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Remarks on the Relations between Non Abelian de Rham Theories with respect to G and ΩG

Dedicated to Professor Shôrô Araki
on his sixtieth birthday

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Abstract Let G be $GL(n, \mathbf{c})$ and ΩG the based loop group over G. Then the (stable) first and second non abelian de Rham sets with respect to G and ΩG are related by the diagram

$$\begin{array}{ccccc} & & H^0(\Omega M_e, \mathscr{M}^1{}_{\Omega\mathfrak{g}\infty})/d^e(H^0(\Omega M_e, \Omega\mathfrak{g}_\infty, d)) & & \\ & \nearrow^{\Omega^*} & | & \searrow^{l} & \\ H^0(M, \mathscr{M}^1{}_\infty)/d^e(H^0(M, \mathfrak{g}_\infty, d)) & \cdots\cdots & \rho^* i \quad\cdots\cdots\quad B & \cdots\cdots\rightarrow & H^1(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\infty}) \\ & \searrow_{B_0} & \downarrow & \nearrow_{B_1} & \\ & & H^1(\Omega M_e, \mathscr{M}^1{}_\infty) & & \end{array}$$

Here, ΩM_e is the space of zero homotopic loops over M, $\mathfrak{g}$ is the Lie algebra of G, $\Omega\mathfrak{g}$ is the loop algebra over $\mathfrak{g}$, and $\mathscr{M}^1$ and $\mathscr{M}^1{}_{\Omega\mathfrak{g}}$ are the sheaves of germs of $\mathfrak{g}$- and $\Omega\mathfrak{g}$-valued integrable forms on M, a smooth Hilbert manifold. The maps ρ^*i, B_0 and B_1 are defined by using Grassmanhian model of loop qroups (B is defined with some additional assumptions at this stage). Geometric characterization of the map from M into ΩG, the basic central extension of ΩG, together with its quantization condition and relations of several characteristic classes of non abelian de Rham sets, including string classes, and the above maps are also given.

Introduction

In our previous papers [3], [4], a G-bundle ξ over a smooth Hilbert manifold M is related to an integrable form $\theta = l(\xi)$ on ΩM_e or an LG-bundle $L^l(\xi)$ on ΩM_e. Here G is a Lie group with the Lie algebra $\mathfrak{g}$ (in the rest, we assume $G = GL(n, \mathrm{C})$), LG is the loop group over G and ΩM_e is the space of zero-homotopic based loops over M. By using notations and terminologies in non abelian de Rham theory ([1], [2]), l and L^l give maps

$$l : \mathrm{H}^1(M, \mathscr{M}^1) \longrightarrow \mathrm{H}^0(\Omega M_e, \mathscr{M}^1) / d^e(\mathrm{H}^0(\Omega M_e, \mathfrak{g}_d)),$$

$$L^! : \mathrm{H}^1(M, \mathscr{M}^1) \longrightarrow \mathrm{H}^1(\Omega M_e, \mathscr{M}^1{}_{L\mathfrak{g}}).$$

Here, $\mathfrak{g}$ is the Lie algebra of G, $L\mathfrak{g}$ is the loop algebra over $\mathfrak{g}$ and d^e is given by

$$d^e f = e^{-f} d(e^f) = df + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} (ad\ f)^n (df),$$

where $(ad\ f)\ (df) = f(df) - (df)\ f$ (cf. [1]). We have also defined characteristic classes $\beta^p(\theta) \in \mathrm{H}^{2p-1}(M, \mathbf{C})$, for $\theta \in \mathrm{H}^0(M, \mathscr{M}^1)$ and $\tilde{c}^p(\langle\omega\rangle) \in \mathrm{H}^{2p+1}(M, \mathbf{C})$ for $\langle\omega\rangle \in \mathrm{H}^1(M, \mathscr{M}^1{}_{L\mathfrak{g}})$ ([3], [4]). $\beta^p(\theta)$ is defined as the de Rham class of

$$(-1)^{p-1} \frac{(p-1)!}{(2\pi\sqrt{-1})^p (2^p - 1)!} \mathrm{tr}\,(\theta^{2p-1}),\quad \phi^q = \phi \wedge \overset{\sim q \sim}{\cdots} \wedge \phi.$$

Definition of $\tilde{c}^p(\langle\omega\rangle)$ is given in [4] and reviewed in Appendix. For these characteristic classes, the followings are shown ([3], [4])

$$Ch^p(\langle\omega\rangle) = (-1)^p \frac{p!\,(p-1)!}{(2p-1)!} \tau(\beta^p(l(\langle\omega\rangle)),$$

$$Ch^p(\langle\omega\rangle) = -\frac{1}{(2\pi\sqrt{-1})^p (p-1)!} \tau(c^{p-1}(L^!(\langle\omega\rangle)).$$

Here $Ch^p(\langle\omega\rangle)$ is the p-th Chern character of $\langle\omega\rangle$ (cf. [1], [2]), and $\tau: \mathrm{H}^{2p-1}(\mathrm{M}, \mathbf{C}) \longrightarrow \mathrm{H}^{2p}(\Omega M_e, \mathbf{C})$ is the transgression map (cf. [5], [6]). In [4] the relation between $c^1(\langle\omega\rangle)$ and the string class (and string structure) of Killingback and Pilch-Warner ([9], [13]) is discussed.

These results suggest that there may exist some map between $\mathrm{H}^0(M, \mathscr{M}^1)/d^e(\mathrm{H}^0(M, \mathfrak{g}_d))$ and $\mathrm{H}^1(M, \mathscr{M}^1{}_{L\mathfrak{g}})$. In this paper, we construct such map by using Grassmannian model of loop groups ([14], [16]). This study also shows that $[M, \tilde{\Omega}G]$, the space of homotopy equivalence classes of based maps from M into $\tilde{\Omega}G$, the basic central extension of the based loop group ΩG, is

$$[M, \tilde{\Omega}G] = \{\textit{Stable } G\textit{-bundles } \xi \textit{ such that } c_1(\xi) = 0\} \times \mathrm{H}^1(M, \mathbf{C}).$$

Here, $c_1(\xi)$ is the first integral Chern class of ξ. If the map $g: M \to \Omega G$ is realized as an $SGL(n, \mathbf{C})$-bundle, then the $\mathrm{H}^1(M, \mathbf{C})$-part of g in the above correspondence is an integral class.

