Non-compact simple Lie group $E_{8(-24)}$ of type $E_8$

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It is known that there exist three simple Lie groups of type $E_8$ up to local isomorphism, one of them is compact and the others are non-compact. We have shown in [7] that the group

$$E_{8} = \{ \alpha \in \text{IsoC}(e_8, e_8) \mid [e_8, e_8] = \{ [\alpha R_1, \alpha R_2], \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle \} \}
$$

(where $e_8$ is a simple Lie algebra over $C$ of type $E_8$ and $\langle R_1, R_2 \rangle$ a positive definite Hermitian inner product in $e_8$) is a simply connected compact simple Lie group of type $E_8$. In this paper, we consider one of the non-compact cases. Our results are as follows. The group

$$E_{8, t} = \{ \alpha \in \text{IsoC}(e_8, e_8) \mid [e_8, e_8] = \{ [\alpha R_1, \alpha R_2], \langle \alpha R_1, \alpha R_2 \rangle_1 = \langle R_1, R_2 \rangle_1 \} \}
$$

(where $\langle R_1, R_2 \rangle_1$ is another inner product in $e_8$) is a connected non-compact simple Lie group of type $E_8$ and its center $z(E_{8, t})$ is trivial:

$$z(E_{8, t}) = \{1\}.
$$

The group $E_{8, t}$ contains, as a subgroup, a special unitary group $SU(2)$ and a simply connected compact simple Lie group $E_7$ of type $E_7$ and the polar decomposition of $E_{8, t}$ is given by

$$E_{8, t} \cong (SU(2) \times E_7)/Z_2 \times R^{412}.
$$

The group $E_{8, t}$ contains also, as a subgroup, a special linear group $SL(2, R)$ and a connected non-compact simple Lie group $E_{7, 1}$ of type $E_{7(-24)}$. In order to show this, we construct another group

$$E_{8, 1} = \{ \alpha \in \text{Iso}(e_8, e_8) \mid [e_8, e_8] = [x, \alpha R_1, \alpha R_2] \}
$$

(where $e_{8, 1}$ is a simple Lie algebra of type $E_{8(-24)}$ and $x$ a submanifold of $e_{8, 1}$)
which is isomorphic to $E_{8,1}$ and find subgroups $SL(2, \mathbb{R})$ and $E_{7,1}$ explicitly in this group $E_{8,1}$

I. Group $E_{8,1}$

1. Preliminaries.

Throughout this paper, we use the same notations as in [7]. However we arrange definitions and some properties of the exceptional Lie algebras $\mathfrak{e}_6^C$, $\mathfrak{e}_7^C$ and $\mathfrak{e}_8^C$.

1.1. Jordan algebra $\mathfrak{J}^C$ [1], [7].

Let $\mathfrak{S}^C$ denote the split Cayley algebra over the field of complex numbers $\mathbb{C}$ and $\mathfrak{J}^C$ the Jordan algebra of all $3 \times 3$ Hermitian matrices with entries in $\mathfrak{S}^C$ with respect to the multiplication $X \circ Y = \frac{1}{2}(XY + YX)$. In $\mathfrak{J}^C$, the inner product $(X, Y)$, the positive definite Hermitian inner product $<X, Y>$, the crossed product $X \times Y$ and the cubic form $(X, Y, Z)$ are defined respectively by

$$(X, Y) = \text{tr}(X \circ Y), \quad <X, Y> = (X, Y),$$
$$X \times Y = \frac{1}{2}(2X \circ Y - \text{tr}(X)Y - \text{tr}(Y)X + (\text{tr}(X) \text{tr}(Y) - (X, Y))E),$$
$$(X, Y, Z) = (X, Y \times Z)$$

where $\overline{X}$ is the complex conjugate of $X$ with respect to the field $\mathbb{C}$ and $E$ the unit matrix.

1.2. Lie algebra $\mathfrak{e}_6^C$ [1], [7].

The exceptional Lie algebra $\mathfrak{e}_6^C$ over $\mathbb{C}$ of type $E_6$ is defined by

$$\mathfrak{e}_6^C = \{ \phi \in \text{Hom}_\mathbb{C}(\mathfrak{g}^C, \mathfrak{g}^C) \mid (\phi, X, X) = 0 \}.$$  

For $A, B \in \mathfrak{J}^C$, we define $A \vee B \in \mathfrak{e}_6^C$ by

$$(A \vee B)X = \frac{1}{2}(B, X)A + \frac{1}{6}(A, B)X - 2B \times (A \times X), \quad X \in \mathfrak{J}^C,$$

then $\{ A \vee B \mid A, B \in \mathfrak{J}^C \}$ generates $\mathfrak{e}_6^C$ additively. In $\mathfrak{e}_6^C$, we define a positive definite Hermitian inner product $<\phi_1, \phi_2>$ by

$$<\phi_1, \phi_2> = \sum_i <\phi_i B_i, A_i>$$

where $\phi = \sum_i A_i \vee B_i$. Finally, for $\phi \in \mathfrak{e}_6^C$, we denote the skew–transposes of $\phi$ by $\phi'$, $\phi'$ with respect to the inner products $(X, Y)$, $<X, Y>$ in $\mathfrak{J}^C$ respectively.
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$\langle \phi X, Y \rangle + \langle X, \phi' Y \rangle = 0, \quad \langle \phi X, Y \rangle + \langle X, \phi Y \rangle = 0,$

then $\phi', \phi \in \mathfrak{e}_6$. 

1.3. Vector space $\mathfrak{g}$ [2], [7].

We define a 56 dimensional vector space $\mathfrak{g}$ by

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h} \oplus \mathfrak{h} \oplus \mathfrak{h} \oplus \mathfrak{h}.$$

In $\mathfrak{g}$, we define a positive definite Hermitian inner product $\langle P, Q \rangle$ and a skew-symmetric inner product $\{ P, Q \}$ respectively by

$$\langle P, Q \rangle = \langle X, Z \rangle + \langle Y, W \rangle + \bar{z} \zeta + \bar{\omega} \omega,$$

$$\{ P, Q \} = \langle X, W \rangle - \langle Z, Y \rangle + \bar{z} \omega - \bar{\omega} \zeta$$

for $P = (X, Y, \xi, \eta), Q = (Z, W, \zeta, \omega) \in \mathfrak{g}$. Finally, for $P = (X, Y, \xi, \eta) \in \mathfrak{g}$, we define $\hat{P} \in \mathfrak{g}$ by

$$\hat{P} = (-X, Y, -\bar{\eta}, \bar{\xi}).$$

1.4. Lie algebra $\mathfrak{e}_7$ [2], [4], [5], [7].

An exceptional Lie algebra $\mathfrak{e}_7$ over $\mathbb{C}$ of type $E_7$ is defined by

$$\mathfrak{e}_7 = \{ \phi(\phi, A, B, \rho) \in \text{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathfrak{g}) \mid \phi \in \mathfrak{g}, A, B \in \mathfrak{h}, \rho \in \mathfrak{h} \},$$

where $\phi(\phi, A, B, \rho)$ is a linear transformation of $\mathfrak{g}$ defined by

$$\begin{pmatrix}
X \\
Y \\
\xi \\
\eta
\end{pmatrix} =
\begin{pmatrix}
\phi - \frac{1}{3} \rho & 2B & 0 & A \\
2A & \phi' + \frac{1}{3} \rho & B & 0 \\
0 & A & \rho & 0 \\
B & 0 & 0 & -\rho
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
\xi \\
\eta
\end{pmatrix}$$

$$= \begin{pmatrix}
\phi X - \frac{1}{3} \rho X + 2B \times Y + \xi A \\
2A \times X + \phi' Y + \frac{1}{3} \rho Y + \xi B \\
(A, Y) + \rho \xi \\
(B, X) - \rho \eta
\end{pmatrix}.$$

The Lie bracket in $\mathfrak{e}_7$ is given by

$$[\phi(\phi_1, A_1, B_1, \rho_1), \phi(\phi_2, A_2, B_2, \rho_2)] = \phi(\phi, A, B, \rho),$$
where

\[
\begin{align*}
\phi &= [\phi_1, \phi_2] + 2A_1 \sqrt{B_2} - 2A_2 \sqrt{B_1}, \\
A &= (\phi_1 + \frac{2}{3} \rho_1)A_1 - (\phi_2 + \frac{2}{3} \rho_2)A_2, \\
B &= (\phi_1' - \frac{2}{3} \rho_1)B_2 - (\phi_2' - \frac{2}{3} \rho_2)B_1, \\
\rho &= (A_1, B_2) - (B_1, A_2).
\end{align*}
\]

