Volume formula of compact simple Lie groups and values of $\tau$

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(Received December, 26, 1994)

This paper is a correction of the volume formula of compact simple Lie groups given by Freudenthal [2]. For a compact center free simple Lie group $G$, Freudenthal [2] p.202 gives the volume formula of $G$ with respect to the metric induced by the Killing form as

$$\mu_0(G) = \frac{k^2 m! D^{1/2}}{(e! \prod_{i=1}^r q_i)} \pi^{e+n} = \frac{c^2 m! D^{1/2}}{\tau} \pi^{e+n}$$

where $k$=order of Weyl group, $r$=dimension, $\ell$=rank, $m=\frac{1}{2}(r-\ell)$, $D=\text{det}(\langle \rho_i, \rho_j \rangle)$ ($\rho_1$, $\cdots$, $\rho_r$ are simple roots), $q_1$, $\cdots$, $q_r$ are the coefficients of the maximal root and $c=\text{order of the center of the universal covering group } \widetilde{G}$ of $G$. (note that $k=c \ell! \times \prod_{i=1}^r q_i$). The value of $\tau$ is given by

$$\tau = (-1)^m \sum (a_{i_1}, a_{i_2}, a_{i_3}, \cdots, a_{i_{2m}})$$

where the summation runs over all permutations of the $2m$ roots $a_1$, $\cdots$, $a_{2m}$. To calculate the volume of a compact simple Lie group $G$, we need to know the value of $\tau$. However it is hard to compute $\tau$ directly because it involves the summation over the symmetric group (even if we use computers in the case $\ell \geq 4$).

Now, we shall correct the Freudenthal’s volume formula, that is, in the proof of Freudenthal-de Vries [2], p.200, 1.2, $\mu(G) = k \nu(D) = k \nu'(D)$ should be change to

$$\mu(G) = k \ell! \prod_{i=1}^r q_i \nu(D) = k \ell! \prod_{i=1}^r q_i \nu'(D).$$

(In [2], the volume of Weyl cell should be taken for that of a parallelootope. (The ratio of their volumes is $1 : \Pi_{i=1}^r q_i$)). Hence the volume $\mu_0(G)$ of $G$ must be $\mu_0(G) = \frac{k 2^r m! D^{-1/2}}{\tau} \pi^{e+n}$. Thus we have

**THEOREM 1.** The natural volume $\mu_0(\widetilde{G})$ of a simply connected compact simple Lie group $\widetilde{G}$ is

$$\mu_0(\widetilde{G}) = \frac{c k 2^r m! D^{-1/2}}{\tau} \pi^{e+n}.$$
Lie groups are calculated, hence we can determine the value of $\tau$ from Theorem 1.

**Theorem 2.** The values $\tau(\mathcal{G})$ of simply connected compact simple Lie groups $\mathcal{G}$ are given as follows.

$$
\tau(SU(n)) = \frac{(n(n-1)/2)!1!!2!!\cdots n!}{n^{n(n-1)/2}},
$$

$$
\tau(Spin(2n+1)) = \frac{1!!3!!\cdots(2n-1)!n!(n^2)!}{(2n-1)^{n^2}},
$$

$$
\tau(Sp(n)) = \frac{1!!3!!\cdots(2n-1)!n!(n^2)!}{2^{n(n-2)}(n+1)^{n^2}},
$$

$$
\tau(Spin(2n)) = \frac{1!!3!!\cdots(2n-3)!n!(n(n-1))!}{2^{(n-1)^2}(n-1)^{(n-1)}}
$$

$$(n \geq 2),
$$

$$
\tau(G_2) = \frac{3.5^2}{2}, \quad \tau(F_4) = \frac{2^{10}5^47^311}{3^{30}}, \quad \tau(E_6) = \frac{5^87^311}{2^{10}3^{28}}36!
$$

$$
\tau(E_7) = \frac{5^{11}7^711^313^217^763!}, \quad \tau(E_8) = \frac{7^{15}11^{8}13^{8}17^{14}19^923^{29}29!}{2^{8}3^{30}5^{8}120!}.
$$

Proof. It follows from the following table using Theorem 1.

$$
\mu_0(SU(n)) = \frac{2^{(n-1)(2n+3)/2}n^{n^2/2}}{1!!2!!\cdots(n-1)!n^{(n-1)(n+2)/2}},
$$

$$
\mu_0(Spin(2n+1)) = \frac{2^{2n(4n+5)/2}(2n-1)^{(2n+1)/2}}{1!!3!!\cdots(2n-1)!n^{n(n+1)}},
$$

$$
\mu_0(Sp(n)) = \frac{2^{n(3n+1)}(n+1)^{(2n+1)/2}}{1!!3!!\cdots(2n-1)!n^{n(n+1)}},
$$

$$
\mu_0(Spin(2n)) = \frac{2^{2n(3n-1)}(n-1)^{(2n-3)/2}}{1!!3!!\cdots(2n-3)!n^{(n-1)!}},
$$

$$
\mu_0(G_2) = \frac{2^{20}3^{3}5}{5}8, \quad \mu_0(F_4) = \frac{2^{20}3^{15}}{5^47^311}n^{26}, \quad \mu_0(E_6) = \frac{2^{134}3^{20}5}{5^57^3}11^{42},
$$

$$
\mu_0(E_7) = \frac{2^{216}3^{16}5}{5^{10}}11^313^217^770!, \quad \mu_0(E_8) = \frac{2^{229}3^{17}5^{103}}{7^{14}11^813^717^919^323^229!}128.
$$

References