The outline of this paper is as follows; In Section 1, we give some basic definitions of sheaf and its cohomology sets of germs of loop algebra valued integrable forms (cf. [4]). In Section 4, first we investigate the relation between loop algebra valued integrable forms and G-bundles. The relation between integrable forms and ΩG-bundles is also studied. Then the relation between several characteristic classes of integrable forms and bundles via the obtained maps are shown. In Appendix, we review differential geometric and topological definitions of $\tilde{c}^p(\langle\omega\rangle)$. Topological definition of characterisitic class of a $Map(X, G)$-bundle is also given. To get differential geometric defini-tion of this class seems to relate the theory of anomaly and its cancellation (cf. [11], [10], [12], [15]). We note that, although we work in smooth category in this paper, it seems interesting to treat similar problem in holomorphic category. Such study may relate to the theory of soliton equations (cf. [16]).

§1 Sheaves of Germs of Loop Algebra valued Integrable Forms

1. Let G be a Lie group with the Lie algebra $\mathfrak{g}$ (We assume $G = GL(n, \mathbf{C})$ in the rest). The free and based loop groups and loop algebras over G and $\mathfrak{g}$ are denoted by LG, ΩG, $L\mathfrak{g}$ and $\Omega\mathfrak{g}$, respectively. Their basic (complexified) central extensions are denoted by $\widetilde{LG}$, $\widetilde{\Omega G}$, $\widetilde{L\mathfrak{g}}$ and $\widetilde{\Omega\mathfrak{g}}$, respectively. By defintions, regarding G and $\mathfrak{g}$ to be the spaces of constant loops, we have the following commutative diagram with exact lines and columns (as sets). (1)

$$
(1)\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbf{C}^* & \xrightarrow{i} & \widetilde{\Omega G} & \xrightarrow{j} & \Omega G & \longrightarrow & 0 \\
 & & \uparrow \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbf{C}^* & \xrightarrow{i} & \widetilde{LG} & \xrightarrow{j} & LG & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & = & \uparrow & & \\
 & & 0 & \longrightarrow & G & \longrightarrow & G & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbf{C} & \xrightarrow{i} & \widetilde{\Omega\mathfrak{g}} & \xrightarrow{j} & \Omega\mathfrak{g} & \longrightarrow & 0 \\
 & & \uparrow \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbf{C} & \xrightarrow{i} & \widetilde{L\mathfrak{g}} & \xrightarrow{j} & L\mathfrak{g} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & = & \uparrow & & \\
 & & 0 & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

A smooth $L\mathfrak{g}$-valued 1-form $\theta = \theta(t)$, $0 \leqq t \leqq 1$ is the loop variable, defined on a smooth Hilbert manifold M, is said to be integrable if it satisfies

$$d\theta + \theta_\wedge\theta = 0.$$

In this case, θ is locally written as $g^{-1}\,dg$, where g is a smooth LG-valued function ([4]). If θ is a $\Omega\mathfrak{g}$-valued form, then we can take this g to be a ΩG-valued function, If $\tilde{\phi}$ is an $\widetilde{L\mathfrak{g}}$-valued 1-form, then we can set $\tilde{\phi} = (\phi, \beta)$, where ϕ is an $L\mathfrak{g}$-valued 1-form and β is a usual 1-form. An $\widetilde{L}$-$\mathfrak{g}$-valued 1-form $\tilde{\phi}$ is said to be integrable if it satisfies $d\theta + {}^1/_2\,[\phi, \phi] = 0$, that is, if $\theta = (\theta, \alpha)$ satisfies

$$d\tilde{\theta} + \tilde{\theta}_\wedge\tilde{\theta} = 0, \quad d\alpha + \frac{1}{2}\int_0^1 tr\,(\theta_\wedge\theta')\,dt = 0.$$

Here θ' means $d\theta/dt$ ([4]). $\tilde{\theta}$ also has a local integration ([4]).

On M, we consider the following sheaves;

$\mathbf{C}^*_t$, G_t, LG_t, ΩG_t, $\tilde{L}G_t$ and $\tilde{\Omega}G_t$: *The sheaves of germs of constant* $\mathbf{C}^*$, *etc., valued maps over* M.

$\mathbf{C}^*_d$, G_d, LG_d, ΩG_d, $\tilde{L}G_d$ and $\tilde{\Omega}G_d$: *The sheaves of germs of smooth* $\mathbf{C}^*$, *etc., valued maps over* M.

$\mathscr{M}^1$, $\mathscr{M}^1_{L\mathfrak{g}}$, $\mathscr{M}^1_{\Omega\mathfrak{g}}$, $\mathscr{M}^1_{\tilde{L}\mathfrak{g}}$ and $\mathscr{M}^1_{\tilde{\Omega}\mathfrak{g}}$: *The sheaves of germs of integrable* $\mathfrak{g}$, *etc., valued 1-forms over* M.

$\mathbf{C}^p$, $\mathfrak{g}^p$, $L\mathfrak{g}^p$, $\Omega_{\mathfrak{g}}{}^p$, $\tilde{L}\mathfrak{g}^p$ and $\Omega\mathfrak{g}^p$: *The sheaves of germs of smooth p-forms and* $\mathfrak{g}$, *etc., valued p-forms over* M.

Θ^p: *The sheaf of germs of closed p-forms over* M.

If $G = GL(1, \mathbf{C})$, then we have $\mathscr{M}^1 = \Theta^1$.

If g is an LG-valued function, then we define a G-valued function $g^\flat$ on $M \times S^1$ by

$$g^\flat(x, t) = (g(x))(t).$$

Similarly, for an $L\mathfrak{g}$-valued form ϕ, we define a $\mathfrak{g}$-valued form $\phi^\flat$ on $M \times S^1$. In the rest, we assume that *g is smooth means $g^\flat$ is smooth (ϕ is smooth means $\phi^\flat$ is smooth)*. If g is an LG-valued function (if ϕ is an $L\mathfrak{g}$-valued form), then *g is smooth means g is smooth in the usual sense and $j(g)$ is smooth in the above sense (ϕ is smooth in the usual sense and $j(\phi)$ is smooth in the above sense).*

2. In [4], commutativity and exactness of each line and column of the following diagram is proved.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Theta^1 & \xrightarrow{i} & \mathscr{M}^1_{\tilde{L}\mathfrak{g}} & \xrightarrow{j} & \mathscr{M}^1_{L\mathfrak{g}} & \longrightarrow & 0 \\
 & & \rho\uparrow & & \rho\uparrow & & \rho\uparrow & & \\
0 & \longrightarrow & \mathbf{C}^*_d & \xrightarrow{i} & \tilde{L}G_d & \xrightarrow{j} & LG_d & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbf{C}^*_t & \xrightarrow{i} & \tilde{L}G_t & \xrightarrow{j} & LG_t & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{2}
$$

ρ is defined by $\rho(g) = g^{-1}dg$. Since LG has no cannonical coordinate, ρ does not have cannonical expression. But it takes the following local form ([4])

$$\rho(g, c) = (g^{-1}dg,\ \alpha + dc), \quad d\alpha + \frac{1}{2}\int_0^1 tr\ (g^{-1}dg \wedge (g^{-1}dg)')\, dt = 0$$

Here (g, c), $g \in LG$, $c \in \mathbf{C}^*$, is a local expression of LG.