For \( P = (X, Y, \xi, \eta) \), \( Q = (Z, W, \zeta, \omega) \in \mathbb{P}^C \), we define \( P \times Q \in \mathfrak{e}_7^C \) by

\[
\begin{align*}
\phi &= -\frac{1}{4}(X \sqrt{W} + Z \sqrt{Y}), \\
A &= -\frac{1}{4}(2Y \times W - \xi Z - \zeta X), \\
B &= \frac{1}{4}(2X \times Z - \eta W - \omega Y), \\
\rho &= \frac{1}{8}(X, W) + (Z, Y) - 3(\xi \omega + \zeta \eta)).
\end{align*}
\]

Then \( \{ P \times Q \mid P, Q \in \mathbb{P}^C \} \) generates \( \mathfrak{e}_7^C \) additively. In \( \mathfrak{e}_7^C \), we define a positive definite Hermitian inner product \( \langle \Phi_1, \Phi_2 \rangle \) by

\[
\langle \Phi_1, \Phi_2 \rangle = 2 \langle \phi_1, \phi_2 \rangle + 4 \langle A_1, A_2 \rangle + 4 \langle B_1, B_2 \rangle + \frac{8}{3} \rho_1 \rho_2,
\]

where \( \Phi_i = \Phi(\phi_i, A_i, B_i, \rho_i) \in \mathfrak{e}_7^C \), \( i = 1, 2 \). Finally, for \( \Phi = \Phi(\phi, A, B, \rho) \in \mathfrak{e}_7^C \), we denote the skew–transpose of \( \Phi \) by \( '\Phi \) with respect to the inner product \( \langle P, Q \rangle \) in \( \mathbb{P}^C : \langle \Phi P, Q \rangle + \langle P, '\Phi Q \rangle = 0 \), then

\[
'\Phi = \Phi(\phi, -B, -A, -\rho).
\]

In particular, \( '\Phi \in \mathfrak{e}_7^C \). And the Lie algebra

\[
\mathfrak{e}_7 = \{ \Phi \in \mathfrak{e}_7^C \mid \Phi = '\Phi \}
\]

is a compact Lie algebra of type \( E_7 \).

1.5. **Lie algebra** \( \mathfrak{e}_8^C \) [2], [7].

An exceptional Lie algebra \( \mathfrak{e}_8^C \) is defined as follows. In a 248 dimensional vector space

\[
\mathfrak{e}_8^C = \mathfrak{e}_7^C \oplus \mathbb{P}^C \oplus \mathbb{P}^C \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C},
\]

we define a Lie bracket \([R_1, R_2]\) by
Non-compact simple Lie group $E_{6(6)}$ of type $E_6$  

$[(\phi_1, P_1, Q_1, r_1, s_1, t_1), (\phi_2, P_2, Q_2, r_2, s_2, t_2)]=([\phi, P, Q, r, s, t])$

where

$$
\begin{align*}
\phi &= [\phi_1, \phi_2] + P_1 \times Q_2 - P_2 \times Q_1, \\
P &= \phi_1 P_2 - \phi_2 P_1 + r_1 P_2 - r_2 P_1 + s_1 Q_2 - s_2 Q_1, \\
Q &= \phi_1 Q_2 - \phi_2 Q_1 - r_1 Q_2 + r_2 Q_1 + t_1 P_2 - t_2 P_1, \\
r &= -\frac{1}{8} \{P_1, Q_2\} + \frac{1}{8} \{P_2, Q_1\} + s t_2 - s t_1, \\
s &= \frac{1}{4} \{P_1, P_2\} + 2 r s_2 - 2 r s_1, \\
t &= \frac{1}{4} \{Q_1, Q_2\} - 2 r t_2 + 2 r t_1.
\end{align*}
$$

Then $\mathfrak{e}_6$ becomes a simple Lie algebra over $C$ of type $E_6$. In $\mathfrak{e}_6$, we use notations

$$
\begin{align*}
(\Phi, 0, 0, 0, 0, 0) &= \Phi, \\
(0, P, 0, 0, 0, 0) &= P, \\
(0, 0, Q, 0, 0, 0) &= Q, \\
(0, 0, 0, 0, 1, 0) &= 1.
\end{align*}
$$

Then the table of the Lie bracket among them is given as follows:

<table>
<thead>
<tr>
<th></th>
<th>$\Phi$</th>
<th>$P$</th>
<th>$Q$</th>
<th>1</th>
<th>$\bar{1}$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi$</td>
<td>$[\Phi_1, \Phi_2]$</td>
<td>$(\Phi_1 P_2)^\text{ad}$</td>
<td>$(\Phi_1 Q_2)^\text{ad}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$P$</td>
<td>$- (\Phi_2 P_1)^\text{ad}$</td>
<td>$\frac{1}{4} {P_1, P_2}^\prime \overline{1}$</td>
<td>$P_1 \times Q_2$</td>
<td>$\overline{1}$</td>
<td>0</td>
<td>$- P_1$</td>
</tr>
<tr>
<td>$Q$</td>
<td>$- (\Phi_2 Q_1)^\text{ad}$</td>
<td>$- P_2 \times Q_1$</td>
<td>$\frac{1}{8} {P_2, Q_1} 1$</td>
<td>$Q_1$</td>
<td>$- Q_1$</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$P_2$</td>
<td>$- Q_2$</td>
<td>0</td>
<td>$2 \overline{1}$</td>
<td>$- 2 \overline{1}$</td>
</tr>
<tr>
<td>$\overline{1}$</td>
<td>0</td>
<td>0</td>
<td>$Q_2$</td>
<td>$- 2 \overline{1}$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$P_2$</td>
<td>0</td>
<td>$2 \overline{1}$</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

For $R = (\Phi, P, Q, r, s, t) \in \mathfrak{e}_6$, we denote the adjoint transformation ad$R$ of $\mathfrak{e}_6$ by $\Theta(\Phi, P, Q, r, s, t)$.
Since \( e_8^C \) is simple, the Lie algebra \( \text{Der}(e_8^C) \) of all derivations of \( e_8^C \) consists of \( \text{ad}R, \quad R \in e_8^C \):

\[
\text{Der}(e_8^C) = \{ \Theta(\Phi, P, Q, r, s, t) \mid \Theta \in e_8^C, \quad P, Q \in \mathbb{C}, \quad r, s, t \in \mathbb{C} \}
\]

and it is also isomorphic to the Lie algebra \( e_8^C \).

In \( e_8^C \), we define a positive definite Hermitian inner product \( \langle R_1, R_2 \rangle \) by

\[
\langle R_1, R_2 \rangle = \langle \Phi_1, \Phi_2 \rangle + \langle P_1, P_2 \rangle + \langle Q_1, Q_2 \rangle + \frac{1}{8} r_1 r_2 + 8 s_1 s_2 + 4 t_1 t_2
\]

where \( R_i = (\Phi_i, P_i, Q_i, r_i, s_i, t_i) \in e_8^C, \quad i = 1, 2 \). Finally, for \( \Theta = \Theta(\Phi, P, Q, r, s, t) \in \text{Der}(e_8^C) \), we denote the skew–transpose of \( \Theta \) with respect to the inner product \( \langle R_1, R_2 \rangle \) by \( ^t\Theta \) with respect to the inner product \( \langle R_1, R_2 \rangle \) by

\[
^t\Theta = \Theta(\Phi, -\Phi, P, -P, Q, -Q, r, -r, s, -s).
\]

2. Group \( E_{8, \epsilon} \).

In \( e_8^C \), we define another inner product \( \langle R_1, R_2 \rangle_{\epsilon} \) by

\[
\langle R_1, R_2 \rangle_{\epsilon} = \langle \Phi_1, \Phi_2 \rangle - \langle P_1, P_2 \rangle - \langle Q_1, Q_2 \rangle + \frac{1}{8} r_1 r_2 + 8 s_1 s_2 + 4 t_1 t_2
\]

where \( R_i = (\Phi_i, P_i, Q_i, r_i, s_i, t_i) \in e_8^C, \quad i = 1, 2 \).

