A Memoir on the Spatially Spherosymmetric Solution for a Nonlinear Parabolic Equation

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Abstract : The paper discusses the temporal behavior of the spatially spherosymmetric solution for a nonlinear parabolic equation, which is a sense related with that of the spatially spherosymmetric solution for the 3-dimentional compressible Burgers' equation.

As for the notation used below, it is conventional, \( H^{2+a}(\Omega), H^{2+a,1+\frac{a}{2}}(\Omega_T) \), etc., denoting Hölder spaces.

1 Introduction

The partial differential equation to be considered is:

\[
\frac{\partial}{\partial t} \psi(x, t) = \phi(x, t)^{-2} \mu \Delta x \psi(x, t) + \psi(x, t)^2 \quad (|x| \leq l (> 0), t \geq 0)
\]  

\[
\psi(x, 0) = \psi_0(x) \quad (\in H^{2+a}(\Omega), \Omega = \{x | x| \leq l \}, 0 < a < 1),
\]

\[
\psi(x, t)|_{|x|=l} = 0 (t \geq 0), \quad \text{(accompanied by compatibility conditions)}.
\]

where \( \mu \) is a positive constant and \( \phi(x, t) \) is defined by

\[
\phi(x, t) = \exp \left\{ \int_0^t \psi(x, r) \, dr \right\}.
\]

Now, without proof, we give two theorems.

Theorem 1 (Temporally local existence). For some \( T \in (0, \infty) \), there exists a unique solution \( \psi(x, t) \) for (1.1)-(1.2)-(1.3) belonging to \( H^{2+a,1+\frac{a}{2}}(\Omega_T) \) \((\Omega_T = \Omega \times [0, T])\). [We note that, if \( \psi_0(x) \geq 0 \), then \( \psi(x, t) \geq 0 \).]

Theorem 2 (Spatial spherosymmetry). If \( \psi(x, t) \in H^{2+a,1+\frac{a}{2}}(\Omega_T) \) \((0 < T < \infty)\) satisfies (1.1)-(1.2)-(1.3) with \( \psi_0(x) = \psi_0(|x|) \), then \( \psi(x, t) \) has a form \( \tilde{\psi}(|x|, t) \).
where \( \tilde{\psi}_0(r) \) depends only on \( r \) and \( \tilde{\psi}(r, t) \) does only on \( r \) and \( t \). Moreover, \( \tilde{\psi}(r, t) \) satisfies

\[
\frac{\partial}{\partial t} \tilde{\psi}(r, t) = \tilde{\psi}(r, t)^2 \mu \left[ \frac{\partial^2}{\partial r^2} \tilde{\psi}(r, t) + \frac{n-1}{r} \frac{\partial}{\partial r} \tilde{\psi}(r, t) \right] + \tilde{\psi}(r, t)^2 \tag{1.5}
\]

\((0 \leq r \leq 1, 0 \leq t \leq T), \quad \left( \frac{\tilde{\psi}}{r} \right)|_{r=0} = \tilde{\psi}_0(0, t), \)

\[
\tilde{\psi}(r, 0) = \tilde{\psi}_0(r), \quad \tilde{\psi}_r(0, t) = \tilde{\psi}(l, t) = 0, \tag{1.6}
\]

(together with compatibility conditions). \( \tilde{\psi}(r, t) = \exp\left[ \int_0^t \tilde{\psi}(r, s) \, ds \right]. \) [In the discussion of the 3-dimensional compressible Burgers' equation, the case \( n=5 \) matters.]

## 2 Blow-up result

In this section, we show that the solution \( \psi(x, t) \) for (1.1)-(1.2)-(1.3) with

\[
\psi(x, 0) = \tilde{\psi}_0(|x|) \quad (\geq 0, \neq 0) \tag{2.1}
\]

blows up in a finite time under a certain condition on \( \tilde{\psi}_0(r) \).

First, we define \( \tilde{\psi}^{(n)}(r) \) \((n = 1, 2, \ldots)\) by

\[
\tilde{\psi}^{(n)}(r) = C_n r^n \tilde{Z}_{n-1} (\beta_n r), \tag{2.2}
\]

\( \left( C_n (\geq 0), \text{const. such that } \int_0^1 \tilde{\psi}^{(n)}(r) \, dr = 1 \right) \),

where \( \tilde{Z}_{n-1}(s) \) is the Bessel function of order \( \frac{n}{2} - 1 \) for each \( n \) and \( \beta_n \) is the first zero-point of \( \tilde{Z}_{n-1}(s) \) \((s>0)\). For example, \( \tilde{\psi}^{(1)}(r) = C_1 \cos \frac{\pi}{2} r, \tilde{\psi}^{(2)}(r) = C_2 r \tilde{J}_2(\beta_2 r) \) \( \left( J_0(\beta_2 l) = 0 \right) \), etc. Next, let \( \psi(x, t) \) belong to \( L^2 \left( \Omega_T \right) \) for some \( T \in (0, \infty) \) and satisfy (1.1)-(1.2)-(2.1). Then, by theorem 2, \( \psi(x, t) \) is expressed as \( \psi(x, t) = \tilde{\psi}(|x|, t) \). Moreover, \( \tilde{\psi}(r, t) \) \((r=|x|)\) satisfies (1.5)-(1.6). Multiplying both sides of (1.5) by \( \tilde{\psi}(r, t)^2 \cdot \tilde{\psi}^{(n)}(r) \) and integrating them in \( r \) over \( [0, l] \), we have easily,

