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Monodromy of a Differential Equation Having a Quadratic Non Linear Term

Dedicated to Professor
Hirosi Toda on his 60th birthday

By AKIRA ASADA

Department of Mathematics, Faculty of Science, Shinshu University
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1 Introduction

In the study of geometry of curves in a manifold, Prof. Abe obtained the following equation ([1])

$$(1)\qquad \frac{dy}{dx}+A(x)y+\tan\ s\sum_{j=1}^{n} y_jB_j(x)\,y=0,\quad y=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}.$$

Here x is the variable of a curve γ, s is a variable in the normal direction of the curve and $y=y(x,\ s)$. A is the torsion matrix of the curve and B_j's are calculated for important manifolds. In any case, they are geometric meaningful.

If the curve γ is a closed curve, that is, if γ is given by a periodic ma $\gamma(x)$ with the period 1, we call the correspondence

$$f(s)=y(0,\ s)\longrightarrow y(k,\ s)=\kappa_\sigma(f)(s),$$

where $y(x,\ s)$ is a solution of (1), to be the *monodromy* of the equation (1). Here k is an integer and represents $\sigma\in\pi_1(\gamma)$. Treaty of the monodromy of (1) is a geometric problem. presented by Prof. Abe. In this note, we treat this problem and show the followings:

Lemma 3. *Let θ and ϕ be matrix valued 1-forms over a smocth manifold M with the universal covering manifold $\widetilde{M}$ such that*

$$(2)\qquad d\theta+\theta_\wedge\theta=0,$$

$$d\phi+\theta_\wedge\phi=0.$$

Then for any $x_0\in M$ and a function $f(s)$ of s, s a (complex) parameter, the equation

$$(3)\qquad dF-F\theta+sF\phi F=0$$

hasa solution $F=F\ (x,\ s)$ such that $F\ (x_0,\ s)=f(s)$ and can be continued as a solution *of* (3) *on* $U(\widetilde{M}\times 0)$, *a neighborhood of* $\widetilde{M}\times 0$ *in* $\widetilde{M}\times C$. $F(x, s)$ *is holomorphic in* s *if* $f(s)$ *is holomorphic. In general,* F *has following form*

$$F=f(I+\sum_{n=1}^{\infty}s^nG_n)F_0,\ I \text{ is the identity matrix.} \tag{4}$$

Here F_0 *is the solution of the linear part of* (3) *such that* $F_0(x_0)=I$ *and each* $G_n=G_n(x, f(s))$ *is homogeneous of degree* n *in* f.

For simple, we regard $x_0\in\widetilde{M}$ when F is continued to be a solution of (3) on $U(\widetilde{M}\times 0)$. In this case, we also have $F(x_0, 0)=I$.

Definition. *We call the correspondence*

$$f(s)=F(x_0, s)\longrightarrow F(\sigma(x_0), s)=\kappa_\sigma(f)(s),\quad \sigma\in\pi_1(M)$$

to be the monodromy of (3). *Here we regard* $f(s)$ *to be a germ of matrix valued function.*

Theorem 3. *To denote the monodromy of the linear part of* (3) *by* χ_σ, *we have the following expansion of* $\kappa_\sigma(f)$.

$$\kappa_\sigma(f)(s)=(I+\sum_{n=1}^{\infty}s^n\lambda_{n,\sigma}(f))\chi_\sigma f(s). \tag{5}$$

Here each $\lambda_{n,\sigma}(f)$ *is homogeneous of degree* n *in* f. $\chi_\sigma f$ *and* $\lambda_{n,\sigma}(f)\chi_\sigma f$ *mean the matrix multiplications of* χ_σ *and* f *and* $\lambda_{n,\sigma}(f)$ *and* $\chi_\sigma f$. *Especially,* $\lambda_{1,\sigma}(f)$ *is linear in* f. *It satisfies following period relation*

$$\lambda_{1,\sigma}(T)=\lambda_{1,\sigma}(\chi_\tau T)+\chi_\sigma\lambda_{1,\tau}(T)\chi_\sigma^{-1},\ T \text{ is a matrix.} \tag{6}$$

Since χ is a representation of $\pi_1(M)$, it defines a local coefficient cohomology $H^*(M, V_\chi)$. (6) shows $\mathrm{Tr}\chi_1$ defines an element of $H^1(M, V_\chi)$. It is a characteristic class of the non-linear part of (3).