(2) follows from the first diagram of (1). By the second diagram of (1), we have the following commutative diagram with exact lines and columns

$$
(3)\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow i & & \uparrow j & & \\
0 & \longrightarrow & \Theta^1 & \longrightarrow & \mathscr{M}^1{}_{\tilde{\mathfrak{g}}} & \longrightarrow & \mathscr{M}^1{}_{\mathfrak{g}} & \longrightarrow & 0 \\
 & & \uparrow \cong & & i \uparrow & & j \uparrow & & \\
0 & \longrightarrow & \Theta^1 & \longrightarrow & \mathscr{M}^1{}_{\tilde{L}\mathfrak{g}} & \longrightarrow & \mathscr{M}^1{}_{L\mathfrak{g}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & = & \uparrow & & \\
 & & 0 & \longrightarrow & \mathscr{M}^1 & \longrightarrow & \mathscr{M}^1 & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

By diagrams (2) and (3), we have the followiug commutative diagrams with exact lines and columns of non abelian cohomology sets (cf., [1], [2] [4]).

$$
\begin{array}{ccccccccc}
0 \to \mathrm{H}^1(M, \mathbf{C}^*) & \longrightarrow & \mathrm{H}^1(M, \tilde{L}G_t) & \longrightarrow & \mathrm{H}^1(M, LG_t) & \longrightarrow & \mathrm{H}^2(M, \mathbf{C}^*) & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 \to \mathrm{H}^0(M, \Theta^1) & \xrightarrow{i} & \mathrm{H}^0(M, \mathscr{M}^1{}_{\tilde{L}\mathfrak{g}}) & \xrightarrow{j} & \mathrm{H}^0(M, \mathscr{M}^1{}_{L\mathfrak{g}}) & \xrightarrow{\delta} & \mathrm{H}^1(M, \Theta^1) & \xrightarrow{\cong} & \mathrm{H}^2(M, \mathbf{C}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 \to \mathrm{H}^0(M, \mathbf{C}^*{}_d) & \longrightarrow & \mathrm{H}^0(M, \tilde{L}G_d) & \xrightarrow{j} & \mathrm{H}^0(M, LG_d) & \xrightarrow{\delta} & \mathrm{H}^1(M, \mathbf{C}^*{}_d) & \xrightarrow{\cong} & \mathrm{H}^2(M, \mathbf{Z}),
\end{array}
$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^1(M, \mathscr{M}^1) & \xrightarrow{=} & \mathrm{H}^1(M, \mathscr{M}^1) & \longrightarrow & 0 & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 \to \mathrm{H}^0(M, \Theta^1) & \longrightarrow & \mathrm{H}^0(M, \mathscr{M}^1{}_{\tilde{\mathfrak{g}}}) & \longrightarrow & \mathrm{H}^0(M, \mathscr{M}^1{}_{\tilde{\mathfrak{g}}}) & \xrightarrow{\delta} & \mathrm{H}^1(M, \Theta^1) & \xrightarrow{\cong} & \mathrm{H}^2(M, \mathbf{C}) \\
\uparrow = & & \uparrow \mathrm{j} & & \uparrow \mathrm{j} & & \uparrow = & & \\
0 \to \mathrm{H}^0(M, \Theta^1) & \longrightarrow & \mathrm{H}^0(M, \mathscr{M}^1{}_{\tilde{L}\mathfrak{g}}) & \longrightarrow & \mathrm{H}^0(M, \mathscr{M}^1{}_{\tilde{L}\mathfrak{g}}) & \xrightarrow{\delta} & \mathrm{H}^1(M, \Theta^1) & \xrightarrow{\cong} & \mathrm{H}^2(M, \mathbf{C}) \\
\uparrow & & \uparrow i & = & \uparrow i & & \uparrow & & \\
0 & \longrightarrow & \mathrm{H}^0(M, \mathscr{M}^1) & \longrightarrow & \mathrm{H}^0(M, \mathscr{M}^1) & \longrightarrow & 0 & &
\end{array}
$$

This second diagram shows

$$
(4)\qquad \delta(\mathrm{H}^0(M, \mathscr{M}^1{}_{\mathfrak{g}})) = \delta(\mathrm{H}^0(M, \mathscr{M}^1{}_{L\mathfrak{g}})).
$$

In [4], we have shown that the representing closed 2-form of $\delta(\theta)$, $\theta \in \mathrm{H}^0(M, \mathscr{M}^1{}_{L\mathfrak{g}})$, is given by $\int_0^1 \mathrm{tr}\,(\theta \wedge \theta')\,dt$. It is also shown that to define $l^p(\theta) \in \mathrm{H}^{2p}(M, \mathbf{C})$ as the de Rham class of

$$
(5)\qquad \left(\frac{1}{2\pi\sqrt{-1}}\right)^p \int_0^1 tr\ (\theta^{2p-1} \wedge \theta')\ dt, \quad \theta \in \mathrm{H}^0(M, \mathscr{M}^1{}_{L\mathfrak{g}}),
$$

we have

$$
(6)\qquad l^p(g^{-1}dg) = C_p g^*(\varepsilon_p), \quad C_p \neq 0 \textit{ is a constant.}
$$

Here ε_p is the $2p$-th generator of $H^*(\Omega G, \mathbf{C})$ (cf. [6], [7], [8], [14]). By (5), we have

(7) $$l^p(\theta) = 0 \quad if\ \theta \in i(H^0(M, \mathscr{M}^1)).$$

Hence we can define l^p as the characteristic class of the elements of $H^0(M, \mathscr{M}^1_{\Omega\mathfrak{g}})$.

3. we denote $GL(n, \mathbf{C})$ by G_n. Its Lie algebra is denoted by $\mathfrak{g}_n$. Then there are inclusions $\varsigma = \varsigma^m{}_n : G_n \to G_m$ and $\varsigma : \mathfrak{g}_n \to \mathfrak{g}_m$ if $m > n$. They induce inclusions $\varsigma = \varsigma^m{}_n : LG_n \to LG_m$ and $\varsigma : L\mathfrak{g}_n \to L\mathfrak{g}_{m'}$ *etc.*. By definitions of ς-s, the following diagrams are commutative.

