Non-compact simple Lie group $E_{7(-25)}$ of type $E_7$

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It is known that there exist four simple Lie groups of type $E_7$ up to local isomorphism, one of them is compact and the others are non-compact. As for the compact case, it is known that the following group

$$E_7 = \left\{ \alpha \in \text{Iso}(\mathbb{R}^7, \mathbb{R}^7) \mid \alpha \mathbb{M} = \mathbb{M}, \{ \alpha_1, \alpha_1 \} = 1, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle \right\}$$

is a simply connected compact simple Lie group of type $E_7$ [4]. As for one of non-compact cases, H. Freudenthal showed in [2] that the Lie algebra of the group

$$E_{7,1} = \{ \alpha \in \text{Iso}(\mathbb{R}, \mathbb{R}) \mid \alpha \mathbb{M} = \mathbb{M}, \{ \alpha P, \alpha Q \} = \{ P, Q \} \}$$

is a simple Lie algebra of type $E_7$, where $\mathbb{M}$ is the Freudenthal's manifold in $\mathbb{R} = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, $\mathbb{M}^c$, $\mathbb{P}^c$ the complexification of $\mathbb{M}$, $\mathbb{P}$ respectively and $\{ P, Q \}$, $\langle P, Q \rangle$ inner products in $\mathbb{P}$ or $\mathbb{P}^c$. In this paper, we shall investigate the structures of this group $E_{7,1}$. Our results are as follows. The group $E_{7,1}$ is a connected non-compact simple Lie group of type $E_7$ and its center is the cyclic group of order 2:

$$\mathbb{Z}(E_{7,1}) = \{ 1, -1 \}.$$

The polar decomposition of the group $E_{7,1}$ is given by

$$E_{7,1} \simeq (U(1) \times E_6)/\mathbb{Z}_3 \times \mathbb{R}^{24}.$$

In order to give the above decomposition, we construct another group

$$E_{7,t} = \left\{ \alpha \in \text{Iso}(\mathbb{P}^c, \mathbb{P}^c) \mid \alpha \mathbb{M}^c = \mathbb{M}^c, \{ \alpha_1, \alpha_1 \} = 1, \langle \alpha P, \alpha Q \rangle_t = \langle P, Q \rangle_t \right\}$$

(\text{where} $\langle P, Q \rangle_t$ \text{is another inner product in $\mathbb{P}^c$} \text{which is isomorphic to} $E_{7,1}$ \text{and find the subgroup $(U(1) \times E_6)/\mathbb{Z}_3$ explicitly in this group $E_{7,t}$}.)
1. Preliminaries.

Let $\mathcal{C}$ denote the Cayley algebra over the field of real numbers $\mathbb{R}$ and $\mathfrak{A}$ the Jordan algebra consisting of all $3 \times 3$ Hermitian matrices in $\mathcal{C}$ with respect to the multiplication $X \circ Y = \frac{1}{2} (XY + YX)$. In $\mathfrak{A}$, the positive definite symmetric inner product $(X, Y)$, the crossed product $X \times Y$, the cubic form $(X, Y, Z)$ and the determinant $\det X$ are defined respectively by

\[
(X, Y) = \text{tr}(X \circ Y),
\]
\[
X \times Y = \frac{1}{2} (2X \circ Y - \text{tr}(X) Y - \text{tr}(Y) X + (\text{tr}(X) \text{tr}(Y) - (X, Y)) E),
\]
\[
(X, Y, Z) = (X \times Y, Z) = (X, Y \times Z),
\]
\[
\det X = \frac{1}{3} (X, X, X)
\]

where $E$ is the $3 \times 3$ unit matrix.

Now we define a 56 dimensional vector space $\mathfrak{B}$ by

\[
\mathfrak{B} = \mathfrak{A} \oplus \mathfrak{A} \oplus \mathbb{R} \oplus \mathbb{R}.
\]

An element $P = \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix}$ of $\mathfrak{B}$ is often denoted by $P = X + Y + \xi + \eta$ briefly. We define a bilinear mapping $\times : \mathfrak{B} \times \mathfrak{B} \rightarrow \mathfrak{A} \oplus \mathfrak{A} \oplus \mathbb{R}$ by

\[
P \times Q = \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} \times \begin{pmatrix} Z \\ W \\ \zeta \\ \omega \end{pmatrix} = \begin{pmatrix} 2X \times Z - \eta W - \omega Y \\ 2Y \times W - \xi Z - \zeta X \\ (X, W) + (Y, Z) + (X, Y) - 3(\xi \omega + \eta \zeta) \end{pmatrix}
\]

and a space $\mathfrak{M}$ by

\[
\mathfrak{M} = \{ L \in \mathfrak{B} \mid L \times L = 0, \ L \neq 0 \}
\]

\[
\mathfrak{M} = \left\{ \begin{pmatrix} M \\ N \\ \mu \\ \nu \end{pmatrix} \in \mathfrak{B} \mid \begin{array}{l}
M \times M = \nu N \\
N \times N = \mu M \\
(M, N) = 3 \mu \nu
\end{array} \right\}.
\]

For example, the following elements of $\mathfrak{B}$
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\[
\begin{pmatrix}
X \\
\frac{1}{\eta}(X \times X) \\
\frac{1}{\xi^2}\det X
\end{pmatrix}, \quad 
\begin{pmatrix}
\frac{1}{\xi}(Y \times Y) \\
Y \\
\frac{1}{\xi^2}\det Y
\end{pmatrix}, \quad 1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

where $\eta \neq 0$, $\xi \neq 0$, belong to $\mathbb{R}$. Finally in $\mathbb{R}$ we define the skew-symmetric inner product $(P, Q)$ by

\[
(P, Q) = (X, W) - (Z, Y) + \xi \omega - \zeta \eta
\]

for $P = X + \hat{Y} + \xi + i\eta$, $Q = Z + \hat{W} + \zeta + i\omega \in \mathbb{R}$.

