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Non-compact Simple Lie Group $E_{6(6)}$ *of Type* E_6

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It is known that there exist five simple Lie groups of type E_6 up to local isomorphism, one of them is obtained as the projective transformation group of the Cayley projective plane Π and defined by $E_6 = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}, \mathfrak{J}) \mid \det \alpha X = \det X\}$ (where $\mathfrak{J}$ is the exceptional Jordan algebra over the Cayley algebra $\mathfrak{C}$) and it is homeomorphic to $F_4 \times \boldsymbol{R}^{26}$ [1] and a simple (in the sense of the center $z(E_6)=1$) Lie group [3]. In this paper, we investigate one of the other non-compact simple Lie groups

$$E_6' = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}') \mid \det \alpha X = \det X\}$$

(where $\mathfrak{J}'$ is the exceptional Jordan algebra over the split Cayley algebra $\mathfrak{C}'$). The results are as follows. The group E_6' is homeomorphic to $Sp(4)/\boldsymbol{Z}_2 \times \boldsymbol{R}^{42}$ and a simple (in the sense of the center $z(E_6')=1$) Lie group, and hence the center $z(\widetilde{E}_6')$ of the non-compact simply connected simple Lie group $\widetilde{E}_6' = E_{6(6)}$ of type E_6 is $\boldsymbol{Z}_2$.

1. Split Jordan algebra $\mathfrak{J}'$

Let $\mathfrak{C}'$ be the split Cayley algebra over the real numbers $\boldsymbol{R}$. This algebra $\mathfrak{C}'$ is defined as follows. In $\mathfrak{C}' = \boldsymbol{H} \oplus \boldsymbol{H}e$, where $\boldsymbol{H}$ is the quaternion field over $\boldsymbol{R}$, the multiplication is defined by

$$(a+be)(c+de) = (ac+\bar{d}b) + (b\bar{c}+da)e.$$

In $\mathfrak{C}'$, the conjugate $\bar{x}'$, the real part $\mathrm{Re}(x)$, the Q-norm $Q(x)$ and the inner product $(x, y)'$ are defined respctively by

$$\overline{a+be} = \bar{a} - be, \qquad \mathrm{Re}(x) = \frac{1}{2}(x+\bar{x}),$$

$$Q(a+be) = |a|^2 - |b|^2, \qquad (a+be, c+de)' = (a,c) - (b,d).$$

Let $\mathfrak{J}' = \mathfrak{J}(3, \mathfrak{C}')$ be the Jordan algebra consisting of all 3×3 Hermitian matrices X with entries in $\mathfrak{C}'$

$$X=X(\xi, x)=\begin{pmatrix}\xi_1 & x_3 & \bar{x}_2\\ \bar{x}_3 & \xi_2 & x_1\\ x_2 & \bar{x}_1 & \xi_3\end{pmatrix}, \qquad \xi_i\in\boldsymbol{R},\ x_i\in\mathfrak{C}'$$

with respect to the multiplication

$$X\circ Y=\frac{1}{2}(XY+YX).$$

In $\mathfrak{J}'$, the crossed product $X\times Y$, the inner product $(X, Y)'$, the trilinear form tr $(X, Y, Z)'$, the cubic form $(X, Y, Z)'$ and the determinant $\det X$ are defined respectively by

$$X\times Y=\frac{1}{2}(2X\circ Y-\mathrm{tr}(X)Y-\mathrm{tr}(Y)X+(\mathrm{tr}(X)\mathrm{tr}(Y)-\mathrm{tr}(X\circ Y))E),$$

$$(X, Y)'=\mathrm{tr}(X\circ Y)=\sum_{i=1}^{3}(\xi_i\eta_i+2(x_i, y_i)'),$$

$$\mathrm{tr}(X, Y, Z)'=(X\circ Y, Z)'=(X, Y\circ Z)',$$

$$(X, Y, Z)'=(X\times Y, Z)'=(X, Y\times Z)',$$

$$\det X=\frac{1}{3}(X, X, X)'=\xi_1\xi_2\xi_3+2\mathrm{Re}(x_1x_2x_3)-\xi_1\mathrm{Q}(x_1)-\xi_2\mathrm{Q}(x_2)-\xi_3\mathrm{Q}(x_3)$$

where $X=X(\xi, x)$, $Y=Y(\eta, y)$ and E is the 3×3 unit matrix.

In $\mathfrak{J}'$ we adopt the following notations.

$$E_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\qquad E_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\qquad E_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},$$

$$F_1(x)=\begin{pmatrix}0&0&0\\0&0&x\\0&\bar{x}&0\end{pmatrix},\quad F_2(x)=\begin{pmatrix}0&0&\bar{x}\\0&0&0\\x&0&0\end{pmatrix},\quad F_3(x)=\begin{pmatrix}0&x&0\\\bar{x}&0&0\\0&0&0\end{pmatrix}.$$

Then these elements generate $\mathfrak{J}'$ additively and the table of the Jordan multiplication and the crossed product among them are given respectively as follows.

$$\begin{cases} E_i\circ E_i=E_i, & E_i\circ E_{i+1}=0,\\ E_i\circ F_i(x)=0, & E_i\circ F_j(x)=\dfrac{1}{2}F_j(x), \qquad i\neq j,\\ F_i(x)\circ F_i(y)=(x, y)'(E_{i+1}+E_{i+2}), & F_i(x)\circ F_{i+1}(y)=\dfrac{1}{2}F_{i+2}(\overline{xy}),\end{cases}$$

$$\begin{cases} E_i\times E_i=0, & E_i\times E_{i+1}=\dfrac{1}{2}E_{i+2},\\ E_i\times F_i(x)=-\dfrac{1}{2}F_i(x), & E_i\times F_j(x)=0, \qquad i\neq j,\\ F_i(x)\times F_i(y)=-(x, y)'E_i, & F_i(x)\times F_{i+1}(y)=\dfrac{1}{2}F_{i+2}(\overline{xy}),\end{cases}$$

where the indexes are considered as mod 3.