(8) $$\begin{array}{ccccccccc} 0 \to H^0(M, \Theta^1) & \to & H^0(M, \mathscr{M}^1{}_{\tilde{\Omega}\mathfrak{g}\,m}) & \to & H^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\,m}) & \to & H^1(M, \Theta^1) & = & H^2(M, \mathbf{C}) \\ = \uparrow & & \varsigma^m{}_n \uparrow & & \varsigma^m{}_n \uparrow & & = \uparrow & & \\ 0 \to H^0(M, \Theta^1) & \to & H^0(M, \mathscr{M}^1{}_{\tilde{\Omega}\mathfrak{g}\,n}) & \to & H^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\,m}) & \to & H^1(M, \Theta^1) & = & H^2(M, \mathbf{C}), \end{array}$$

(8') $$\begin{array}{ccccccccc} 0 \to H^0(M, \mathbf{C}^*{}_d) & \to & H^0(M, \tilde{\Omega}G_{m,d}) & \to & H^0(M, \Omega G_{m,d}) & \to & H^1(M, \mathbf{C}^*{}_d) & = & H^2(M, \mathbf{Z}) \\ = \uparrow & & \varsigma^m{}_n \uparrow & & \varsigma^m{}_n \uparrow & & = \uparrow & & \\ 0 \to H^0(M, \mathbf{C}^*{}_d) & \to & H^0(M, \tilde{\Omega}G_{n,d}) & \to & H^0(M, \Omega G_{n,d}) & \to & H^1(M, \mathbf{C}^*{}_d) & = & H^2(M, \mathbf{Z}). \end{array}$$

Hence we can define stable non abelian de Rham sets $H^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\,\infty})$, *etc.*, by

$$H^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\,\infty}) = \lim\ [H^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\,n}) \mid \varsigma^m{}_n], \ etc..$$

Then by (8) and (8)', the following sequences are exact.

(9) $$0 \to H^0(M, \Theta^1) \xrightarrow{i} H^0(M, \mathscr{M}^1{}_{\tilde{\Omega}\mathfrak{g}\,\infty}) \xrightarrow{j} H^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\,\infty}) \xrightarrow{\delta} \\ \to H^1(M, \Theta^1) = H^2(M, \mathbf{C}),$$

(9') $$0 \to H^0(M, \mathbf{C}^*{}_d) \xrightarrow{i} H^0(M, \tilde{\Omega}G_{\infty,d}) \xrightarrow{j} H^0(M, \Omega G_{\infty,d}) \xrightarrow{\delta} \\ \to H^1(M, \mathbf{C}^*{}_d) = H^2(M, \mathbf{Z}).$$

By definitions of $l^p : H^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\,m}) \to H^{2p}(M, \mathbf{C})$ and $\varsigma^m{}_n$, the diagram

$$\begin{array}{ccc} H^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\,m}) & \xrightarrow{l^p} & H^{2p}(M, \mathbf{C}) \\ \varsigma^m{}_n \uparrow & & = \uparrow \\ H^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\,n}) & \xrightarrow{l^p} & H^{2p}(M, \mathbf{C}) \end{array}$$

is commutative. Hence l^p is defined on $H^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\,\infty})$ (and on $H^0(M, \mathscr{M}^1{}_{L\mathfrak{g}\,\infty})$. Then we have

$$\delta(\theta) = l^1(\theta), \quad \theta \in H^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\,\infty}) \ (or\ \theta \in H^0(M, \mathscr{M}^1{}_{L\mathfrak{g}\,\infty})).$$

Similarly, we can define $H^0(M, \mathscr{M}^1{}_\infty)$ by

$$H^0(M, \mathscr{M}^1{}_\infty) = lim\ [H^0(M, \mathscr{M}^1{}_n) \mid \varsigma^m{}_n].$$

Here $\mathscr{M}^1{}_n$ is the sheaf of germs of complex (n, n) -matrix valued inte-grable forms. Then we can define the map $\beta^p : H^0(M, \mathscr{M}^1{}_\infty) \to H^{2p-1}(M, C)$ by using the map $\beta^p : H^0(M, \mathscr{M}^1{}_n) \to H^{2p-1}(M, C)$ (cf. [3]).

In [3], it is noted that $H^0(M, \mathscr{M}^1) / d^e (H^0(M, \mathfrak{g}_d))$ is more natural cohomology set than $H^0(M, \mathscr{M}^1)$ from the point of view of view of non abelian de Rham theory (cf. [1]). Here d^e is given by

$$d^e f = e^{-f} d(e^f) = df + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} (ad\ f)^n (df).$$

By using $H^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\,n})/d^e\ (H^0(M, \Omega\mathfrak{g}_n))$, *etc.*, we get the exact sequence

$$(10)_n \qquad 0 \longrightarrow H^1(M, C) \longrightarrow H^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\,n}))/d^e(H^0(M, \Omega\mathfrak{g}_n))\delta \longrightarrow$$
$$\longrightarrow H^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\,n})/d^e(H^0(M, \Omega\mathfrak{g}_n)) \longrightarrow H^1(M, \Theta^1) = H^2(M, C).$$

This sequence induces the exact sequence

$$(10)_\infty . \qquad 0 \longrightarrow H^1(M, C) \longrightarrow H^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\,\infty})/d^e(H^0(M, \Omega\mathfrak{g}_{\infty,d})) \longrightarrow$$
$$\longrightarrow H^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}})/d^e(H^0(M, \Omega\mathfrak{g}_{\infty, d})) \longrightarrow H^1(M, \Theta^1) = H^2(M, C).$$

§2 $\Omega\mathfrak{g}$-valued Integrable Forms and G-bundles

4. Let Gr be the universal Grassmann manifold. Then there is an inclusion $i = i_n : \Omega G_n \longrightarrow Gr$ such that $i_n{}^* : \pi_r\ (\Omega G_n) \cong \pi_r(Gr)$, if $r < 2n-2$ ([14], [16]). Hence if $\theta \in H^0\ (M, \mathscr{M}^1{}_{\Omega\mathfrak{g}n})$ is integrated on M, that is, if we have

$$(11) \qquad \theta = g^{-1} dg, \quad g \in H^0(M, \Omega G_{n,d}),$$

then $i_n g$ gives a smooth map from M into Gr. g is determined uniquely by θ if we determine its value at a (fiexd) point of M. Hence we may consider θ defines a based smooth map from M into Gr. Therefore θ defines a (stable) vector bundle $\xi = \xi(\theta)$ on M. Computations of characteristic classes of LG-bundles in [4] (cf. [6], [7], [8]) show

$$(12) \qquad l^p(\theta) = (p-1)!\ Ch^p\ (\xi(\theta)),$$

where $Ch^p(\xi)$ is the p-th Chern character of (stable) G-bundle ξ.