The group \( E_{8, \epsilon} \) is defined to be the group of automorphisms of \( e_8^C \) leaving the inner product \( \langle R_1, R_2 \rangle_{\epsilon} \) invariant:

\[
E_{8, \epsilon} = \{ \alpha \in \text{Isoc}(e_8^C) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2], \quad \langle \alpha R_1, \alpha R_2 \rangle_{\epsilon} = \langle R_1, R_2 \rangle_{\epsilon} \}.
\]

The Lie algebra \( e_{8, \epsilon} \) of the group \( E_{8, \epsilon} \) is

\[
e_{8, \epsilon} = \{ \Theta \in \text{Der}(e_8^C) \mid \langle \Theta R_1, R_2 \rangle_{\epsilon} + \langle R_1, \Theta R_2 \rangle_{\epsilon} = 0 \}.
\]

We define an involutive automorphism \( \epsilon \) of \( e_8^C \) by
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\[
\begin{pmatrix}
1 \\
-1 \\
-1 \\
1 \\
1 \\
1
\end{pmatrix}
\]

Then \( \iota \in E_{8,1} \). And the two inner products \( \langle R_1, R_2 \rangle, \langle R_1, R_2 \rangle_\iota \) in \( \mathfrak{e}_8 \mathbb{C} \) are combined with relations

\[
\begin{align*}
\langle R_1, R_2 \rangle_\iota &= \langle \iota R_1, \iota R_2 \rangle = \langle R_1, \iota R_2 \rangle, \\
\langle R_1, R_2 \rangle &= \langle \iota R_1, \iota R_2 \rangle_\iota = \langle R_1, \iota R_2 \rangle_\iota.
\end{align*}
\]

We can define an automorphism \( \iota \) of \( E_{8,1} \) by

\[
\iota \alpha = \alpha \iota, \quad \alpha \in E_{8,1}.
\]

And for \( \Theta = \Theta(\Phi, P, Q, r, s, t) \in \mathfrak{e}_{8,1} \), we have \( \Theta \iota \in \mathfrak{e}_{8,1} \), more explicitly

\[
\Theta \iota = \Theta(\Phi, -P, -Q, r, s, t).
\]

**Theorem 1.** Any element \( \Theta \) of the Lie algebra \( \mathfrak{e}_{8,1} \) is represented by the form

\[
\Theta = \Theta(\Phi, P, -P, -Q, r, s, -t), \quad \Phi \in \mathfrak{e}_8, \quad P \in \mathfrak{m}_8, \quad r, s \in \mathbb{C}, \quad r + r = 0.
\]

In particular, the type of the group \( E_{8,1} \) is \( E_8 \).

**Proof.** Put \( \Theta = \Theta(\Phi, P, Q, r, s, t) \in \mathfrak{e}_{8,1} \), \( \Phi \in \mathfrak{e}_8 \), \( P, Q \in \mathfrak{m}_8 \), \( r, s, t \in \mathbb{C} \).

From the condition \( \langle \Theta R_1, R_2 \rangle_\iota + \langle R_1, \Theta \iota R_2 \rangle = 0 \), that is,

\[
\langle \Theta R_1, R_2 \rangle_\iota + \langle R_1, \Theta \iota R_2 \rangle = 0,
\]

we have \( \Theta \iota = \Theta \iota \), i.e.,

\[
\Theta(\Phi, -P, -Q, r, s, t) = \Theta(\Phi, -Q, -P, \bar{r}, -r, -t).
\]

hence \( \Phi = \Phi, \quad Q = -Q, \quad r = -r, \quad t = -t \). Therefore we see that the complexification of \( \mathfrak{e}_{8,1} \) is \( \mathfrak{e}_8 \mathbb{C} \), so the Lie algebra \( \mathfrak{e}_{8,1} \) is also of type \( E_8 \).

**3. Subgroups \( E_7 \) and \( SU(2) \) of \( E_{8,1} \).**

We have proved in [4], [6] that the group

\[
E_{7(-133)} = \{ \beta \in \text{Isoc}(\mathfrak{m}_8, \mathfrak{m}_8) \mid \beta(P \times Q)\beta^{-1} = \beta P \times \beta Q, \quad \langle \beta P, \beta Q \rangle = \langle P, Q \rangle \}
\]
is a simply connected compact simple Lie group of type $E_7$. Now, we shall show that the group $E_{8,1}$ contains compact subgroups of type $E_7$ and $A_2$.

**Theorem 2.** The group $E_{8,1}$ contains a subgroup

$$E_7 = \{ \alpha \in E_{8,1} \mid \alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1 \}$$

which is a simply connected compact simple Lie group of type $E_7$.

**Proof.** The mapping

$$E_{7(-133)} \ni \beta \mapsto \beta = \begin{pmatrix} \text{Ad}\beta & \\ \beta & 1 \end{pmatrix} \in E_7 \subset E_{8,1},$$

(\text{where } \text{Ad} \beta : e_7 \rightarrow e_7 \text{ is defined by } (\text{Ad} \beta)\phi = \beta \phi \beta^{-1})

gives an isomorphism between $E_{7(-133)}$ and $E_7$. The analogy of this proof is in [7] Theorem 25, so we omit here. (This Theorem follows also from the following Theorem 4).

**Theorem 3.** The group $E_{8,1}$ contains a subgroup

$$SU(2) = \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a_{1} & -b_{1} & 0 & 0 \\ 0 & b_{1} & a_{1} & 0 & 0 \\ 0 & 0 & 0 & |a|^2 - |b|^2 & -ab - \overline{ab} \\ 0 & 0 & 0 & 2ab & a^2 - b^2 \end{pmatrix} \in SU(2) \right\}$$

which is isomorphic to the special unitary group $SU(2) = \{ A \in M(2, \mathbb{C}) \mid A^*A = E, \det A = 1 \}$.

**Proof.** It is easy to verify that $SU(2)$ is a subgroup of $E_{8,1}$ (or see the following Theorem 4).

In the followings, we identify these groups $E_{7(-133)}$ with $E_7$, $SU(2)$ with $SU(2)$ under the above correspondences.

4. Involution automorphism $\iota$ and subgroup $(SU(2) \times E_7)/Z_2$ of $E_{8,1}$.

**Theorem 4.** The subgroup $\{ \alpha \in E_{8,1} \mid \alpha \iota = \alpha \}$ of the group $E_{8,1}$ is isomorphic to the group $(SU(2) \times E_7)/Z_2$, where $Z_2 = \{(E, 1), (-E, \iota)\}$.

**Proof.** We define a mapping $\phi : SU(2) \times E_7 \rightarrow \{ \alpha \in E_{8,1} \mid \alpha \iota = \alpha \}$ by
Since $A \in SU(2)$ and $\beta \in E_7$ commute in $E_{8,7} : A\beta = \beta A$, obviously $\phi$ is a homomorphism. We shall prove that $\psi$ is onto. If $a \in E_{8,7}$ satisfies $\alpha\alpha = \alpha$, then $\alpha$ has the form

$$\alpha = \begin{pmatrix}
\beta_1 & 0 & 0 & \Psi_1 & \Psi_2 & \Psi_3 \\
0 & \beta_2 & \beta_{23} & 0 & 0 & 0 \\
0 & \beta_{23} & \beta_3 & 0 & 0 & 0 \\
l_1 & 0 & 0 & r_1 & r_2 & r_3 \\
l_2 & 0 & 0 & s_1 & s_2 & s_3 \\
l_3 & 0 & 0 & t_1 & t_2 & t_3
\end{pmatrix}$$

where $\beta_1 : \mathbb{C} \rightarrow \mathbb{C}$, $\beta_2, \beta_3, \beta_{23} : \mathbb{C} \rightarrow \mathbb{C}$, $l_i : \mathbb{C} \rightarrow \mathbb{C}$ are linear mappings, $\Psi_i \in \mathbb{C}$ and $r_i, s_i, t_i \in \mathbb{C}$, $i = 1, 2, 3$.

1. $[1, \overline{1}] = 2\overline{1}$ implies $[\alpha I, \alpha \overline{1}] = 2\overline{1}$, that is,

$$[[\Psi_1, 0, 0, r_1, s_1, t_1], (\Psi_2, 0, 0, r_2, s_2, t_3)] = [[\Psi_1, \Psi_2], 0, 0, s_1t_2-s_2t_1, 2r_1s_3-2r_3s_1, -2r_1t_2+2r_2t_1] = 2(\Psi_2, 0, 0, r_2, s_2, t_2).$$

Hence we have

1. $[\Psi_1, \Psi_2] = 2\Psi_2$,  
2. $s_1t_2-s_2t_1 = 2r_2$, 
3. $r_1s_2-r_2s_1 = s_2$,  
4. $-r_1t_2+r_2t_1 = t_2$.