Finally we define the positive definite inner products (x, y) in $\mathfrak{C}'$ and (X, Y) in $\mathfrak{J}'$ respectively by

$$(a+be,\ c+de)=(a,\ c)+(b,\ d),$$

$$(X,\ Y)=\sum_{i=1}^{3}(\xi_i\eta_i+2(x_i,\ y_i))$$

where $X=X(\xi, x)$, $Y=Y(\eta, y)$, and we denote by $'\alpha$ and $^t\alpha$ the transpose of $\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}')$ relative to $(X, Y)'$ and (X, Y) respectively:

$$(\alpha X,\ Y)'=(X,\ '\alpha Y)', \qquad (\alpha X,\ Y)=(X,\ {}^t\alpha Y).$$

2. Groups E_6' and F_4'.

The group E_6' is defined to be the gronp of linear isomorphisms of $\mathfrak{J}'$ leaving the determinant $\det X$ invariant:

$$\begin{aligned}E_6'&=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}',\ \mathfrak{J}')\,|\,\det\alpha X=\det X\}\\&=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}',\ \mathfrak{J}')\,|\,(\alpha X,\ \alpha Y,\ \alpha Z)'=(X,\ Y,\ Z)'\}\\&=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}',\ \mathfrak{J}')\,|\,\alpha X\times\alpha X={}'\alpha^{-1}(X\times X)\}\end{aligned}$$

and $F_4'(=F_{4,2})$ the group of automorphisms of $\mathfrak{J}'$:

$$\begin{aligned}F_4'&=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}',\ \mathfrak{J}')\,|\,\alpha(X\circ Y)=\alpha X\circ\alpha Y\}\\&=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}',\ \mathfrak{J}')\,|\,\alpha(X\times Y)=\alpha X\times\alpha Y\}\\&=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}',\ \mathfrak{J}')\,|\,\mathrm{tr}(\alpha X,\ \alpha Y,\ \alpha Z)'=\mathrm{tr}(X,\ Y,\ Z)',(\alpha X,\ \alpha Y)'=(X,\ Y)'\}\\&=\{\alpha\in E_6'\,|\,(\alpha X,\ \alpha Y)'=(X,\ Y)'\}\\&=\{\alpha\in E_6'\,|\,\alpha E=E\}.\end{aligned}$$

Then the group F_4' is homeomorphic to $(Sp(1)\times Sp(3))/\boldsymbol{Z}_2\times\boldsymbol{R}^{28}$ and a simple (in the sense of the center $z(F_4')=1$) Lie group [6].

Remark. In [6], the group F_4' $(=F_{4,2})$ is defined to be the group of automorphisms of $\mathfrak{J}'$ leaving the trace invariant. However the condition of the trace-preserving can be omitted, that is, the condition $\alpha(X\circ Y)=\alpha X\circ\alpha Y$ implies the condition $\mathrm{tr}(\alpha X)=\mathrm{tr}(X)$. In fact, any element $X\in\mathfrak{J}'$ satisfies the Cayley-Hamilton identity

$$X\circ(X\times X)=X^{\circ 3}-\mathrm{tr}(X)X^2+\frac{1}{2}(\mathrm{tr}(X)^2-\mathrm{tr}(X^2))X=(\det X)E. \qquad \text{(i)}$$

Now, apply (i) to αX and then operate $\alpha^{-1}\in\mathrm{Aut}(\mathfrak{J}')$ on it. Then we have

$$X^{\circ 3}-\mathrm{tr}(\alpha X)X^2+\frac{1}{2}(\mathrm{tr}(\alpha X)^2-\mathrm{tr}(\alpha X^2))X=(\det\alpha X)E. \qquad \text{(ii)}$$

Thus we get by subtraction (i)$-$(ii)

$$(\mathrm{tr}(\alpha X)-\mathrm{tr}(X))X^2+\frac{1}{2}(\mathrm{tr}(X)^2-\mathrm{tr}(\alpha X)^2+\mathrm{tr}(\alpha X^2)-\mathrm{tr}(X^2))X=(\det X-\det\alpha X)E.$$

So, put $X=F_i(e_j)$[1], $i=1, 2, 3$, $j=0, 1, \cdots, 7$, then we have

$$(e_j, e_j)'\mathrm{tr}(\alpha F_i(e_j))(E_{i+1}+E_{i+2})+\frac{1}{2}(-\mathrm{tr}(\alpha F_i(e_j))^2+\mathrm{tr}(\alpha F_i(e_j)^2)-\mathrm{tr}(F_i(e_j)^2))F_i(e_j)$$
$$=-(\det\alpha F_i(e_j))E.$$