In (11), we assume $g = j(g)$, where g is a smooth map from M into ΩG. Then we get $\Delta^1(\theta) = 0$. Hence by (12) and (8)′, we have

(13) $\quad c_1(\xi(\theta)) = 0$, *$c_1(\xi)$ is the fiirst integral Chern class of ξ.*

Conversely, since $i_n : \pi_r(\Omega G_n) \cong \pi_r(Gr)$, $r < 2n-2$, if a vector bundle satisfies $c_1(\xi) = 0$, its stable class is represented by $\xi(j(\theta))$, where $\tilde{\theta} \in \mathrm{H}^0(M, \mathscr{M}^1{}_{\mathfrak{gl}\, m})$ for some m, by the exactness of (8)'. Therefore we have the first part of the following Theorem.

Theorem 1. *There is a 1 to 1 correspondence between* $\tilde{p}$ $(\mathrm{H}^0(M, \tilde{\Omega}G_{\infty,d}))$ *and the set of pairs* (ξ, ϕ), *where* ξ *is a (stable) G-bundle such that* $c_1(\xi) = 0$ *and* ϕ *is a 1-form on* M *such that*

$$\phi | U_1 = tr(A_i), \quad \{A_i\} \text{ is a connection of } \xi.$$

proof. By (7) and (11), if $\theta \in \rho(\mathrm{H}^0(M, \Omega G_{\infty,d}))$, then to set $\theta = (\theta, \alpha)$, the 1-form α satisfies

$$tr(F_{\xi(\theta)}) = d\alpha.$$

Here $F_{\xi(\theta)}$ is a curvature form of $\xi(\theta)$. we denote $\{A_i\}$ the connection of $\xi(\theta)$ whose curvature is $F_{\xi(\theta)}$. Then we get

$$\alpha | U_i = tr(A_i) + \beta_i, \quad d\beta_i = 0.$$

Since β_i is a 1-form, we set $\beta_i = dh_i'$ where h_i is a matrix valued function. Then we get

$$\alpha | U_i = tr\Big(\mathrm{e} - \frac{1}{m} h_i \ A_i \ \mathrm{e} \frac{1}{m} \mathrm{h}_i + \frac{1}{m} \beta_i \mathrm{I}_m\Big), \quad m = rank\ \xi(\theta),$$

I_m *is the unit* (m, m)*-matrix*

Hence we have the second part of Theorem. Because if $\{A_i\}$ is a connection of $\xi(\theta)$, then another connection $\{A_i'\}$ is given by $\{A_i + B_i\}$, where $B_i = g_{ij} B_j g_{ij}^{-1}$, so $tr(B_i)$ defines a global 1-form on M.

By this Theorem and exactness of $(10)_\infty$, we have

(14) $$\rho(\mathrm{H}^0(M, \Omega G_{\infty,\, l})) / d^e(\mathrm{H}^0(M, \Omega \mathfrak{g}_{\infty\ d})) = \{\textit{Stable class of G-bundles } \xi \textit{ such that } c_1(\xi) = 0\} \times \mathrm{H}^1(M, \mathbf{C}).$$

Since the kernel of this left hand side is the set of zero-homotopic maps from M into ΩG, we have

(14)' $$[M, \tilde{\Omega} G] = \{\textit{Stable G-bundles } \xi \textit{ with } c_1(\xi) = 0\} \times \mathrm{H}^1(M, \mathbf{C}).$$

If $c_1(\xi) = 0$, the structure group of ξ is reduced to $SGL(n, \mathbf{C})$. Hence ξ has a connection $\{A_i\}$ such that $tr(A_i) = 0$. Therefore, in the correspondence (14), we can take $(\xi, 0)$ to be the cannonical element. Other c-s, $c \in \mathrm{H}^1(M, \mathbf{C})$, measure the difference between the connection $\{A_i\}$, $tr(A_i)$ represents c by the de Rham correspondence, and $\mathfrak{sgl}(n, \mathbf{C})$-valued connections of ξ. On the other hand, by using sheaf exact sequences

$$0 \longrightarrow SGL_d \longrightarrow G_d \xrightarrow{det} \mathbf{C}^* \longrightarrow 0,$$

$$0 \longrightarrow \mathscr{M}^1{}_{\mathfrak{sgl}} \longrightarrow \mathscr{M}^1 \xrightarrow{\beta^1} \Theta^1 \longrightarrow 0,$$

where SGL_d and $\mathscr{M}^1{}_{\mathfrak{sgl}}$ are the sheaves of germs of smooth $SGL(n, \mathbf{C})$-valued functions and $\mathfrak{sgl}(n, \mathbf{C})$-valued integrable 1-forms over M, we have the following commutative diagram.

$$\begin{array}{ccccccc}
\mathrm{H}^1(M, \mathbf{Z}) & \longleftarrow & \mathrm{H}^0(M, \mathbf{C}^*{}_d) & \longrightarrow & \mathrm{H}^1(M, SGL_d) & \longrightarrow & \mathrm{H}^1(M, G_d) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^1(M, \mathbf{C}) & \longleftarrow & \mathrm{H}^0(M, \Theta^1) & \longrightarrow & \mathrm{H}^1(M, \mathscr{M}^1{}_{\mathfrak{sgl}}) & \longrightarrow & \mathrm{H}^1(M, \mathscr{M}^1).
\end{array}$$

Hence if θ corresponds to an SGL-bundle, then c is an integral class.