2. Group $E_{7,i}$ and its Lie algebra $\mathfrak{e}_{7,i}$.

The group $E_{7,i}$ is defined to be the group of linear isomorphisms of $\mathbb{R}$ leaving the space $\mathbb{R}^8$ and the skew-symmetric inner product $(P, Q)$ invariant:

\[
E_{7,i} = \{ \alpha \in \text{IsoR}(\mathbb{R}, \mathbb{R}) | \alpha \mathbb{R} = \mathbb{R}, \{ \alpha P, \alpha Q \} = \{ P, Q \} \}.
\]

We define a subgroup $E_{6,i}$ of $E_{7,i}$ by

\[
E_{6,i} = \{ \alpha \in E_{7,i} | \alpha 1 = 1, \alpha i = i \}.
\]

**Proposition 1.** The group $E_{6,i}$ is a simply connected non-compact simple Lie group of type $E_{6(-26)}$.

**Proof.** We define a group $E_{6(-26)}$ by

\[
E_{6(-26)} = \{ \beta \in \text{IsoR}(\mathbb{R}, \mathbb{R}) | \det \beta X = \det X \}
= \{ \beta \in \text{IsoR}(\mathbb{R}, \mathbb{R}) | \beta X \times \beta Y = \beta^{-1}(X \times Y) \}
\]

where $^t\beta$ is the transpose of $\beta$ with respect to the inner product $(X, Y) : (\beta X, Y) = (X, ^t\beta Y)$. Then $E_{6(-26)}$ is a simply connected simple Lie group of type $E_6$ [1] and moreover of type $E_{6(-26)}$, since its polar decomposition is given by

\[
E_{6(-26)} \cong F_4 \times R^{26}
\]

where $F_4$ is a simply connected compact simple Lie group of type $F_4$ [1]. We shall show that the group $E_{6,i}$ is isomorphic to the group $E_{6(-26)}$. It is easy to verify that, for $\beta \in E_{6(-26)}$, the linear transformation $\alpha$ of $\mathbb{R}$ defined by

\[
\alpha = \begin{pmatrix}
\beta & 0 & 0 & 0 \\
0 & ^t\beta^{-1} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
belongs to $E_{7,i}$. Conversely suppose $\alpha \in E_{7,i}$ satisfies $\alpha \hat{1} = 1$ and $\alpha \hat{1} = 1$. Then from the conditions $\{\alpha X, \alpha \hat{1}\} = \{\alpha X, \alpha \hat{1}\} = 0$ and $\{\alpha \hat{X}, \alpha \hat{1}\} = \{\alpha \hat{X}, \alpha \hat{1}\} = 0$, we see that $\alpha$ has the form

$$\alpha = \begin{pmatrix}
\beta & \varepsilon & 0 & 0 \\
\delta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

where $\beta$, $\gamma$, $\delta$, $\varepsilon$ are linear transformations of $\mathbb{G}$. Since

$$\alpha \left( \begin{array}{c}
X \\
\frac{1}{\eta} (X \times X) \\
\frac{1}{\eta^2} \text{det}X \\
\eta
\end{array} \right) = \left( \begin{array}{c}
\beta X + \frac{1}{\eta} \varepsilon (X \times X) \\
\delta X + \frac{1}{\eta} \gamma (X \times X) \\
\frac{1}{\eta^2} \text{det}X \\
\eta
\end{array} \right) \in \mathbb{G},$$

we have

$$(\beta X + \frac{1}{\eta} \varepsilon (X \times X)) \times (\beta X + \frac{1}{\eta} \varepsilon (X \times X)) = \eta (\delta X + \frac{1}{\eta} \gamma (X \times X))$$

for all $0 \neq \eta \in \mathbb{R}$. Hence we have $\delta X = 0$ for all $X \in \mathbb{G}$ as the coefficient of $\eta$, therefore $\delta = 0$. Similarly $\varepsilon = 0$. Thus

$$\alpha = \begin{pmatrix}
\beta & 0 & 0 & 0 \\
0 & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$ 

Again the condition $\alpha (X + (X \times X) + \text{det}X + \hat{1}) = \beta X + (\gamma (X \times X)) + \text{det}X + \hat{1} \in \mathbb{G}$ implies

$$\begin{cases}
\beta X \times \beta X = \gamma (X \times X), \\
(\beta X, \gamma (X \times X)) = 3 \text{det}X.
\end{cases}$$

Hence $\text{det} \beta X = \frac{1}{3} (\beta X, \beta X \times \beta X) = \frac{1}{3} (\beta X, \gamma (X \times X)) = \text{det}X$, therefore $\beta \in E_{6(-26)}$ and $\gamma = \hat{\beta}^{-1} \in E_{6(-26)}$. Thus Proposition 1 is proved.

The group $E_{7,i}$ contains also a subgroup
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\[ R^* = \begin{cases} \begin{pmatrix} r^{-1} & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^2 \end{pmatrix} & \text{if } r \neq 0 \in \mathbb{R} \\ \end{cases} \]

(where 1 denotes the identity mapping of $\mathbb{S}$) which is isomorphic to the group $R^* = \{ r \in \mathbb{R} | r \neq 0 \}$.

From now on, we identify these groups $E_{6(-26)}$ and $E_{6,1}$, $R^*$ and $R^*$ under the above correspondences.

We consider the Lie algebra $\mathfrak{e}_{7,1}$ of the group $E_{7,1}$

\[ \mathfrak{e}_{7,1} = \left\{ \phi \in \text{Hom}_\mathbb{R}(\mathbb{S}, \mathbb{R}) \mid \phi L \times L = 0 \text{ for } L \in \mathbb{R} \right\} \cap \left\{ \phi P, Q \mid \{ P, Q \} = 0 \text{ for } P, Q \in \mathbb{S} \right\} \]

H. Freudenthal proved in [2] the following

**Theorem 2.** Any element $\phi$ of the Lie algebra $\mathfrak{e}_{7,1}$ of the group $E_{7,1}$ is represented by the form

\[
\phi = \phi(\phi, A, B, \rho) = \begin{pmatrix}
\phi - \frac{1}{3} \rho & 2B & 0 & A \\
2A & \phi' + \frac{1}{3} \rho & B & 0 \\
0 & A & \rho & 0 \\
B & 0 & 0 & -\rho
\end{pmatrix}
\]

where $\phi \in \mathfrak{e}_{6,1} = \{ \phi \in \text{Hom}_\mathbb{R}(\mathbb{S}, \mathbb{S}) \mid \langle \phi X, X, X \rangle = 0 \}$ (which is the Lie algebra of the group $E_{6,1}$), $\phi'$ is the skew-transpose of $\phi$ with respect to the inner product $\langle X, Y \rangle : \langle \phi X, Y \rangle + \langle X, \phi' Y \rangle = 0$, $A, B \in \mathbb{S}$, $\rho \in \mathbb{R}$ and the action of $\phi$ on $\mathbb{S}$ is defined by

\[
\phi \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \phi X - \frac{1}{3} \rho X + 2B \times Y + \gamma A \\ 2A \times X + \phi' Y + \frac{1}{3} \rho Y + \xi B \\ (A, Y) + \rho \xi \\ (B, X) - \rho \eta \end{pmatrix}
\]

And the type of the Lie algebra $\mathfrak{e}_{7,1}$ is $E_7$.

