Realization of automorphisms $\sigma$ of order 3 and $G^\sigma$ of compact exceptional Lie groups $G$, I,

$G = G_2, F_4, E_6$

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

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J. A. Wolf and A. Gray [1] classified automorphisms $\sigma$ of order 3 and the fixed subgroups $G^\sigma$ of connected compact simple Lie groups $G$ of center-free. In this paper, we find these automorphisms $\sigma$ and realize $G^\sigma$ for simply connected compact exceptional Lie groups $G = G_2, F_4$ and $E_6$. (As for $E_7$ and $E_8$, they will appear in the next issue). Our result is the following second column. The first column is the chart of involutive automorphisms and the fixed subgroups which are connected our cases.

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Notations. (1) Let $G$ be a group and $\sigma$ an automorphism of $G$. $G^\sigma$ denotes $\{g \in G \mid sg = g\}$. If $\sigma$ is an inner automorphism $Ad_s$ induced by $s \in G$, $G^{Ad_s}$ is briefly denoted by $G^s : G^s = \{g \in G \mid sg = gs\}$. Moreover, for a subset $S$ of $G$, the centralizer of $S$ in $G$ is denoted by $G^S : G^S = \{g \in G \mid sg = gs \text{ for all } s \in S\}$.

(2) When two groups $G, G'$ are isomorphic: $G \cong G'$, we often identify these groups: $G = G'$.

(3) For an $R$-vector space $V$, its complexification $\{u+iv \mid u, v \in V\}$ is denoted by $V^C$. The complex conjugation in $V^C$ is denoted by $\tau : \tau(u+iv) = u-iv$.

(4) The definitions of classical Lie groups $U(n), SU(n)$ and $Sp(n), n = 1, 3$
appeared in this paper are usual ones: $U(n) = \{ A \in M(n, \mathbb{C}) \mid A^* A = E \}$, $SU(n) = \{ A \in U(n) \mid \det A = 1 \}$ and $Sp(n) = \{ A \in M(n, \mathbb{H}) \mid A^* A = E \}$.

1. The group $G_2$

Let $\mathbb{C} = \sum_{i=0}^{7} \mathbb{R} e_i$ be the Cayley division algebra with the multiplication such that $e_0 = 1$ is the unit, $e_i^2 = -1$, $1 \leq i \leq 7$, $e_i e_j = -e_j e_i$, $1 \leq i \neq j \leq 7$ and $e_i e_7 = e_7$, $e_7 e_8 = e_8$, $e_8 e_7 = e_7$ etc. In $\mathbb{C}$, the conjugation $\bar{x}$, the inner product $(x, y)$ and the length $|x|$ are naturally defined. The Cayley algebra $\mathbb{C}$ contains the field of real numbers $\mathbb{R}$ naturally, furthermore the fields of complex numbers $\mathbb{C}$, and quaternions $\mathbb{H}$:

$$
\mathbb{C} = \{ \xi + \eta e_0 \mid \xi, \eta \in \mathbb{R} \}, \quad \mathbb{C}_i = \{ \xi + \eta e_i \mid \xi, \eta \in \mathbb{R} \}, \\
\mathbb{H} = \{ \xi + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 \mid \xi, \xi_i \in \mathbb{R} \}.
$$

Hereafter $e_i$ is briefly denoted by $e_i$.

The automorphism group $G_2$ of the Cayley algebra $\mathbb{C}$,

$$
G_2 = \{ \alpha \in \text{Iso}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \mid \alpha(x y) = (\alpha x)(\alpha y) \}
$$

is a simply connected compact simple Lie group of type $G_2$ [8]. To find some subgroups of $G_2$, we will give alternative definitions of the Cayley algebra $\mathbb{C}$.

1. In $\mathbb{C} = \mathbb{H} \oplus \mathbb{H} e$, we define a multiplication, a conjugation $\bar{\cdot}$ and an inner product $(\cdot, \cdot)$ respectively by

$$
(a + be)(c + de) = (ac - db) + (bc + da)e, \\
\bar{a + be} = \bar{a} - be, \\
(a + be, c + de) = (a, c) + (b, d).
$$

2. In $\mathbb{C} = \mathbb{C} \oplus \mathbb{C}^3$, we define a multiplication etc. by

$$
(a + m)(b + n) = (ab - m^* n) + (an + bm + m \times n), \\
\bar{a + m} = \bar{a} - m, \\
(a + m, b + n) = (a, b) + (m, n)
$$

where $m \times n \in \mathbb{C}^3$ is the exterior product of $m, n \in \mathbb{C}^3$ and $(m, n) = \frac{1}{2} (m^* n + n^* m)$.

1.1. Automorphism $\gamma_3$ of order 3 and subgroup $(U(1) \times Sp(1))/\mathbb{Z}_3$ of $G_2$

We define an $\mathbb{R}$-linear transformation $\gamma$ of $\mathbb{C}$ by

$$
\gamma(a + be) = a - be, \quad a + be \in \mathbb{H} \oplus \mathbb{H} e = \mathbb{C}.
$$

Then we have $\gamma \in G_2$ and $\gamma^3 = 1$.

**Known result 1.1** [2]. *The group $(G_2)^{\gamma}$ is isomorphic to the group $(Sp(1) \times Sp)^{\gamma_3}$.*
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by an isomorphism induced from the homomorphism \( \psi : \text{Sp}(1) \times \text{Sp}(1) \rightarrow (G_2)^{\gamma} \)

\[
\psi(p, q)(a + be) = qa\bar{q} + (p\bar{q})e,
\]

with \( \text{Ker}\psi = Z_2 = \{(1, 1), (-1, -1)\} \).

Let \( \omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} e_1 \in \text{Sp}(1) \subset H \subset \mathbb{C} \).
Denote \( \psi(\omega_1, 1) \) by \( \gamma_3 : \gamma_3(a + be) = a + (\omega_1 b)e, \ a + be \in H \oplus He = \mathbb{C} \).

Of course \( \gamma_3 \in G_2 \) and \( \gamma_3^3 = 1 \).

**Theorem 1.2.** The group \( (G_2)^{\gamma} \) is isomorphic to the group \( (U(1) \times \text{Sp}(1))/Z_2 (\cong U(2)) \) where \( Z_2 = \{(1, 1), (-1, -1)\} \).