Similarly, from $[1, \overline{1}] = -2\overline{1}$, $[\overline{1}, \overline{1}] = 1$, we have

1. $[\Psi_1, \Psi_3] = -2\Psi_3$,  
2. $s_1t_3-s_3t_1 = -2r_3$, 
3. $r_1s_3-r_3s_1 = -s_3$,  
4. $-r_1t_3+r_3t_1 = -t_3$, 
5. $[\Psi_2, \Psi_3] = \Psi_1$,  
6. $s_2t_3-s_3t_2 = r_1$, 
7. $r_2s_3-r_3s_2 = s_1$,  
8. $-r_2t_3+r_3t_2 = t_3$. 
9. $r_1s_3-r_3s_1 = s_2$,  
10. $-r_1t_3+r_3t_1 = -t_3$, 
11. $2r_2s_3-2r_3s_2 = s_1$,  
12. $-2r_2t_3+2r_3t_2 = t_0$. 

\[ \phi(A, \beta) = A\beta = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & a_1 & -b_1 & 0 & 0 & 0 \\
0 & b_1 & a_1 & 0 & 0 & 0 \\
0 & 0 & 0 & |a|^2-|b|^2 & -ab & -\overline{ab} \\
0 & 0 & 0 & 2\overline{ab} & a^2 & -\overline{b^2} \\
0 & 0 & 0 & -\overline{b^2} & a^2 & -b^2
\end{pmatrix} \begin{pmatrix}
\text{Ad} \beta & 0 & 0 & 0 & 0 \\
0 & \beta & 0 & 0 & 0 \\
0 & 0 & \beta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \beta_1 & 0 & 0 & 0 \\
0 & 0 & \beta_2 & 0 & 0 \\
0 & 0 & 0 & \beta_3 & 0 \\
l_1 & 0 & 0 & r_1 & r_2 \\
l_2 & 0 & 0 & s_1 & s_2 \\
l_3 & 0 & 0 & t_1 & t_2
\end{pmatrix}. \]
\([\mathcal{F}, 1]=0\) implies \([\alpha \mathcal{F}, \alpha 1]=0\), that is,
\[
\left(\beta_i \mathcal{F}, 0, 0, l_i \mathcal{F}, l_2 \mathcal{F}, l_3 \mathcal{F}\right), (\mathcal{W}_i, 0, 0, r_i, s_i, t_i)
\]
\[=\left(\beta_i \mathcal{F}, \mathcal{W}_i\right), 0, 0, -s_i l_i \mathcal{F} + t_i l_2 \mathcal{F}, -2 r_i l_3 \mathcal{F} + 2 s_i l_i \mathcal{F}, 2 r_i l_3 \mathcal{F} - 2 t_i l_i \mathcal{F}\] = 0.

Hence we have
\[
(13) \ [\beta_i \mathcal{F}, \mathcal{W}_i]=0,
(14) \ s_i l_3 = t_i l_2,
(15) \ r_i l_2 = s_i l_1,
(16) \ r_i l_3 = t_i l_1.
\]

Similarly, from \([\mathcal{F}, \mathcal{W}]=0\), \([\mathcal{F}, \mathcal{W}]=0\), we have
\[
(17) \ [\beta_i \mathcal{F}, \mathcal{W}_i]=0,
(18) \ s_i l_3 = t_i l_2,
(19) \ r_i l_2 = s_i l_1,
(20) \ r_i l_3 = t_i l_1,
(21) \ [\beta_i \mathcal{F}, \mathcal{W}_i]=0,
(22) \ s_i l_3 = t_i l_2,
(23) \ r_i l_2 = s_i l_1,
(24) \ r_i l_3 = t_i l_1.
\]

And \([\alpha \mathcal{F}_1, \mathcal{F}_2]=\alpha \mathcal{F}_1, \alpha \mathcal{F}_2]\) implies
\[
(25) \ \beta_i \left[\alpha \mathcal{F}_1, \mathcal{F}_2\right]=\alpha \left[\beta_i \mathcal{F}_1, \mathcal{F}_2\right].
\]

We shall prove that \(\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = 0\) and \(l_1 = l_2 = l_3 = 0\).

Case (i) : \[
\begin{bmatrix}
r_1 & r_2 & r_3 \\
s_1 & s_2 & s_3 \\
t_1 & t_2 & t_3
\end{bmatrix}
\]
is not zero. For example, assume \(r_1 \neq 0\). First we show that \(\beta_i\) is non-degenerate. Suppose \(\beta_i\) is degenerate, then there exists \(0 \neq \Phi_0 \in \mathfrak{c}_i \mathbb{C}\) such that \(\beta_i \Phi_0 = 0\). From \(<\alpha \Phi_0, \alpha 1>=<\Phi_0, 1>=0\), we have
\[
<\beta_i \Phi_0, \mathcal{W}_i> + 8 \overline{\Phi}_0 r_1 + 4 \overline{\Phi}_0 s_2 + 4 \overline{\Phi}_0 t_3 = 0.
\]

Since \(l_2 = s_i l_1, l_3 = t_i l_1\) from (15), (16), we have
\[
\overline{\mathcal{W}}_0 (8 |r_1|^2 + 4 |s_1|^2 + 4 |t_1|^2) = 0.
\]

Therefore \(l_1 \Phi_0 = 0\), and hence \(l_2 \Phi_0 = l_3 \Phi_0 = 0\). Therefore \(\alpha \Phi_0 = 0\) for \(\Phi_0 \neq 0\). This contradicts to the non-degeneracy of \(\alpha\). Thus we see that \(\beta_i\) is non-degenerate, so \(\beta_i \mathfrak{c}_i \mathbb{C} = \mathfrak{c}_i \mathbb{C}\). Hence (15) shows that \(\mathcal{W}_i\) is a central element of \(\beta_i \mathfrak{c}_i \mathbb{C} = \mathfrak{c}_i \mathbb{C}\). Since the Lie algebra \(\mathfrak{c}_i \mathbb{C}\) is simple, we have
\[
\mathcal{W}_1 = 0, \quad \text{and hence} \quad \mathcal{W}_2 = \mathcal{W}_3 = 0
\]
from (1), (5). Again using \(<\alpha \mathcal{F}, \alpha 1>=<\Phi, 1>=0\), that is, \(\overline{\mathcal{W}} (8 |r_1|^2 + 4 |s_1|^2 + 4 |t_1|^2)\)
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= 0, we have $L_\Phi = 0$ for all $\Phi \in \mathfrak{e}_C$. Hence

$$L_1 = 0,$$

and hence

$$L_2 = L_3 = 0$$

from (15), (16).

case (ii). $r_i = s_i = t_i = 0$, $i = 1, 2, 3$ (which doesn't occur). In this case, $\Psi_1 \neq 0$, $\Psi_2 \neq 0$, $\Psi_3 \neq 0$ from the non-degeneracy of $\alpha$. 133 = dim $\mathfrak{e}_C$ = dim $(\beta_1 \mathfrak{e}_C + \mathfrak{e}_1 + \mathfrak{e}_2 + \mathfrak{e}_3)$ implies dim $\beta_1 \mathfrak{e}_C \geq 130$, and from $\langle \beta_i \Phi, \Psi_j \rangle = 0$, $i = 1, 2, 3$, $\langle \Psi_i, \Psi_j \rangle = 0$, $i \neq j$, we see that dim $\beta_1 \mathfrak{e}_C$ is just 130, so

$$\mathfrak{e}_C = \beta_1 \mathfrak{e}_C \oplus \mathfrak{w}_1 \oplus \mathfrak{w}_2 \oplus \mathfrak{w}_3.$$

However (13), (17), (21), (25) show that $\beta_1 \mathfrak{e}_C$ is an ideal of $\mathfrak{e}_C$. So $\beta_1 \mathfrak{e}_C = \mathfrak{e}_C$ from the simplicity of the Lie algebra $\mathfrak{e}_C$. This contradicts to dim $\mathfrak{e}_C$ = dim $\beta_1 \mathfrak{e}_C = 130 < 133 = \dim \mathfrak{e}_C$.

Thus $\alpha$ has the form

$$\alpha = \begin{pmatrix}
\beta_1 & 0 & 0 & 0 & 0 & 0 \\
0 & \beta_2 & \beta_{23} & 0 & 0 & 0 \\
0 & 0 & \beta_{32} & \beta_3 & 0 & 0 \\
0 & 0 & 0 & r_1 & r_2 & r_3 \\
0 & 0 & 0 & s_1 & s_2 & s_3 \\
0 & 0 & 0 & t_1 & t_2 & t_3
\end{pmatrix}$$

II. $[P, 1] = -P$ implies $[\alpha P, \alpha 1] = -\alpha P$, that is,

$$[(0, \beta_2 P, \beta_{23} P, 0, 0, 0), (0, 0, 0, r_1, s_1, t_1)]$$

$$= (0, -r_1 \beta_2 P - s_1 \beta_{23} P, r_1 \beta_{32} P - t_1 \beta_3 P, 0, 0, 0)$$

$$= (0, \beta_2 P, \beta_{23} P, 0, 0, 0).$$

Hence we have

$$1 - r_1 \beta_2 = s_1 \beta_{23}, \quad 1 + r_1 \beta_3 = t_1 \beta_2.$$

Similarly, from $[P, 1] = 0$, $[P, 1] = -P$, we have

$$r_2 \beta_2 = -s_2 \beta_{23}, \quad r_2 \beta_3 = t_2 \beta_2,$$

$$r_3 \beta_2 + s_3 \beta_{23} = \beta_{23}, \quad r_3 \beta_3 = t_3 \beta_2 = \beta_3.$$

And from $[Q, 1] = Q$, $[Q, 1] = -\bar{Q}$, $[Q, 1] = 0$, we have
(32) \((1+r_1)\beta_{23} = -s_1\beta_2,\) \hspace{1cm} (33) \((1-r_1)\beta_3 = -t_1\beta_{23},\)