5. In general, denoting M the universal covering space of M with the projection π, we have

$$(11)' \qquad \pi^*(\theta) = g^{-1}dg, \;\; \theta \in \mathrm{H}^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}}), \;\; g \in \mathrm{H}^0(\widetilde{M}, \Omega G_d).$$

Since $\pi^*(\theta)$ is invariant under the action of $\pi_1(M)$, g is a representative function with rəsopect to the action of $\pi^1(M)$. Hence the transition function $\{g_{ij}\}$ of the induced bundle of the universal bundle of Gr by the map $i_n g$ satisfies

$$\rho(g_{ij}) = \rho(g_{i'j'}), \;\; if \;\; \pi(U_i) = \pi(U_{i'}).$$

Therefore, $\{\rho(g_{ij})\}$ defines a cocycle in $\mathrm{Z}^1(\mathfrak{U}, \mathscr{M}^1{}_\infty)$. If $g = \mathrm{e}^{\pi^*(f)}$, f is a smooth $\mathfrak{g}$-valued function on M, then $\{\rho(g_{ij})\}$ defines a coboundary in $\mathrm{B}^1(\mathfrak{U}, \mathscr{M}^1{}_\infty)$. Hence we have the map

$$\rho^* i : \mathrm{H}^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\infty})/d^e(\mathrm{H}^0(M, \Omega\mathfrak{g}, {}_{d\infty})) \longrightarrow \mathrm{H}^1(M, \mathscr{M}^1{}_\infty).$$

On the other hand, if $g = \mathrm{e}^f$ on M, then $i_n g$ is a zero-homotopic map from M into Gr. Hence we have the map

$$i : \mathrm{H}^0(M, \Omega G_{\infty, d})/\exp\,(\mathrm{H}^0(M, \Omega\mathfrak{g}_{\infty, d})) \longrightarrow \mathrm{H}^1(M, G_{\infty, d}).$$

This map is a bijection, because $i_n^* : \pi_r(\Omega G_n) \cong \pi_r(Gr)$, if $r < 2n - 2$. Therefore we obtain the first part of the following Theorem.

Theorem 2. *we have the following commutative diagram with exact lines.*

$$(15) \qquad \begin{array}{lcl}
0 \;\; \mathrm{H}^0(M, \Omega G_{\infty, d})/\exp\,(\mathrm{H}^0(M, \Omega\mathfrak{g}_{\infty, d})) & \xrightarrow{\rho^*} & \\
\quad i \downarrow = & & \\
\mathrm{H}^1(M, G_{\infty, t}) \longrightarrow \mathrm{H}^1(M, G_{\infty, d}) & \xrightarrow{\rho^*} & \\
\longrightarrow \mathrm{H}^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\infty})/d^e(\mathrm{H}^0(M, \Omega\mathfrak{g}_{\infty, a})) & \xrightarrow{\delta} & \mathrm{H}^1(M, \Omega G_{\infty, t}) = \mathrm{Hom}(\pi_1(M), \;\; \Omega G_\infty) \\
\qquad \rho^* i \downarrow & & \\
\longrightarrow \mathrm{H}^1(M, \mathscr{M}^1{}_\infty) & \xrightarrow{\delta} & \mathrm{H}^2(M, G_{\infty, t}).
\end{array}$$

In this diagram, we have

$$(16) \qquad lp(\theta) = (p-1)! \;\; Ch^p \;\; (\rho^*(\xi(\theta))). \;\; \theta \in \mathrm{H}^0(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\infty}).$$

Proof. We need only to show (16). But it follows from (12).

By (15), if the map $\delta: \mathrm{H}^0(M, \mathscr{M}^1_{\Omega\mathfrak{g}\infty})/d^e(\mathrm{H}^0(M, \Omega\mathfrak{g}_{\infty,d})) \longrightarrow \mathrm{H}^1(M, \Omega G_{\infty,t}) = \mathrm{Hom}\,(\pi_1(M), \Omega G_\infty)$ is onto, then we can define the map

$$i^*: \mathrm{Hom}\,(\pi_1(M),\ \Omega G_\infty) \longrightarrow \mathrm{H}^2(M, G_{\infty,t}),$$

by

(17) $$i^*(\chi) = \delta(\rho^*(\xi(\theta))),\ \ if\ \chi = \delta(\theta).$$

Note. $i: \Omega G \longrightarrow Gr$ induces the map $i^*: \mathrm{H}^0(M, \mathscr{M}^1_{\Omega\mathfrak{g}\infty}) \longrightarrow \mathrm{H}^1\,(M, G_{\infty,d})$. Then by (12), (15) and the definition of the k-group $\mathrm{K}^0\,(M)$, we have the following commutative diagram

$$\begin{array}{ccccc}
 & & \mathrm{K}^0(M) & \xrightarrow{Ch^p} & \mathrm{H}^{2p}(M, \mathbf{Q}) \\
 & & \uparrow & & \uparrow = \\
\mathrm{H}^0(M, \mathrm{M}^1_{\Omega\mathfrak{g}\infty}) & \xrightarrow{i^*} & \mathrm{H}^1\,(M, G_{\infty,d}) & \xrightarrow{Ch^p} & \mathrm{H}^{2p}(M, \mathbf{Q}) \\
\rho^* \downarrow & & \pi^* \downarrow & & \pi^* \uparrow \\
\mathrm{H}^0(M, \Omega G_{\infty,d}) & \xrightarrow{i^*} & \mathrm{H}^1(M, G_{\infty,d}) & \xrightarrow{Ch^p} & \mathrm{H}^{2p}(M, \mathbf{Q}) \\
 & & \downarrow & & \downarrow = \\
 & & \mathrm{K}^0(M) & \xrightarrow{Ch^p} & \mathrm{H}^{2p}(M, \mathbf{Q}).
\end{array}$$

6. we denote by ΩM_e the space of based zero-homotopic loops over M. If g is a smooth G-valued function on M, then we define a smooth ΩG-valued function g^Ω on ΩM_e by

$$(g^\Omega(\gamma))\ (t) = g(*)^{-1}g(\gamma(t)),\ \ * = \gamma(o).$$

The correspondence $g \longrightarrow g^\Omega$ induces maps

$$\Omega^!: \mathrm{H}^0\,(M, \mathscr{M}^1) \longrightarrow \mathrm{H}^0\,(\Omega M_e, \mathscr{M}^1_{\Omega\mathrm{g}}),$$

$$\Omega^!: \mathrm{H}^0\,(M, \mathscr{M}^1)\,/\,d^e\,(\mathrm{H}^0\,(M, \mathfrak{g}_d)) \longrightarrow \mathrm{H}^0\,(\Omega M_e, \mathscr{M}^1_{\Omega\mathfrak{g}}\,)\,d^e\,(\mathrm{H}^0\,(\Omega M_e, \Omega\mathfrak{g}_d)),$$