**Proof.** Let \( U(1) = \{s \in \mathbb{C} | |s| = 1\} \subset \text{Sp}(1) \subset H \subset \mathbb{C} \).
We define a homomorphism \( \psi : U(1) \times \text{Sp}(1) \rightarrow (G_2)^{\gamma} \) by the restriction of \( \psi \) of Known result 1.1.
Clearly \( \gamma_3 \psi(s, q) = \psi(s, q) \gamma_3 \) for \((s, q) \subset U(1) \times \text{Sp}(1)\), so \( \psi \) is well-defined. We shall show that \( \psi \) is onto. Let \( a \in (G_2)^{\gamma} \).
Since \( a \) commutes with \( \gamma_3 \), \( \gamma_3 a \gamma_3 = \gamma_3 \), so \( \psi(\omega_1 s, q) = \psi(s \omega_1, q) \), \( s \omega_1 = s \omega_1 \), therefore \( s \subset U(1) \).
Hence \( \psi \) is onto. Obviously \( \text{Ker}\psi = Z_2 \).
Thus we have the isomorphism \( (U(1) \times \text{Sp}(1))/Z_2 \cong (G_2)^{\gamma} \).

**Corollary 1.3.** \((G_2)^{\gamma} = S \) where \( S = \psi(U(1), 1) \). In particular, the manifold \( G_2/(G_2)\gamma \) has a homogeneous complex structure.

1.2. Automorphism \( w \) of order 3 and subgroup \( SU(3) \) of \( G_2 \)

Let \( \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} e \in C \subset \mathbb{C} \).
We define an \( \mathbb{R} \)-linear transformation \( w \) of \( \mathbb{C} \) by

\[
w(a + m) = a + \omega m, \ a + m \in C \oplus C^3 = \mathbb{C}.
\]

Then we have \( w \in G_2 \) and \( w^3 = 1 \).

**Remark.** We have the following

**Proposition 1.4.** For \( a \in \mathbb{S} \) such that \( |a| = 1 \), the condition that the mapping \( \alpha_a : \mathbb{S} \rightarrow \mathbb{S}, \alpha_a x = ax \alpha \) belongs to the group \( G_2 \) is \( a^g = \pm 1 \).

Now, \( w \) is nothing but the mapping \( \alpha_w : wx = \bar{w}x \alpha , x \in \mathbb{S} \).

**Known result 1.5** [7], [8]. The group \( \langle G_2 \rangle_e = \{a \in G_2 | ae = e\} \) is isomorphic to the group \( SU(3) \) by the isomorphism \( \psi : SU(3) \rightarrow (G_2)_e \),

\[
\psi(A)(a + m) = a + Am, \ a + m \in C \oplus C^3 = \mathbb{C}.
\]

**Theorem 1.6.** The group \( (G_2)_w \) coincides with the group \( (G_2)_e \), so it is isomorphic to the group \( SU(3) \).

**Proof.** We shall show \( (G_2)_w = (G_2)_e \).
Clearly \( (G_2)_e = \psi(SU(3)) \subset (G_2)_w \).
Conversely, let \( a \in (G_2)_w \).
Since \( a \) commutes with \( w \), \( \mathbb{C}_w = \{x \in \mathbb{S} |wx = x\} = C \) is invariant under \( a \).
So \( a \) induces an automorphism of \( C \), hence
\[ a\varepsilon = e \quad \text{or} \quad a\varepsilon = -e. \]

In the latter case, consider a mapping \( \gamma : \frak{g} \to \frak{g}, \gamma(a+m) = \overline{a} + \overline{m}. \) Then \( \gamma \in G_2 \) and \( \gamma e = -e. \) (This \( \gamma \) is the same one as \( \gamma \) of the preceding section 1.1). Put \( \beta = \gamma a. \) Since \( \beta e = e, \) we have \( \beta \in (G_2)_e \subset (G_2)^e. \) Therefore \( \gamma = \beta a^{-1} \in (G_2)^e. \) However this is a contradiction. In fact, \( \omega m = \overline{\omega m} = \omega (\gamma m) = \gamma (\omega m) = \overline{\omega m} = \omega m \) for all \( m \in C^3 \) which is false. Hence \( a\varepsilon = e, \) so \( a \in (G_2)_e. \) Thus we have \( (G_2)^e \subset (G_2)_e. \)

2. The group \( F_4 \)

Let \( \mathfrak{G} = \{ X \in M(3, \frak{g}) \mid X^* = X \} \) be the exceptional Jordan algebra with the Jordan multiplication

\[ X \cdot Y = \frac{1}{2}(XY + YX). \]

In \( \mathfrak{G}, \) we define a positive definite inner product \( (X, Y) \) by \( \text{tr}(X \cdot Y). \) Moreover, in \( \mathfrak{G}, \) we define a multiplication \( X \times Y \) called the Freudenthal multiplication, a trilinear form \( (X, Y, Z) \) and the determinant \( \text{det}X \) respectively by

\[
X \times Y = \frac{1}{2} (2X \cdot Y - \text{tr}(X)Y - \text{tr}(Y)X + (\text{tr}(X)\text{tr}(Y) - (X, Y))E), \\
(X, Y, Z) = (X, Y \times Z), \\
\text{det}X = \frac{1}{3}(X, X, X).
\]

The algebra \( \mathfrak{G} \) with the multiplication \( X \times Y \) and the inner product \( (X, Y) \) will be called the Freudenthal algebra.

The automorphism group \( F_4 \) of the Jordan algebra \( \mathfrak{G}, \)

\[ F_4 = \{ \alpha \in \text{Iso}_R(\mathfrak{G}, \mathfrak{G}) \mid \alpha(X \cdot Y) = aX \cdot aY \} \]

\[ = \{ \alpha \in \text{Iso}_R(\mathfrak{G}, \mathfrak{G}) \mid \text{det}aX = \text{det}X, (aX, aY) = (X, Y) \} \]

\[ = \{ \alpha \in \text{Iso}_R(\mathfrak{G}, \mathfrak{G}) \mid \alpha(X \times Y) = aX \times aY \} \]

is a simply connected compact simple Lie group of type \( F_4 \) [3], [8]. The group \( F_4 \) contains \( G_2 \) as a subgroup naturally, that is, any \( \alpha \in G_2 \) is regarded as \( \alpha \in F_4 \) by

\[
\alpha \left( \begin{array}{ccc}
\xi_1 & x_3 & \bar{x}_2 \\
\bar{x}_3 & \xi_2 & x_1 \\
x_2 & \bar{x}_1 & \xi_3
\end{array} \right) = \left( \begin{array}{ccc}
\xi_1 & ax_3 & \bar{ax}_2 \\
\bar{ax}_3 & \xi_2 & ax_1 \\
ax_2 & \bar{ax}_1 & \xi_3
\end{array} \right).
\]

To find some subgroups of \( F_4, \) we will give alternative definitions of Freudenthal algebra \( \mathfrak{G}. \) For \( K = R, C, \) let \( \mathfrak{G}_K = \mathfrak{G}(3, K) = \{ X \in M(3, K) \mid X^* = X \} \) be the Freudenthal algebra with the multiplication \( X \times Y \) and the inner product \( (X, Y) \) as analogous to ones in \( \mathfrak{G}. \)

1. In \( \mathfrak{G}_R = \mathfrak{G}(3, H) \oplus H^3 \) (where \( H^3 = \{ (a_1, a_2, a_3) \mid \text{"row vector"}, a_i \in H\} \)), we define a multiplication and an inner product respectively by

\[
(X + a) \times (Y + b) = (X \times Y - \frac{1}{2}(a^* b + b^* a)) - \frac{1}{2}(aY + bX), \\
(X + a, Y + b) = (X, Y) + 2(a, b)
\]
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where $(a, b) = \frac{1}{2}(ab^* + ba^*) = \frac{1}{2} \text{tr}(a^*b + b^*a)$.