(34) \(r_2\beta_{23} + s_2\beta_2 = \beta_2,\) \hspace{1cm} (35) \(r_3\beta_3 - t_2\beta_{23} = -\beta_{32},\)

(36) \(r_3\beta_{23} = -s_2\beta_2,\) \hspace{1cm} (37) \(r_3\beta_3 = t_3\beta_{23}.\)

We shall prove that there exist \(a, b, c, d \in \mathbb{C}\) and \(\beta, \gamma \in \text{Isoc}(\varepsilon_7\mathbb{C}, \varepsilon_7\mathbb{C})\) such that

\[
\begin{cases}
\beta_2 = a\beta, & \beta_{23} = c\gamma, \\
\beta_{32} = b\beta, & \beta_3 = d\gamma,
\end{cases}
\quad
\begin{cases}
r_2 = -ab, & r_3 = cd, \\
s_2 = a^2, & t_3 = d^2.
\end{cases}
\tag{38}
\]

Case (i) : \(s_2 \neq 0, s_2 \neq 0\) implies \(t_3 \neq 0\). In fact, suppose \(t_3 = 0\). Then we have \(s_2t_1 = 2r_3, r_3t_1 = 0\) from (6), (8), hence \(r_3 = 0\). So \(s_2 \neq 0\) (because \(\alpha\) is non-degenerate) and hence \(t_1 = 0\). Hence \(r_1 = -1\) from (7). From \(<\alpha_1, \alpha_1> = <1, 1> = 8\), that is, \(8 + 4|s_1|^2 = 8\), hence \(s_1 = 0\). And \(r_2 = 0\) from (2) and finally \(s_2 = 0\). This contradicts to the hypothesis \(s_2 \neq 0\). Now, choose \(a, d \in \mathbb{C}\) such that

\[
a^2 = s_2, \quad d^2 = t_3
\]

and put

\[
b = -\frac{r_2}{a}, \quad c = \frac{r_3}{d},
\]

\[
\beta = \frac{1}{a}\beta_2, \quad \gamma = \frac{1}{d}\beta_3.
\]

Then \(
\beta_{32} = \frac{r_2}{s_2} \beta_2 = b\beta\)
from (28) and \(
\beta_{23} = \frac{r_3}{t_3} \beta_3 = c\gamma\)
from (37). Obviously \(\beta, \gamma \in \text{Isoc}(\varepsilon_7\mathbb{C}, \varepsilon_7\mathbb{C})\), because \(\alpha\) is non-degenerate.

Case (ii) : \(s_2 = 0, s_2 \neq 0\) implies \(t_2 = 0\) and \(r_2 = r_3 = 0, t_2 \neq 0, s_3 \neq 0\) from the same arguments as Case (i). Hence \(\beta_2 = \beta_3 = 0\) from (29), (36). Now, choose \(b, c \in \mathbb{C}\) such that

\[
-b^2 = t_2, \quad -c^2 = s_3
\]

and put

\[
a = 0, \quad d = 0,
\]

\[
\beta = \frac{1}{b}\beta_{32}, \quad \gamma = \frac{1}{c}\beta_{23}.
\]

Then (38) is also valid in this case.

III. \([P, Q] = \frac{1}{4} [P, Q] \bar{\alpha}\) implies \([aP, aQ] = \frac{1}{4} [P, Q] a \bar{\alpha}\), that is,
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\[ [(0, \beta_2 P, \beta_3 P, 0, 0, 0), (0, \beta_2 Q, \beta_3 Q, 0, 0, 0)] \]
\[ = (\beta_2 P \times \beta_3 Q - \beta_3 P \times \beta_2 Q, 0, 0, -\frac{1}{8} \{ \beta_3 P, \beta_2 Q \} - \frac{1}{8} \{ \beta_2 P, \beta_3 Q \}, \]
\[ \frac{1}{4} \{ \beta_2 P, \beta_2 Q \} - \frac{1}{4} \{ \beta_3 P, \beta_3 Q \}) \]
\[ = (0, 0, 0, 0, \frac{1}{4} \{ P, Q \} r_2, \frac{1}{4} \{ P, Q \} s_2, \frac{1}{4} \{ P, Q \} t_2) \]

Hence we have

\[ \beta_2 P \times \beta_3 Q = \beta_3 P \times \beta_2 Q, \]
\[ \{ \beta_2 P, \beta_2 Q \} + \{ \beta_3 P, \beta_3 Q \} = -2r_2 \{ P, Q \}, \]
\[ \{ \beta_2 P, \beta_3 Q \} = s_2 \{ P, Q \}, \]
\[ \{ \beta_3 P, \beta_3 Q \} = -t_2 \{ P, Q \}. \]

Similarly, from $[P, Q] = P \times Q - \frac{1}{8} \{ P, Q \} 1$, $[P, Q] = -\frac{1}{4} \{ P, Q \} 1$, we have

\[ \beta_1 (P \times Q) = \beta_2 P \times \beta_3 Q - \beta_3 P \times \beta_2 Q, \]
\[ \{ \beta_2 P, \beta_2 Q \} + \{ \beta_3 P, \beta_3 Q \} = r_1 \{ P, Q \}, \]
\[ 2\{ \beta_2 P, \beta_3 Q \} = -s_1 \{ P, Q \}, \]
\[ 2\{ \beta_3 P, \beta_3 Q \} = t_1 \{ P, Q \}, \]
\[ \beta_3 P \times \beta_3 Q = \beta_3 P \times \beta_3 Q, \]
\[ \{ \beta_3 P, \beta_3 Q \} + \{ \beta_3 P, \beta_3 Q \} = 2r_3 \{ P, Q \}, \]
\[ \{ \beta_2 P, \beta_3 Q \} = -s_3 \{ P, Q \}, \]
\[ \{ \beta_3 P, \beta_3 Q \} = t_3 \{ P, Q \}. \]

From either one of (41), (42) and either one of (49), (50), we have

\[ \{ \beta P, \beta Q \} = \{ P, Q \}, \]
\[ \{ \gamma P, \gamma Q \} = \{ P, Q \}. \]

Since there exists $\lambda \in C$ such that $\gamma = \lambda \beta$ from (31), so $\lambda^2 = 1$ from (51). If $\lambda = -1$, then by considering $-b$ instead of $b$, we may assume that

\[ \beta = \gamma. \]

Now, from (44), (45), (46), (49), we have

\[ r_1 = ad + bc, \quad \{ r_2 = -ab \}, \quad \{ r_3 = cd \}, \]
\[ s_1 = -2ac, \quad \{ s_2 = a^2 \}, \quad s_3 = -c^2, \]
\[ t_1 = 2bd, \quad \{ t_2 = -b^2 \}, \quad (t_3 = d^2). \]

IV. $[\Phi, \overline{P}] = (\Phi P)^\alpha$ implies $[\alpha \Phi, \alpha \overline{P}] = \alpha (\alpha P)^\alpha$, that is,
\[
\begin{align*}
[(\beta_1, \phi, 0, 0, 0, 0, 0, 0, 0, 0, 0), (0, \beta_1 P, \beta_2 P, 0, 0, 0, 0, 0, 0, 0, 0)] \\
= (0, (\beta_1, \phi, \beta_1 P, \beta_1, \phi, \beta_1 P), 0, 0, 0) \\
= (0, \beta_1 (\phi P), \beta_2 (\phi P), 0, 0, 0).
\end{align*}
\]

Hence we have
\[
\begin{align*}
(53) & \quad \beta_1 \phi \beta_2 = \beta_2 \phi, \\
(54) & \quad \beta_1 \phi \beta_2 = \beta_2 \phi.
\end{align*}
\]

Similarly, from \([\phi, Q] = (\phi Q)_-\), we have
\[
\begin{align*}
(55) & \quad \beta_1 \phi \beta_3 = \beta_3 \phi, \\
(56) & \quad \beta_1 \phi \beta_3 = \beta_3 \phi.
\end{align*}
\]

Now, from either one of (53), (54), we have \(\beta_1 \phi = \beta \phi \beta^{-1}\), in particular,
\[
\beta_1 (P \times Q) = \beta (P \times Q) \beta^{-1}.\tag{57}
\]

From (43) we have
\[
\beta_1 (P \times Q) = (ad - bc) \beta P \times \beta Q.\tag{58}
\]

Since \(ab - bc \neq 0\), choose \(p \in C\) such that \(p^2 = ad - bc\) and rewrite again
\[
\frac{1}{p} \beta \rightarrow \beta, \quad pa \rightarrow a, \quad pb \rightarrow b, \quad pc \rightarrow c, \quad pd \rightarrow d.
\]

Then, with respect to these new \(\beta, a, b, c, d\), the above statements (especially (38)) are also valid and from (57), (58) we have
\[
\beta (P \times Q) \beta^{-1} = \beta P \times \beta Q.\tag{59}
\]