(cf. [4]). Since there is the map $\rho^*\mathrm{i}: \mathrm{H}^0\,(\Omega M_e, \mathscr{M}^1_{\Omega\mathfrak{g}\infty})\,/\,d^e\,(\mathrm{H}^0\,(\Omega M_e, \Omega\mathfrak{g}_{\infty,d})) \longrightarrow \mathrm{H}^1\,(\Omega M_e, \mathscr{M}^1_\infty)$, to set

$$B_0 = \rho^* i \Omega^!,$$

B_0 gives the map

(18) $$B_0: \mathrm{H}^0\,(M, \mathscr{M}^1_\infty)\,/\,d^e\,(\mathrm{H}^0(M, \mathfrak{g}_{\infty,d})) \longrightarrow \mathrm{H}^1\,(\Omega M_e, \mathscr{M}^1_\infty).$$

B_0 is a kind of non abelian de Rham version of Bott map with respect to the space. As for the relation between characteristic classes, we obtain by the results in [3], [4] and (16)

(19) $$\beta^p\,(\theta) = (-1)^p \frac{(2p-1)!}{((p-1)!)^2}\,\tau^{-1}\,(Ch^p\,(B_0\,(\theta))).$$

Here, $\tau^{-1}: \mathrm{H}^{2p}(M, \mathbf{C}) \longrightarrow \mathrm{H}^{2p-1}(M, \mathbf{C})$ is the inverse of the transgression map.

In [3], we have defined the map

$$l: \mathrm{H}^1(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\infty}) \longrightarrow \mathrm{H}^0(\Omega M_e, \mathscr{M}^1{}_{\Omega\mathfrak{g}\infty}) \,/\, d^e(\mathrm{H}^0(\Omega M_e, \Omega\mathfrak{g}_{\infty,d})).$$

Hence to set

$$B_1 = \rho^* i l,$$

we obtain the map

$$(20) \qquad B_1: \mathrm{H}^1(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\infty}) \longrightarrow \mathrm{H}^1(\Omega M_e, \mathscr{M}^1{}_{\infty}).$$

By (20), if $B_1(\mathrm{H}^1(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\infty})) \supset B_0(\mathrm{H}^0(M, \mathscr{M}^1{}_{\infty}) / d^e(\mathrm{H}^0(M, \mathfrak{g}_{\infty,d})))$ and $B_1{}^{-1}$ is defined, then we can define the map

$$\mathrm{B}: \mathrm{H}^0(M, \mathscr{M}^1{}_{\infty}) \,/\, d^e(\mathrm{H}^0(M, \mathfrak{g}_{\times,d})) \longrightarrow \mathrm{H}^1(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\infty})$$

by $B = B_1{}^{-1} B_0$ (similarly, if $B_0(\mathrm{H}^0(M, \mathscr{M}^1{}_{\infty}) / d^e(\mathrm{H}^0(M, \mathfrak{g}_{\infty,d}))) \supset B_1(\mathrm{H}^1(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\infty}))$ and $B_1{}^{-1}$ is defined, then $B^{-1}: \mathrm{H}^1(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\infty}) \longrightarrow \mathrm{H}^0(M, \mathscr{M}^1{}_{\infty}) / d^e(\mathrm{H}^0(M, \mathfrak{g}_{\infty,d}))$ is defined by $B_0{}^{-1} B_1$). If B is defined, we have

$$(21) \qquad \beta^p(\theta) = \frac{(p-2)!}{(2\pi\sqrt{-1})^p\,(2p-1)!}\, c_{p-1}\,(B(\xi)), \quad p \geqq 2.$$

we may consider B *to be a kind of non abelian de Rham version of Bott periodicity map with respect to the coefficients.*

We summarlize the results of this Section as the following Theorem

Theorem 3. *The following diagram is commutative. By these maps, characteristic classes* Ch^p, β^p *and* c_{p-1} *are mapped each other via the transgression map.*

$$\begin{array}{ccccc}
 & \overset{\Omega^*}{\nearrow} & \mathrm{H}^0(\Omega M_e, \mathscr{M}^1{}_{\Omega\mathfrak{g}\infty}) / d^e(\mathrm{H}^0(\Omega M_e, \Omega\mathfrak{g}_{\infty,d})) & \overset{l}{\searrow} & \\
\mathrm{H}^0(M, \mathscr{M}^1{}_{\infty}) / d^e(\mathrm{H}^0(M, \mathfrak{g}_{\infty,d})) & \cdots\cdots\cdots & \downarrow \rho^* i & \cdots\cdots\cdots\rightarrow & \mathrm{H}^1(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}\infty}) \\
 & \underset{B_0}{\searrow} & \mathrm{H}^1(\Omega M_e, \mathscr{M}^1{}_{\infty}) & \underset{B^1}{\nearrow} &
\end{array}$$

Appendix. Characteristic Classes of ΩG-bundles and Elements of $\mathrm{H}^1(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}})$.

Let $\xi = \{g_{ij}\}$ be a smooth ΩG-bundle over a smooth Hilbert manifold M. Its connection form $\{\theta_i\}$ is a collection of $\Omega\mathfrak{g}$-valued 1-forms such that

$$g_{ij}{}^{-1} dg_{ij} = \theta_j - g_{ij}{}^{-1} \theta_i g_{ij}.$$

It is known that $\{\theta_i\}$ exists if $\{\omega_{ij}\} = \{g_{ij}{}^{-1}\, dg_{ij}\}$ satisfies

$$(1) \qquad \omega_{jk} - \omega_{ik} + g_{jk}{}^{-1}\, \omega_{ij}\, g_{jk} = 0.$$

(1) is weaker than the condition $g_{ij}\, g_{jk}\, g_{ki} = 1$, and we call $\{\omega_{ij}\}$ to be a 1-cocycle with respect to $\mathscr{M}^1{}_{\Omega\mathfrak{g}}$, if $\{\omega_{ij}\}$ satisfies (1). Then we can define cohomology set

$H^1(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}})$. The cohomology class of $\{\omega_{ij}\}$ is denoted by $\langle\omega\rangle$. We call a collection of $\Omega\mathfrak{g}$-valued 1-forms $\{\theta_i\}$ such that $\omega_{ij} = \theta_j - g_{ij}^{-1}\theta_i g_{ij}$, $\omega_{ij} = g_{ij}^{-1}dg_{ij}$, to be a connection (form) of $\langle\omega\rangle$. The curvature (form) $\{\Theta_i\}$ of $\{\theta_i\}$ is defined by $\Theta_i = d\theta_i + \theta_i \wedge \theta_i$.