To apply these results to the original equation (1) of Abe, (3) is a little restrictive. So we consider the following equation

$$\frac{dY}{dx}+A(x)Y+sF(Y, Y)=0. \tag{1'}$$

Here $Y(x, s)$ and $A(x)$ are matrix valued functions, $F(U, V)$ is a matrix valued bilinear function such that

$$F(U, VC)=F(U, V)C,\ C \text{ a matrix.} \tag{7}$$

Then we have

Lemma 1. *For any* $a>0$ *and a matrix valued continuous function* $f(s)$ *of* s, *there exists an* $\varepsilon=\varepsilon(a, f)>0$ *such that* (1)$'$ *has a solution* $Y(x, s)$ *with the initial data*

$f(s)$ on $\{|x|<a, |s|<\varepsilon\}$. If $f(s)$ is C^k-class, this $Y(x, s)$ is C^k-class in s and if $f(s)$ is holomorphis, $Y(x, s)$ is holomorphic in s. Precisely, $Y(x, s)$ takes the following form

$$Y(x, s)=U(x)\,(I+\sum_{n=1}^{\infty} s^n V_n(f))\,f(s).$$

Here $U(x)$ is the unitary solution of the linear part of (1)′, *that is, $U(x)$ is the solution of the equation*

$$\frac{dY}{dx}+A(x)Y=0, \tag{8}$$

with the initial data $Y(0)=I$ and each $V_n(x, f)$ is homogeneous of degree n in f.

By Lemma 1, if (1)′ is defined on a closed curve γ, we can define its monodromy $\kappa_\sigma=\kappa_\sigma(f)$. Then we have

Theorem 1. *Let σ be in $\pi_1(\gamma)$. Then the monodromy $\kappa_\sigma(f)$ allows the following expansion*

$$\kappa_\sigma(f)(s)=\chi_\sigma(I+\sum_{n=1}^{8} s^n\lambda_{n,\sigma}(f)(s))\,f(s). \tag{5′}$$

Here χ_σ is the monodromy of (8), *$\lambda_{n,\sigma}(f)$ is homogeneous of degree n in f. Especially, λ_1 is linear in f and satisfies following period relation*

$$\lambda_{1,\sigma\tau}(T)=\lambda_{1,\tau}(T)+\chi_\tau^{-1}\lambda_{1,\sigma}(\chi_\tau T)\,\chi_\tau. \tag{6′}$$

It seems main informations from the non-linear part of (1)′ (or (3)) are contained in λ_1. In Abe's equation (1), λ_1 is determined by A and B_j's and does not depend on the choice of the normal direction.

Lemma 1 is proved in Section 2. Theorem 2 is proved in Section 3 together with the C^0-estimate of κ_σ(regarding κ_σ to be a map in $C^0(-\varepsilon, \varepsilon)$) and the period relation of λ_2. Considering (1)′ on a complex domain, we have similar results. These are remarked in Section 4. These may concern non-linear Riemann-Hilbert problem (cf. [5], [6]). But our main interest is its (non-linear) monodromy. Except the use of the integrability condition ($d\phi+\theta\wedge\phi=0$), Lemma 3 and Theorem 3 follow Lemma 1 and Theorem 1. So in Section 5, we show how to use the integrability condition to construct a local solution of (3). This integrability condition is chiral to the linear part of (3) and relate some works inspired recent particle physics and field theory (cf. [3]).

