Non-compact Simple Lie Groups $E_6(-14)$ and $E_6(2)$
of Type $E_6$

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(Received May 17, 1979)

It is known that there exist five simple Lie groups of type $E_6$ up to local
isomorphism, one of them is compact and the others are non-compact. The compact
simple Lie group is given by

$$E_6 = \{ a \in \text{Iso}(\mathfrak{g}_C, \mathfrak{g}_C) \mid \text{det} aX = \text{det} X, \left< aX, aY \right> = \left< X, Y \right> \}$$

where $\mathfrak{g}_C$ is the split exceptional Jordan algebra over the complex numbers $C$ and
$\left< X, Y \right>$ the positive definite Hermitian inner product in $\mathfrak{g}_C$, and it is simply
connected and its center is $Z_8$ [8]. Two of the non-compact simple Lie groups are
given respectively by

$$E_6(-26) = \{ a \in \text{Iso}(\mathfrak{g}_R, \mathfrak{g}_R) \mid \text{det} aX = \text{det} X \},$$

$$E_6(6) = \{ a \in \text{Iso}(S\mathfrak{g}_R, S\mathfrak{g}_R) \mid \text{det} aX = \text{det} X \}$$

where $\mathfrak{g}_R$ (resp. $S\mathfrak{g}_R$) is the exceptional (resp. split exceptional) Jordan algebra over
the real numbers $R$, and their polar decompositions are given respectively by

$$E_6(-26) \cong F_4 \times R^{26}, \quad E_6(6) \cong Sp(4)/Z_4 \times R^{40},$$

and both centers are trivial [1], [3], [5].

In this paper, we find out explicitly the two other non-compact simple Lie
groups. The results are as follows. These groups are given respectively by

$$E_{6,a} = \{ a \in \text{Iso}(\mathfrak{g}_C, \mathfrak{g}_C) \mid \text{det} aX = \text{det} X, \left< aX, aY \right> \sigma = \left< X, Y \right> \sigma \},$$

$$E_{6,r} = \{ a \in \text{Iso}(\mathfrak{g}_C, \mathfrak{g}_C) \mid \text{det} aX = \text{det} X, \left< aX, aY \right> r = \left< X, Y \right> r \}$$

where $\left< X, Y \right> \sigma$ and $\left< X, Y \right> r$ are the Hermitian inner products in $\mathfrak{g}_C$. Their polar
decompositions are given respectively by

$$E_{6,a} \cong (U(1) \times Spin(10))/Z_4 \times R^{26},$$

$$E_{6,r} \cong (Sp(1) \times SU(6))/Z_4 \times R^{40}.$$
and both centers are given by the cyclic group $Z_3 = \{1, \omega, \omega^2\}$, $\omega \in \mathbb{C}$, $\omega^3 = 1$, $\omega \neq 1$, of order 3:

$$z(E_{6,\omega}) = Z_3, \quad z(E_{6,\tau}) = Z_3.$$

I. Non-compact simple Lie group $E_{6,\tau}$ of type $E_6$

1. Jordan algebras $\mathfrak{g}$ and $\mathfrak{g}_c$.

Let $\mathfrak{g}$ be the Cayley algebra over the real numbers $\mathbb{R}$. In this algebra $\mathfrak{g} = H \oplus H_0$ (where $H$ is the quaternion field over $\mathbb{R}$), the multiplication $xy$, the conjugate $\bar{x}$, the scalar part $t(x)$, the inner product $(x, y)$ and the norm $|x|$ are defined respectively by

$$(a+bc)(c+de) = (ac-db) + (bb+da)e,$$

$$\bar{a+be} = \bar{a} - \bar{b}e,$$

$$(a+be, c+de) = (a, c) + (b, d), \quad |x| = \sqrt{(x, x)}.$$

Let $\mathfrak{g}^C = \{x_1 + ix_2 | x_1, x_2 \in \mathbb{C}\}$ be the complexification algebra of $\mathfrak{g}$. In $\mathfrak{g}^C$, the conjugate $\bar{x}$, the scalar part $t(x)$ and the inner product $(x, y)$ are also defined naturally.