2. In $\mathbb{H} = (3, C) \oplus M(3, C)$, we define a multiplication etc. by

$$(X + M) \times (Y + N) = (X \times Y - \frac{1}{2}(M^*N + N^*M)) - \frac{1}{2}(MY + NX + M \times N)$$

where, for $M = (m_1, m_2, m_3), N = (n_1, n_2, n_3) \in M(3, C), M \times N \in M(3, C)$ is defined by

$$M \times N = \begin{pmatrix} m_1 \times n_3 & m_2 \times n_1 & m_3 \times n_2 \\ n_2 \times m_3 & n_3 \times m_1 & n_1 \times m_2 \end{pmatrix}$$

and $(M, N) = \frac{1}{2} \text{tr}(M^*N + N^*M)$.

2.1. Automorphism $\gamma_2$ of order 3 and subgroup $(U(1) \times \text{Sp}(3))/\mathbb{Z}_2$ of $F_i$

We consider $R$-linear transformations $\gamma, \gamma_2 \in \mathbb{H}$ which are extensions of $\gamma, \gamma_2 \in G_2$ to $F_i$ respectively. Of course $\gamma, \gamma_2 \in F_i$ and $\gamma^2 = 1, \gamma_2^2 = 1$.

**Known result 2.1** [5]. The group $(F_i)^g$ is isomorphic to the group $(\text{Sp}(1) \times \text{Sp}(3))/\mathbb{Z}_2$ by an isomorphism induced from the homomorphism $\psi : \text{Sp}(1) \times \text{Sp}(3) \rightarrow (F_i)^g$,

$$\psi(b, A)(X + a) = AXA^* + bA^*, \quad X + a \in \mathbb{H}$$

with $\text{Ker} \psi = \mathbb{Z}_2 = \{(1, E), (-1, -E)\}$.

**Theorem 2.2.** The group $(F_i)^{g_2}$ is isomorphic to the group $(U(1) \times \text{Sp}(3))/\mathbb{Z}_2$ where $\mathbb{Z}_2 = \{(1, E), (-1, -E)\}$.

**Proof.** Let $U(1) = \{s \in C \mid |s| = 1\} \subset \text{Sp}(1) \subset H \subset g$. We define a homomorphism $\psi : U(1) \times \text{Sp}(3) \rightarrow (F_i)^{g_2}$ by the restriction of $\psi$ of Known result 2.1. Then $\psi$ induces an isomorphism $(U(1) \times \text{Sp}(3))/\mathbb{Z}_2 \cong (F_i)^{g_2}$ whose proof is similar to Theorem 1.2.

**Corollary 2.3.** $(F_i)^{g_2} = (F_i)^S$ where $S = G_2(U(1), 1)$. In particular, the manifold $F_i/(F_i)^{g_2}$ has a homogeneous complex structure.

2.2. Automorphism $\sigma_3$ of order 3 and subgroup $(U(1) \times \text{Spin}(7))/\mathbb{Z}_2$ of $F_i$

Let $U(1) = \{a \in C \mid |a| = 1\}$. For $a \in U(1)$, we define an $R$-linear transformation $D_a$ of $\mathbb{H}$ by

$$D_a \begin{pmatrix} \xi_1 \\ \xi_3 \\ \xi_2 \\ \xi_1 \\ \xi_3 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_3 \\ \xi_2 \\ \xi_1 \\ \xi_3 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

Then we have $D_a \in F_i$. Denote $D_{-1}$ by $\sigma$. Of course $\sigma \in F_i$ and $\sigma^2 = 1$.

Hereafter we use the following notations in $\mathbb{H}$ [6].
\[
E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
F_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ \bar{x} & 0 & 0 \end{pmatrix}, \quad F_2(x) = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}, \quad F_3(x) = \begin{pmatrix} 0 & x & 0 \\ \bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

**Known result 2.4.** [4], [8]. The group \( (F_3)^\sigma \) coincides with the group \( (F_3)_{E_1} = \{ \alpha \in F_3 | \alpha E_1 = E_1 \} \), so it is isomorphic to the group \( \text{Spin}(9) \) which is the universal covering group of \( SO(9) = SO(V^9) \) where \( V^9 = \{ X \in \mathbb{C} | E_* X = 0, \text{tr}(X) = 0 \} \).

Let \( \omega = -\frac{1}{2} + \sqrt{3} \frac{e}{2} \in U(1) \subset \mathbb{C} \subset \mathbb{C} \) and denote \( D_\omega \) by \( \sigma \). Of course \( \sigma \in F_1 \) and \( \sigma^3 = 1 \). To investigate the group \( (F_3)^\sigma \), we consider \( \mathbb{R} \)-vector subspaces \( \mathfrak{z}_{\sigma}, (\mathfrak{z}_{\sigma})^\perp \)

of \( \mathfrak{z} \):

\[
\mathfrak{z}_{\sigma} = \{ X \in \mathfrak{z} | \sigma X = X \} = \{ \xi E_1 + \xi_2 E_2 + \xi_3 E_3 + F_1(t) | \xi_i \in \mathbb{R}, t \in \mathbb{C}^\perp \},
\]

\( (\mathfrak{z}_{\sigma})^\perp \) is the orthogonal complement of \( \mathfrak{z}_{\sigma} \) in \( \mathfrak{z} \):

\[
(\mathfrak{z}_{\sigma})^\perp = \{ F_1(s) + F_2(x_0) + F_3(x_0) | s \in \mathbb{C}, x_0 \in \mathbb{C} \}
\]

where \( \mathbb{C}^\perp \) is the orthogonal complement of \( C \) in \( \mathfrak{z} \). Then \( \mathfrak{z} = \mathfrak{z}_{\sigma} \oplus (\mathfrak{z}_{\sigma})^\perp \) and \( (\mathfrak{z}_{\sigma})^\perp \) are invariant under the group \((F_3)^\sigma\).

**Lemma 2.5.** For \( \alpha \in (F_3)^\sigma \), we have \( \alpha E_1 = E_1 \). Hence \( (F_3)^\sigma \) is a subgroup of \( (F_3)_{E_1} = \text{Spin}(9) \).