2 Proof of Lemma 1

For the convenience to get the informations about monodromy, we apply the method developed in [2]. For the coefficients of (1)′, we assume

$$\|A(x)\| \leqq k, \ |x| \leqq a, \tag{9}$$

$$\|F(U, V)(x, s)\| \leqq L_k \|U(x, s)\| \ \|V(x, s)\|, \ |x| \leqq a, \ |s| < b.$$

Here $\|(a_{ij})\|$ means $(\sum_{ij} |a_{ij}|^2)^{1/2}$, $a>0$ is a given constant and $b>0$ is a suitable constant.

Let $U=U(x)$ be the unitary solution of (8). Then $Y_0(x, s) = U(x) f(s)$ is the solution of (8) with the initial data $f(s)$. Starting Y_0, we define a series of matrix valued functions $Y_0, Y_1, \cdots, Y_n, \cdots$, successively by the equation

$$\frac{dY_n}{dx} + A(x) Y_n + \sum_{k=0}^{n=1} F(Y_k, Y_{n-k-1}) = 0, \ Y_n(0)(s) = 0, n \geqq 1. \tag{10$_n$}$$

Explicitly, Y_n is given by

$$Y_n(x, s) = -U(x) \int_0^x U(\xi)^{-1} (\sum_{k=0}^{n-1} F(Y_k(\xi, s), Y_{n-k-1}(\xi, s)) d\xi. \tag{11}$$

By (11), if $Y_k(x, s) = U(x) V_k(x, s) f(s)$, $k \geqq n-1$, where $V_k(x, s) = V_k(x, s, f)$ is homogeneous of degree k *in* f, then to set

$$V_n(x, s) = -\int_0^x U(\xi)^{-1} (\sum_{k=0}^{n-1} F(Y_k(\xi, s), U(x) V_{n-k-1}(\xi, s)) d\xi, \tag{11$'$}$$

$Y_n(x, s)$ is equal to $U(x) V_n(x, s) f(s)$ by (7). Since $F(U, V)$ is bilinear in U, V, $V_n(x, s, f)$ is homogeneous of degree n in f because $Y_k(x, s, f)$ is homogeneous of degree $k+1$ in f.

By (9), we have $\|U(x)\| \leqq e^{k|x|} \leqq e^{ka}$ (cf. [2], [4]). Hence we have

$$\|Y_0(x, s)\| \leqq e^{ka} |f(s)|, \ |x| \leqq a. \tag{12$_0$}$$

By (12)$_0$ and (11), we get $\|Y_1(x, s)\| \leqq L |f(s)|^2 e^{4ka} |x|$ if $|x| \leqq a$. Hence we assume the inequality

$$\|Y_k(x, s)\| \leqq L^k |f(s)|^{k+1} e^{(3k+1)ka} |x| \tag{12$_k$}$$

is hold if $|x| \leqq a$ and $k \leqq n-1$, Then we have

$$\|\sum_{k=0}^{n-1} F(Y_k(x, s), Y_{n-k-1}(x, s))\|$$

$$\leqq \sum_{k=0}^{n-1} L \cdot L^k |f(s)|^{k+1} e^{(3k+1)Ka} L^{n-k-1} |f(s)|^{n-k} e^{(3n-3k-2)Ka} |x|^{n-1}$$

$$\leqq n L^k |f(s)|^{n+1} e^{(3n-1)Ka} |x|^{n-1},$$

if $|x|\leqq a$. Hence we obtain $(12)_n$ by (11). Therefore the series

$$Y(x, s)=\sum_{n=0}^{\infty} s^n Y_n(x, s)$$

converges absolutely and uniformly on $\{|x|\leqq a, |s|<\varepsilon\}$ if $|sLf(s)e^{3Ka}|<1$ for $|s|<\varepsilon$. Hence to take ε to satisfy

$$\varepsilon<\frac{1}{L||f||}e^{-3Ka}, \quad ||f||=\max_{|s|\leqq b}|f(s)|, \text{ for suitable } b>0,$$