Let $\mathfrak{s} = \mathfrak{s}(3, \mathbb{C})$ be the Jordan algebra consisting of all $3 \times 3$ Hermitian matrices with entries in $\mathbb{C}$

$$X = X(\xi, x) = \begin{pmatrix} \xi_1 & x_2 & \bar{x}_3 \\ \bar{x}_2 & \xi_2 & x_1 \\ x_3 & \bar{x}_1 & \xi_3 \end{pmatrix}, \quad \xi_i \in \mathbb{R}, x_i \in \mathbb{C}$$

with respect to the multiplication

$$X \circ Y = \frac{1}{2} (XY + YX).$$

In $\mathfrak{s}$, the inner product $(X, Y)$, the crossed product $X \times Y$, the cubic form $(X, Y, Z)$ and the determinant $\det X$ are defined respectively by

$$(X, Y) = \text{tr}(X \circ Y),$$

$$X \times Y = -\frac{1}{2} (2X \circ Y - \text{tr}(X) Y - \text{tr}(Y) X + (\text{tr}(X) \text{tr}(Y) - (X, Y)) E),$$

$$(X, Y, Z) = (X \times Y, Z) = (X, Y \times Z),$$

$$\det X = -\frac{1}{3} (X, X, X) = \xi_1 \xi_2 \xi_3 + t(x_1 x_2 \bar{x}_3) - \xi_1 x_2 \bar{x}_1 - \xi_2 x_3 \bar{x}_2 - \xi_3 x_1 \bar{x}_3,$$

where $X = X(\xi, x)$ and $E$ is the $3 \times 3$ unit matrix.

Let $\mathfrak{s}^C = \mathfrak{s}(3, \mathbb{C})^C$ be the split exceptional Jordan algebra over the complex numbers $\mathbb{C}$. This Jordan algebra $\mathfrak{s}^C$ may be considered as the complexification of
the Jordan algebra $\mathcal{A}$. Especially any element $X$ of $\mathcal{A}^C$ can be uniquely represented by the form

$$X=X_1+iX_2, \quad X_1, X_2 \in \mathcal{A}, \quad i^2=-1.$$ 

In $\mathcal{A}^C$, the inner product $(X, Y)$, the crossed product $X \times Y$, the cubic form $(X, Y, Z)$ and the determinant $\det X$ are also defined naturally. Moreover we define a mapping, called the complex conjugation, $\tau: \mathcal{A}^C \to \mathcal{A}^C$ by

$$\tau(X_1+iX_2)=X_1-iX_2, \quad X_1, X_2 \in \mathcal{A}$$

and the positive definite Hermitian inner product $\langle X, Y \rangle$ in $\mathcal{A}^C$ by

$$\langle X, Y \rangle=\langle \tau X, Y \rangle.$$

Next, let $\mathfrak{A}_1$ be the Jordan algebra consisting of all $3 \times 3$ $\Gamma$-Hermitian matrices, i.e. $\Gamma X^* \Gamma = X$, where $\Gamma=\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, with entries in $\mathbb{C}$

$$X=X(\xi, x)=\begin{pmatrix} \xi_1 & x_3 & -\bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ x_3 & \bar{x}_1 & \xi_3 \end{pmatrix}, \quad \xi_i \in \mathbb{R}, x_i \in \mathbb{C}$$

with respect to the multiplication $X \circ Y = \frac{1}{2}(XY+YX)$. In $\mathfrak{A}_1$ also, the inner product $(X, Y)$, the crossed product $X \times Y$, the cubic form $(X, Y, Z)$ and the determinant $\det X$ are defined by the quite analogous formulae as in $\mathcal{A}$ (e.g. $\det X=\frac{1}{3}(X, X, X)-\xi_1\xi_2\xi_3+(x_1x_3x_2)-\xi_1x_3\bar{x}_2+\xi_2x_2\bar{x}_1+\xi_3x_1\bar{x}_3$).

Furthermore let $\mathfrak{A}_1^C$ be the complexification of the Jordan algebra $\mathfrak{A}_1$ and also in $\mathfrak{A}_1^C$ the inner product $(X, Y)$, the crossed product $X \times Y$, the cubic form $(X, Y, Z)$ and the determinant $\det X$ are naturally defined. Finally we define the Hermitian inner product $\langle X, Y \rangle$ in $\mathfrak{A}_1^C$ by

$$\langle X, Y \rangle=\langle \tau X, Y \rangle$$

where $\tau(X_1+iX_2)=X_1-iX_2$ for $X_1, X_2 \in \mathfrak{A}_1$.

From now on, we will use the same notations for the same operations in $\mathcal{A}$ and $\mathfrak{A}_1$, but as occasion demands the notations in $\mathfrak{A}_1$ will be indexed by the figure 1.