**Proof** is similar to [6, Lemma 9], however we need some modifications. To show \( \alpha E_1 \in \mathfrak{z}(2, \mathbb{C}) \), \( \{ \xi E_1 + \xi_2 E_2 + \xi_3 E_3 + F_1(t) | \xi_i \in \mathbb{R}, t \in \mathbb{C}^\perp \} \), put \( \alpha E_1 = \xi E_1 + \xi_2 E_2 + \xi_3 E_3 + F_1(t), \xi_i \in \mathbb{R}, t \in \mathbb{C}^\perp \) and suppose \( \xi_i \neq 0 \). From \( \alpha E_2 \times \alpha E_2 = 0 \), we see that \( \xi_2 = \xi_3 = t = 0 \), that is, \( \alpha E_2 = \xi_1 E_1 \). Next use \( \alpha E_2 \times \alpha F_1(1) = 0 \), then we see that \( \alpha F_1(1) = \eta E_1 \) for some \( 0 \neq \eta \in \mathbb{R} \) which contradicts to \( \alpha E_2 = \xi_1 E_1 \). Hence \( \xi_1 = 0 \). Thus we have \( \alpha E_2 \in \mathfrak{g}(2, \mathbb{C}) \). Similarly \( \alpha E_3 \in \mathfrak{g}(3, \mathbb{C}) \). Therefore \( \alpha E_1 \notin \mathfrak{g}(2, \mathbb{C}) \). Moreover \( \alpha E_1 \) must be \( 1 \).

Thus we have \( \alpha E_1 = E_1 \).

From Lemma 2.5, we see that \( \mathbb{R} \)-vector subspaces

\[
\{ \xi E_1 + \xi_2 E_2 + F_1(t) | \xi_i \in \mathbb{R}, t \in \mathbb{C}^\perp \}, \{ F_2(x_0) + F_3(x_0) | x_0 \in \mathbb{C} \}, \{ F_i(s) | s \in \mathbb{C} \}
\]

of \( \mathfrak{z} \) are invariant under the group \((F_3)^\sigma\).

We define a subgroup \( (F_3)_{E_1, F_1(s)} \) of the group \( F_1 \) by

\[
(F_3)_{E_1, F_1(s)} = \{ \alpha \in (F_3)^\sigma | \alpha E_1 = E_1, \alpha F_1(s) = F_1(s) \text{ for all } s \in \mathbb{C} \}
\]

\[
= \{ \alpha \in \text{Spin}(9) | \alpha F_1(i) = F_1(i), \alpha F_1(e) = F_1(e) \}.
\]

This group \( (F_3)_{E_1, F_1(s)} \) is isomorphic to the group \( \text{Spin}(7) \) which is the universal covering group of \( SO(7) = SO(V^7) \) where \( V^7 = \{ \xi (E_2 - E_3) + F_1(t) | \xi \in \mathbb{R}, t \in \mathbb{C}^\perp \} \).

Furthermore we use the following notation.
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\((E_4)^{U(1)} = \{ \alpha \in F_4 | D_0 \alpha = aD_0 \text{ for all } a \in U(1) \} \).

**Lemma 2.6.** \( Spin(7) = (E_4)_{F_4, F(1)} \) is a subgroup of \((E_4)^{U(1)} \).

**Proof.** Let \( \beta \in Spin(7) \). Then for \( D_0, \alpha \in U(1) \) we have

\[
\beta D_0 F_1(z) = \beta F_1(\alpha z) = \beta F_1(\alpha^2 s + t) \quad (z = s + t, s \in R, t \in C^\perp)
\]

\[
= F_1(\alpha^2 s) + \beta F_1(\alpha^2 t) = F_1(\alpha^2 s) + (\xi E_2 + \xi E_3 + F_1(t')).
\]

(\text{for some } \xi, \eta \in R, t' \in C^\perp). On the other hand,

\[
D_0 \beta F_1(z) = D_0 \beta F_1(s + t) = D_0 (F_1(s) + \beta F_1(t))
\]

\[
= D_0 (F_1(s) + \xi E_2 + \xi E_3 + F_1(t')) = F_1(\alpha^2 s) + \xi E_2 + \xi E_3 + F_1(t').
\]

Thus we have \( \beta D_0 F_1(z) = D_0 \beta F_1(z) \), \( z \in G \). Next, for \( z \in G \),

\[
\beta D_0 F_1(z) = \beta F_1(az) = 4 \beta (F_1(1) \times F_1(z)) \times F_1(\alpha) = 4 (F_1(1) \times F_1(z) \times F_1(\alpha)
\]

\[
= 4 (F_1(1) \times F_1(z_0) \times F_1(x_0)) \times F_1(\alpha) \quad \text{(for some } x_0 \in G \).
\]

\[
F_2(x_0) = F_2(x_0) \times F_1(\alpha) = D_0 (F_1(x_0) \times F_1(z_0)) = D_0 F_2(z).
\]

Similarly \( \beta D_0 F_1(z) = D_0 \beta F_1(z) \). Clearly \( D_0 \beta = \beta D_0 \) on \( E_i \). Finally

\[
D_0 \beta E_2 = D_0 (\xi E_2 + \xi E_3 + F_1(t)) = \xi E_2 + \xi E_3 + F_1(t) = \beta E_2 = \beta D_0 E_2
\]

(\text{for some } \xi, \eta \in R, t \in C^\perp). Similarly \( D_0 \beta E_3 = \beta D_0 E_3 \). Thus we have \( D_0 \beta = \beta D_0 \), that is, \( \beta \in (E_4)^{U(1)} \).

**Theorem 2.7.** The group \((E_4)^{\mathbb{Z}_2}\) is isomorphic to the group \((U(1) \times Spin(7)) / \mathbb{Z}_2\) where \( \mathbb{Z}_2 = \{(1, 1), (-1, -1)\} \).

**Proof.** We define a mapping \( \psi : U(1) \times Spin(7) \to (E_4)^{\mathbb{Z}_2} \) by

\[
\psi(a, \beta) = D_0 \beta.
\]

Obviously \( \psi \) is well-defined: \( \psi(a, \beta) \in (E_4)^{\mathbb{Z}_2} \) (Lemma 2.6). Since \( D_0(a \in U(1)) \) and \( \beta \in Spin(7) \) commute (Lemma 2.6), \( \psi \) is a homomorphism. We shall show that \( \psi \) is onto. Let \( \alpha \in (E_4)^{\mathbb{Z}_2} \). Put \( aF_1(1) = F_1(s_0), s_0 \in C \). Then we have

\[
aF_1(\omega) = aF_1(\tilde{\omega} \tilde{\omega}) = aD_0 F_1(1) = D_0 aF_1(1) = D_0 F_1(s_0) = F_1(\omega s_0),
\]

\[
aF_1(\tilde{\omega}) = aD_0 D_0 F_1(1) = D_0 D_0 aF_1(1) = D_0 D_0 F_1(s_0) = F_1(\omega s_0).
\]