The characterisitic class $c_p(\langle\omega\rangle) \in H^{2p+1}(M, \mathbf{C})$ of $\langle\omega\rangle \in H^1(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}})$ is defined as the de Rham class of a closed (2p+1)-form whose local form is $\int_0^1 \mathrm{tr}(\Theta_i{}^p \wedge \theta_{i'})\, \mathrm{dt}$. Here $\Theta_i{}^p = \Theta_i \wedge \overset{\sim p \sim}{\cdots\cdots} \wedge \Theta_i$ and$\theta_{i'} = d\theta/dt$. But in general, $\int_0^1 tr(\Theta_i{}^p \wedge \theta_i{}')\mathrm{dt}$ behaves anomalously by the change of coordinates. To cancell this anomaly, we assume

$$(2) \qquad \int_0^1 tr\ (\Theta_j{}^p g_{jk}{}' g_{jk}{}^{-1})\ dt - \int_0^1 tr\ (\Theta_i{}^p g_{ik}{}' g_{ik}{}^{-1})\ dt + \\ + \int_0^1 tr\ (\Theta_i{}^p g_{ij}{}' g_{ij}{}^{-1})\ dt = 0.$$

(2) is satisfied if $(d/dt)\ (g_{ij}g_{jk}g_{ki}) = 0$. Especially, if $\{g_{ij}\}$ defines a ΩG-bundle, then (2) is satisfied. If (2) is hold, we can set

$$\int_0^1 tr\ (\Theta_i{}^p g_{ij}{}' g_{ij}{}^{-1})\ dt = \Psi_{p,j} - \Psi_{p,i}.$$

Then, it is shown that

$$\int_0^1 tr\ (\Theta_i{}^p \wedge \theta_i{}')\ dt - d\Psi_{p,i} = \int_0^1 tr\ (\Theta_i{}^p \wedge \theta_j{}')\ dt - d\Psi_{p,j'}$$

on $U_i \wedge U_j$. Hence it defines a global closed $(2p+1)$-form on M and whose de Rham class is $\tilde{c}_p\ (\langle\omega\rangle)$.

Instead of the above differential geometric definition, we can give topological definition of $\tilde{c}_p(\langle\omega\rangle)$ as follows; If $\omega_{ij} = g_{ij}{}^{-1}\ dg_{ij}$ and $(d/dt)(g_{ij}g_{jk}g_{ki}) = 0$, then we can associate an element $\langle\omega\rangle^b$ of $H^1(M \times S^1, \mathscr{M}^1)$ for $\langle\omega\rangle \in H^1(M, \mathscr{M}^1{}_{\Omega\mathfrak{g}})$. Especially, if $\xi = \{g_{ij}\}$ defines a ΩG-bundle, then we can define a G-bundle ξ^{b} over $M \times S$ by

$$\xi^{\mathrm{b}} = \{g_{ij}{}^{\mathrm{b}}\},\quad g_{ij}{}^{\mathrm{b}}(x, t) = (g_{ij}(x))(t),$$

where the coordinate system of ξ^{b} is $\{U_i \times S^1\}$. We denote the integration along the fibre S^1 (in $H^*(M \times S^1, \mathbf{C})$) by $\int_{S^1} \phi$ (cf. [5], [6]). Then we obtain

$$(3) \qquad \tilde{c}_p(\langle\omega\rangle) = -(2\pi\sqrt{-1})^p p! \int_{S^1} \mathrm{Ch}^{p+1}(\langle\omega\rangle^{\mathrm{b}}).$$

(3) and the properties of the evaluation map $ev : \Omega M_e \times S^1 \longrightarrow M$ (cf. [5]) shows the following formula on (generalized) string classes ([4])

$$(4) \qquad \tilde{c}_p\,(\Omega^!(\xi)) = (2\pi\sqrt{-1})^{p+1}\ p!\,\tau^{-1}(Ch^{p+1}(\xi)).$$

Note. Topological definition of $c_p(\xi)$ $(=c_p(\rho^*(\xi)))$ is generalized for a *Map* (X, G)-bundle ξ, X is a smooth compact manifold, as follows: Let γ be a fixed generator of integral homology group H_q $(M, \mathbf{Z})$ of X. Then we define a characteristic class $\tilde{c}_{\gamma,p}(\xi) \in H^{2p-q}(M, \mathbf{Q})$ of ξ by

$$(5) \qquad \tilde{c}_{\gamma,p}(\xi) = \int_\gamma Ch^p(\xi^b).$$

Here ξ^b is defined similarly as above. Especially, if $X = S^m$, a *Map* (S^m, G)-bundle has even dimensional characteristic classes if m is even, and has odd dimensional characteristic classes if m is odd (cf. [11]). It is shown that ξ^b has the following form curvature $\{F_i\}$

$$F_i = \Theta_i^b + d^X\theta_i^b + D_{\theta_i}\eta_i, \quad D_\theta\phi = d\phi + [\theta, \phi].$$

Here, d^X is the derivation on X, d is the derivation on M, $\{\theta_i\}$ is a connection of ξ and $\Theta_i = d\theta_i + \theta_i \wedge \theta_i$. Hence, if $\gamma \in H_1(M, \mathbf{Z})$, we can give differential geometric definition of $\tilde{c}_{\gamma,p}(\xi)$ (cf. [4]). But other cases to get differential geometric definition of $\tilde{c}_{\gamma,p}(\xi)$, it seems to need some considerations like anomaly cancellation (cf. [10], [12], [15]).

Added in Proof. : Dr Terazawa kindly taught the author the book "*Group of Paths, Observations, Fields, and Particles*" by MENSKY, M. B. : Moscow 1983 (Japanese translation, *Keiro-Gun no Kikagaku to Soryusi-Ron*, transl. by SUGANO, K. : Tokyo, 1988). In this book, Mensky emphasized the importance of the study of representation theory of the group of paths ΩM (multiplication is defined by the composition of paths, cf. [3]). Results of this paper together with results in [3], [4] (and Theorem of Milnor-Lashof, cf. [3]), *such representations divide two classes, one is representations in* $U(n)$ *and the other is representations in* $\Omega U(n)$. In Chap. 8 (of Japanese translation) of above book, representations of $\Omega^2 M$, the double loop space over M, is connected to the study of strings. Results of this paper show that *such representations may be considered as gauge theory on* ΩM *(in stable range), and it turns out representation theory of* ΩM *in* $\Omega U(n)$. In [3], we remarked that the third non abelian de Rham theory (cf. [2]) may be regarded as gauge theory on ΩM. The third non abelian de Rham theory produces 2-form connection ([2]), which appear in Chap. 8 of the above book to describe interaction of strings. So this paper (and [2], [3], [4]) give some answers (and mathematical backgrounds) of the problems raised in the above book (cf. Chap. 12). I would like to thank Dr. Terazawa to teach me Mensky's book. We also note that we have defined B. Details will appear soon (cf. [3])

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