On Rings Satisfying the Polynomial Identity \((x+x^2)^2=0\)

By Masayuki Ohori

Department of Mathematics, Faculty of Science
Shinshu University
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Throughout, \(R\) will represent a ring with center \(C\). Let \(N\) be the set of nilpotents in \(R\), and \(E\) the set of idempotents in \(R\). If \(E\) is contained in \(C\), \(R\) is said to be normal. It is well known that \(R\) is normal if and only if \([E, N]=0\) (resp. \([E, E]=0\)).

Let \(n\) be a positive integer, and consider the following property:
\[(*)_n (x+x^2+\cdots+x^n)^n = (1+x+x^2+\cdots+x^n)^n - 1 = 0 \quad \text{for all } x \in R.\]
If \(2R=0\) then \((*)_2\) becomes \((x+x^2)^2 = -x^2+x^2 = 0\). In [2], \(R\) is called a generalized Boolean-like ring if \(2R=0\) and \((x+x^2)(y+y^2)=0\) for all \(x, y \in R\). According to [2, Theorem 2], a ring with \(2R=0\) is a generalized Boolean-like ring if and only if \(N\) is an ideal with \(N^2=0\) and \(R/N\) is a Boolean ring.

The present objective is to generalize [2, Theorems 3 and 4] as follows:

**Theorem 1.** Suppose that \(2R=0\) and \(R\) satisfies the polynomial identity \((x+x^2)^2=0\). Then the following are equivalent:
1) \(R\) is commutative.
2) \(E\) is an additive subsemigroup of \(R\).
3) \(E\) is a subring of \(R\).
4) \([E, N]=0\).
5) \(N\) is central.
6) Every element of \(R\) can be uniquely written as the sum of an element in \(E\) and an element in \(N\).

In preparation for proving the theorem, we state four lemmas. First, we quote [1, Lemma 3].

**Lemma 1.** If \(R\) satisfies \((*)_{nk}\) and \(2^n R=0\), then \(N\) is an ideal and \(R/N\) is a Boolean ring.

**Corollary 1** (cf. [2, Theorem 2]). Suppose that \(2R=0\). If \(R\) satisfies the polynomial identity \((x+x^2)^2=0\), then \(N\) is a commutative nil ideal of bounded index 2 and \(R/N\) is a Boolean ring (and conversely).

**Lemma 2.** Suppose that \(2^n R=0\). If \(E\) is an additive subsemigroup of \(R\), then \(R\)
is normal and \( E \) is a Boolean ring.

**Proof.** We claim first that \( E \) is a group. In fact, if \( e \in E \) then \( 2e \in E \cap N = 0 \), and therefore \( -e = e \in E \). Moreover, for any \( x \in R \) we have \( e + ex(1-e) \in E \), and therefore \( ex(1-e) \in E \cap N = 0 \), namely \( ex = exe \). Similarly, \( xe = exe \). Hence, \( R \) is normal and \( E \) is a Boolean ring.

**Lemma 3.** Suppose that \( N \) is an ideal and \( R/N \) is a Boolean ring. Then the following are equivalent:
1) \( R \) is commutative.
2) \( R \) is normal and \( N \) is commutative.
3) \([E,N]=0\) and \( N \) is commutative.
4) \( N \) is central.

**Proof.** As is well known, every idempotent of \( R/N \) can be lifted to an idempotent of \( R \). Thus, every element of \( R \) is the sum of an element in \( E \) and an element in \( N \), and the equivalence of 1) - 4) is almost clear.

**Lemma 4.** Suppose that \( N \) is commutative and every element of \( R \) can be uniquely written as the sum of an element in \( E \) and an element in \( N \). Then \( R \) is commutative.

**Proof.** Given \( e \in E \) and \( x \in R \), we have \( e + ex(1-e) \in E \) and \( e + (1-e)xe \in E \). Hence, by the uniqueness, \( ex(1-e) = 0 = (1-e)xe \), namely \( ex = exe = xe \). This proves that \( R \) is normal, and therefore \( R \) is commutative.

We are now ready to complete the proof of our theorem.

**Proof of Theorem 1.** According to Corollary 1, \( N \) is a commutative ideal and \( R/N \) is a Boolean ring. Obviously, 1), 4) and 5) are equivalent by Lemma 3, and 1), 2) and 3) are so by Lemmas 2 and 3. Finally, if \( R \) is commutative, then \( E \) is a Boolean ring and \( R=E \oplus N \) (grouptheoretic direct sum). Hence, 1) and 6) are equivalent by Lemma 4.

**References**