Taking (1) - (2), we have \( aF_1(e) = F_1(es_0) \). Now, choose \( \omega_0 \in C \) such that \( \tilde{\omega}_0^2 = s_0 \). Then

\[
\alpha F_1(1) = F_1(s_0) = F_1(\tilde{\omega}_0^2) = D_0 F_1(1), \quad aF_1(e) = F_1(es_0) = F_1(\tilde{\omega}_0^2 e) = D_0 F_1(e).
\]

Put \( \beta = D_0^{-1} \alpha \), then \( \beta F_1(1) = F_1(1), \beta F_1(e) = F_1(e) \) and \( \beta E_i = E_i \) (Lemma 2.5), so \( \beta \in Spin(7) \). Thus we have

\[
\alpha = D_0 \beta, \quad D_0 \in U(1), \beta \in Spin(7),
\]

that is, \( \psi \) is onto. Obviously \( \text{Ker} \psi = \mathbb{Z}_2 \). Thus we have the isomorphism \((U(1) \times Spin(7)) / \mathbb{Z}_2 \cong (E_4)^{\mathbb{Z}_2} \).
Corollary 2.8. \((F_4)^8 = (F_4)^5\) where \(S = \psi(U(1), 1)\). In particular, the manifold \(F_4/(F_4)^5\) has a homogeneous complex structure.

2.3. Automorphism \(w\) of order 3 and subgroup \((SU(3) \times SU(3))/\mathbb{Z}_3\) of \(F_4\)

Let \(\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i e \in C \subset G\) and we define an \(R\)-linear transformation \(w\) of \(3\) by

\[
w(X + M) = X + \omega M, \quad X + M \in \mathbb{S}(3, C) \oplus M(3, C) = 3.
\]

This \(w\) is the same one as \(w \in G_2 \subset F_4\). Of course \(w^3 = 1\).

Theorem 2.9. The group \((F_4)^w\) is isomorphic to the group \((SU(3) \times SU(3))/\mathbb{Z}_3\) where \(\mathbb{Z}_3 = \{(E, E), (\omega E, \omega E), (\omega^2 E, \omega^3 E)\}\).

Proof. We define a mapping \(\psi : SU(3) \times SU(3) \to (F_4)^w\) by

\[
\psi(P, E)(X + M) = AXA^* + PMA^*, \quad X + M \in \mathbb{S}(3, C) \oplus M(3, C) = 3.
\]

\(\psi\) is well-defined: \(\psi(P, E) \in F_4\) [6] moreover \(\psi\) is a homomorphism. We shall show that \(\psi\) is onto. Let \(E \in (F_4)^w\). Since the restriction \(\alpha'\) of \(\alpha\) to \(3_w = \{X \in 3 \mid \omega X = X\} = 3(3, C)\) belongs to the group \(F_4, C = \{\alpha \in Is_1(3, C, \mathbb{C}) \mid \alpha(X + Y) = \alpha X + \alpha Y\}\), there exists \(A \in SU(3)\) such that

\[
\alpha X = AXA^* \text{ or } \alpha X = A\bar{X}A^*, \quad X \in \mathbb{S}(3, C)
\]

[7]. In the former case, put \(\beta = \psi(E, A)^{-1} \alpha\), then \(\beta \mid 3(3, C) = 1\). Hence \(\beta \in G_2\), moreover \(\beta \in (G_2)^w = (F_4)^w\) (Theorem 1.1) = \(SU(3)\). Hence there exists \(P \in SU(3)\) such that

\[
\beta(X + M) = X + PM = \psi(P, E)(X + M), \quad X + M \in 3(3, C) \oplus M(3, C) = 3.
\]

Therefore we have \(\alpha = \psi(E, A) \beta = \psi(E, A) \psi(P, E) = \psi(P, A)\). In the latter case, consider the mapping \(\gamma : 3 \to 3, \gamma(X + M) = \bar{X} + \bar{M}, X + M \in 3\) and recall \(\gamma \in G_2 \subset F_4\). Put \(\beta = \alpha^{-1} \psi(E, A) \gamma\), then \(\beta \in F_4\) and \(\beta \mid 3(3, C) = 1\). Hence \(\beta \in (G_2)^e = (G_2)^w \subset (F_4)^w\). Since \(\beta, \alpha, \psi(E, A) \in (F_4)^w\), \(\gamma\) also \(\in (F_4)^w\), so \(\gamma \in (G_2)^w\) which is a contradiction (Theorem 1.6). Thus we see that \(\psi\) is onto. Ker\(\psi = \mathbb{Z}_3\) is easily obtained. Thus we have the isomorphism \((SU(3) \times SU(3))/\mathbb{Z}_3 \cong (F_4)^w\).

3. The group \(E_6\)

Let \(\mathbb{S}^\mathbb{C} = \{X_i + iX_i \mid X_i \in \mathbb{S}\}(\text{called the complex exceptional Jordan algebra})\) be the complexification of \(3\). As in \(3\), in \(\mathbb{S}^\mathbb{C}\) also, we define multiplications \(X \cdot Y, X \times Y\), the inner product \((X, Y)\), the trilinear form \((X, Y, Z)\) and the determinant \(\det X\). Finally, in \(\mathbb{S}^\mathbb{C}\), we define a positive definite Hermitian inner product \(<X, Y>\) by \((\tau X, Y)\).

The group

\[
E_6 = \{a \in \text{Iso}_C(\mathbb{S}^\mathbb{C}, \mathbb{S}^\mathbb{C}) \mid \det a X = \det X, \quad <aX, aY> = <X, Y> \}
\]
Realization of automorphisms $\sigma$ of order 3

$$= \{ \alpha \in \text{Iso}_c(3^C, 3^C) \mid \alpha X \times aY = \tau a\tau(X \times Y), <\alpha X, \alpha Y> = <X, Y> \}$$

is a simply connected compact simple Lie group of type $E_6$ [6]. For $\alpha \in F_4$, its complexification $\alpha^C : 3^C \to 3^C$ belongs to $E_6$, so we can regard $F_4$ as a subgroup of $E_6$ under the complexification.

3.1. Automorphism $\gamma_3$ of order 3 and subgroup $(U(1) \times SU(6))/\mathbb{Z}_2$ of $E_6$

We consider $C$-linear transformations $\gamma, \gamma_3$ of $3^C$ which are the complexifications of $\gamma, \gamma_3 \in G_2 \subset F_4$, respectively. Of course $\gamma, \gamma_3 \in E_6$ and $\gamma^2 = 1, \gamma^3 = 1$.