$Y(x, s)$ converges on $\{|x|<a, |s|<\varepsilon.\}$ Then, since $Y_n'(x, s)=-A(x)Y_n(x, s)-$ $-\sum_{k=0}^{n-1}F(Y_k, Y_{n-k-1})(x, s)$, we obtain by (12)

$$\begin{aligned}&||\frac{dY_n}{dx}(x, s)||\\ &\leqq KL^n|f(s)|^{n+1}e^{(3n+1)Ka}|x|^n+nL^{n-1}|f(s)|^{n+1}e^{(3n-1)Ka}|x|^{n-1}.\end{aligned}$$

Hence $\sum_{n=0}^{\infty}s^nY_n'(x, s)$ converges absolutely and uniformly on the same domain. Therefore $Y(x, s)$ is a solution of $(1)'$. Since $Y_n(x, s)=U(x)V_n(x, s)f(s)$, where $V_n(x,s)=V_n(x, s, f)$ is homogeneous of degree n in f, we have

$$\begin{aligned}&Y(x, s)=U(x)(I+\sum_{n=1}^{\infty}s^nV_n(x, s, f))f(s),\\ &Y(x, 0)=f(s).\end{aligned}$$

Since $Y_n(x, s)$ is holomorphic in s if $f(s)$ is holomorphic and $F(U, V)$ is holomorphic in s, $Y(x, s)$ is holomorphic in s in this case.

If $F(U, V)$ is C^k-class in s, we use the notation

$$(\frac{\partial^k F}{\partial s^k})(U, V)=(\frac{\partial^k}{\partial s^k}F(U(x, t), V(x, t))|_{t=s}.$$

Then we assume

$$(9)' \qquad ||\left(\frac{\partial^k F}{\partial s^k}\right)(U, V)(x, s)||\leqq L_k||U(x, s)||\ ||V(x, s)||.$$

To show the C^1-regularity for C^1-class f, we set

$$|f(s)|_1=\max(|f(s)|, |f'(s)|), \quad L_{(1)}=\max(L, L_1).$$

Then, since

$$\frac{\partial F(U, V)}{\partial s}=\left(\frac{\partial F}{\partial s}\right)(U, V)+F\left(\frac{\partial U}{\partial s}, V\right)+F\left(U, \frac{\partial V}{\partial s}\right),$$

we get $\|(\partial Y_1/\partial s)(x, s)\| \leqq 3L_{(1)}|f(s)|_1{}^2 e^{4Ka}|x|$. Hence we assume the inequality

$$(13)_k \qquad \left\| \frac{\partial Y_k}{\partial s}(x, s) \right\| \leqq 3^k (L_{(1)})^k (|f(s)|_1)^{k+1} e^{(3k+1)Ka} |x|^k$$

is hold if $k \leqq n-1$. Then we have

$$\left\| \frac{\partial}{\partial s} \left(\sum_{k=0}^{n-1} F(Y_k(x, s), Y_{n-k-1}(x, s)) \right) \right\|$$
$$\leqq \sum_{k=0}^{n-1} \left(\left\| \left(\frac{\partial F}{\partial s} \right) (Y_k, Y_{n-k-1}) \right\| + \left\| F\left(\frac{\partial Y_k}{\partial s}, Y_{n-k-1} \right) \right\| + \right.$$
$$\left. + \left\| F\left(Y_k, \frac{\partial Y_{n-k-1}}{\partial s} \right) \right\| \right)$$
$$\leqq (nL_{(1)} L_{(1)}{}^{n-1} |f(s)|_1{}^{n+1} e^{(3n-1)Ka} +$$
$$+ 2 \sum_{k=0}^{n-1} L_{(1)} 3^k L_{(1)}{}^{n-1} |f(s)|_1{}^{n+1} e^{(3n-1)Ka}) |x|^n$$
$$< n(3L_{(1)})^n |f(s)|_1{}^{n+1} e^{(3n-1)Ka} |x|^n.$$

Hence we obtain $(13)_n$ by (11). Therefore $Y(x, s)$ is C^1-class in s. Higher regularities are similarly proved.