\[
\int_0^l \tilde{\psi}^{(n)}(r) \frac{\partial}{\partial r} \tilde{\psi}(r, t) J^{(n)}(r) \, dr = -\mu \beta_n \int_0^l \tilde{\psi}(r, t) J^{(n)}(r) \, dr + \int_0^l \tilde{\psi}^{(n)}(r, t) \tilde{\psi}(r, t)^2 J^{(n)}(r) \, dr, \tag{2.3}
\]

\( \left( \text{N.B.: } \frac{d^2}{dx^2} J^{(n)}(x) = -(n-1) \frac{d}{dx} \left( \frac{J^{(n)}(x)}{x} \right) = -\beta_n J^{(n)}(x) \right) \).

By the strength of the equality \( \tilde{\psi} = \tilde{\psi} \), it is seen that

\[
\int_0^l \tilde{\psi}^{(n)}(r) \frac{\partial}{\partial r} \tilde{\psi}(r, t) J^{(n)}(r) \, dr = \frac{d}{dt} \int_0^l \tilde{\psi}^{(n)}(r, t) J^{(n)}(r) \, dr - 2 \int_0^l \tilde{\psi}^{(n)}(r, t) \tilde{\psi}(r, t)^2 J^{(n)}(r) \, dr. \tag{2.4}
\]

Thus, it holds that

\[
\frac{d}{dt} \int_0^l \tilde{\psi}^{(n)}(r, t) J^{(n)}(r) \, dr = -\mu \beta_n \int_0^l \tilde{\psi}(r, t) J^{(n)}(r) \, dr + 3 \int_0^l \tilde{\psi}^{(n)}(r, t) \tilde{\psi}(r, t)^2 J^{(n)}(r) \, dr \\
\geq -\mu \beta_n \int_0^l \tilde{\psi}(r, t) J^{(n)}(r) \, dr
\]
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\[ + \frac{3}{2} \frac{\int_0^t \tilde{\phi}^2 \tilde{\psi} (r, t) J^{(n)}(r) dr \int_0^t \tilde{\phi}^2 J^{(n)}(r)}{\int_0^t \tilde{\phi}^2 J^{(n)}(r)} \tag{2.5} \]

(N.B.: \( \tilde{\phi}^2 \tilde{\psi} J^{(n)}(r) = \tilde{\phi} (J^{(n)}) \tilde{\psi} (J^{(n)}) \tilde{\phi} \geq 1, J^{(n)} \geq 0 \equiv 0 \)).

Let \( y_n(t) \) denote \( \int_0^t \tilde{\phi}^2 \tilde{\psi} J^{(n)}(r) dr \equiv 0 \). Then from (2.5) follows an inequality,

\[ \frac{d}{dt} y_n(t) \geq -\mu \beta_n y_n(t) + 3 y_n(t)^2 \left( 2 \int_0^t y_n(r) dr + 1 \right)^{-1}, \tag{2.6} \]

(N.B.: \( \frac{d}{dt} \int_0^t \tilde{\phi}^2 J^{(n)}(r) dr = 2 \int_0^t \tilde{\phi}^2 \tilde{\psi} J^{(n)}(r) dr = 2 y_n, \int_0^t \tilde{\phi} (r, 0) \tilde{\psi} (r, 0)^2 J^{(n)}(r) dr = 1 \)).

Hereafter, we write \( y_n(t), J^{(n)}(r) \) and \( \beta_n \) simply \( y(t), J(r) \) and \( \beta \), respectively. Defining \( Q(t) \) by

\[ Q(t) = 2 \int_0^t y(\tau) d\tau + 1 \geq 1, \tag{2.7} \]

we rewrite (2.6) in the following way:

\[ \frac{Q}{2} \frac{d}{dt} Q \geq -\mu \beta Q + \frac{3}{4} Q^2 \left( Q = \frac{d}{dt} Q \right), \tag{2.8} \]

\[ Q(0) = 1, Q(0) = 2y(0) > 0, \tag{2.9} \]

According to the relation \( \frac{d}{dt} Q = Q \frac{d}{dQ} Q (N.B.: \dot{Q} = 2y > 0) \), from (2.8) we have an inequality

\[ \frac{d}{dQ} Q \geq -\nu + \frac{3}{2} \frac{Q}{Q} \quad (\nu = \mu \beta), \tag{2.10} \]

or, what is the same,

\[ \frac{d}{dQ} Q - \frac{3}{2} \frac{Q}{Q} \geq -\nu. \tag{2.11} \]

Hence,

\[ Q^{-\frac{3}{4}} \dot{Q} - Q(0)^{-\frac{3}{4}} \dot{Q}(0) = Q^{-\frac{3}{4}} \dot{Q} - 2y(0) \geq -\nu \int_{q(0)=1}^Q Q^{-\frac{3}{4}} dQ = 2\nu(Q^{-\frac{3}{4}} - 1). \tag{2.12} \]