Let $C = \mathbb{R}^C = \{ e^{i\phi} \mid \phi \in \mathbb{R} \}$ and we define an $R$-linear mapping $k : H \to M(2, C)$ by

$$k((e_i + ie_j) + e_k(e_i + e_j)) = \begin{pmatrix} e_i + ie_j & -e_i + ie_j \\ e_k + ie_j & e_k - ie_j \end{pmatrix}, e_i \in C.$$

This $k$ is naturally extended to $R$-linear mappings

$$k : M(3, H) \to M(6, C), \quad k : H^3 \to M(2, 6, C).$$

Moreover these $k$ are extended to $C$-linear isomorphisms $k : M(3, H)^C \to M(6, C), k : (H^3)^C \to M(2, 6, C)$ respectively by

$$k(X_i + iX_j) = k(X_i) + i(k(X_j), \quad X_i \in M(3, H),$$

$$k(a_i + ia_j) = k(a_i) + i(k(a_j), \quad a_i \in H^3.$$

Finally, we define a $C$-vector space $(6, C)$ by

$$(6, C) = \{ S \in M(6, C) \mid S = -S \}$$

and a $C$-linear isomorphism $k : 3(3, H)^C \to (6, C)$ by

$$k_j(X_i + iX_j) = k(X_i)j + i(k(X_j)J, \quad X_i \in 3(3, H)$$

where $J = \begin{pmatrix} J' & 0 & 0 \\ 0 & J' & 0 \\ 0 & 0 & J' \end{pmatrix}, J' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

**Known result 3.1.** [6]. The group $(E_6)^\gamma$ is isomorphic to the group $(Sp(1) \times SU(6))/\mathbb{Z}_2$ by an isomorphism induced from the homomorphism $\psi : Sp(1) \times SU(6) \to (E_6)^\gamma,$

$$\psi(p, A)(X + a) = k^{-1}(Ak(X)^A + pk^{-1}(k(a)A^*), \quad X + a \in 3H \oplus (H^3)^C = 3^C$$

with $\text{Ker} \psi = Z_2 = \{(1, E), (-1, -E)\}.$

**Theorem 3.2.** The group $(E_6)^{\gamma_3}$ is isomorphic to the group $(U(1) \times SU(6))/\mathbb{Z}_2$ where $Z_2 = \{(1, E), (-1, -E)\}.$

**Proof.** Let $U(1) = \{ s \in C \mid |s| = 1 \} \subset Sp(1) \subset H \subset \mathfrak{g}$, where we define a homomorphism $\psi : U(1) \times SU(6) \to (E_6)^{\gamma_3}$ by the restriction of $\psi$ of Known result 3.1. Then $\psi$ induces an isomorphism $(U(1) \times SU(6))/\mathbb{Z}_2 \cong (E_6)^{\gamma_3}$ whose proof is similar to Theorems 1.2, 2.2.
Corollary 3.3. \((E_6)^{n}_S = (E_6)^S\) where \(S = \psi(U(1), 1)\). In particular, the manifold \(E_6/(E_6)^{n}_S\) has a homogeneous complex structure.

3.2. Automorphism \(\gamma_3\) of order 3 and subgroup \((Sp(1) \times S(U(1) \times U(5)))/Z_6\) of \(E_6\).

Let \(v = \exp \frac{-2\pi i}{9} \in \mathbb{C}\) and put \(A_v = \begin{pmatrix} v^5 & v^{-1} \\ v^{-1} & v^5 \end{pmatrix} \in SU(6) \subset M(6, \mathbb{C})\). Put \(\gamma' = \psi(1, A_v)\) where \(\psi\) is the mapping \(\psi : Sp(1) \times SU(6) \rightarrow (E_6)^{\gamma}\) defined in Known result 3.1. Of course \(\gamma' \in E_6\) and \(\gamma'^n = 1\). Since \(A_v^3 = v^3 \in \mathbb{Z}(SU(6))\) (the center of SU(6)) and \(\psi(1, A_v^3) = \omega 1\) (where \(\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} \in \mathbb{C}\) \(\in \mathbb{Z}(E_6)\) (the center of \(E_6\)), \(\gamma'\) induces an automorphism \(\gamma_3\) of \(E_6\) of order 3,

\[ \gamma_3'( \alpha ) = \gamma' \alpha \gamma'^{-1}, \quad \alpha \in E_6. \]

In order to investigate the group \((E_6)^{n}_\gamma\), we consider \(\mathbb{C}\)-eigen vector subspaces \((\mathfrak{H}^C)_\nu, i = 0, 1, \ldots, 8\) of \(\mathfrak{H}^C\) with respect to \(\gamma'\):

\[
(\mathfrak{H}^C)_\nu = \{ X + a \in \mathfrak{H}^C \oplus (\mathfrak{H}^C)^C \mid \gamma'(X + a) = \nu(X + a) \}
= \{0 + (a_1 \epsilon_1, a_2, a_3) \mid a_1 \in H, a_2, a_3 \in H^C \},
\]

\[
(\mathfrak{H}^C)_\nu = \{ X + a \in \mathfrak{H}^C \oplus (\mathfrak{H}^C)^C \mid \gamma'(X + a) = \nu(X + a) \}
= \left\{ \begin{bmatrix} \xi_1 \\ (e_1 + i)a_1 \\ a_2(e_1 - i) \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2(e_1 + i) \\ (e_1 + i)a_1 \end{bmatrix} \right\} \quad \text{for } \gamma'(X + a) = \nu(X + a)
= \begin{bmatrix} 0 \\ (e_1 + i)a_1 \\ a_2(e_1 - i) \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2(e_1 + i) \\ (e_1 - i)a_1 \end{bmatrix} \right\} \quad \text{for } \gamma'(X + a) = \nu(X + a)
= \{ 0 \}, \quad i = 0, 2, 3, 5, 6, 8.
These spaces are invariant under the group \((E_6)^{\gamma}\).

Theorem 3.4. The group \((E_6)^{n}_\gamma\) is isomorphic to the group \((Sp(1) \times S(U(1) \times U(5)))/Z_6\) where \(Z_6 = \{(1, (1, E)), (1, (-1, -E))\}\).