3 Proof of Theorem 1

Since the uniqueness is hold for the Cauchy problem of the equation (1)', if $Y(x, s)$ is a solution of (1)' such that $Y(x, 0) = f(s)$, we have $Y(x, s) = U(x)(I + \sum_{n=1}^{\infty} s^n V_n(x, s)) f(s)$. Hence if $\sigma \in \pi_1(\gamma) \cong \mathbf{Z}$ is represented by an integer k, we have

$$(14) \qquad \kappa_\sigma(f)(s) = \chi_\sigma (I + \sum_{n=1}^{\infty} s^n V_n(k, s)) f(s).$$

Because $U(k) = \chi_\sigma$, the monodromy of (8). Since $V_n(x, s) = V_n(x, s, f)$ is homogeneous of degree n in f, we have the first assertion of Theorem 1. We note that, since V_1 is given by

$$V_1(x, s) = -U(x) \int_0^x U(\xi)^{-1} F(U(\xi) f(s), U(\xi)) d\xi,$$

we get

$$(15) \qquad V_1(x, s) = -U(x) \int_0^x U(\xi)^{-1}\, {}^t f(s)\, F(U(\xi), U(\xi)) d\xi,$$

for the original equation (1) of Abe. Because F satisfies

$$(7)' \qquad F(UB, VC) = {}^t B F(U, V) C, \text{ B, } V \text{ are matrices}$$

in this case. (15) shows that V_1 is independent to the choice of normal direction in the equation of Abe.

By (5), we have

$$\begin{aligned}
\kappa_{\sigma\tau}(f)(s) &= \chi_{\sigma\tau}\,(I+\sum_{n=1}^{\infty} s^n\lambda_{n,\sigma\tau}(f))(f)(s) \\
&= \kappa_\sigma(\kappa_\tau(f))(\kappa_\tau(f))(s) \\
&= \chi_\sigma\,(I+\sum_{n=1}^{\infty} s^n\lambda_{n,\sigma}\,(\kappa_\tau(f)))\ (\kappa_\tau(f))\,(s) \\
&= \chi_\sigma\,(I+\sum_{n=1}^{\infty} s^n\lambda_{n,\sigma}\ (\kappa_\tau\,(f)))\ (\chi_\tau\,(I+\sum_{n=1}^{\infty} s^n\lambda_{n,\tau}(f))\ f(s) \\
&= \chi_{\sigma\tau} f(s)+s(\chi_\sigma\lambda_{1,\sigma}(\kappa_\tau(f))\chi_\tau+\chi_{\sigma\tau}\lambda_{1,\tau}(f))f(s)+ \\
&\quad +\sum_{n=2}^{\infty} s^n(\chi_\sigma\lambda_{n,\sigma}\,(\kappa_\tau\,(f))\chi_\tau+\sum_{k=1}^{n-1}\chi_\sigma\lambda_{k,\sigma}\ (\kappa_\tau(f))\chi_\tau\lambda_{n-k,\tau}(f)+ \\
&\quad +\chi_{\sigma\tau}\lambda_{n,\tau}(f))f(s).
\end{aligned}$$

We set $f(s)=f_0+sf_1+\cdots$, $f_0=f(0)$, where each f_i is a (constant) matrix. Then by the linearity of λ_1, we have

$$\lambda_{1,\sigma\tau}(f_0)=\chi_\tau^{-1}\lambda_{1,\sigma}(\chi_\tau f_0)\chi_\tau+\lambda_{1,\tau}(f_0).$$

Since f_0 is an arbitrary matrix, this shows (6). By (15), λ_1 is independent to the choice of normal directions in the equation of Abe.