We shall determine the Cartan index of the group $E_{7,1}$. For this purpose we use the following

**Lemma 3** ([3], p.345). Let $G$ be an algebraic subgroup of the general linear group $GL(n, \mathbb{R})$ such that the condition $A \in G$ implies $^tA \in G$. Then $G$ is homeomorphic to the topological product of the group $G \cap O(n)$ (which is a maximal compact
subgroup of \( G \) and a Euclidean space \( \mathbb{R}^d \):

\[
G \cong (G \cap O(n)) \times \mathbb{R}^d
\]

where \( O(n) \) is the orthogonal subgroup of \( GL(n, \mathbb{R}) \). In particular, the Cartan index of \( G \) is \( \dim G - 2 \dim (G \cap O(n)) \).

**Theorem 4.** The group \( E_{7,1} \) is a simple Lie group of type \( E_7(-27) \).

**Proof.** We define in \( \mathfrak{g} \) a positive definite symmetric inner product \( (P, Q) \) by

\[
(P, Q) = (X, Z) + (Y, W) + \xi + \gamma
\]

for \( P = X + Y + \xi + \eta, \ Q = Z + W + \zeta + \omega \in \mathfrak{g} \) and denote the transpose of \( \Phi \) with respect to this inner product \( (P, Q) \) by \( ^t\Phi : (\Phi P, Q) = (P, ^t\Phi Q). \) Then for

\[
\Phi = \begin{pmatrix}
\phi - \frac{1}{3}\rho^1 & 2B & 0 & A \\
2A & \phi' + \frac{1}{3}\rho^1 & B & 0 \\
0 & A & \rho & 0 \\
B & 0 & 0 & -\rho
\end{pmatrix} \in \mathfrak{e}_{7,1},
\]

we see easily that

\[
^t\Phi = \begin{pmatrix}
-\phi' - \frac{1}{3}\rho^1 & 2A & 0 & B \\
2B & -\phi + \frac{1}{3}\rho^1 & A & 0 \\
0 & B & \rho & 0 \\
A & 0 & 0 & -\rho
\end{pmatrix},
\]

therefore \( ^t\Phi \) also belongs to \( \mathfrak{e}_{7,1} \). Since \( E_{7,1} \) is an algebraic subgroup of the general linear group \( \text{Iso}_\mathbb{R}(\mathfrak{g}, \mathfrak{g}) = GL(56, \mathbb{R}) \), from Lemma 3, the Lie algebra \( \mathfrak{e}_{7,1} \cap o(\mathfrak{g}) \) (where \( o(\mathfrak{g}) = o(56) = \{ \Phi \in \text{Hom}_\mathbb{R}(\mathfrak{g}, \mathfrak{g}) | \Phi + ^t\Phi = 0 \} \)) of the group \( E_{7,1} \cap O(\mathfrak{g}) \) (where \( O(\mathfrak{g}) = O(56) = \{ \alpha \in \text{Iso}_\mathbb{R}(\mathfrak{g}, \mathfrak{g}) | (\alpha P, \alpha Q) = (P, Q) \} \)) is a maximal compact Lie subalgebra of \( \mathfrak{e}_{7,1} \). Now if \( \Phi \in \mathfrak{e}_{7,1} \) satisfies \( \Phi + ^t\Phi = 0 \), then

\[
\Phi = \begin{pmatrix}
\delta & -2A & 0 & A \\
2A & \delta & -A & 0 \\
0 & A & 0 & 0 \\
-A & 0 & 0 & 0
\end{pmatrix},
\]

where \( \delta \in \mathfrak{f}_4 = \{ \delta \in \mathfrak{e}_{7,1} | \delta' = \delta \} \) (which is the Lie algebra of \( F_4 \)). Therefore \( \dim (\mathfrak{e}_{7,1} \cap o(\mathfrak{g})) = \dim \mathfrak{f}_4 + \dim \mathfrak{g}_3 = 52 + 27 = 79. \) Hence

The Cartan index of \( \mathfrak{e}_{7,1} = \dim \mathfrak{e}_{7,1} - 2 \dim (\mathfrak{e}_{7,1} \cap o(\mathfrak{g})) \)
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$$= 133 - 2 \times 79 = -25.$$  

Thus we see that the type of the Lie algebra $e_{7,1}$ is $E_{7(-25)}$.

### 3. Connectedness of $E_{7,1}$.

We shall prove that the group $E_{7,1}$ is connected. We denote, for a while, the connected component of $E_{7,1}$ containing the identity $1$ by $(E_{7,1})_0$.

**Lemma 5.** For $A \in \mathfrak{g}$, the linear transformation $\exp_1(A)$ of $\mathfrak{g}$ defined by

$$\exp_1(A) = \begin{pmatrix} 1 & 0 & 0 & A \\ 2A & 1 & 0 & A \times A \\ A \times A & A & 1 & \det A \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(the action of $\exp_1(A)$ on $\mathfrak{g}$ is as similar to that of Theorem 2) belongs to $(E_{7,1})_0$.

Similarly for $B \in \mathfrak{g}$ we can define

$$\exp_2(B) = \begin{pmatrix} 1 & 2B & B \times B & 0 \\ 0 & 1 & B & 0 \\ 0 & 0 & 1 & 0 \\ B & B \times B & \det B & 1 \end{pmatrix} \in (E_{7,1})_0.$$

**Proof.**

For $\Phi_1(A) = \begin{pmatrix} 0 & 0 & 0 & A \\ 2A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{e}_{7,1}$, we have $\exp_1(A) = \exp \Phi_1(A)$, hence $\exp_1(A) \in (E_{7,1})_0$. Similarly $\exp_2(B) \in (E_{7,1})_0$.

**Proposition 6.** The subgroup $G_1 = \{ \alpha \in E_{7,1} \mid \alpha^1 = 1 \}$ is the semi-direct product of the group $\exp(\mathfrak{g}) = \{ \exp(A) \mid A \in \mathfrak{g} \}$ (which is an abelian group) and the group $E_{6,1}$:

$$G_1 = \exp(\mathfrak{g})E_{6,1}, \quad \exp(\mathfrak{g}) \cap E_{6,1} = \{1\}.$$

Therefore $G_1$ is homeomorphic to

$$G_1 \simeq E_{6,1} \times \mathbb{R}^{27} \simeq F_4 \times \mathbb{R}^{33}.$$

In particular, the group $G_1$ is simply connected.