Hence we have

$$\mathrm{tr}(\alpha F_i(e_j))=0=\mathrm{tr}(F_i(e_j)), \quad i=1, 2, 3, \ j=0, 1, \cdots, 7,$$

and also $\mathrm{tr}(\alpha F_i(e_j)^2)=\mathrm{tr}(F_i(e_j)^2)$ hence

$$\mathrm{tr}(\alpha E_i)=\mathrm{tr}(\alpha(E-F_i(1)^2))=\mathrm{tr}(E-F_i(1)^2)=\mathrm{tr}(E_i), \quad i=1, 2, 3.$$

Consequently we obtain

$$\mathrm{tr}(\alpha X)=\mathrm{tr}(X), \qquad \text{for all } X\in\mathfrak{J}'.$$

We define the involution γ in $\mathfrak{J}'$ by

$$\gamma X(\xi, a+be)=X(\xi, a-be).$$

Then $\gamma\in F_4'$, $'\gamma={}^t\gamma=\gamma^{-1}=\gamma$ and two inner products $(X, Y)'$, (X, Y) in $\mathfrak{J}'$ are combined with the following relations

$$(X, Y)'=(X, \gamma Y), \qquad (X, Y)=(X, \gamma Y)'.$$

And we have

$${}^t\alpha=\gamma'\alpha\gamma, \qquad \text{for } \alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}', \mathfrak{J}'),$$

because it holds that $({}^t\alpha X, Y)'=({}^t\alpha X, \gamma Y)=(X, \alpha\gamma Y)=(X, \gamma\alpha\gamma Y)'=(\gamma'\alpha\gamma X, Y)'$ for all $X, Y\in\mathfrak{J}'$.

3. Jordan algebra $\mathfrak{J}(4, \boldsymbol{H})$ and Symplectic group $Sp(4)$.

Before we consider the group E_6', we prepare the several spaces $\mathfrak{J}(4, \boldsymbol{H})$, $\mathfrak{J}(4, \boldsymbol{H})_0$, $\boldsymbol{H}P_3$ and the group $Sp(4)$.

Let $\mathfrak{J}(4, \boldsymbol{H})$ be the Jordan algebra consisting of all 4×4 Hermitian matrices S with entries in $\boldsymbol{H}$:

$$\mathfrak{J}(4, \boldsymbol{H})=\{S\in M(4, \boldsymbol{H}) \mid S^*=S\}$$

with respect to the multiplication

1) $\{e_0, e_1, \cdots, e_7\}$ is the canonical basis of $\mathfrak{C}'$:
$$e_0=1, \ e_1=i, \ e_2=j, \ e_3=k, \ e_4=e, \ e_5=ie, \ e_6=je, \ e_7=ke$$
where $\{1, i, j, k\}$ is the canonical basis of $\boldsymbol{H}$. Then $\{E_i, F_i(e_j), i=1, 2, 3, j=0, 1, \cdots, 7\}$ is a basis of $\mathfrak{J}'$.

$$S\circ T=\frac{1}{2}(ST+TS).$$

And we define the positive definite inner product (S, T) in $\mathfrak{J}(4, \boldsymbol{H})$ by

$$(S, T)=\mathrm{tr}(S\circ T).$$

Let $\mathfrak{J}(4, \boldsymbol{H})_0$ be the vector space of all $S\in\mathfrak{J}(4, \boldsymbol{H})$ such that $\mathrm{tr}(S)=0$:

$$\mathfrak{J}(4, \boldsymbol{H})_0=\{S\in\mathfrak{J}(4, \boldsymbol{H})\,|\,\mathrm{tr}(S)=0\}.$$

Now, we define a mapping $f:\mathfrak{J}'\to\mathfrak{J}(4, \boldsymbol{H})$ by

$$f\begin{pmatrix}\xi_1 & a_3+b_3e & \bar{a}_2-b_2e\\ \bar{a}_3-b_3e & \xi_2 & a_1+b_1e\\ a_2+b_2e & \bar{a}_1-b_1e & \xi_3\end{pmatrix}=\begin{pmatrix}\lambda_1 & b_1 & b_2 & b_3\\ \bar{b}_1 & \lambda_2 & a_3 & \bar{a}_2\\ \bar{b}_2 & \bar{a}_3 & \lambda_3 & a_1\\ \bar{b}_3 & a_2 & \bar{a}_1 & \lambda_4\end{pmatrix}$$

where $\lambda_1=\frac{1}{2}(\xi_1+\xi_2+\xi_3)$, $\lambda_2=\frac{1}{2}(\xi_1-\xi_2-\xi_3)$, $\lambda_3=\frac{1}{2}(\xi_2-\xi_3-\xi_1)$, $\lambda_4=\frac{1}{2}(\xi_3-\xi_1-\xi_2)$. Then we have the following key

Lemma 1. *The mapping $f:\mathfrak{J}'\to\mathfrak{J}(4, \boldsymbol{H})_0$ is an isometry, i.e. f is a linear isomorphism which satisfies*

$$(fX, fY)=(X, Y), \qquad X, Y\in\mathfrak{J}'$$

Moreover we have the following identity in $\mathfrak{J}(4, \boldsymbol{H})$

$$fX\circ fY=f(\gamma(X\times Y))+\frac{1}{4}(X, Y)I, \qquad X, Y\in\mathfrak{J}'$$

where I is the 4×4 unit matrix.