Since $\lambda_{n,\sigma}(f)$ is homogeneous of degree n in f, we have

$$\lambda_{n,\sigma}(f_0+sg)=\lambda_{n,\sigma}(f_0)+o(s),\quad n\geqq 1. \tag{16}$$

Hence we get

$$\begin{aligned}
&\chi_{\sigma\tau}(\lambda_{1,\sigma\tau}(f_1)+\lambda_{2,\sigma\tau}(f_0)) \\
&=\chi_{\sigma\tau}(\lambda_{1,\tau}(f_1)+\lambda_{2,\tau}(f_0))+\chi_\sigma\lambda_{1,\sigma}(\chi_\tau f_1+\chi_\tau\lambda_{1,\tau}(f_0))\chi_\tau+ \\
&+\chi_\sigma\lambda_{2,\sigma}(\chi_\tau f_0)\chi_\tau+\chi_\sigma\lambda_{1,\sigma}(\chi_\tau f_0)\chi_\tau\lambda_{1,\tau}(f_0) \\
&=\chi_{\sigma\tau}\lambda_{1,\tau}(f_1)+\chi_\sigma\lambda_{1,\sigma}(\chi_\tau f_1)\chi_\tau+ \\
&+\chi_{\sigma\tau}\lambda_{2,\tau}(f_0)+\chi_\sigma\lambda_{1,\sigma}(\chi_\tau\lambda_{1,\tau}(f_0))\chi_\tau+\chi_\sigma\lambda_{2,\sigma}(\chi_\tau f_0)\chi_\tau+ \\
&+\chi_\sigma\lambda_{1,\sigma}(\chi_\tau f_0)\chi_\tau\lambda_{1,\tau}(f_0).
\end{aligned}$$

Therefore by (6), we obtain the following period relation of λ_2.

$$\begin{aligned}
\lambda_{2,\sigma\tau}(T)&=\lambda_{2,\tau}(T)+\chi_\tau^{-1}\lambda_{2,\sigma}(\chi_\tau T)\chi_\tau+ \\
&+\chi_\tau^{-1}(\lambda_{1,\sigma}(\chi_\tau\lambda_{1,\tau}T)\chi_\tau+\lambda_{1,\sigma}(\chi_\tau T)\chi_\tau\lambda_{1,\tau}(T)).
\end{aligned}\tag{17}$$

We consider κ to be a map of the space of germs of functions in Theorem 1. But with a suitable $\varepsilon>0$, we can regard κ to be a map from $\{f \mid f\in C^0[-\varepsilon,\varepsilon], ||f|| \leqq a\}$ into $C^0[-\varepsilon, \varepsilon]$. Then we have

$$||\kappa_\sigma(f)|| \leqq \frac{||f||}{1-kL||f||e^{3kK}}. \tag{18}$$

Note. By definition, we have

$$\lambda_{n,e}(f)=0, \tag{19}$$

for all $n\geqq 1$. Here e means the identity of $\pi_1(\gamma)$. Especially we have

$$\lambda_{1,\sigma^{-1}}(T)=-\chi_\sigma{}^{-1}\lambda_{1,\sigma}(\chi_\sigma T)\chi_\sigma.$$

4 Equation on a Complex Domain

On a domain D in $\boldsymbol{C}$, the complex plane, we consider the equation

$$\frac{dY}{dz}+A(z)Y+sF(Y,\ Y)0,\ z\in D. \tag{20}$$

Here $A(z)$ and $F(U, V)$ (s, z) are holomorphic in z (and s). We denote the universal covering space of D by $\widetilde{D}$. Then denote the s-space by $\boldsymbol{C}$, $\widetilde{D}\times\boldsymbol{C}$ is the universal covering space of $D\times\boldsymbol{C}$. By Lemma 1, we have

Lemma 2. *For any holomorphic function $f(s)$ near the origin of $\boldsymbol{C}$, there exists a neighborhood $U(\widetilde{D}\times 0)=U(\widetilde{D}\times 0,\ f)$ of $\widetilde{D}\times 0$ in $\widetilde{D}\times\boldsymbol{C}$ such that* (20) *has a holomorphic solution $Y(z,\ s)$ on $U(\widetilde{D}\times 0)$ such that $f(z_0,\ s)=f(s)$, where z_0 is a fixed point of $\widetilde{D}$. Precisely, this $Y(z)$ has the following form*

$$Y(z,\ s)=U(z)\,(I+\sum_{n=1}^{\infty} s^n V_n\,(z,\ s,\ f))f(s). \tag{21}$$

Here $U(z)$ is the solution of the linear part of (20) *such that $U(z_0)=I$ and each $V_n(f)$ is homogeneous of degree n in f.*