**Proof.** Let $\alpha \in G$ and put $\alpha \hat{1} = M + \hat{N} + \mu + \hat{\nu}$. Then the conditions $\{\alpha^1, \alpha \hat{1}\} = 1$ and $\alpha \hat{1} \in \mathfrak{g}_2$ imply $\nu = 1$ and $N = M \times M$, $\mu = \det M$ respectively. Therefore we have

$$\exp_1(M) \hat{1} = M + (M \times M) + \det M + \hat{1} = \alpha \hat{1}, \quad \exp_1(M)1 = 1 = \alpha 1.$$
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so \((\exp_1(M))^{-1} \in E_{6,1}, \) i.e.

\[
\alpha \in \exp_1(\mathfrak{g})E_{6,1},
\]

and conversely. Since the Lie subalgebra \(\{ \Phi_1(A) = \Phi(0, A, 0, 0) \in \mathfrak{e}_{6,1} | A \in \mathfrak{g} \}\) of \(\mathfrak{e}_{6,1}\) is abelian, the group \(\exp_1(\mathfrak{g})\) is also abelian. Moreover \(\exp_1(\mathfrak{g})\) is a normal subgroup of \(G_1\), because it holds that

\[
\beta \exp_1(A) \beta^{-1} = \exp_1(\beta A) \quad \text{for } \beta \in E_{6,1}, A \in \mathfrak{g}.
\]

Therefore we have the following split exact sequence

\[
1 \longrightarrow \exp_1(\mathfrak{g}) \longrightarrow G_1 \longrightarrow E_{6,1} \longrightarrow 1
\]

Thus we see that \(G_1\) is the semi-direct product of \(\exp_1(\mathfrak{g})\) and \(E_{6,1}\).

**Theorem 7.** The group \(E_{7,1}\) acts transitively on the manifold \(\mathfrak{M}\) (which is connected) and the isotropy subgroup \(G_1\) of \(E_{7,1}\) at \(1 \in \mathfrak{M}\) is \(\exp_1(\mathfrak{g})E_{6,1}\) (Proposition 6). Therefore the homogeneous space \(E_{7,1}/\exp_1(\mathfrak{g})E_{6,1}\) is homeomorphic to \(\mathfrak{M}\):

\[
E_{7,1}/\exp_1(\mathfrak{g})E_{6,1} \simeq \mathfrak{M}.
\]

In particular, the group \(E_{7,1}\) is connected.

**Proof.** Obviously the group \(E_{7,1}\) acts on \(\mathfrak{M}\). We shall prove that the group \((E_{7,1})_0\) acts transitively on \(\mathfrak{M}\). Since

\[
\exp_1(-E) \exp_2(E) = I, \quad \exp_1(E) \exp_2(-E) = -I, \quad \exp_2(-E) \exp_1(E) \exp_2(-E) = -1,
\]

it is sufficient to show that any element \(L \in \mathfrak{M}\) can be transformed in either of 1, -1, I, -I. Let \(L = M + N + \mu + \nu \in \mathfrak{M}\). First assume \(\mu > 0\). Then \(M = \frac{1}{\mu} (N \times N)\), \(\nu = \frac{1}{\mu^3} \det N\). Choose \(0 < r \in R\) such that \(r^2 = \mu\), then for

\[
\begin{pmatrix}
 r^{-1} & 0 & 0 & 0 \\
 0 & r & 0 & 0 \\
 0 & 0 & r^3 & 0 \\
 0 & 0 & 0 & r^{-3}
\end{pmatrix} \in (E_{7,1})_0
\]

we have \(r1 = \mu\), and hence

\[
\exp_2\left(\frac{N}{\mu}\right) r1 = \mu \left(\frac{N}{\mu} \times \frac{N}{\mu}\right) + \mu \left(\frac{N}{\mu}\right) + \mu \left(\frac{\det N}{\mu}\right) = \frac{1}{\mu} (N \times N) + N + \mu + \frac{1}{\mu^3} (\det N) = L.
\]

If \(\mu < 0\), \(L\) can be transformed in \(-1\). Similarly in the case \(\nu \neq 0\) the statement is also valid. Next we consider the case \(L = M + N \in \mathfrak{M}\), \(N \neq 0\). Then \(M \times M = N\)
\[ \times N = 0, \quad \det M = 0, \quad (N, N) \neq 0 \] and so
\[ \exp(\mathbf{N}) = e^* + e^* (N, N) + e^* . \]

So we can reduce to the first case \( \mu \neq 0 \). In the case of \( M \neq 0 \), the statement is also valid. Thus the transitivity of \((E_{7,1})\) on \( \mathcal{M} \) is proved. Therefore we have \( \mathcal{M} = (E_{7,1})_{p1} \), hence \( \mathcal{M} \) is connected. Since the group \( E_{7,1} \) acts transitively on \( \mathcal{M} \) and the isotropy subgroup of \( E_{7,1} \) is \( \exp(\mathfrak{g})E_{6,1} \), we have the following homeomorphism
\[ E_{7,1}/\exp(\mathfrak{g})E_{6,1} \cong \mathcal{M} . \]

Since \( \exp(\mathfrak{g})E_{6,1} \) is connected, \( E_{7,1} \) is also connected. Thus the proof of Theorem 7 is completed.

4. Center \( z(E_{7,1}) \) of \( E_{7,1} \).

Theorem 8. The center \( z(E_{7,1}) \) of the group \( E_{7,1} \) is isomorphic to the cyclic group \( \mathbb{Z}_2 \) of order 2:
\[ z(E_{7,1}) = \{1, -1\} \cong \mathbb{Z}_2 . \]

Proof. Let \( \alpha \in z(E_{7,1}) \). From the commutativity with \( \beta \in E_{6,1} \subset E_{7,1} \), we have
\[ \beta \alpha = \alpha \beta = \alpha \beta = \beta \alpha . \]
If we denote \( \alpha \beta = M + N + \mu + \nu, \) then \( \beta M + (\beta^{-1} N) + \mu + \nu = M + N + \mu + \nu, \) hence
\[ \beta M = M, \quad \beta^{-1} N = N \quad \text{for all } \beta \in E_{6,1} . \]

Therefore \( M = N = 0, \) so \( \alpha \beta = \mu + \nu, \) where \( \mu \nu = 0 \) (since \( \alpha \beta \in \mathcal{M} \)). Suppose that \( \mu = 0, \) i.e. \( \alpha \beta = \nu \neq 0, \) then from the commutativity with
\[ r = \begin{pmatrix} r^{-1} & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r^3 & 0 \\ 0 & 0 & 0 & r^{-3} & \end{pmatrix} \in \mathbb{R}^* \subset E_{7,1} , \]
we have
\[ (r^{-3} \nu)^* = r \nu = r \alpha \beta = a r \beta = a r^3 = (r^3 \nu)^* \quad \text{for all } r \in \mathbb{R}^* . \]