Finally, we have
\[
\begin{align*}
\left\{ \begin{array}{l}
|a|^2 + |b|^2 = 1 \quad \text{from } \langle a \tilde{I}, a \tilde{I} \rangle = \langle \tilde{1}, \tilde{1} \rangle, \\
|c|^2 + |d|^2 = 1 \quad \text{from } \langle a \tilde{I}, a \tilde{I} \rangle = \langle \tilde{1}, \tilde{1} \rangle, \\
ac + bd = 0 \quad \text{from } \langle a \tilde{I}, a \tilde{I} \rangle = \langle \tilde{1}, \tilde{1} \rangle = 0, \\
ad - bc = 1.
\end{array} \right.
\end{align*}
\]

So \[
\begin{bmatrix}
    a & c \\
    b & d
\end{bmatrix} = \begin{bmatrix}
    a & -b \\
    b & a
\end{bmatrix} \in SU(2). \]

And from \(\langle a \tilde{P}, a \tilde{Q} \rangle = \langle \tilde{P}, \tilde{Q} \rangle\), that is, \(\langle \beta_2 P, \beta_2 Q \rangle + \langle \beta_2 P, \beta_2 Q \rangle = \langle P, Q \rangle\), i.e., \(|a|^2 + |b|^2 \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\), hence we have
\[
\langle \beta P, \beta Q \rangle = \langle P, Q \rangle.\tag{60}
\]

So \(\beta \in E\) from (59), (60) and \(\beta_1 = Ad \beta\) from (57). Thus
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$$\begin{pmatrix}
\text{Ad} \beta & 0 & 0 & 0 & 0 \\
0 & a \beta & -b \beta & 0 & 0 \\
0 & b \beta & a \beta & |a|^2 - |b|^2 & -ab & ab \\
0 & 0 & 0 & 2ab & a^2 & -b^2 \\
0 & 0 & 0 & 2ab & -b^2 & a^2
\end{pmatrix}
$$

\[= \phi\left(\begin{pmatrix} a & -\overline{b} \\ \overline{b} & a \end{pmatrix}, \beta\right) \in \phi(SU(2) \times E_7).\]

Hence $\phi$ is onto. It is easy to verify that $\ker\phi = \{(E, 1), (-E, 1)\}$. Thus the proof of Theorem 4 is completed.

5. Polar decomposition of $E_{6,t}$.

In order to give a polar decomposition of the group $E_{6,t}$, we use the following

**Lemma 5** ([3] p. 345). Let $G$ be a pseudoalgebraic subgroup of the general linear group $GL(n, \mathbb{C})$ such that the condition $A \in G$ implies $A^* \in G$. Then $G$ is homeomorphic to the topological product of the group $G \cap U(n)$ and a Euclidean space $\mathbb{R}^d$

$$G \cong (G \cap U(n)) \times \mathbb{R}^d$$

where $U(n)$ is the unitary subgroup of the general linear group $GL(n, \mathbb{C})$.

**Lemma 6.** $E_{6,t}$ is a pseudoalgebraic subgroup of the general linear group $GL(248, \mathbb{C}) = \text{Isoc}(e_6^C, e_6^C)$, and satisfies the condition $a \in E_{6,t}$ implies $a^* \in E_{6,t}$, where $a^*$ is the transpose of $a$ with respect to the inner product $\langle R_1, R_2 \rangle = \langle aR_1, aR_2 \rangle = \langle R_1, a^*R_2 \rangle = \langle a^{-1}R_1, R_2 \rangle = \langle a^{-1}tR_1, R_2 \rangle$ for $a \in E_{6,t}$, we have

$$a^* = a^{-1}t \in E_{6,t}.$$

And it is obvious that $E_{6,t}$ is pseudoalgebraic, because $E_{6,t}$ is defined by pseudoalgebraic relations $a[R_1, R_2] = [aR_1, aR_2]$ and $\langle aR_1, aR_2 \rangle = \langle R_1, R_2 \rangle$.

Let $U(248) = U(e_6^C) = \{ a \in \text{Isoc}(e_6^C, e_6^C) \mid \langle aR_1, aR_2 \rangle = \langle R_1, R_2 \rangle \}$ denote the unitary subgroup of the general linear group $GL(248, \mathbb{C}) = \text{Isoc}(e_6^C, e_6^C)$, then we have

$$E_{6,t} \cap U(e_6^C) = \{ a \in E_{6,t} \mid a\alpha = \alpha \} \cong (SU(2) \times E_7)/\mathbb{Z}_2 \quad \text{(Theorem 4)}$$

Since $E_{6,t}$ is a simple Lie group of type $E_6$, the dimension of $E_{6,t}$ is 248. Hence
the dimension $d$ of the Euclidean part of $E_{6,1}$ is
\[ d = \dim E_{6,1} - \dim (SU(2) \times E_7) = 248 - (3 + 133) = 112. \]

Thus we get the following

**Theorem 7.** The group $E_{6,1}$ is homeomorphic to the topological product of the group $(SU(2) \times E_7)/\mathbb{Z}_2$ and a 112 dimensional Euclidean space $\mathbb{R}^{112}$:
\[ E_{6,1} \cong (SU(2) \times E_7)/\mathbb{Z}_2 \times \mathbb{R}^{112}. \]

In particular, the group $E_{6,1}$ is a connected non-compact simple Lie group of type $E_6(-26)$.

6. **Center $z(E_{6,1})$ of $E_{6,1}$.**

**Theorem 8.** The center $z(E_{6,1})$ of the group $E_{6,1}$ is trivial: $z(E_{6,1}) = \{1\}$.

**Proof.** Let $a \in z(E_{6,1})$. From the commutativity with $e \in E_{6,1}$, $a$ has the form
\[ a = A\beta, \quad A \in SU(2), \quad \beta \in E_7 \]
from Theorem 4. Furthermore, from the commutativity with all $A \in SU(2)$, we see $A \in z(SU(2)) = \{E, -E\}$. Similarly we see $\beta \in z(E_7) = \{1, \iota\}$ [4]. Hence $\alpha = 1$ or $\iota$. However $\iota \notin z(E_{6,1})$ from Theorem 4. Thus $z(E_{6,1}) = \{1\}$.

II. **Group $E_{6,1}$**

In order to investigate the group $E_{6,1}$ more detail, we shall construct one more group $E_{6,1}$ which is isomorphic to $E_{6,1}$.

7. **Preliminaries.**

We consider the real restriction of the preceding chapter. The statements are similar to the complex cases. In the real case, the inner products $<,>$ will be denoted by $( , )$.

7.1. **Jordan algebra $\mathfrak{J}$** [1].

Let $\mathfrak{C}$ denote the non-split Cayley algebra over the field of real numbers $\mathbb{R}$ and $\mathfrak{J} = \mathfrak{J}(3, \mathfrak{C})$ the Jordan algebra consisting of all $3 \times 3$ Hermitian matrices with entries in $\mathfrak{C}$ with respect to the multiplication $X \circ Y = \frac{1}{2}(XY + YX)$.

7.2. **Lie algebra $\mathfrak{e}_{6,1}$** [1].

The Lie algebra $\mathfrak{e}_{6,1}$ is defined by
\[ \mathfrak{e}_{6,1} = \{ \phi \in \text{Hom}_\mathbb{R}(\mathfrak{J}, \mathfrak{J}) \mid \langle \phi X, X, X \rangle = 0 \}. \]

Then $\mathfrak{e}_{6,1}$ is a simple Lie algebra of type $E_6(-26)$. This $\mathfrak{e}_{6,1}$ is the Lie algebra of a Lie group.
Non-compact simple Lie group $E_{6(-26)}$ of type $E_6$.

$$E_{6,1} = \{ \alpha \in \text{Iso}_R(\mathbb{S}, \mathbb{S}) \mid \det \alpha X = \det X \}$$

which is a simply connected non-compact simple Lie group of type $E_{6(-26)}$.

7.3. Lie algebra $\mathfrak{e}_{7,1}$ [2], [5], [6].

We define a vector space $\mathfrak{g}$ by

$$\mathfrak{g} = \mathbb{S} \oplus \mathbb{S} \oplus R \oplus R.$$ 

And the Lie algebra $\mathfrak{e}_{7,1}$ is defined by

$$\mathfrak{e}_{7,1} = \{ \Phi \in \text{Hom}_R(\mathfrak{g}, \mathfrak{g}) \mid \Phi = \Phi(A, B, \rho), \Phi \in \mathfrak{e}_{6,1}, A, B \in \mathfrak{g}, \rho \in R \}$$

as in I. 1. 2. Then $\mathfrak{e}_{7,1}$ is a simple Lie algebra of type $E_{7(-2s)}$. This $\mathfrak{e}_{7,1}$ is the Lie algebra of a Lie group

$$E_{7(-2s)} = \{ \alpha \in \text{Iso}_R(\mathfrak{g}, \mathfrak{g}) \mid \alpha(\mathfrak{g} \times \mathfrak{g}) = \alpha \mathfrak{g} \times \mathfrak{g}, \{ \alpha \mathfrak{g}, \alpha \mathfrak{g} \} = \{ \mathfrak{g}, \mathfrak{g} \} \}$$

(where $\mathfrak{g} = \{ P \in \mathfrak{g} \mid P \times P = 0, P \neq 0 \}$) which is a connected non-compact simple Lie group of type $E_{7(-2s)}$.

