\to+0}|D+aI|^{-1}(D+aI)$, so that ν and ν_- defined by this operator both become integers. Since this operator is an elliptic pseudodifferential operator, such selection of m and n is always possible. But global definition of half infinite forms on Map(X, M) seems difficult unless M is parallelisable. Because global existence of the polarization operator implies triviality of the tangent bundle of Map(X, M) ([3],[4]).

$(\infty-p)$-forms on an infinite dimensional space has been defined by Nikolaishvili under the assumption of the existence of a filtration of the space ([10]). Contrary to the definition of Nikolaishvili, our definition (applied to the mapping space Map(X, M), does not use any filtration of the space. On the other hand, Nikolaishvili does not use metrical structure of the space, but out difinition crucially depends on the metrical structure (not only the Sobolev structure, but also to the Sobolev metric). As already stated in [2], if D is possitive definite, then our local study is naturally extended to the global case. While if D is the Dirac operator, there arise several topological and geometric problems related to the global study. These will discussed elsewhere.

1. Finite Type $(\infty-p)$-Forms

Let f be an $(\infty-p)$-form on U, an open set of $W^{-k}(X)$ such that

$$f=\sum_{i_1<\cdots<i_p} f_{i_1\cdots i_p}\Lambda^{\infty-\{i_1,\cdots,i_p\}}dx. \tag{1}$$

If this right hand side is finite sum, then taking N to be

$$N>max\{i_p|f_{i_1\ldots i_p}\neq 0\},$$

we can write

$$f=f_N\wedge\Lambda^{\infty-N}dx,\quad \Lambda^{\infty-N}dx=\Lambda^{\infty-\{1,\cdots,N\}}dx. \tag{2}$$

Here f_N is an $(N-p)$-form involving only $dx_1,\ldots,dx_N$. Then, since $\Lambda^{\infty-N}dx$ is a closed form, we have

$$df=df_\wedge\Lambda^{\infty-N}dx.$$

We divide d as $d^N+d^{\infty-N}$, where

$$d^N=\sum_{i=1}^{N}\frac{\partial}{\partial x_i}dx_i,\qquad d^{\infty-N}=\sum_{j=N+1}^{N}\frac{\partial}{\partial x_j}dx_j.$$

By definition, we have

$$d^{\infty-N}f_\wedge\Lambda^{\infty-N}dx=0.$$

Hence we get

(3) $$df = d^N f_\wedge \Lambda^{\infty-N} dx.$$

(3) shows df is divisible by $\Lambda^{\infty-N}\,dx$ if f is divisible by $\Lambda^{\infty-N}dx$. That is, if f is divisible by $\Lambda^{\infty-N}dx$, then exterior derivation of f is carried on the $\{x_1,\dots,x_N\}$-space. Hence we can apply finite dimensional space differential and integral calculation to f.Therefore we obtain

Theorem 1. *If an $(\infty-p)$-form f on U is expressed as a finite sum, then df is also expressed as a finite sum and we have*

(4) $$d^2 f = 0.$$

Definition 1. *An $(\infty-p)$-form f on U is said to be of finite type if it is expressed as a finite sum.*

Let $\{g_{uv}\}$ be the transition function of the tangent bundle of Map(X, M) and $\{A_u\}$ is a selfadjoint connection of D with respect to $\{g_{uv}\}$. Then the orthonormal system $\{e_{u,\lambda}\} = \{e_{u,\lambda}(p)\}$ of the proper functions of $D+A_u = D+A_u(p)$, $p \in U$, is mapped to the orthonormal system $\{g_{vu} e_{u,\lambda}\} = \{e_{v,\lambda}\}$ of $D+A_v$, if $p \in U \cap V$. Hence finite type $(\infty-p)$-form has the global meaning. We set

$C^p_f(Map(X, M))$: The space of finite type $(\infty-p)$-forms on Map(X, M),

$$C_f(Map(X,\ M)) = \sum_{p=0}^{\infty} C^p_f(Map(X, M)).$$

$C^p_f(U)$, $C_f(U)$, *etc*., are similarly defined if finite type $(\infty-p)$-forms are defined.