Proof. Noting that $\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2=\xi_1^2+\xi_2^2+\xi_3^2$, it is easy to verify the formula $(fX, fY)=(X, Y)$. Next, to prove the identity $fX\circ fY=f(\gamma(X\times Y))+\frac{1}{4}(X, Y)I$, it is sufficient to show that

(1) $fE_i\circ fE_i=\frac{1}{4}I$,

(2) $fE_i\circ fE_{i+1}=f(E_i\times E_{i+1})$,

(3) $fE_i\circ fF_i(x)=f(E_i\times\gamma F_i(x))$,

(4) $fE_i\circ fF_j(x)=0=f(E_i\times\gamma F_j(x))$, $\quad i\neq j$,

(5) $fF_i(x)\circ fF_i(y)=f(\gamma(F_i(x)\times F_i(y)))+\frac{1}{4}(F_i(x), F_i(y))I$,

(6) $fF_i(x)\circ fF_{i+1}(y)=f(\gamma(F_i(x)\times F_{i+1}(y)))$.

Proof of (3). $\qquad fE_1\circ fF_1(x) \qquad (x=a+be)$

$$=\frac{1}{2}\begin{pmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix}\circ\begin{pmatrix}0&b&0&0\\\bar{b}&0&0&0\\0&0&0&a\\0&0&\bar{a}&0\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}0&b&0&0\\\bar{b}&0&0&0\\0&0&0&-a\\0&0&-\bar{a}&0\end{pmatrix}=-\frac{1}{2}fF_1(a-be)=f(E_1\times\gamma F_1(x)).$$

Proof of (5). $\quad fF_1(x)\circ fF_1(y) \qquad (x=a+be,\ y=c+de)$

$$=\begin{pmatrix}0&b&0&0\\\bar{b}&0&0&0\\0&0&0&a\\0&0&\bar{a}&0\end{pmatrix}\circ\begin{pmatrix}0&d&0&0\\\bar{d}&0&0&0\\0&0&0&c\\0&0&\bar{c}&0\end{pmatrix}$$

$$=\begin{pmatrix}(b,d)&&&\\&(b,d)&&\\&&(a,c)&\\&&&(a,c)\end{pmatrix}$$

$$=(b,\ d)(fE_1+\frac{1}{2}I)+(a,\ c)(-fE_1+\frac{1}{2}I)$$

$$=-((a,\ c)-(b,\ d))fE_1+\frac{1}{2}((a,\ c)+(b,\ d))I$$

$$=-(x,\ y)'fE_1+\frac{1}{2}(x,\ y)I$$

$$=f(\gamma(F_1(x)\times F_1(y)))+\frac{1}{4}(F_1(x),\ F_1(y))I.$$

The other formulae are also proved by calculations similar to the above.

Let $Sp(4)$ be the symplectic group:

$$Sp(4)=\{A\in M(4,\ \boldsymbol{H})\,|\,AA^*=I\}.$$

The group $Sp(4)$ acts on $\mathfrak{J}(4,\ \boldsymbol{H})$ by the way μ: $Sp(4)\times\mathfrak{J}(4,\ \boldsymbol{H})\to\mathfrak{J}(4,\ \boldsymbol{H})$,

$$\mu(A,\ S)=ASA^*.$$

Then this action induces an automorphism of $\mathfrak{J}(4,\ \boldsymbol{H})$ and an isometry of $\mathfrak{J}(4,\ \boldsymbol{H})$ (and $\mathfrak{J}(4,\ \boldsymbol{H})_0$):

$$A(S\circ T)A^*=ASA^*\circ ATA^*,$$

$$(ASA^*,\ ATA^*)=(S,\ T).$$

Finally, let $\boldsymbol{H}P_3$ be the 3-dim. quaternion projective space:

$$HP_3=\{P\in\mathfrak{J}(4,\ \boldsymbol{H})\,|\,P^2=P,\ \mathrm{tr}(P)=1\}$$
$$=\{AI_1A^*\,|\,A\in Sp(4)\}$$

where $I_1=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\in M(4,\ \boldsymbol{H})$.

4. Compact subgroup $(E_6')_K$ of E_6'.

We shall consider the following subgroup $(E_6')_K$ of E_6'

$$(E_6')_K=\{\alpha\in E_6'\,|\,(\alpha X,\ \alpha Y)=(X,\ Y)\}.$$

Proposition 2. *The group $(E_6')_K$ is isomorphic to the group $Sp(4)/\boldsymbol{Z}_2$, where $\boldsymbol{Z}_2=\{I,\ -I\}$.*

Proof. We define a mapping $\varphi: Sp(4)\to(E_6')_K$ by

$$\varphi(A)X=f^{-1}(A(fX)A^*),\qquad X\in\mathfrak{J}'.$$

First of all, we have to show $\varphi(A)\in E_6'$.

$$\begin{aligned}
&3\det(\varphi(A)X)\qquad(\text{denote } \varphi(A)X \text{ by } Y)\\
&=(Y,\ Y,\ Y)'=(Y\times Y,\ Y)'=(\gamma(Y\times Y),\ Y)\\
&=(f(\gamma(Y\times Y)),\ fY)\\
&=(fY\circ fY-\frac{1}{4}(Y,\ Y)I,\ fY)\\
&=(fY\circ fY-\frac{1}{4}(fY,\ fY)I,\ fY)\\
&=(A(fX)A^*\circ A(fX)A^*-\frac{1}{4}(A(fX)A^*,\ A(fX)A^*)I,\ A(fX)A^*)\\
&=(fX\circ fX-\frac{1}{4}(X,\ X)I,\ fX)\\
&=(f(\gamma(X\times X)),\ fX)\\
&=(\gamma(X\times X),X)=(X\times X,\ X)'=(X,\ X,\ X)'=3\det X.
\end{aligned}$$

Hence $\varphi(A)\in E_6'$. And it is obviously obtained that $(\varphi(A)X,\ \varphi(A)Y)=(X,\ Y)$, then $\varphi(A)\in(E_6')_K$.