By Lemma 2, we have

Theorem 2. (20) *has the monodromy $\kappa_\sigma=\kappa_\sigma(f)$, $\sigma\in\pi_1(D)$. It has the following form*

$$\kappa_\sigma(f)(s)=\chi_\sigma(I+\sum_{n=1}^{\infty} s^n\lambda_{n,\sigma}(f))\,f(s). \tag{22}$$

Here χ_σ is the monodromy of the linear part of (20), *$\lambda_{n,\sigma}(s)=\lambda_{n,\sigma}(f)$ is holomorphic in s and homogeneous of degree n in f. Especially, $\lambda_{1,\sigma}(f)$ is linear in f and satisfies the periodic relation* (6)

χ_σ defines aflat vector bundle $|\chi|$. It defines a local coefficient cohomology $H^1(D, V_\chi)$ of D. Since Tr λ_1 satisfies

$$\mathrm{Tr}(\lambda_{1,\sigma\tau}(T))=\mathrm{Tr}(\lambda_{1,\tau}(T))+\mathrm{Tr}(\lambda_{1,\sigma}(\chi_\tau T))$$

by (6), Trλ_1 defines an element of $H^1(D, V_\chi)$. It is a characteristic class of the non-linear part of (20). The meaning of this characteristic class is discussed in Section 5.

5 Proof of Lemma 3 and Characteristic Classes of Non-Linear Part

As in Section 2, we solve (3) as follows : Let $U=U(x)$ be the unitary solution of the linear part of (3). That is, U satisfies

$$dU-U\theta=0,\ U(x_0)=I,\ \textit{the identity matrix},$$

and set $F_0=F_0(x, s)=f(s)U(x)$. Here $f(s)$ is the initial data at x_0. Starting this F_0, we want to define a series of matrix valued functions $F_0, F_1, \cdots$, inductively by

$$dF_n-F_n\theta+\sum_{k=0}^{n-1}F_k\phi F_{n-k-1}=0,\ F_n(x_0)=0,\ n\geqq 1. \tag{23}$$

To solve (23), we set $F_n=G_nU$ ($G_0=f(s)$). Then (23) becomes

$$dG_n+\sum_{k=0}^{n-1}F_k\phi G_{n-k-1}=0. \tag{23$'$}$$

(23)$'$ has a local soloution G_n if and only if $d\,(\sum_{k=0}^{n-1} F_k\phi G_{n-k-1})=0$. If $n=1$, this condition is

$$dF_0\wedge\phi+F_0d\phi=f(s)(dU\wedge\phi+Ud\phi)=0.$$

Hence it is the integrability condition $d\phi+\theta\wedge\phi=0$. So we assume

$$dG_k=-\sum_{j=0}^{k-1}F_j\phi G_{k-j-1},\ 0\leqq k\leqq n-1.$$

Then we have

$$\begin{aligned}
d\,(\sum_{k=0}^{n-1}F_k\phi G_{n-k-1}) &= d\,(\sum_{k=0}^{n-1}G_kU\phi G_{n-k-1})\\
&= \sum_{k=0}^{n-1}(dG_kU\phi G_{n-k-1}+G_k(dU\phi+Ud\phi)G_{n-k-1}-G_kU\phi G_{n-k-1})\\
&= -\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}F_j\phi F_{k-j-1}\phi G_{k-j-1}+\sum_{k=0}^{n-1}\sum_{i=0}^{n-k-1}F_k\phi F_i\phi G_{n-k-1}\\
&= 0.
\end{aligned}$$

Therefore we can define G_n by

$$G_n = -J\left(\sum_{k=0}^{n-1} F_k\phi G_{n-k-1}\right),\ n\geqq 1,$$

$$J\beta(x)=\int_0^1 t\sum_i x_i\beta_i(xt)\,dt,\ \beta=\sum_i \beta_i dx_i.$$

Then, since J satisfies

$$\|J\beta(x)\|\leqq \frac{K}{m+1}\|x-x_0\|^{m+1},\ \text{if}\ \|\beta(x)\|\leqq K\|x-x_0\|^m,$$

and defined along a curve, we have Lemma 3.