It was shown in [1], any exterior differentiable $(\infty-p)$-form can be written globally as

(5) $$f = d^m g,$$

for any m. Since smooth partition of unity subordinate to locally finite open covering of Map(X, M) exists, provided the regularity of the elements of Map(X, M) to be Sobolev k-class, this result holds on Map(X, M). Hence to set

$$C_m(Map(X, M)) = \{f \mid d^{m-2} f \in C_f(Map(X, M))\},\ m \geqq 3,$$

we have

(6) $$d^{m-2}\, C_m(Map(X, M)) = C_f(Map(X, M)).$$
$$d^r\, C_m(Map(X, M)) = C_{m-r}(Map(X, M)),\quad m-r \geqq 3.$$

Therefore $C_f(Map(X, M))$ is a non-trivial space. Since $d^2 = 0$ on $C_f(Map(X, M))$, we have

$$d^m f = d^2(d^{m-2}f) = 0, \quad f \in C_m(Map(X, M)),$$
$$d^{m-1} f = d(d^{m-2}f) = 0, \quad if\ d^{m-2}f\ is\ not\ closed,$$

Hence we get

Theorem 2. *We have*

(7) $$d^m = 0, \quad d^{m-1} \neq 0 \ on\ C_m(Map(X, M)),\ m \geqq 3.$$

Hence we obtain a geometric example of Kerner's higher gauge theory ([6]).

2. Clifford Algebra with an ∞-spinor.

In [6], extended Clifford algebra corresponding to the higher gauge theory is also discussed. We do not discuss this argument. But Clifford argument of (∞-p)-forms (spinors) will be discussed.

Taking $\{e_\lambda\}$ to be he ortho-normal basis of $L^2(X)$, there Clifford multiplications are

(8) $$e_\lambda \vee e_\mu = -e_\mu \vee e_\lambda,\ \lambda \neq \mu, \quad e_\lambda \vee e_\mu = 1.$$

In this case, the infinite spinor e^∞ should be

(9) $$e^\infty \vee e^\infty = (-1)^{\nu(\nu+1)/2}, \quad e^\infty \vee e_\lambda^\infty = e_\lambda \vee e^\infty,\ \nu \equiv 1\ mod.\ 2$$
$$e^\infty \vee e_\lambda = -e_\lambda \vee e^\infty,\ \nu \equiv 0\ mod.\ 2$$

According to the $mod.\ 4$ classification of the virtual dimension $\nu = \zeta_{|D|}(0)$, (9) is rewritten as follows;

(9)' $$e^\infty \vee e^\infty = 1, \quad e^\infty \vee e_\lambda = -e_\lambda \vee e^\infty, \quad \nu \equiv 0, \quad mod.\ 4,$$
$$e^\infty \vee e^\infty = 1, \quad e^\infty \vee e_\lambda = e_\lambda \vee e^\infty, \quad \nu \equiv 3, \quad mod.\ 4,$$
$$e^\infty \vee e^\infty = -1, \quad e^\infty \vee e_\lambda = -e_\lambda \vee e^\infty, \quad \nu \equiv 2, \quad mod.\ 4,$$
$$e^\infty \vee e^\infty = -1, \quad e^\infty \vee e_\lambda = e_\lambda \vee e^\infty, \quad \nu \equiv 1, \quad mod.\ 4.$$

Next we move this discussion to the Clifford algebra over $W^{-k}(X)$ whose Grassmann counter part is the algebra of finite degree forms on $W^k(X)$. By definition, we have

$$(e_\lambda, e_\lambda) = |\lambda|^{2k}.$$

Hence we modify (8) as follows;