This is contradiction. Hence \( \alpha \beta = \mu. \) Similarly \( \alpha \beta = \lambda. \) The condition \( \{ \alpha \beta, \alpha \beta \} = 1 \) implies \( \mu \lambda = 1, \) hence
\[ \alpha \beta = \mu, \quad \alpha \beta = (\mu^{-1}) \cdot . \]

Next note that
belongs to $E_{7,i}$. Then the commutativity condition $t'\alpha = \alpha t'$ implies 

$$\mu = t'\mu = t'\alpha 1 = \alpha t' 1 = \alpha 1 = (\mu^{-1}),$$

hence $\mu = \mu^{-1}$, i.e. $\mu = \pm 1$. In the case of $\mu = 1$, $\alpha \in E_{8,i}$ so $\alpha \in z(E_{8,i}) = \{1\}$ [5] i.e. $\alpha = 1$. In the case of $\mu = -1$, $-\alpha \in z(E_{8,i}) = \{1\}$, i.e. $\alpha = -1$. Thus we see that $z(E_{7,i}) = \{1, -1\}$.

### 5. Group $E_{7,i}$ and its Lie algebra $e_{7,i}$.

We construct another simple Lie group of type $E_{7(-25)}$. Let $C$ denote the field of complex numbers and $\mathbb{C}$ the complexification of $\mathfrak{g}$. In $\mathbb{C}$ also, the inner product $(X, Y)$, crossed product $X \times Y$, the cubic form $(X, Y, Z)$ and the determinant $\det X$ are defined as similar in $\mathfrak{g}$. Let $\mathfrak{g}^C$ be also the complexification of $\mathfrak{g}$:

$$\mathfrak{g}^C = \mathfrak{g}^C \oplus \mathfrak{g}^C \oplus C \oplus C.$$

We define a mapping $\times : \mathfrak{g}^C \times \mathfrak{g}^C \rightarrow \mathfrak{g}^C \oplus \mathfrak{g}^C \oplus C \oplus C$ as similar as the case $\mathfrak{g}$ and a space $\mathfrak{g}^C$ by

$$\mathfrak{g}^C = \{ L \in \mathfrak{g}^C \mid L \times L = 0, \ L \neq 0 \}.$$

Finally in $\mathfrak{g}^C$, $\mathfrak{g}^C$, positive definite Hermitian inner products $<X, Y>$, $<P, Q>$ and the inner product $<P, Q>$, the skew-symmetric inner product $(P, Q)$ are defined respectively by

$$<X, Y> = (\tau X, Y) = (X, Y),$$

$$<P, Q> = <X, Z> + <Y, W> + \bar{\zeta} + \bar{\eta} \omega,$$

$$<P, Q> = <X, Z> - <Y, W> + \bar{\zeta} - \bar{\eta} \omega,$$

$$\{ P, Q \} = (X, W) - (Z, Y) + \bar{\xi} \omega - \bar{\zeta} \eta,$$

where $\tau : \mathfrak{g}^C \rightarrow \mathfrak{g}^C$ is the complex conjugate ($\tau X$ is also denoted by $\bar{X}$) and $P = X + Y + \xi + \eta$, $Q = Z + W + \xi + \omega \in \mathfrak{g}^C$.

Now the group $E_{7,i}$ is defined to be the group of linear isomorphisms of $\mathfrak{g}^C$ leaving the space $\mathfrak{g}^C$, some skew–symmetric inner product $(P, Q)$ and the inner product $<P, Q>$ invariant:

$$E_{7,i} = \left\{ \alpha \in \text{Iso}(\mathfrak{g}^C, \mathfrak{g}^C) \mid \alpha \mathfrak{g}^C = \mathfrak{g}^C, \{ \alpha 1, \alpha \hat{1} \} = 1 \right\}.$$
Non-compact simple Lie group $E_{7(-20)}$ of type $E_7$

We define a subgroup $E_6$ of $E_{7,t}$ by

$$E_6 = \{ \alpha \in E_{7,t} \mid \alpha 1 = 1, \alpha i = i \}. $$

**Proposition 9.** The group $E_6$ is a simply connected compact simple Lie group of type $E_6$ and isomorphic to the group $E_6(-7s) = \{ \beta \in \text{Isoc}(\mathcal{G}, \mathcal{G}) \mid \text{det} \beta X = \text{det} X, \quad \langle \beta X, \beta Y \rangle = \langle X, Y \rangle \}

(see [7]) by the correspondence

$$E_{6(-7s)} \ni \beta \rightarrow \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \tau \beta \tau & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in E_{7,t}.$$  

**Proof.** It is seen by the analogous proof of Proposition 1 (or see [4] Proposition 2).

The group $E_{7,t}$ contains also a subgroup

$$U(1) = \left\{ \theta = \begin{pmatrix} \theta^{-1} & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta^{-1} \\ 0 & 0 & 0 \end{pmatrix} \mid \theta \in \mathbb{C}, \quad |\theta| = 1 \right\}$$

which is isomorphic to the unitary group $U(1) = \{ \theta \in \mathbb{C} \mid |\theta| = 1 \}.$

From now on, we identify these group $E_{6(-7s)}$ and $E_6,$ $U(1)$ and $U(1)$ under the above correspondences.

We consider the Lie algebra $\mathfrak{e}_{7,t}$ of the group $E_{7,t}:$

$$\mathfrak{e}_{7,t} = \left\{ \Phi \in \text{Hom}_c(\mathfrak{g}^c, \mathfrak{g}^c) \right\}
\begin{array}{c}
\Phi L \times L = 0 \quad \text{for} \quad L \in \mathfrak{g}^c \\
\{ \Phi_1, \; \Phi_2 \} + \{ 1, \; \Phi \} = 0 \\
\langle \Phi P, \; Q \rangle + \langle P, \; \Phi Q \rangle = 0 \quad \text{for} \quad P, \; Q \in \mathfrak{g}^c \end{array}.$$