8. Lie algebra $\mathfrak{e}_{8,1}$.

We define a Lie algebra

$$\mathfrak{e}_{8,1} = \mathfrak{e}_{7,1} \oplus \mathfrak{g} \oplus \mathfrak{g} \oplus R \oplus \mathfrak{g} \oplus R$$

as in I.2.

Proposition 9. $\mathfrak{e}_{8,1}$ is a simple Lie algebra of type $E_{8(-24)}$.

Proof. Since the complexification Lie algebra of $\mathfrak{e}_{8,1}$ is $\mathfrak{e}_{8}$, $\mathfrak{e}_{8,1}$ is a simple Lie algebra of type $E_8$. A maximal compact subalgebra of $\mathfrak{e}_{8,1}$ is

$$\mathfrak{l} = \{ \theta \in \mathfrak{e}_{8,1} \mid '\theta = \theta \}$$

$$= \{ \theta(A, P, \tilde{P}, 0, s, -s) \in \mathfrak{e}_{8,1} \mid \theta \in \mathfrak{e}_{7,1}, '\theta = \Phi, P \in \mathfrak{g}, s \in \mathfrak{g} \}.$$ 

Hence the Cartan index of $\mathfrak{e}_{8,1}$ is

$$\dim \mathfrak{e}_{8,1} - 2 \dim \mathfrak{l} = 248 - 2(79 + 56 + 1) = -24,$$

that is, the type of $\mathfrak{e}_{8,1}$ is $E_{8(-24)}$.

9. Manifold $\mathbb{X}$ and group $E_{8,1}$.

We define a subspace of $\mathfrak{e}_{8,1}$ by
\[
\mathcal{X} = \left\{ \begin{array}{l}
\Phi \\
 P \\
 Q \\
 r \\
 s \\
t 
\end{array} \right\} \in \mathfrak{e}_{8,1} \\
\left\{ \begin{array}{l}
2t\Phi + Q \times Q = 0 \\
t^3P - trQ + \frac{1}{6}(Q \times Q)Q = 0 \\
st^a + rt^3 - \frac{1}{96}[Q, (Q \times Q)Q] = 0 \\
t > 0
\end{array} \right\}.
\]

Now, the group $E_{8,1}$ is defined to be the group of all automorphisms of the Lie algebra $\mathfrak{e}_{8,1}$ leaving $\mathcal{X}$ invariant:

\[
E_{8,1} = \left\{ \alpha \in \text{Iso}_R(\mathfrak{e}_{8,1}, \mathfrak{e}_{8,1}) \mid \alpha \mathcal{X} = \mathcal{X}, \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2] \right\}.
\]

**Proposition 10.** \(\mathcal{X} = \{ \exp(\Theta(0, P_1, 0, r_1, s_1, 0) | P_1 \in \mathfrak{g}, r_1, s_1 \in \mathfrak{g} \}.\)

In particular, $\mathcal{X}$ is connected.

**Proof.** The same as [7] Proposition 27.

**Theorem 11.** $E_{8,1}$ is a Lie group of type $E_{6(-24)}$.

**Proof.** The Lie algebra $\mathfrak{e}_{8,1}$ of $E_{8,1}$ is the derivation Lie algebra $\text{Der}(\mathfrak{e}_{8,1})$ (its proof is the same as [7] Proposition 28) which is isomorphic to $\mathfrak{e}_{8,1}$. Hence the type of the group $E_{8,1}$ is $E_{6(-24)}$ from Proposition 9.

10. Subgroups $E_{7,1}$ and $SL(2, \mathbb{R})$ of $E_{8,1}$.

We shall show that the group $E_{8,1}$ contains non-compact subgroups of type $E_7$ and $A_2$.

**Theorem 12.** The group $E_{8,1}$ contains a subgroup

\[
E_{7,1} = \{ \alpha \in E_{8,1} \mid \alpha 1 = 1, \alpha \mathfrak{g} = \mathfrak{g}, \alpha \mathfrak{a} = \mathfrak{a} \}
\]

which is a connected non-compact simple Lie group of type $E_{7(-25)}$.

**Proof.** The mapping

\[
E_{7(-25)} \ni \beta \mapsto \beta = \begin{pmatrix}
\text{Ad} \beta \\
\beta \\
1 \\
1
\end{pmatrix} \in E_{7,1} \subseteq E_{8,1}
\]

gives an isomorphism between $E_{7(-25)}$ and $E_{7,1}$. Its proof is analogous to [7] Theorem 25 (in [7], in order to prove that $\alpha \in E_{7,1}$ is a digonal form, we used the
properties of the inner product $\langle \, , \rangle$, but it follows only from the condition $\alpha[R_1, R_2]=[\alpha R_1, \alpha R_2])$.

**Proposition 13.** The group $E_{6,i}$ contains a subgroup

$$SL(2, \mathbb{R}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & c_1 & 0 & 0 & 0 \\ 0 & b_1 & d_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+2bc & -ab & cd \\ 0 & 0 & 0 & -2ac & a^2 & -c^2 \\ 0 & 0 & 0 & 2bd & -b^2 & d^2 \end{bmatrix} \in SL(2, \mathbb{R})$$

which is isomorphic to the special linear group $SL(2, \mathbb{R}) = \{ A \in M(2, \mathbb{R}) \mid \det A = 1 \}$.

We identify these groups $E_7(-25)$ with $E_{7,1}$, $SL(2, \mathbb{R})$ with $SL(2, \mathbb{R})$ under the above correspondences.

**11. Connectedness of $E_{6,i}$.**

We shall prove that the group $E_{6,i}$ is connected.

**Proposition 14.** The isotropy subgroups $G_A = \{ \alpha \in E_{6,i} \mid \alpha_1 = 1 \}$ of the group $E_{6,i}$ at $1 \in \mathbb{S}$ is the semi-direct product of groups $\exp(\mathfrak{g})\exp(\mathfrak{r})$ and $E_{7,1}$:

$$G_A = (\exp(\mathfrak{g})\exp(\mathfrak{r}))E_{7,1}, \quad (\exp(\mathfrak{g})\exp(\mathfrak{r})) \cap E_{7,1} = \{1\},$$

where

$$\exp(\mathfrak{g})\exp(\mathfrak{r}) = \{ \exp(\theta(0, 0, Q, 0, 0, t)) \mid Q \in \mathfrak{g}, t \in \mathbb{R} \}.$$ 

In particular, $G_A$ is connected.

**Proof.** First, note that $\mathfrak{g} \oplus \mathfrak{r} = \{ Q + t = (0, 0, Q, 0, 0, t) \mid Q \in \mathfrak{g}, t \in \mathbb{R} \}$ is a subalgebra of $\mathfrak{e}_{6,i}$ and $[Q, t] = 0$, so $\exp(\mathfrak{g})\exp(\mathfrak{r})$ is a connected subgroup of $E_{6,i}$ and $\exp(\mathfrak{Q}) = \exp(\theta(0, 0, Q, 0, 0, 0))$, $\exp(t) = \exp(\theta(0, 0, 0, 0, 0, 0))$ commute to each other. Now, let $\alpha \in G_A$ and put

$$\alpha_1 = (\theta, P, Q, r, s, t), \quad \alpha_1 = (\theta, P_1, Q_1, r_1, s_1, t_1).$$

Then, $[1, \mathfrak{g}] = -2\mathfrak{g}$, $[\mathfrak{g}, 1] = 1$ implies $[\alpha_1, 1] = -2\mathfrak{g}$, $[\alpha_1, 1] = \alpha_1$, that is,

$$\langle 0, 0, -P, s, 0, -2r \rangle = \langle 0, 0, 0, 0, 0, -2 \rangle,$$

$$\langle 0, 0, -P_1, s_1, 0, -2r_1 \rangle = \langle \theta, P, Q, r, s, t \rangle$$

respectively. Hence we have
Furthermore $[1, 1] = 21$ implies $[\alpha_1, \alpha_1^T] = 2\alpha_1$, that is,

$$
[(0, 0, Q, 1, 0, t), (\phi_1, -Q, Q_1, -\frac{t}{2}, 1, t_1)]
$$

$$
= (Q \times Q, -2Q, -\phi_1Q - Q_1 - \frac{3}{2}tQ, -t, 2, -\frac{1}{4}(Q, Q_1) - t^2 - 2t).
$$

$$
= 2(\phi_1, -Q, Q_1, -\frac{t}{2}, 1, t).
$$

Hence we have

$$\phi_1 = \frac{1}{2}Q \times Q, \quad Q_1 = -\frac{t}{2}Q - \frac{1}{3}\phi_1Q, \quad t_1 = -\frac{t^2}{4} - \frac{1}{16}(Q, Q_1).
$$