(8)' $$e_\lambda \vee e_\lambda = |\lambda|^{2k}.$$

$e^\infty \vee e^\infty$ is modified by using the zeta determinant

(9)'' $$e^\infty \vee e^\infty = (-1)^{\nu(\nu+1)/2}\ (det\,|D|)^{2k}.$$

Note. If we want to remain signature contribution, it must be

$$e_\lambda \vee e_\lambda = sgn\lambda |\lambda|^{2k}, \quad e^\infty \vee e^\infty = (-1)^{\nu(\nu+1)/2+\nu_-} (det|D|)^{2k}.$$

But in this case, we must consider half infinite spinors corresponding to products of proper spinors corresponding to positive and negative proper values of D (cf, [11]). That is, we must consider two elements extension $C(W^{-k}(X))[e^\infty{}_+, e^\infty{}_-]$, where the commutation relations are

$$\begin{aligned}
&e^\infty{}_+ \vee e^\infty{}_- = (-1)^{\nu_+\nu_-} e^\infty{}_- \vee e^\infty{}_+,\\
&e^\infty{}_+ \vee e^\infty{}_+ = (-1)^{\nu_+(\nu_++1)/2} (det\, D_+)^{2k},\\
&e^\infty{}_- \vee e^\infty{}_- = (-1)^{\nu_-(\nu_-+3)/2} (det\, D_-)^{2k},\\
&e^\infty{}_+ \vee e_\lambda = (-1)^{\nu_+-1} e_\lambda \vee e^\infty{}_+,\ \lambda \text{ is positive},\\
&e^\infty{}_+ \vee e_\lambda = (-1)^{\nu_+} e_\lambda \vee e^\infty{}_+,\ \lambda \text{ is negative},\\
&e^\infty{}_- \vee e_\lambda = (-1)^{\nu_-} e_\lambda \vee e^\infty{}_-,\ \lambda \text{ is positive},\\
&e^\infty{}_- \vee e_\lambda = (-1)^{\nu_-} e_\lambda \vee e^\infty{}_-,\ \lambda \text{ is negative},
\end{aligned}$$

Here $D_+ = (D + |D|)/2$ and $D_- = (D - |D|)/2$.

To give a representation of e^∞, set $\Lambda W^{-k}(X)$ the (Sobolev-k)-completion of $\sum \Lambda^p W^{-k}(X)$. $\Lambda W^k(X)$ is similarly defined. Since we have

$$D^{2k} W^k(X) = W^{-k}(X), \quad G^{2k} W^{-k}(X) = W^k(X),$$

there are isometries between $\Lambda W^{-k}(X)$ and $\Lambda W^k(X)$ which are also denoted by D^{2k} and G^{2k}:

(10) $$D^{2k}\Lambda W^k(X) = \Lambda W^{-k}(X), \quad G^{2k}\Lambda W^{-k}(X) = \Lambda W^k(X).$$

By using super Poisson structure induced from the Sobolev $-k$-norm, there is an isomorphism $r : C(W^{-k}(X)) \to B(\Lambda W^{-k}(X))$, the algebra of bounded linear operators on $\Lambda W^{-k}(X)$ ([9], strictly speaking, the topology of $C(W^{-k}(X))$ is defined by this representation, and we consider $C(W^{-k}(X))$ is complete by this topology). We extend this representation to a repesentation $R : C(W^{-k}(X)) \to B(\Lambda W^{-k}(X) \oplus \Lambda W^k(X))$ as follows;

(11) $$R(e_\lambda) = \begin{pmatrix} r(e_\lambda) & 0 \\ 0 & (-1)^{\nu-1}\, G^{2k} r(e_\lambda) D^{2k} \end{pmatrix}.$$

We also set

(12) $$R(e^\infty) = \begin{pmatrix} 0 & (det|D|)^k D^{2k} \\ (-1)^{\nu(\nu-1)/2} (det|D|)^k\, G^{2k} & 0 \end{pmatrix}.$$