**Theorem 10.** Any element $\Phi$ of the Lie algebra $\mathfrak{e}_{7,t}$ is represented by the form

$$\phi = \begin{pmatrix} \phi - \frac{1}{3} \rho 1 & 2A & 0 & \bar{A} \\ 2 \bar{A} & \tau \phi \tau + \frac{1}{3} \rho 1 & A & 0 \\ 0 & \bar{A} & \rho & 0 \\ A & 0 & 0 & -\rho \end{pmatrix}.$$
where \( \phi \in \mathfrak{e} = \{ \phi \in \text{Hom}_C(\mathfrak{g}, \mathfrak{g}) \mid \langle \phi X, X, X \rangle = 0, \langle \phi X, Y \rangle + \langle X, \phi Y \rangle = 0 \} \)
which is the Lie algebra of the group \( E_6 \), \( A \in \mathfrak{g}, \rho \in C \) such that \( \rho + \overline{\rho} = 0 \) and
the action of \( \Phi \) on \( \mathfrak{g} \) is defined by

\[
\phi \left( \begin{array}{c} X \\ Y \\ \xi \\ \eta \\ \tau \\ \gamma \\ \zeta \\ A \\ Y \\ X \end{array} \right) = \left( \begin{array}{c} \phi X - \frac{1}{3} \rho X + 2A \tau Y + 2X \eta \xi + 2A \xi A \\ 2A \tau X + \tau \phi \xi Y + \frac{1}{3} \rho Y + \xi A \\ \xi A, Y + \rho \xi \\ (A, X) - \rho \eta \\ \end{array} \right).
\]

In particular, the type of the Lie group \( E_7 \), is \( E_7 \). [2]

**Proof.** It is obtained by the analogous argument as Theorem 3 of [4].

6. Involutive automorphism \( \iota \) and subgroup \( (U(1) \times E_6) / \mathbb{Z}_3 \).

We define an involutive linear isomorphism \( \iota \) of \( \mathfrak{g} \) by

\[
\iota = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \end{pmatrix}.
\]

Then two inner products \( <P, Q>, <P, Q>^\iota \) in \( \mathfrak{g} \) are combined with relations

\[
<P, Q>^\iota = <\iota P, Q> = <P, \iota Q>, \quad <P, Q> = <\iota P, Q>^\iota = <P, \iota Q>^\iota.
\]

The following Lemma is easily verified.

**Lemma 11.** For \( \alpha \in E_{7,t} \), we have \( \iota \alpha \in E_{7,t} \).

Therefore we can define an automorphism \( \iota : E_{7,t} \longrightarrow E_{7,t} \) by

\[
\alpha \iota = \iota \alpha \quad \alpha \in E_{7,t}.
\]

**Proposition 12.** The subgroup \( \{ \alpha \in E_{7,t} \mid \alpha \iota = \alpha \} \) of the group \( E_{7,t} \) is isomorphic to the group \( (U(1) \times E_6) / \mathbb{Z}_3 \):

\[
\{ \alpha \in E_{7,t} \mid \alpha \iota = \alpha \} \cong (U(1) \times E_6) / \mathbb{Z}_3
\]

where \( \mathbb{Z}_3 = \{ (1, 1), (\omega, \omega^1), (\omega^2, \omega^3) \} \), \( \omega \in C, \omega^3 = 1, \omega^4 = 1 \) and

\[
\omega = \begin{pmatrix} \omega^{-1} & 0 & 0 & 0 \\
0 & \omega & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix} \in U(1), \quad \omega_1 = \begin{pmatrix} \omega & 0 & 0 & 0 \\
0 & \omega^{-1} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix} \in E_6.
\]
Non-compact simple Lie group $E_{7(-25)}$ of type $E_7$

**Proof.** We define a mapping $\psi : U(1) \times E_{7(-25)} \rightarrow \{ \alpha \in E_{7}, \ alf = \alpha \}$ by

$$
\psi(\theta, \beta) = \begin{pmatrix}
\theta^{-1} & 0 & 0 & 0 \\
0 & \theta I & 0 & 0 \\
0 & 0 & \theta I & 0 \\
0 & 0 & 0 & \theta^{-3}
\end{pmatrix}
\begin{pmatrix}
\beta & 0 & 0 & 0 \\
0 & \beta \tau & 0 & 0 \\
0 & 0 & \beta \tau & 0 \\
0 & 0 & 0 & \beta \tau^{-1}
\end{pmatrix}
= \beta \theta.
$$

Then obviously $\psi$ is a homomorphism. We shall prove that $\psi$ is onto. If $\alpha \in E_{7}$, satisfies $\alpha \ell = \alpha$, then $\alpha$ has the form

$$
\alpha = \begin{pmatrix}
\beta & 0 & M & 0 \\
0 & \tau & 0 & N \\
a & 0 & \mu & 0 \\
b & 0 & \nu & 0
\end{pmatrix}
$$

where $\beta, \tau$ are linear transformations of \( C \), $a, b$ linear functionals of \( C \), $M, N \in C$ and $\mu, \nu \in C$. The conditions $a \ell, a \in \mathfrak{g} \mathfrak{o}$ imply

$$
\mu M = 0, \quad \nu N = 0
$$

respectively. We shall show that $M = N = 0$. Assume $M \neq 0$, then $\mu = 0$, so $\alpha$ is not identically 0. And then from $\{ a \ell, a \} = 1$, we have

$$
(M, N) = 1,
$$

hence $N \neq 0$, so $\nu = 0$. Furthermore the condition

$$
\begin{pmatrix}
X \\
\frac{1}{\eta}(X \times X) \\
\frac{1}{\eta^2}\det X \\
\eta
\end{pmatrix}
= \begin{pmatrix}
\beta X + \frac{1}{\eta^2}(\det X)M \\
\frac{1}{\eta}\tau(X \times X) + \eta N \\
\frac{1}{\eta}a(X) \\
\frac{1}{\eta}b(X \times X)
\end{pmatrix}
\in \mathfrak{g} \mathfrak{o}
$$

implies

$$
\begin{cases}
\left( \frac{1}{\eta} \tau(X \times X) + \eta N \right) \times \left( \frac{1}{\eta^2} \tau(X \times X) + \eta N \right) = a(X)(\beta X + \frac{1}{\eta^2}(\det X)M), \\
(\beta X + \frac{1}{\eta^2}(\det X)M, \frac{1}{\eta} \tau(X \times X) + \eta N) = 3a(X)\frac{1}{\eta} - b(X \times X)
\end{cases}
$$

for all $0 \neq \eta \in C$. Hence we have
\[
\begin{align*}
2\gamma(X \times X) \times N &= a(X)\beta X, \\
\gamma(X \times X) \times r(X \times X) &= a(X)(\det X)M, \\
(\beta X, r(X \times X)) + \det X &= 3a(X)b(X \times X).
\end{align*}
\]