Obviously φ is a homomorphism. We shall prove that φ is onto. For a given $\alpha\in(E_6')_K$, we consider the element $\alpha E\in\mathfrak{J}'$. This αE satisfies

$$(f(\alpha E))^2=f(\alpha E)+\frac{3}{4}I.$$

In fact, $(f(\alpha E))^2=f(\gamma(\alpha E\times\alpha E))+\frac{1}{4}(\alpha E,\ \alpha E)I=f(\gamma'\alpha^{-1}(E\times E))+\frac{1}{4}(E,E)I=f(\gamma'\alpha^{-1}E)$

$+\frac{3}{4}I=f({}^t\alpha^{-1}\gamma E)+\frac{3}{4}I=f({}^t\alpha^{-1}E)+\frac{3}{4}I=f(\alpha E)+\frac{3}{4}I$. Therefore $P=\frac{1}{4}(2f(\alpha E)+I)$ satisfies $P^2=P$, $\mathrm{tr}(P)=1$, that is, P is an element of $\boldsymbol{H}P_3$. Hence there exists $A\in Sp(4)$ such that

$$P=AI_1A^*.$$

Then we have $\varphi(A)E=f^{-1}(A(fE)A^*)=f^{-1}(A(2I_1-\frac{1}{2}I)A^*)=f^{-1}(2P-\frac{1}{2}I)=f^{-1}(f(\alpha E))=\alpha E$. So, put $\beta=\varphi(A)^{-1}\alpha$, then $\beta E=E$, that is, $\beta\in F_4{}'$ and satisfies

$$(\beta X,\ \beta Y)=(X,\ Y),\qquad X,\ Y\in\mathfrak{J}'.$$

(If we use the notation in [6], $\beta\in(F_4{}')_K$.) Hence, from [6], there exists $B=\begin{pmatrix} p & 0\\ 0 & C\end{pmatrix}\in Sp(4)$, where $p\in Sp(1)=\{p\in\boldsymbol{H}\,|\,|p|=1\}$ and $C\in Sp(3)=\{C\in M(3,\boldsymbol{H})\,|\,CC^*=E\}$ such that

$$\beta=\varphi(B).$$

(In [6], we have proved that the group $(F_4{}')_K$ is isomorphic to the group $(Sp(1)\times Sp(3))/\boldsymbol{Z}_2$. It is easy to see that the mapping to prove this isomorphism coincides with φ, if we note the following

$$fX=f\begin{pmatrix} \xi_1 & a_3+b_3e & \bar{a}_2-b_2e\\ \bar{a}_3-b_3e & \xi_2 & a_1+b_1e\\ a_2+b_2e & \bar{a}_1-b_1e & \xi_3\end{pmatrix}$$

$$=\begin{pmatrix} \mathrm{tr}(X) & b_1 & b_2 & b_3\\ \bar{b}_1 & \xi_1 & a_3 & \bar{a}_2\\ \bar{b}_2 & \bar{a}_3 & \xi_2 & a_1\\ \bar{b}_3 & a_2 & \bar{a}_1 & \xi_3\end{pmatrix}-\frac{1}{2}\mathrm{tr}(X)I$$

$$=\begin{pmatrix} \mathrm{tr}(X_{\boldsymbol{H}}) & \boldsymbol{b}\\ \boldsymbol{b}^* & X_{\boldsymbol{H}}\end{pmatrix}-\frac{1}{2}\mathrm{tr}(X_{\boldsymbol{H}})I$$

and

$$B(fX)B^*=\begin{pmatrix} \mathrm{tr}(CX_{\boldsymbol{H}}C^*) & p\boldsymbol{b}C^*\\ (p\boldsymbol{b}C^*)^* & CX_{\boldsymbol{H}}C^*\end{pmatrix}-\frac{1}{2}\mathrm{tr}(CX_{\boldsymbol{H}}C^*)I).$$

Hence we have

$$\alpha=\varphi(A)\varphi(B)=\varphi(AB),\qquad AB\in Sp(4),$$

that is, φ is onto. Finally $\mathrm{Ker}\varphi=\{I,\ -I\}$ is easily obtained. Thus the proof of Proposition 2 is completed.

5. Polar decomposition of $E_6{}'$.

To give a polar decomposition of $E_6{}'$ we use the following

Lemma 3 ([2] *p.* 345). *Let G be a real algebraic subgroup of the general linear* *group $GL(n, \mathbf{R})$ such that the condition $A\in G$ implies ${}^tA\in G$. Then G is homeomorphic to the topological product of $G\cap O(n)$ (which is a maximal compact subgroup of G) and a Euclidean space $\mathbf{R}^d$:*

$$G\simeq(G\cap O(n))\times \mathbf{R}^d, \qquad d=\dim(\mathfrak{g}\cap\mathfrak{h}(n))$$

where $O(n)$ is the orthogonal subgroup of $GL(n,\mathbf{R})$, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{h}(n)$ the vector space of all real symmetric matrices of degree n.