Then we get

$$R(e_\infty) R(e_\lambda) = (-1)^{\nu-1} R(e_\lambda) R(e_\infty),$$

$$R(e_\infty)R(e_\infty) = (-1)^{\nu(\nu+1)/2}(det\,|D|)^{2k} I,$$

where I is the identity of $B(\Lambda W^{-k}(X) \oplus \Lambda W^{k}(X))$. Hence we have a representation R : $C(W^{-k}(X))[e^\infty] \to B(\Lambda W^{k}(X) \oplus \Lambda W^{-k}(X))$. By using the distinguished element $1 \in \Lambda W^{-k}(X)$, we difine a map s : $C(W^{-k}(X))[e^\infty] \to \Lambda W^{-k}(X) \oplus \Lambda W^{k}(X)$ by

$$s(a) = R(a)1.$$

Since we have

$$s(e_\lambda \vee e^\infty) = (-1)^{\nu(\nu+1)/2+1}\,(det\,|D|)^{2k}\, r(e_\lambda) G^{2k},$$

$s(C(W^{-k}(X)e^\infty))$ is mapped on $\Lambda W^{k}(X)$.

We denote C_0 and C_1 the subspaces of $C(W^{-k}(X))$ consisted by even and odd elements, respectively. The subspace of C_a generated the elements expressed at most multiple of p-elements, $p \equiv a \ mod.\,2$, is denoted by C^p. Since $s(C(W^{-k}(X))) = \Lambda W^{-k}(X)$, we may regard $C(W^{-k}(X))$ to be a Sobolev space and C_0 and C_1 are orthogonal each other. The orthogonal complement of C^p in C_a, $a \equiv p$, $mod.\,2$, is denoted by $C^{p\perp}$. As modules, we have

(13) $$C^{p\perp} = \sum_{q \geqq p+2,\, q \equiv p \ mod.2} \Lambda^q W^{-k}(X).$$

Here we identified $C^{p\perp}$ and $s(C^{p\perp})$. By the same identifycation, we have

(14) $$C^{p\perp} e^\infty = \sum_{q \geqq p+2,\, q \equiv p \ mod.2} \Lambda^q W^{k}(X).$$

Hence we get

(15) $$C^{p-2\perp} e^\infty / C^{p\perp} e^\infty = \Lambda^q W^{k}(X).$$

Definition 2. *We define the Grassmann map* gr : $C(W^{-k}(X))[e^\infty] \to \Lambda W^{-k}(X) \oplus \Lambda W^{k}(X)$ *by*

$$gr : C^p / C^{p+2} \leqq \Lambda^q W^{-k}(X),$$
$$gr : C^{p-2\perp} e^\infty / C^{p\perp} e^\infty \leqq \Lambda^q W^{k}(X) = \Lambda^{\infty - p} W^{-k}(X), \quad p \geqq 1$$
$$gr\,(e^\infty) = (-1)^{\nu(\nu+1)/2}(det\,|D|)^k \in \Lambda^0 W^{k}(X).$$

Note. This definition of the Grassmann map depends on the metric. As a module map, there is an alternative expression of $\Lambda^p W^{k}(X)$, namely $C^p e^\infty / C^{p-2} e^\infty$, which is independent to the metric. But this definition is not appropriate to the definition of the Grassmann product.