Therefore
\[
a(X)\det X = a(X)(\det X)(M, N) = (r(X \times X) \times \gamma(X \times X), N)
\]

\[
= (\gamma(X \times X), \gamma(X \times X) \times N) = \frac{1}{2}a(X)(\gamma(X \times X), \beta X)
\]

\[
= \frac{1}{2}a(X)(3a(X)b(X \times X) - \det X).
\]

Hence
\[
a(X)\det X = (a(X))^2b(X \times X).
\]

Thus we have
\[
\det X = a(X)b(X \times X)
\]
(since \(a : \mathbb{F} \rightarrow \mathbb{C}\) is a linear functional and \(\det X = a(X)b(X \times X)\) is continuous with respect to \(X\), even if for \(X\) such that \(a(X) = 0\). This contradicts to the irreducibility of the determinant \(\det X\) with respect to the variables of its components. Thus we have \(M = 0\). Similarly \(N = 0\). So
\[
\alpha_1 = \mu, \quad \alpha^*_1 = (\mu^{-1}), \quad \mu \in \mathbb{C}, |\mu| = 1.
\]

Choose \(\theta \in \mathbb{C}\) such that \(\theta^0 = \mu\) and put \(\beta = \theta^*\alpha\), then \(\beta_1 = 1\) and \(\beta^*_1 = \beta\), therefore \(\beta \in E_0\). Thus we have
\[
\alpha = \theta\beta \quad \theta \in U(1), \ \beta \in E_0.
\]

So \(\phi\) is onto. \(\text{Ker}\phi = \{(1, 1), (\omega, \omega_1), (\omega^2, \omega^2_1)\}, \ \omega \in \mathbb{C}, \ \omega^3 = 1, \ \omega \neq 1\), is easily obtained. Thus the proof of Proposition 12 is completed.

7. Polar decomposition of \(E_{7, \ell}\).

In order to give a polar decomposition of the group \(E_{7, \ell}\), we use the following

**Lemma 13** ([3] p. 345). Let \(G\) be a pseudoalgebraic subgroup of the general linear group \(GL(n, \mathbb{C})\) such that the condition \(A \in G\) implies \(A^* \in G\). Then \(G\) is homeomorphic to the topological product of the group \(G \cap U(n)\) (which is a maximal compact subgroup of \(G\)) and a Euclidean space \(\mathbb{R}^d\):

\[
G \cong (G \cap U(n)) \times \mathbb{R}^d
\]

where \(U(n)\) is the unitary subgroup of \(GL(n, \mathbb{C})\).
Lemma 14. \( E_{7,\tau} \) is a pseudoalgebraic subgroup of the general linear group \( GL(56, C) = \text{Isoc}(\mathcal{R}^C, \mathcal{P}^C) \) and satisfies the condition \( \alpha \in E_{7,\tau} \) implies \( \alpha^* \in E_{7,\tau} \) where \( \alpha^* \) is the transpose of \( \alpha \) with respect to the inner product \( \langle P, Q \rangle : \langle \alpha P, Q \rangle = \langle P, \alpha^* Q \rangle \).

Proof. Since \( \langle \alpha^* P, Q \rangle = \langle P, \alpha Q \rangle = \langle \alpha P, Q \rangle \), we have

\[
\alpha^* = \alpha^{-1} \in E_{7,\tau} \quad \text{(Lemma 11)}.
\]

And it is obvious that \( E_{7,\tau} \) is pseudoalgebraic, because \( E_{7,\tau} \) is defined by pseudoalgebraic relations \( \alpha^C, \alpha^C, \{ \alpha 1, \alpha 1 \} = 1 \) and \( \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle \).

Let \( U(56) = U(\mathcal{R}^C) = \{ \alpha \in \text{Isoc}(\mathcal{R}^C, \mathcal{P}^C) | \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle \} \) denote the unitary subgroup of the general linear group \( GL(56, C) = \text{Isoc}(\mathcal{R}^C, \mathcal{P}^C) \), then we have

\[
E_{7,\tau} \cap U(\mathcal{R}^C) = \{ \alpha \in E_{7,\tau} | \alpha \tau = \alpha \} \cong (U(1) \times E_6)/\mathbb{Z}_3 \quad \text{(Proposition 12)}.
\]

Since \( E_{7,\tau} \) is a simple Lie group of type \( E_7 \), the dimension of \( E_{7,\tau} \) is 133. Hence the dimension \( d \) of the Euclidean part of \( E_{7,\tau} \) and the Cartan index \( i \) are calculated as follows:

\[
d = \dim E_{7,\tau} - \dim (U(1) \times E_6) = 133 - (1 + 78) = 54,
\]

\[
i = \dim E_{7,\tau} - 2\dim (U(1) \times E_6) = 133 - 2(1 + 78) = -25.
\]

Thus we get the following

Theorem 15. The group \( E_{7,\tau} \) is homeomorphic to the topological product of the group \( (U(1) \times E_6)/\mathbb{Z}_3 \) and a 54 dimensional Euclidean space \( \mathbb{R}^{54} \):

\[
E_{7,\tau} \cong (U(1) \times E_6)/\mathbb{Z}_3 \times \mathbb{R}^{54}.
\]

In particular, the group \( E_{7,\tau} \) is a connected non-compact simple Lie group of type \( E_{7,\text{-}2s} \).

8. Center \( z(E_{7,\tau}) \) of \( E_{7,\tau} \).

Lemma 16. For \( a \in C \), the transformation of \( \mathcal{R}^C \) defined by

\[
\alpha_1(a) = \begin{pmatrix}
1 + (\cosh |a| - 1)p_1 & 2\sinh |a| |E_1| & 0 & -\sinh |a| |E_1| \\
2\sinh |a| |E_1| & 1 + (\cosh |a| - 1)p_1 & a\sinh |a| |E_1| & 0 \\
0 & a\sinh |a| |E_1| & \cosh |a| & 0 \\
\sinh |a| |E_1| & 0 & 0 & \cosh |a|
\end{pmatrix}
\]
(if $a = 0$, then $a^\frac{\sinh |a|}{|a|}$ means $0$) belongs to $E_{7,t}$, where $E_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{C}$,

the mapping $p_1: \mathbb{C} \to \mathbb{C}$ is defined by

$$p_1 \begin{pmatrix} \xi_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ 0 \\ x_1 \end{pmatrix}$$

and the action of $\alpha_1(a)$ on $\mathbb{C}$ is defined as similar to that of Theorem 10.

Proof.