Proof. First we shall show that \((\mathfrak{H}^C)^{n}_\gamma\) is invariant under the group \((E_6)^{n}_\gamma\). From the form of \((\mathfrak{H}^C)_\nu\), it is sufficient to show that we have \(a = \alpha (e_1 + i)\), \(0, 0) = F_1((\alpha (e_1 + i)))\), \(a \in H\). Now, in fact,

\[
\alpha F_1((\alpha (e_1 + i))) = -4\alpha (\alpha F_1((e_1 - i)\alpha)) \times F_1(e)
= -4((\alpha F_1((e_1 - i)\alpha)) \times \alpha F_1(e))
\subset -4((\mathfrak{H}^C) \times (\mathfrak{H}^C)) \subset (\mathfrak{H}^C)^C \subset (\mathfrak{H}^C)^C.
\]
Thus we see that \((H^3)^C\) is invariant under the group \((E_6)^{\gamma'}\), hence \(\exists H^C = (H^3)^C\) is \(\{X \in \mathbb{C} \mid <X, Y> = 0 \text{ for all } Y \in (H^3)^C\}\) is also invariant under \((E_6)^{\gamma'}\). Consequently, \(\alpha \in (E_6)^{\gamma'}\) commutes with \(\gamma : (E_6)^{\gamma'} \subset (E_6)^Y\). Now, we define a homomorphism \(\psi : Sp(1) \times S(U(1) \times U(5)) \to (E_6)^{\gamma'}\) by the restriction of \(\psi\) of Known result 3.1. Clearly \(\psi\) is well-defined. We shall show that \(\psi\) is onto. Let \(\alpha \in (E_6)^{\gamma'}\). Since \((E_6)^{\gamma'} \subset (E_6)^Y\), from Known result 3.1, there exist \(p \in Sp(1), A \in SU(6)\) such that \(\alpha = \psi(p, A)\). From the commutativity \(\gamma ' \alpha = \alpha \gamma '\), that is, \(\psi(p, A, A) = \psi(p, AA')\), we have \(A, A = AA'\). Hence \(A \in S(U(1) \times U(5)) (\cong U(5))\). Thus \(\psi\) is onto. Obviously \(\text{Ker} \psi = \mathbb{Z}_2\). Thus we have the isomorphism \((Sp(1) \times S(U(1) \times U(5)))/\mathbb{Z}_2 \cong (E_6)^{\gamma'}\).

**Corollary 3.5.** \((E_6)^{\gamma'} = (E_6)^S\) where \(S = \{\psi(1, A) \mid A = \left[\begin{array}{cc} a & b \\ \cdot & a \end{array}\right] \in SU(6), a \in U(1)\}\). In particular, the manifold \(E_6/(E_6)^{\gamma'}\) has a homogeneous complex structure.

3.3. Automorphism \(\sigma_5\) of order 3 and subgroup \((U(1) \times U(1) \times Spin(8))/(Z_4 \times Z_2)\) of \(E_6\)

Let \(U(1) = \{\theta \in C \mid |\theta| = 1\}\) and we define an embedding \(\phi : U(1) \to E_6\) by

\[
\phi(\theta) = \begin{pmatrix}
\xi_1 & x_1 \\
x_3 & \xi_2
\end{pmatrix} = \begin{pmatrix}
\theta^4 \xi_1 & \theta x_1 & \theta \xi_2 \\
\theta x_0 & \theta^{-2} x_2 & \theta^{-2} x_1
\end{pmatrix}.
\]

Now, we regard \(\sigma, \sigma_5 \in F_4\) as elements of \(E_6\). Of course \(\sigma^2 = 1, \sigma_5^8 = 1\).

**Known result 3.6** [6]. (1) The group \((E_6)_E = \{\alpha \in E_6 \mid aE_i = E_i\}\) is isomorphic to the group \(Spin(10)\) which is the universal covering group of \(SO(10) = SO(V^{10})\) where \(V^{10} = \{X \in \mathbb{C} \mid 2E_i \times X = -\tau X\}\).

(2) The group \((E_6)^{\sigma}\) is isomorphic to the group \((U(1) \times Spin(10))/Z_4\) by an isomorphism induced from the homomorphism \(\psi : U(1) \times Spin(10) \to (E_6)^{\sigma}\),

\[\psi(\theta, \beta) = \phi(\theta) \beta\]

with \(\text{Ker} \psi = Z_4 = \{(1, \phi(1)), (-1, \phi(-1)), (i, \phi(i)), (-i, \phi(-i))\}\).

**Lemma 3.7.** For \(\alpha \in (E_6)^{\sigma}\), there exists \(\xi \in U(1)\) such that \(aE_i = \xi E_i\).

**Proof** is similar to Lemma 2.5 and see [6, Lemma 9].

We define a subgroup \((E_6)_{E_i, F_i(s)}\) of the group \(E_6\) by

\[
(E_6)_{E_i, F_i(s)} = \{\alpha \in E_6 \mid aE_i = E_i, aF_i(s) = F_i(s) \text{ for all } s \in C\}
\]

\[
= \{\alpha \in Spin(10) \mid aF_i(1) = F_i(1), aF_i(e) = F_i(e)\}.
\]

This group \((E_6)_{E_i, F_i(s)}\) is isomorphic to the group \(Spin(8)\) which is the universal covering group of \(SO(8) = SO(V^8)\) where \(V^8 = \{\xi E_2 - \tau \xi E_3 + F_i(t) \mid \xi \in C, t \in C\}\). Furthermore we use the following notation.

\[
(E_6)^{U(1)} = \{a \in E_6 \mid D_a a = a D_a \text{ for all } a \in U(1)\}.
\]
Lemma 3.8. \(\text{Spin}(8) = (E_8)_{E_8, F_4(8)}\) is a subgroup of \((E_8)^{U(1)}\).

Proof is similar to Lemma 2.6.

Theorem 3.9. The group \((E_8)^{\alpha_8}\) is isomorphic to the group \((U(1) \times U(1) \times \text{Spin}(8)) / (\mathbb{Z}_4 \times \mathbb{Z}_4)\) where \(\mathbb{Z}_4 = \{(1, 1, 1), (i, e, \phi(i)D_e), (-1, -1, 1), (-i, -e, \phi(i)D_e)\}\) and \(\mathbb{Z}_4 = \{(1, 1, 1), (1, -1, \sigma)\}\).

Proof. We define a mapping \(\psi : U(1) \times U(1) \times \text{Spin}(8) \to (E_8)^{\alpha_8}\) by

\[
\psi(\theta, \alpha, \beta) = \phi(\theta)D_\alpha \beta.
\]

Obviously \(\psi\) is well-defined: \(\psi(\theta, \alpha, \beta) \in (E_8)^{\alpha_8}\) (Lemma 3.7). Since \(\phi(\theta)(\theta \in U(1))\), \(D_\alpha \in U(1)\) and \(\beta \in \text{Spin}(8)\) commute with one another (Lemma 3.8), \(\psi\) is a homomorphism. We shall show that \(\psi\) is onto. Let \(\alpha \in (E_8)^{\alpha_8}\). From Lemma 3.7, there exists \(\theta \in U(1)\) such that

\[
a E_1 = \theta^a E_1 = \phi(\theta)E_1.
\]