To use the above Lemma, first of all we show the following

Lemma 4. *E_6' is a real algebraic subgroup of the general linear group $GL(27,\mathbf{R})$ $=\mathrm{Iso}_{\mathbf{R}}(\mathfrak{J}', \mathfrak{J}')$ and satisfies the condition $\alpha\in E_6'$ implies ${}^t\alpha\in E_6'$.*

Proof. We use the following identity

$$(Z\times Z)\times(Z\times Z)=(\det Z)Z, \qquad Z\in\mathfrak{J}'.$$

For $\alpha\in E_6'$ and $Y\in\mathfrak{J}'$, we have

$$\begin{aligned}'\alpha^{-1}(Y\times Y)\times{}'\alpha^{-1}(Y\times Y)&=(\alpha Y\times\alpha Y)\times(\alpha Y\times\alpha Y)\\ &=(\det\alpha Y)\alpha Y=(\det Y)\alpha Y=\alpha((\det Y)Y)\\ &=\alpha((Y\times Y)\times(Y\times Y)).\end{aligned}$$

Put $Y=X\times X$ for any $X\in\mathfrak{J}'$ in the above, then we have

$$'\alpha^{-1}((\det X)X)\times{}'\alpha^{-1}((\det X)X)=\alpha((\det X)X)\times(\det X)X).$$

Hence, if $\det X\neq 0$ then $'\alpha^{-1}X\times{}'\alpha^{-1}X=\alpha(X\times X)$. This implies $\det{}'\alpha^{-1}X=\det X$, i.e. (considering α^{-1} instead of α)

$$\det{}'\alpha X=\det X$$

for $X\in\mathfrak{J}'$ such that $\det X\neq 0$. The same holds also for $X\in\mathfrak{J}'$ such that $\det X=0$. In fact, assume that $\det{}'\alpha X\neq\det X=0$, then applying the above result to $'\alpha X$, we have $\det{}'\alpha X=\det{}'\alpha^{-1}({}'\alpha X)=\det X$, which contradicts the assumption. Therefore $'\alpha\in E_6'$, hence

$${}^t\alpha=\gamma{}'\alpha\gamma\in E_6'.$$

Finally, it is obvious that E_6' is real algebraic, because E_6' is defined by the algebraic relation $\det\alpha X=\det X$.

Let $O(\mathfrak{J}')$ be the orthogonal subgroup of $\mathrm{Iso}_{\mathbf{R}}(\mathfrak{J}', \mathfrak{J}')$:

$$O(27)=O(\mathfrak{J}')=\{\alpha\in\mathrm{Iso}_{\mathbf{R}}(\mathfrak{J}',\mathfrak{J}')\,|\,(\alpha X, \alpha Y)=(X, Y)\}.$$

Then by Proposition 2 we have

$$E_6'\cap O(\mathfrak{J}')=(E_6')_K\cong Sp(4)/\mathbf{Z}_2.$$

Next we shall determine the Euclidean part $\mathfrak{e}_6'\cap\mathfrak{h}(\mathfrak{J}')$ of E_6' where

$$\mathfrak{e}_6'=\{\zeta\in\mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{J}',\ \mathfrak{J}')\,|\,(\zeta X,\ X,\ X)'=0\},$$

$$\mathfrak{h}(27)=\mathfrak{h}(\mathfrak{J}')=\{\zeta\in\mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{J}',\ \mathfrak{J}')\,|\,(\zeta X,\ Y)=(X,\ \zeta Y)\}.$$

(The dimension of the Euclidean part of E_6' is obtained by $\dim E_6'-\dim Sp(4)=78-36=42$. However we investigate the structure of $\mathfrak{e}_6'\cap\mathfrak{h}(\mathfrak{J}')$ directly.)

Lemma 5. *Any element ζ of the Lie algebra $\mathfrak{e}_6'$ of E_6' is uniquely represented by the form*

$$\zeta=\delta+\tilde{T},\qquad \delta\in\mathfrak{f}_4',\ T\in\mathfrak{J}',\ \mathrm{tr}(T)=0$$

where $\mathfrak{f}_4'=\{\delta\in\mathfrak{e}_6'\,|\,\delta E=0\}$ is the Lie algebra of F_4' and $\tilde{T}\in\mathfrak{e}_6'$ is defined by $\tilde{T}X=T\circ X$ for $X\in\mathfrak{J}'$.

Proof. For a given $\zeta\in\mathfrak{e}_6'$, put

$$T=\zeta E\quad\text{and}\quad\delta=\zeta-\tilde{T},$$

then the required results are obtained quite analogously in [1].