By the discussions of Section 3, we obtain Theorem 3 from Lemma 3. In the above calculations, G_1 is given by

(24) $$G_1(x,\ s)=-J(F_0(x,\ s)\phi G_0(x,\ s))=-f(s)J(U(x)\phi)f(s).$$

By the integrability condition (2), $U\phi$ is a global closed 1-form on $\widetilde{M}$. Hence we can set $U\phi=dH$ on $\widetilde{M}$. Then, since $(U\phi)^\sigma=\chi_\sigma U\phi$, we have

(25) $$H(x)^\sigma=\chi_\sigma H(x)+h_\sigma,$$

$$h_{\sigma\tau}=\chi_\sigma h_\tau+h_\sigma.$$

Here each h_σ is a constant matrix. Using this h_σ, $\lambda_{1,\sigma}$ is given by

(26) $$\lambda_{1,\sigma}T=h_\sigma T\chi_\sigma^{-1}.$$

On M, we may regard $U\phi$ to be a collection $\{U_i\phi\}$, where $U_i^{-1}dU_i=\phi$ on V_i, an open set of M. Since $U_iU_j^{-1}$ is the transition function of the flat bundle $[\chi]$, $\{U_i\phi\}$ is a cross-section of $[\chi]$ and closed by (2). Hence in the sence of de Rham, $\{U_i\phi\}$ defines an element of $H^1(M,\ \mathfrak{g}[\chi])$, where $\mathfrak{g}$ is the module of matrices. But this class does not reflect the influence of the linear part of (3). While (26) shows the class Tr λ_1 reflects influence from the linear part.

We note that, using above $H=H_1$, G_k is given by

(27) $$G_k=(-1)^k(fH)^k f.$$

To show this, we set $H_k=(-1)^k(Hf)^{k-1}H$, $k\geqq 2$. Since $G_1=fHf$, we have

$$dG_2=fd(HfH)f=f(U\phi fH+HfU\phi)f=F_0\phi G_1+F_1\phi G_0.$$

Hence we use the induction about n. Then, since $G_k=fH_kf$, we have

$$(-1)^n d(fH_kf)=(-1)^n f dH_k f$$
$$=(-1)^n f\sum_{k=0}^{n-1}(Hf)^k(dH)f(Hf)^{n-k-1}$$

$$= -\sum_{k=0}^{n-1}(-1)^k f(Hf)^k U\phi(-1)^{n-k-1} f(Hf)^{n-k-1}$$

$$= -\sum_{k=0}^{n-1} F_k\phi G_{n-k-1}.$$

Hence we obtain (27).

By (27), we have

Theorem 3′ *Let U be the unitary solution of the linear part of* (3), *H, the matrix valued function on $\widetilde{M}$ such that*

$$dH = U\phi,\quad H(x_0) = 0.$$

Then on $U(\widetilde{M})$, a suitable neighborhood of $\widetilde{M}\times 0$ in $\widetilde{M}\times C$, the solution $F(x, s)$ of (3) *with the initial data $f(s)$ is given by*

$$F(x,\ s) = f(s)(I + \sum_{n=1}^{\infty}(-1)^n s^n (H(x)f(s))^n)U(x). \tag{28}$$

Note 1. By (28), $\lambda_{n,\sigma}$'s, $n \geqq 2$, are written by h_σ and χ_σ.

Note 2. Even the solution of the linear part of (20) has regular singularity at $z_0 \in \overline{D}$, the solution of (20) may not have regular singularity at z_0 by (28).

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