Put \(\beta = \phi(\theta)^{-1} \alpha\), then \(\beta E_1 = E_1\), that is, \(\beta \in ((E_8)^{\alpha_8})_{E_1} = \{\alpha \in (E_8)^{\alpha_8} | a E_1 = E_1\}\). From Lemma 3.7, we see that the vector space

\[
\{F_i(s) \mid s \in C\} = \{X \in \{(3, s)_{\phi}\}_{\sigma_i} \mid E_1 \times X = 0, \langle E_1, X \rangle = 0, 2E_2 \times X = -\tau X\}
\]

is invariant under the group \((E_8)^{\alpha_8}_{E_1}\). So we can put \(\beta F_i(1) = F_i(s_0), s_0 \in C\). Then we have also \(\beta F_1(e) = F_i(e)\) (cf. Theorem 2.7). Choose \(a_0 \in C\) such that \(e_0^2 = s_0\). Then \(\beta F_i(1) = D_{a_0} F_i(1), \beta F_1(e) = D_{a_0} F_1(e)\). Put \(\delta = D_{a_0}^{-1} \beta\), then \(\delta \in \text{Spin}(8)\). Hence we have

\[
\alpha = \phi(\theta)D_{a_0} \beta, \quad \theta \in U(1), \quad a_0 \in U(1), \quad \delta \in \text{Spin}(8).
\]

Thus \(\psi\) is onto. Finally we shall determine \(\text{Ker}\psi\). Let \(\phi(\theta)D_\alpha \delta = 1, \theta \in U(1), \alpha \in U(1), \delta \in \text{Spin}(8)\). From \(\phi(\theta)D_\alpha \delta E_1 = E_1\), we have \(\theta^a = 1\). Hence \(\theta = \pm 1, \pm i\). In the case of \(\theta = -1\), we have \(\delta E_1 = F_1(1), \delta = -D_{-1} = -\sigma\). So \(\langle 1, 1, 1 \rangle, \langle 1, -1, \sigma \rangle \notin \text{Ker}\psi\). In other cases of \(\theta\), we can similarly determine elements of \(\text{Ker}\psi\). Thus

\[
\text{Ker}\psi = \{(1, 1, 1), (i, e, \phi(i)D_e), (-1, -1, 1), (-i, -e, \phi(-i)D_e)\}
\]

Thus we have the isomorphism \((U(1) \times U(1) \times \text{Spin}(8)) / (\mathbb{Z}_4 \times \mathbb{Z}_4) \cong (E_8)^{\alpha_8}\).

Corollary 3.10. \((E_8)^{\alpha_8} = (E_8)^{S_1}\) where \(S_1 = \psi(1, U(1), 1)\)

\[
= (E_8)^{S_2} \text{ where } S_2 = \psi(U(1), U(1), 1).
\]

In particular, the manifold \(E_8 / (E_8)^{\alpha_8}\) has a homogeneous complex structure.

3.4. Automorphism \(\alpha_8'\) of order 3 and subgroup \((U(1) \times \text{Spin}(10)) / \mathbb{Z}_4\) of \(E_8\)

Let \(\phi : U(1) \to E_8\) be the imbedding defined in Known result 3.6. Now, let \(v = \exp\)
$\frac{2\pi i}{9} \in \mathbb{C}$ and denote $\phi(v)$ by $\sigma'$. Of course $\sigma' \in E_0$ and $\sigma'^3 = 1$. Since $\sigma'^3 = \omega 1 \in z(E_0)$, $\sigma'$ induces an automorphism $\sigma'$ of $E_0$ of order 3,

$$\sigma'(\alpha) = \sigma' \alpha \sigma'^{-1}, \quad \alpha \in E_0.$$

**Theorem 3.11.** The group $(E_0)^{\sigma'}$ coincides with the group $(E_0)^{\sigma}$, so it is isomorphic to the group $(\mathbb{U}(1) \times \text{Spin}(10))/\mathbb{Z}_3$.

**Proof.** Since

$$\sigma'(\xi) = \begin{pmatrix} \xi_1 & x_2 & \bar{x}_2 \\ x_1 & \xi_2 & \bar{x}_3 \\ \bar{x}_3 & \bar{x}_1 & \xi_3 \end{pmatrix},$$

$\mathbb{C}$-vector subspaces $\{\xi E_1 | \xi \in \mathbb{C}\}$, $\{x_2 F_2(x_1) + x_3 F_3(x_1) | x_1 \in \mathbb{C}\}$ and $\{\xi_2 E_2 + \xi_3 E_3 + F_1(x) | x_1 \in \mathbb{C}\}$ of $\mathbb{C}^3$ are invariant under the group $(E_0)^{\sigma'}$. In particular, $\alpha \in (E_0)^{\sigma'}$ commutes with $\sigma : (E_0)^{\sigma'} \subset (E_0)^{\sigma}$. The converse inclusion $(E_0)^{\sigma} \subset (E_0)^{\sigma'}$ is clear because $(E_0)^{\sigma} = \phi(\mathbb{U}(1)) \times \text{Spin}(10)$. Thus we have $(E_0)^{\sigma} = (E_0)^{\sigma} \cong (\mathbb{U}(1) \times \text{Spin}(10))/\mathbb{Z}_3$.

**Corollary 3.12.** $(E_0)^{\sigma'} = (E_0)^{S}$ where $S = \psi(\mathbb{U}(1), 1)$. In particular, the manifold $E_0/ (E_0)^{\sigma'}$ has a homogeneous complex structure.

**3.5. Automorphism $w$ of order 3 and subgroup $(SU(3) \times SU(3) \times SU(3))/\mathbb{Z}_3$ of $E_0$**

Let $\omega = -\frac{1}{2} + \sqrt{3} \in \mathbb{C}$ and we define a $\mathbb{C}$-linear transformation $w$ of $\mathbb{C}^3$ by

$$w(X + M) = X + \omega M, \quad X + M \in \mathbb{C}^3.$$

This $w$ is the same one as $w \in G_2 \subset F_4 \subset E_6$. Of course $w^3 = 1$.

**Theorem 3.5.** The group $(E_0)^w$ is isomorphic to the group $(SU(3) \times SU(3) \times SU(3))/\mathbb{Z}_3$ where $\mathbb{Z}_3 = \{(1, E, E), (\omega 1, \omega E, \omega E), (\omega^2 1, \omega^2 E, \omega^3 E)\}$.

**Proof.** We define a mapping $\psi : SU(3) \times SU(3) \times SU(3) \rightarrow (E_0)^w$ by

$$\psi(P, A, B)(X + M) = h(A, B)Xh(A, B)^* + PMrh(A, B)^*,$$

where $h : M(3, C) \times M(3, C) \rightarrow M(3, C)$ is the mapping defined by $h(A, B) = \frac{A + B}{2} + \frac{1}{2}(A - B)^\omega$. $\psi$ is well-defined: $\psi(P, A, B) \in E_0$ [7] moreover $\psi(E_0)^w$. Obviously $\psi$ is a homomorphism. The proof that $\psi$ is onto is similar to Theorem 2.9. Thus we have the isomorphism $(SU(3) \times SU(3) \times SU(3))/\mathbb{Z}_3 \cong (E_0)^w$.

**References**