Let $\zeta=\delta+\tilde{T}\in\mathfrak{e}_6'\cap\mathfrak{h}(\mathfrak{J}')$. Then it holds

$$(\delta X,\ Y)+(\tilde{T}X,\ Y)=(X,\ \delta Y)+(X,\ \tilde{T}Y),\qquad X,\ Y\in\mathfrak{J}'.$$

Put $Y=E$, then $\mathrm{tr}(\delta X)+\mathrm{tr}(\tilde{T}X)=0+(X,\ T)$. Since $\mathrm{tr}(\delta X)=0$ [6], we have $(T,\ X)'=(T,\ X)$ for all $X\in\mathfrak{J}'$. This implies $\gamma T=T$, that is,

$$T\in\mathfrak{J}(3,\ \boldsymbol{H}),\qquad \mathrm{tr}(T)=0.$$

Furthermore we have $(\tilde{T}X,\ Y)=(\tilde{T}X,\ \gamma Y)'=\mathrm{tr}(T,\ X,\ \gamma Y)'=\mathrm{tr}(\gamma T,\ X,\ \gamma Y)'=\mathrm{tr}(T,\ \gamma X,\ Y)'=\mathrm{tr}(T,\ Y,\ \gamma X)'=(\tilde{T}Y,\ \gamma X)'=(\tilde{T}Y,\ X)=(X,\ \tilde{T}Y)$, therefore $\tilde{T}\in\mathfrak{e}_6'\cap\mathfrak{h}(\mathfrak{J}')$ and $\delta\in\mathfrak{f}_4'\cap\mathfrak{h}(\mathfrak{J}')$. Hence any $\zeta\in\mathfrak{e}'_6\cap\mathfrak{h}(\mathfrak{J}')$ has a form

$$\zeta=\delta+\tilde{T},\qquad \delta\in\mathfrak{f}_4'\cap\mathfrak{h}(\mathfrak{J}'),\ T\in\mathfrak{J}(3,\ \boldsymbol{H}),\ \mathrm{tr}(T)=0,$$

and conversely. The structure of $\mathfrak{f}_4'\cap\mathfrak{h}(\mathfrak{J}')$ has been already seen in [6] and its dimension is 28. Hence we have

$$\dim(\mathfrak{e}_6'\cap\mathfrak{h}(\mathfrak{J}'))=28+14=42.$$

Thus we have the following

Theorem 6. *The group E_6' is homeomorphic to the topological product of the group $Sp(4)/\boldsymbol{Z}_2$ and a 42-dim. Euclidean space $\boldsymbol{R}^{42}$:*

$$E_6'\simeq Sp(4)/\boldsymbol{Z}_2\times\boldsymbol{R}^{42}.$$

In particular, E_6' is a connected (but not simply connected) Lie group.

6. Simplicity of E_6'.

Proposition 7. *The center $z(E_6')$ of E_6' is trivial:*

$$z(E_6')=1.$$

Proof. We define the linear transformations β_i, $i=1, 2, 3$ of $\mathfrak{J}'$ by

$$\beta_1 X=\begin{pmatrix} \xi_1 & -x_3 & -\overline{x}_2 \\ -\overline{x}_3 & \xi_2 & x_1 \\ -x_2 & \overline{x}_1 & \xi_3 \end{pmatrix},\quad \beta_2 X=\begin{pmatrix} \xi_1 & -x_3 & \overline{x}_2 \\ -\overline{x}_3 & \xi_2 & -x_1 \\ x_2 & -\overline{x}_1 & \xi_3 \end{pmatrix},\quad \beta_3 X=\begin{pmatrix} \xi_2 & x_1 & \overline{x}_3 \\ \overline{x}_1 & \xi_3 & x_2 \\ x_3 & \overline{x}_2 & \xi_1 \end{pmatrix}$$

for $X=X(\xi, x)\in\mathfrak{J}'$. Then as readily seen thay are elements of F_4'. Now, let $\alpha\in z(E_6')$. By the commutativity of $\beta\in F_4'\subset E_6'$, we have $\beta\alpha E=\alpha\beta E=\alpha E$. Hence if we denote αE by $Y=Y(\eta, y)$, then

$$\beta Y=Y, \qquad \text{for all } \beta\in F_4'.$$

From this, putting $\beta=\beta_1, \beta_2$, we get $y_1=y_2=y_3=0$, that is, $Y=\eta_1 E_1+\eta_2 E_2+\eta_3 E_3$. Furthermore, putting $\beta=\beta_3$, we get $\eta_1=\eta_2=\eta_3(=\eta)$, that is, $Y=\eta E$. Since $\alpha\in E_6'$, we have $\eta^3=\det Y=\det\alpha E=\det E=1$. Thus $\eta=1$, that is, $\alpha E=E$, which means that $\alpha\in F_4'$, then α is an element of the center $z(F_4')$ of F_4'. Since $z(F_4')=1$ by [6], we get $\alpha=1$, that is, $z(E_6')=1$.

It is well known that the Lie algebra $\mathfrak{e}_6'$ of E_6' is simple [1], [4]. Now, since E_6' is a connected group from Theorem 6 and a simple Lie group, any normal subgroup of E_6' is contained in the center $z(E_6')$ except E_6' itself. Thus Proposition 7 implies the following

Theorem 8. *The group E_6' is simple (in the algebraic sense) Lie group.*

Since the fundamental group of E_6' is $\boldsymbol{Z}_2$ from Theorem 6 and E_6' is a simple group, we have the following

Theorem 9. *The center $z(\widetilde{E}_6')$ of the non-compact simply connected Lie group $\widetilde{E}_6'=E_{6(6)}$ of type E_6 is $\boldsymbol{Z}_2$.*