If a is a representative of $gr(a) \in \Lambda^p W^{-k}(X)$, then the highest order term of a has the meaning $mod.\ C^{p-2}$. On the other hand, if $a \vee e^\infty$ is a representative of $gr(a \vee e^\infty) \in \Lambda^{\infty - p} W^{-k}(X)$, then the least order term of a has the meaning $mod.\ C^{p\perp} e^\infty$. Consequently, if a and $b\vee e^{\infty}$ are the representatives of $gr(a)\in\Lambda^{p}W^{-k}(X)$ and $\Lambda^{q}W^{-k}(X)=\Lambda^{\infty-q}W^{-k}(X)$, then only the order $(q-p)$-term of $a\vee b$ has invariant meaning. That is, the Grassmann product between the elements of $\Lambda^{p}W^{-k}(X)$ and $\Lambda^{\infty-q}W^{-k}(X)$ is induced from the Clifford product of $C(W^{-k}(X))[e^{\infty}]$ by

(16) $$gr(a)_{\wedge}\, gr(b^{\vee}e^{\infty})=(a^{\vee}b)^{\vee}e^{\infty}\ mod.\ C^{q-p-2\perp}e^{\infty}.$$

Especially, we have

(16)' $$x_{\wedge}y=0,\quad if\ x\in\Lambda^{p}W^{-k}(X),\quad y\in\Lambda^{q}W^{-k}(X)\ and\ q<p.$$

This definition of the Grassmann product coincides to our former definition ([1], [2]).

Note. In (16), $q-p$ depends on the order of a and $b^{\vee}e^{\infty}$. Since $(a^{\vee}b)^{\vee}e^{\infty}=\pm(b^{\vee}a)^{\vee}e^{\infty}$, the Grassmann product defined by (16) violate the associative law in $C(W^{-k}(X))[e^{\infty}]$, although resulting Grassmann algebra satisfies associative law (cf. [5], [7]).

We define the modules $C[e^{\infty}]_0$ and $C[e^{\infty}]_1$ of even elements and odd elements of $C(W^{-k}(X))[e^{\infty}]$ by

$$\begin{aligned} C[e^{\infty}]_0 &= C_0\oplus C_0e^{\infty}, && if\ \nu\ is\ even,\\ &\ C_0\oplus C_1e^{\infty}, && if\ \nu\ is\ odd,\\ C[e^{\infty}]_1 &= C_1\oplus C_1e^{\infty}, && if\ \nu\ is\ even,\\ &\ C_1\oplus C_0e^{\infty}, && if\ \nu\ is\ odd. \end{aligned}$$

The Hodge $*$ operator on $C(W^{-k}(X))[e^{\infty}]$ is defined to be

(17) $$*a=a^{\vee}e^{\infty}=(-1)^{c(\nu-c)}e^{\infty\vee}a,\quad a\in C[e^{\infty}]_c,\quad c=0 \text{ or } 1.$$

As the map on the Grassmann algebra $\Lambda W^{-k}(X)\oplus\Lambda W^{k}(X)$, this is the map

(18) $$\begin{aligned} *u &= (-1)^{\nu(\nu+1)/2+p(\nu-p)}(det\,|D|)^{k}\,G^{2k}u,\quad u\in\Lambda^{p}W^{-k}(X),\\ *f &= (-1)^{p(\nu-p)}(det\,|D|)^{k}\,D^{2k}f,\quad f\in\Lambda^{p}W^{k}(X). \end{aligned}$$

Except signature convention, this definition of the Hodge $*$ operator coincides to our former definition ([2]).

3. Supplementary Remarks

Since the Grassmann map $gr:\ C(W^{-k}(X))[e^{\infty}]\to\sum\Lambda^{p}W^{-k}(X)\oplus\sum\Lambda^{\infty-p}W^{-k}(X)$, $\Lambda^{\infty-p}W^{-k}(X)=\Lambda^{p}W^{k}(X)$, is defined, $\sum\Lambda^{p}W^{-k}(X)\oplus\sum\Lambda^{\infty-p}W^{-k}(X)$ has a super Poisson strructure ([9]). We give some explicit computation of the Poisson bracket. If $u\in C_a$ and $v\in C_b$, then the commutator of u and $v^{\vee}e^{\infty}$ in $C(W^{-k}(X))[e^{\infty}]$ is defined to be

$$\begin{aligned} [u,\,v^{\vee}e^{\infty}] &= u^{\vee}(v^{\vee}e^{\infty})-(-1)^{a(b+\nu)}(v^{\vee}e^{\infty})^{\vee}u^{\infty}\\ &= (u^{\vee}v-(-1)^{ab+a^2}v^{\vee}u)^{\vee}e^{\infty}. \end{aligned}$$