For $\Phi_1(a) = \begin{pmatrix} 0 & 2aE_1 & 0 & aE_1 \\ 2aE_1 & 0 & aE_1 & 0 \\ aE_1 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{E}_t$, we have $\alpha_1(a) = \exp \Phi_1(a)$, hence

$$\alpha_1(a) \in E_{7,t}.$$

Theorem 17. The center $z(\mathcal{E}_{7,t})$ of the group $\mathcal{E}_{7,t}$ is isomorphic to the cyclic group of order 2:

$$z(\mathcal{E}_{7,t}) = \{1, -1\}.$$

Proof. Let $a \in z(\mathcal{E}_{7,t})$. From the commutativity with $\beta \in E_6 \subset E_{7,t}$, we have $\beta \alpha_1 = \alpha_1 \beta = \alpha_1$. If we denote $\alpha_1 = M + \hat{N} + \mu + \nu$, then $\beta M + (\tau \beta \tau M) + \mu + \nu = M + \hat{N} + \mu + \nu$, hence we have

$$\beta M = M, \quad \tau \beta \tau N = N \quad \text{for all } \beta \in E_6.$$

Therefore $M = N = 0$, so $\alpha_1 = \mu + \nu$. Similarly $\hat{\alpha}_1 = \lambda + \hat{\nu}$. The conditions $\alpha_1 \hat{\alpha}_1 = \lambda + \hat{\nu}$ imply

$$\mu \nu = 0, \quad \lambda \nu = 0, \quad \mu \lambda + \mu = 1, \quad |\mu|^2 - |\nu|^2 = 1$$

respectively, hence

$$\alpha_1 = \mu, \quad \hat{\alpha}_1 = (\mu^{-1})^*, \quad \mu \in \mathbb{C}, |\mu|^2 = 1.$$ 

Choose $\theta \in \mathbb{C}$ such that $\theta^2 = \mu$ and then put $\beta = \theta^{-1} \alpha$, where

$$\theta = \begin{pmatrix} \theta^{-1} & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta^2 \end{pmatrix} \in U(1),$$

Choose $\theta \in U(1)$ such that $\theta^2 = \mu$ and then put $\beta = \theta^{-1} \alpha$.
Then $\beta_1 = \theta^{-1}\alpha_1 = \theta^{-1}\mu = \theta^{-2}\beta_3 = 1$, similarly $\beta_1 = 1$, hence $\beta \in E_6$. Moreover $\beta \in z(E_6)$ (which denotes the center of $E_6$), in fact, $\beta\beta' = \theta^{-1}\alpha\beta' = \theta^{-1}\beta'\alpha = \beta'\theta^{-1}\alpha = \beta'\beta$ for all $\beta' \in E_6$. Thus we have

$$
\alpha = \theta \beta \quad \theta \in U(1), \quad \beta \in z(E_6).
$$

Since $z(E_6) = \{1, \omega, \omega^2\}$, $\omega \in C$, $\omega^3 = 1$, $\omega \neq 1$ [7], we have

$$
\alpha = \begin{pmatrix}
\theta^{-1}\omega & 0 & 0 & 0 \\
0 & \theta\omega^{-1} & 0 & 0 \\
0 & 0 & \theta^3 & 0 \\
0 & 0 & 0 & \theta^{-2}
\end{pmatrix} \quad \omega \in C, \quad \omega^3 = 1.
$$

Again from the commutativity with $\alpha(a)$ of Lemma 16 : $\alpha(1)a = \alpha(a)(1)$, we have

$$
\theta\omega^{-1}\cosh E_2 + \theta^{-1}\omega(\sinh E_3) = \alpha(1)(\theta^{-1}\omega E_2) = \alpha(1)\alpha E_2
$$

$$
= \alpha(1)(E_2 + (\sinh E_3)) = \theta^{-1}\omega\cosh E_2 + (\theta\omega^{-1}\sinh E_3)
$$

where $E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, hence $\theta^{-1}\omega = \theta\omega^{-1}$, i.e. $\theta^{-1}\omega = \pm 1$.

Therefore $\alpha = \pm 1$, i.e. $z(E_{7,t}) = \{1, -1\}$. Thus the proof of Theorem 17 is completed.

9. Isomorphism $E_{7,t} \cong E_{7,s}$.

From Theorems 4, 7 and 15, we see that the groups $E_{7,t}$ and $E_{7,s}$ are both connected and their Lie algebras have the same type $E_{7(-23)}$. Therefore there exist central normal subgroups $N_i, N_t$ of the simply connected simple Lie group $\bar{E}_{7(-23)}$ of type $E_{7(-23)}$ such that

$$
E_{7,t} \cong \bar{E}_{7(-23)}/N_i, \quad E_{7,s} \cong \bar{E}_{7(-23)}/N_t.
$$

We shall show $N_i = N_t$. From the general theory of Lie groups, we know that the center $z(\bar{E}_{7(-23)})$ of $\bar{E}_{7(-23)}$ is the infinite cyclic group $Z$ [6]. Now assume that $N_1 \neq N_t$. Since the centers of $E_{7,t}$ and $E_{7,s}$ are both $Z_1$ (Theorems 8, 17), we may assume that $2Z = N_1 \subset N_t = Z$ without loss of generality. Consider the natural homomorphism

$$
f : E_{7,t} \cong \bar{E}_{7(-23)}/N_i \rightarrow E_{7(-23)}/N_t \cong E_{7,s}.
$$

Then $f^{-1}(z(E_{7,s})) = f^{-1}(Z_2)$ is a discrete (because $E_{7,s}$ is simple Lie group) normal subgroup, therefore $f^{-1}(z(E_{7,t}))$ is a central (because $E_{7,t}$ is connected) normal subgroup of $E_{7,t}$ : $f^{-1}(z(E_{7,t})) \subset z(E_{7,t})$ and the order of $f^{-1}(z(E_{7,t}))$ is not less than 4. This contradicts to $z(E_{7,t}) = Z_2$. Therefore $N_1 = N_t$ and we see that the groups $E_{7,t}$ and $E_{7,s}$...
are isomorphic:

$$E_{7,1} \cong E_{7,\mu}.$$  

Thus from the preceding arguments we have the following main

**Theorem 18.** The group $E_{7,1} = \{ \alpha \in \text{Iso}_R(\mathbb{R}, \mathbb{R}) | \alpha \mathbb{R} = \mathbb{R}, \{ \alpha P, \alpha Q \} = \{ P, Q \} \}$

is a connected non-compact simple Lie group of type $E_7$, its center $z(E_{7,1})$ is the cyclic group of order 2:

$$z(E_{7,1}) = \{ 1, -1 \}$$

and the polar decomposiion is given by

$$E_{7,1} \cong (U(1) \times E_6)/\mathbb{Z}_2 \times \mathbb{R}^{34}.$$  

**References**