7. Generators of E_6'.

Analogously in the case of the non-split type, we define the split Cayley plane Π' by

$$\Pi'=\{A\in\mathfrak{J}' \mid A^2=A,\ \mathrm{tr}(A)=1\}$$
$$=\{A\in\mathfrak{J}' \mid A\times A=0, \mathrm{tr}(A)=1\}.$$

Then, from the straightforward calculations, we have the following formulae

(I) $A\times(Y\times(A\times X))=\dfrac{1}{4}(A, Y)'A\times X,$

(II) $X\times(Y\times(X\times X))=\dfrac{1}{4}((\det X)Y+(X, Y)'X\times X),$

(III) $A\times(A\times X)=\dfrac{1}{4}(X-2A\circ X+(A, X)'A),$

for $A\in\Pi'$, $X, Y\in\mathfrak{J}'$. Therefore, following [5], we can define a mapping ψ':

$\{(A, B)\in\Pi'\times\Pi' \mid (A, B)'\neq 0\}\to E_6'$ $(\psi':\Pi'\to F_4')$ as follows

$$\psi'(A, B)X=\frac{1}{(A, B)'}(8B\times(A\times X)+2(B, X)'A-(A, B)'X)$$

$$(\psi'(A)X=\psi'(A, A)X=X-4A\circ X+4(A, X)'A).$$

Then ψ' (ψ') has the analogous properties of [5], especially it holds

$$\alpha\psi'(A, B)\alpha^{-1}=\psi'(\alpha A, {}'\alpha^{-1}B), \qquad \text{for } \alpha\in E_6'$$

$$(\alpha\psi'(A)\alpha^{-1}=\psi'(\alpha A), \qquad \text{for } \alpha\in F_4').$$

This implies that the subgroup generated by $\{\psi'(A, B)\mid A, B\in\Pi', (A, B)'\neq 0\}$ is a normal subgroup of E_6' (so is F_4'). Hence by Theorem 8 (by [6] Theorem 12), we have the following

Theorem 10. *The group E_6' is generated by $\{\psi'(A, B)\mid A, B\in\Pi', (A, B)'\neq 0\}$ (The group F_4' is generated by $\{\psi'(A)\mid A\in\Pi'\}$).*

8. Homogeneous space E_6'/F_4'.

We consider the space $\mathfrak{J}_1'$ consisting of all elements $X\in\mathfrak{J}'$ such that $\det X=1$:

$$\mathfrak{J}_1'=\{X\in\mathfrak{J}'\mid \det X=1\}.$$

Theorem 11 ([3] Theorem 7). *The group E_6' acts transitively on $\mathfrak{J}_1'$ and the isotropy subgroup of E_6' at E is F_4'. Therefore the homogeneous space E_6'/F_4' is homeomorphic to $\mathfrak{J}_1'$:*

$$E_6'/F_4'\simeq\mathfrak{J}_1'.$$

Proof. We define the linear transformations σ and $\tau=\tau(\lambda_1, \lambda_2, \lambda_3)$ of $\mathfrak{J}'$ respectively by

$$\sigma X=\begin{pmatrix} \xi_1 & x_3e & \overline{ex_2} \\ \overline{x_3e} & -\xi_2 & ex_1e \\ ex_2 & \overline{ex_1e} & -\xi_3 \end{pmatrix},$$

$$\tau X=\begin{pmatrix} \lambda_1\xi_1\lambda_1 & \lambda_1x_3\lambda_2 & \lambda_1\overline{x}_2\lambda_3 \\ \lambda_2\overline{x}_3\lambda_1 & \lambda_2\xi_2\lambda_2 & \lambda_2x_1\lambda_3 \\ \lambda_3x_2\lambda_1 & \lambda_3\overline{x}_1\lambda_2 & \lambda_3\xi_3\lambda_3 \end{pmatrix}, \qquad \lambda_1\lambda_2\lambda_3=1, \ \lambda_i\in\boldsymbol{R}$$

for $X=X(\xi, x)\in\mathfrak{J}'$. Then as readily seen they are elements of E_6'. Now, we shall prove that E_6' acts transitively on $\mathfrak{J}_1'$. To do this, it is sufficient to show that any element of $\mathfrak{J}_1'$ can be transformed to E by some element of E_6'. For any element $Y\in\mathfrak{J}_1'$, as well known there exists $A\in Sp(4)$ such that $fY\in\mathfrak{J}(4, \boldsymbol{H})_0$ is transformed to a diagonal form by the action μ. Namely, there exists $\alpha\in(E_6')_K$ such that αY is a diagonal form

$$\alpha Y=Z=\zeta_1E_1+\zeta_2E_2+\zeta_3E_3, \qquad \zeta_1\zeta_2\zeta_3=1.$$

Here, if there exist $\zeta_i<0$, then we may assume $\zeta_1>0$, $\zeta_2<0$ $\zeta_3<0$ by choosing a suitable element $A\in Sp(4)$ in the above. Hence, transforming Z by σ if necessary, we may assume $\zeta_i>0$, $i=1, 2, 3$. Therefore operate $\tau=\tau(1/\sqrt{\zeta_1}, 1/\sqrt{\zeta_2}, 1/\sqrt{\zeta_3})$ on Z, then we have

$$\tau Z=E.$$

Thus we have proved the transitivity of E_6'. Since the isotropy subgroup of E_6' at E is F_4', we have the following homeomorphism

$$E_6'/F_4'\simeq\mathfrak{J}_1'.$$

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