Hence we have

$$[u, v^{\vee}e^{\infty}] = [u, v]^{\vee}e^{\infty}, \quad \textit{if} \quad a = 0.$$

If $u = gr(s)$ and $f = gr(t^{\vee}e^{\infty})$, s, $t \in C(W^{-k}(X))$, then there Poissn bracket $\{u, f\}$ is defined to be

(19) $$\{u, f\} = [s, t^{\vee}e^{\infty}] \ mod. \ C^{q-p\perp}e^{\infty}, \quad s \in C^{p}, \quad t \in C^{q}.$$

By (19), we have

(20) $$\begin{aligned} &\{dx_i, \Lambda^{\infty-A} dx\} = 2sgn(i, A)\Lambda^{\infty-\{i, A\}} dx, \quad A = \{j_1, \dots, j_p\}, \\ &sgn(i, A) = 1, \quad \textit{if} \quad i < j_1 (< j_2 < \dots < j_p), \\ &sgn(i, A) = (-1)^k, \quad \textit{if} \quad j_k < i < j_{k+1}, \\ &sgn(i, A) = (-1)^p, \quad \textit{if} \quad i > j_p, \\ &sgn(i, A) = 0, \quad \textit{if} \quad i \in A. \end{aligned}$$

On the other hand, we define

(21) $$\{f, g\} = 0, \quad \textit{if} \quad f, \ g \in \Lambda^{p}W^{k}(X).$$

We note $\{dx_i, \Lambda^{\infty} dx\} = 2\Lambda^{\infty-\{i\}} dx \neq 0$ by (20). That is, $\Lambda^{\infty} dx$ is not a central element of $\sum\Lambda^{p}W^{k}(X) \oplus \Lambda^{\infty-p}W^{-k}(X)$ by this Poisson structure. Further properties of this super Poisson structure related to the Clifford and Grassmann algebras with infinite degree elements together with the Clifford and Grassmann algebras with half infinite degree elements wil be discussed in future.

To define $(\infty-p)$-forms or spinor fields on the mapping space $Map(X, M)$, we need to add connection term $\{A_u\}$ to the original D so that

(22) $$(D + A_u) g_{uv} = g_{uv} (D + A_v),$$

where $\{g_{uv}\}$ is the transition function of the tangent bundle of $Map(X, M)$ ([2]). A_u is a smooth map from U into the space order 0 pseudodifferential operators acting on the same space that acts D. We can take A_u taking the values in the space of selfadjpoint operators. If D is possitive definite, then we can take $\{A_u\}$ such that $D + A_u(p)$, $p \in U$, is non-degenerate for any p and U. While if D is the Dirac operator, such selection is impossible unless $Map(X, M)$ is parallelisable (providing the structure group of the tangent bundle is contained in GL_p, [3]).

Since $gr(e^{\infty}) = (-1)^{\nu(\nu-1)/2} (det|D|)^k \in \Lambda^{0}W^{k}(X)$, and the associated $\Lambda^{0}W^{k}(X)$-bundle of the tangent bundle of $Map(X, M)$ is the rank 1 trivial bundle, we need the triviality of the determinant bundle of $Map(X, M)$ to the global definition of e^{∞} (or $\Lambda^{\infty}dx$) (for the definition of the determinant bundle, see [2]). If D is possitive definite, the determinant bundle of $Map(\mathrm{X}, \mathrm{M})$ is trivial. Moreover, selecting $\{A_u\}$ such that $D+A_u$ always non-degenerate, $det|D+A_u|$, $D+A_u$ and $G_{Au}(=(D+A_u)^{-1})$ are all non-vanishing. $R(\mathrm{e}^\infty)$ given by (12) is globally defined on $Map(\mathrm{X}, \mathrm{M})$.