Thus, we have

\[ \dot{Q} \geq 2(y(0) - 2\nu) Q^\downarrow + 2\nu Q > (y(0) - 2\nu) Q^\downarrow, \tag{2.13} \]

\[ Q(0) = 1, \tag{2.14} \]

which leads us to the assertion that, if \( y(0) \) satisfies

\[ y(0) \left( = \int_0^t \tilde{\psi} (r) J(r) dr \right) > \nu = \mu \beta, \tag{2.15} \]

then

\[ Q(t) > \frac{1}{1 - (y(0) - \nu) t} \left( 0 \leq t < \frac{1}{y(0) - \nu} \right). \tag{2.16} \]
From the discussion made above, we have:

**Theorem 3.** The solution \( \psi(x,t) \) for (1.1)-(1.2)-(1.3)-(2.1) blows up in a finite time under the condition (2.15).

### 3 Global existence

Here, we give a theorem which asserts the temporally global existence of a unique solution for our problem (1.1)-(1.2)-(1.3)-(2.1) under an additional condition on \( \tilde{\psi}(r) \). (In the case of \( \tilde{\psi}(r) \leq 0 \), the existence of such a solution is obvious.)

**Theorem 4.** There exists a unique temporally global solution \( \psi(x,t) \) of the problem (1.1)-(1.2)-(1.3)-(2.1) belonging to \( H^{\alpha+\frac{\epsilon}{2},\alpha+\frac{\epsilon}{2}}(\Omega_T) \) for an arbitrary \( T \in (0,\infty) \) under the condition on \( \tilde{\psi}(r) \),

\[
| \tilde{\psi}_0 |^\alpha + | \psi_0 |^\alpha + \frac{1}{\beta} \| \psi \|_{\Omega_T}^\alpha < \frac{B_0}{6\beta^2}, \quad (I = [0,1]).
\]

Moreover, it holds that

\[
| \tilde{\psi}(\cdot,t) |^\alpha = | \psi(\cdot,t) |^\alpha \leq \frac{2B_0}{1+4B_0}, \quad (t \geq 0).
\]

**Outline of the proof.** Let \( \psi(x,t) \) satisfy (1.1)-(1.2)-(1.3)-(2.1)-(3.1) in \( \Omega_T \) for some \( T \in (0,\infty) \). (N.B.: \( \psi \) has the form \( \psi(x,t) = \tilde{\psi}(|x|,t) \).) Now, we define \( w(x,t;a, k) \) (or, simply, \( w(x,t) \)) by

\[
w(x,t;a,k) = (1 + akut) \psi(x,t) = (1 + akut) \tilde{\psi}(|x|,t)
\]

\((a \text{ and } k, \text{ positive constants}).

Then \( w(x,t) \) satisfies

\[
w_t(x,t) = \tilde{\psi}(|x|,t)^{-2} \mu \Delta_n w + \frac{w(x,t)^2 + akutw}{1 + akut},
\]

\( w(x,0) = \tilde{\psi}(|x|), \quad w(x,t)|_{|x|=1} = \tilde{\psi}(1,t) = 0 \quad (0 \leq t \leq T).
\]

Moreover, \( \tilde{w}(x,t) \) defined by

\[
\tilde{w}(x,t) = w(x,t) + \frac{k}{2n} |x|^2.
\]

satisfies

\[
\tilde{w}_t(x,t) = \tilde{\psi}^{-2} \Delta_n \tilde{w} + \left( \frac{w^2 + akutw}{1 + akut} - k\mu \tilde{\psi}^2 \right),
\]

\( \tilde{w}(x,0) = \tilde{\psi}(|x|) + \frac{k}{2n} |x|^2, \quad \tilde{w}(x,t)|_{|x|=1} = \frac{k}{2n} l^2.
\]

After a somewhat lengthy calculation concerning (3.7)-(3.8), on the basis of the maximum principle, we obtain a sufficient condition on \( | \tilde{\psi}_0 |^\alpha \) under which \( | \tilde{w} |^\alpha \) is...
bounded from above by known quantities and the term \( \{\cdots \}_A \) is smaller than 0. As a final result, we have our assertion.

Q.E.D

4 Concluding remark

The blowup-nonblowup problem of the spatially spherosymmetric solution for the 3-dimensional compressible Burgers' equation

\[
\begin{align*}
v_t(x,t) &= \frac{\mu}{\rho(x,t)} \left( \Delta + \frac{1}{3} \nabla \cdot \text{div} \right) v(x,t) - (v \cdot \nabla) v(x,t), \\
\rho_t(x,t) + \text{div}(\rho(x,t) v(x,t)) &= 0
\end{align*}
\]

is closely related to our problem (1.1)-(1.2)-(1.3)-(2.1) \((n=5)\) in a technical sense, although the former is more complicated and more difficult than the latter. Our discussion above will be useful in treating the former problem, whose settlement consists in estimating \( \rho(x,t) = \rho(|x|,t) \) in an a priori way.

References