Thus we see that $\alpha$ has the form

$$
\alpha = \begin{pmatrix}
\ast & \ast & \ast & 0 & \frac{1}{2}Q \times Q & 0 \\
\ast & \ast & \ast & 0 & -Q & 0 \\
\ast & \ast & \ast & Q & -\frac{t}{2}Q - \frac{1}{6}(Q \times Q)Q & 0 \\
\ast & \ast & \ast & 1 & -\frac{t}{2} & 0 \\
\ast & \ast & \ast & 0 & 1 & 0 \\
\ast & \ast & \ast & t & -\frac{t^2}{4} + \frac{1}{36}(Q, (Q \times Q)Q) & 1
\end{pmatrix}
$$

On the other hand, $\exp\left(\frac{t}{2}\exp(Q)\exp(\frac{t}{2})\right)$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{t}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{t}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & t & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -Q & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -Q & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{8}Q & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{4}Q & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$
Non-compact simple Lie group $E_{8(-24)}$ of type $E_8$

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & \frac{t}{2} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -\frac{t}{2} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & t & -\frac{t^2}{4} & 1
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2}Q \times Q \\
-\cdot Q \\
-\frac{\cdot t}{2} - \frac{1}{6}(Q \times Q)Q \\
\cdot -\frac{\cdot t}{2} \\
\cdot 1 \\
\cdot \frac{1}{96} (Q, (Q \times Q) Q)
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{1}{2}Q \times Q \\
-\cdot Q \\
-\frac{\cdot t}{2} - \frac{1}{6}(Q \times Q)Q \\
\cdot -\frac{\cdot t}{2} \\
\cdot 1 \\
\cdot \frac{1}{96} (Q, (Q \times Q) Q)
\end{pmatrix}
= \alpha_1,
\]

and

\[
\exp \left( \frac{t}{2} \right) \exp (Q) 1 = \exp \left( \frac{t}{2} \right)(0, 0, Q, 1, 0, 0) = (0, 0, Q, 1, 0, t) = \alpha_1,
\]

\[
\exp \left( \frac{t}{2} \right) \exp (Q) 1 = 1 = \alpha_2.
\]

Therefore $\exp(-Q)\exp \left( -\frac{t}{2} \right)\beta \in E_{7,1}$, hence we have

\[G_2 = (\exp (P) \exp (R))E_{7,1}.\]

Next, for $\beta \in E_{7,1}$, we have

\[\beta(\exp(Q))\beta^{-1} = \exp(\beta Q), \quad \beta(\exp(P))\beta^{-1} = \exp(P).
\]

In fact,

\[
\beta(\exp(Q))\beta^{-1} R = \beta(\exp(Q))\beta^{-1}(\phi_1, P_1, Q_1, r_1, s_1, t_1)
\]

\[= \beta(\exp(Q))(\beta^{-1} \phi_1 \beta, \beta^{-1} P_1, \beta^{-1} Q_1, r_1, s_1, t_1).
\]
\[
\begin{align*}
\beta^{-1} \Phi_1 \beta - Q \times \beta^{-1} P_1 + \frac{1}{2} s_1 Q \times Q \\
\beta^{-1} P_1 - s_1 Q \\
\beta^{-1} Q_1 - \beta^{-1} \Phi_1 \beta Q + r_1 Q + \frac{1}{2} (Q \times \beta^{-1} P_1) Q - \frac{1}{16} (Q, \beta^{-1} P_1) Q - \frac{1}{6} s_1 (Q \times Q) Q \\
r_1 - \frac{1}{8} (Q, \beta^{-1} Q_1) \\
t_1 - \frac{1}{4} (Q, \beta^{-1} Q_1) + \frac{1}{8} (Q, \beta^{-1} \Phi_1 \beta Q) - \frac{1}{24} (Q, (Q \times \beta^{-1} P_1) Q) + \frac{1}{96} s_1 (Q, (Q \times Q) Q)
\end{align*}
\]

\[\exp [\beta Q] R,\]

an similarly \( \beta (\exp (t)) \beta^{-1} = \exp (t) \). This shows that \( \exp (\mathfrak{g}) \exp (\mathfrak{R}) \) is a normal subgroup of \( G_\perp \). Thus we have a split exact sequence

\[1 \to \exp (\mathfrak{g}) \exp (\mathfrak{R}) \to G_\perp \to E_{7,1} \to 1.\]

Hence \( G_\perp \) is the semi-direct product of \( \exp (\mathfrak{g}) \exp (\mathfrak{R}) \) and \( E_{7,1} \).

**Theorem 15.** The group \( E_{8,1} \) acts on \( \Xi \) transitively and the isotropy subgroup at \( \underline{1} \in \Xi \) of \( E_{8,1} \) is the semi-direct product of subgroups \( \exp (\mathfrak{g}) \exp (\mathfrak{R}) \) and \( E_{7,1} \).

Therefore we have the following homeomorphism

\[E_{8,1}/(\exp (\mathfrak{g}) \exp (\mathfrak{R})) E_{7,1} \simeq \Xi.\]

In particular, the group \( E_{8,1} \) is connected.

**Proof** is the direct consequence of Propositions 10, 14.

From the above theorem we have

**Theorem 16.** The group \( E_{8,1} \) is the connected component containing the identity of the automorphism group \( \text{Aut}(\underline{c}_{8,1}) = \{ a \in \text{Iso}_R(\underline{c}_{8,1}, \underline{c}_{8,1}) \mid [aR_1, R_2] = [aR_1, aR_2] \} \).

12. Center of \( E_{8,1} \).
Theorem 17. The center \( z(E_{8,1}) \) of the group \( E_{8,1} \) is trivial : \( z(E_{8,1}) = \{1\} \).

Proof. Let \( \alpha \in z(E_{8,1}) \). From the commutativity with \( \beta \in E_{7,1} \),

\[
\alpha \beta 1 = \alpha 1, \quad \alpha \beta_1 = \alpha 1, \quad \alpha \beta_1 = \alpha 1.
\]

From this we see that \( \alpha \) has the form

\[
\alpha = \begin{bmatrix} \beta & 0 \\ 1 & B \end{bmatrix}, \quad B \in M(3, \mathbb{R}).
\]

Next, from the commutativity with \( A \in SL(2, \mathbb{R}) \),

\[
B \begin{pmatrix} 1+2bc & -ab & cd \\ -2ac & a^2 & -c^2 \\ 2bd & -b^2 & d^2 \end{pmatrix} = \begin{pmatrix} 1+2bc & -ab & cd \\ -2ac & a^2 & -c^2 \\ 2bd & -b^2 & d^2 \end{pmatrix} B
\]

where \( \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL(2, \mathbb{R}) \), we see \( B = \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{pmatrix}, \ r \neq 0 \). Furthermore, from

\[
[a_1, a_1] = a_1,
\]

we have \( r^2 = r \), hence \( r = 1 \), so \( B = I \). Hence \( \alpha \in E_{7,1} \), moreover \( \alpha \in z(E_{7,1}) \) which is \( \{1, e^{\frac{1}{2}}\} \). And we see easily

\[
\exp(Q) \neq \exp(Q)/t, \quad \text{for} \ Q \in \mathfrak{h}
\]

(see Proposition 14). Therefore \( \alpha = 1 \). Thus we have \( z(E_{8,1}) = \{1\} \).

13. Isomorphism \( E_{8,1} \cong E_{8,1} \).

From Theorems 7, 11, 15, we see that the groups \( E_{8,1} \) and \( E_{8,1} \) are both connected and their Lie algebras have the same type \( E_{8(-24)} \). Therefore there exist central normal subgroups \( N, N_1 \) of the simply connected simple Lie group \( E_{8(-24)} \) such that

\[
E_{8,1} \cong E_{8(-24)}/N, \quad E_{8,1} \cong E_{8(-24)}/N_1.
\]

From Theorem 7, we know that the center of the group \( E_{8(-24)} \) is the cyclic group of order 2 : \( z(E_{8(-24)}) = \mathbb{Z}_2 \). And the centers of \( E_{8,1}, E_{8,1} \), are both trivial (Theorems 8, 17). Hence it must be \( N = N_1 = \mathbb{Z}_2 \). Therefore the groups \( E_{8,1} \) and \( E_{8,1} \) are isomorphic :

\[
E_{8,1} \cong E_{8,1}.
\]
References