When D is the Dirac operator, we need to use $det\ D=(-1)^{\nu_-}\ det\ |D|$ to the definition of the determinant bundle. If we select $\{A_u\}$ such that the virtual dimension $(\zeta_{|D+A_u|}(0))$ of $\mathrm{W}^k(\mathrm{X})$ to be invariant, then the topological information of the determinant bundle comes from the discontinuity of $\zeta_{|D+A_u|}(0)$ and it was shown this discontinuity gives the (complete) obstruction to the reduction of the structure group of the tangent bundle to its connected component of the identity ([4]). In other word, the determinant bundle is trivial if and only if there exists a smooth function f on $Map(\mathrm{X}, \mathrm{M})$ such that

$$\textit{The divisor of } f=\sum m\{p,\ dim\ ker(D+A_u(p))=m\}.$$

As a cohomology class, this obstruction is expressed as an element of $\mathrm{H}^1(Map(\mathrm{X}, \mathrm{M}), \mathbf{R})$.

For the global definition of $\mathrm{e}^\infty(\Lambda^\infty dx)$, we also need to consider resolving singularities comes from G_{Au}^{2k} or $(det|D+A_u|)^k\ (D+A_u)^{2k}$, when D is the Dirac operator.

Appendix. Finite Rank Forms

If $f: U\to\Lambda^p\mathrm{W}^k(\mathrm{X})$ is an $(\infty-p)$-form of finite type, then there is a finite dimensional subspace V such that f maps U into V. There is another condition on $(\infty-p)$-forms which seems similar to finite type condition. To state the condition, we review the definition of exterior differential of $(\infty-p)$-forms ([1], [2]).

Let $d^\wedge f$ be the Frechet differential of $f: U\to\Lambda^p\mathrm{W}^k(\mathrm{X})$, $p\geqq 1$. Then we may regard $d^\wedge f$ to be a map from U to $\mathrm{B}(\mathrm{W}^k(\mathrm{X}))$, the algebra of bounded linear operators on $\mathrm{W}^k(\mathrm{X})$, with the $(p$-$1)$-parameters $x_1,\ldots,x_{p-1}$. If $d^\wedge f$ takes the values in I_1, the ideal of trace class opearators, then we say f is exterior differentiable and define the exterior differential df of f by

$$df(x)(x_1,\ldots,x_{p-1})=(-1)^{p-1}\ tr(d^\wedge f(x, x_p))(x_1,\ldots,x_{p-1}).$$

By the same notations, we say f is afinite rank form if $d^\wedge f$ takes the values in I_0, the ideal of finite rank operators.

Let $f=\sum f_A\Lambda^{\infty-A}dx$ be an $(\infty-p)$-form. If the coefficients f_A of f satisfies

$$\frac{\partial}{\partial x_i}f_{\{B,i\}}=0,\quad \textit{for large } i,\ B=\{i_1,\ldots,i_{p-1}\},\quad i_1<\ldots<i_{p-1},$$

where $\{B, i\}=A\ (=\{i_1,\ldots,i_p\},\ i_1<\ldots<i_{p-1})$, then f is a finite rank form.

By definition, if f is a finite type form, then we have

$$f_{\{B,i\}}=0,\quad \textit{if } i \textit{ is sufficiently large.}$$

Hence a finite type form is a finite rank form. On the other hand, a finite rank form is not necessarily a finite form. Example 1 in Introduction gives an example of finite rank form which is not a finite type form. Forms in example 2 are also finite rank forms. They give examples of the element in $C_3(U)$, $C_4(U)$ and $C_5(U)$.

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Added in Proof. Considering diagonalization of e^∞, it is shown *mod.* 8 class of ν also has meanings. This will cliscussed in forthcoming papers.