**Proposition 1.** $\mathfrak{A}_1^C$ is isomorphic to $\mathcal{A}^C$ as Jordan algebra over $\mathbb{C}$ by an isomorphism $f: \mathfrak{A}_1^C \to \mathcal{A}^C$ defined as follows:

$$fX=\Gamma_1 X \Gamma_1^*, \quad \Gamma_1=\begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

And $f$ satisfies the following properties.
(i) \((X, Y)_1 = (fX, fY)\),
(ii) \(\det X = \det fX\),
(iii) \(\langle X, Y \rangle_1 = \langle fX, fY \rangle_\sigma\)

where \(\sigma: \mathbb{C}^2 \to \mathbb{C}^2\) is the linear involution defined by
\[
\sigma \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1 & -x_2 \\ -x_3 & x_4 \end{pmatrix}
\]
and the inner product \(\langle X, Y \rangle_\sigma\) in \(\mathbb{C}^2\) is defined by
\[
\langle X, Y \rangle_\sigma = \langle \sigma X, Y \rangle.
\]

**Proof.** It is easy to see that \(f\) is a linear isomorphism over \(\mathbb{C}\) and satisfies
\(f(X \circ Y) = fX \circ fY\). And
(i) \((X, Y)_1 = \text{tr}(X \circ Y) = \text{tr}(f(X \circ Y)) = \text{tr}(fX \circ fY) = \langle fX, fY \rangle\).
(ii) We have immediately \(\det X = \det fX\).
(iii) Since we have \(f \tau X = \tau fX\), we have
\[
\langle X, Y \rangle_1 = \langle \tau X, Y \rangle_1 = \langle f \tau X, fY \rangle = \langle \tau fX, fY \rangle = \langle \sigma fX, fY \rangle = \langle fX, fY \rangle_\sigma.
\]

2. Groups of type \(E_6\) and \(F_4\).

The group \(E_{6,\sigma}\) is defined to be the group of linear isomorphisms of \(\mathbb{C}^6\) leaving the determinant \(\det X\) and the Hermitian inner product \(\langle X, Y \rangle_\sigma\) invariant:
\[
E_{6,\sigma} = \{\alpha \in \text{Isoc}(\mathbb{C}^6, \mathbb{C}^6) | \det \alpha X = \det X, \langle \alpha X, \alpha Y \rangle_\sigma = \langle X, Y \rangle_\sigma\}
\]
\[
= \{\alpha \in \text{Isoc}(\mathbb{C}^6, \mathbb{C}^6) | \langle \alpha X, \alpha Y, \alpha Z \rangle = \langle X, Y, Z \rangle, \langle \alpha X, \alpha Y \rangle_\sigma = \langle X, Y \rangle_\sigma\}
\]
and \(F_{4,\sigma}\) the subgroup of \(E_{6,\sigma}\) preserving the inner product \((X, Y)\):
\[
F_{4,\sigma} = \{\alpha \in E_{6,\sigma} | \langle \alpha X, \alpha Y \rangle = (X, Y)\}
\]
\[
= \{\alpha \in E_{6,\sigma} | \alpha E = E\}.
\]

Next, to consider the group \(E_{6,\sigma}\) we need to define the group \(E_6_{0,1}\) and the subgroup \(F_{4,1}\) of \(E_{6,1}\):
\[
E_{6,1} = \{\alpha \in \text{Isoc}(\mathbb{C}^6, \mathbb{C}^6) | \det \alpha X = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}
\]
\[
= \{\alpha \in \text{Isoc}(\mathbb{C}^6, \mathbb{C}^6) | \langle \alpha X, \alpha Y, \alpha Z \rangle = \langle X, Y, Z \rangle, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\},
\]
\[
F_{4,1} = \{\alpha \in E_{6,1} | \alpha X = (X, Y)\}
\]
\[
= \{\alpha \in E_{6,1} | \alpha E = E\}.
\]

Finally we shall recall the compact group \(E_6\) and the compact subgroup \(F_4\) of \(E_6\):
\[
E_6 = \{\alpha \in \text{Isoc}(\mathbb{C}^6, \mathbb{C}^6) | \det \alpha X = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}
\]
\[
= \{\alpha \in \text{Isoc}(\mathbb{C}^6, \mathbb{C}^6) | \langle \alpha X, \alpha Y, \alpha Z \rangle = \langle X, Y, Z \rangle, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\},
\]
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\[ F_4 = \{ \alpha \in E_6 | \alpha(X, \alpha Y) = (X, Y) \} \]
\[ = \{ \alpha \in E_6 | \alpha E = E \}. \]

**Lemma 2.** The group $F_{4,41}$ is homeomorphic to $\text{Spin}(9) \times \mathbb{R}^{16}$ and a simple (in the sense of the center $z(F_{4,41}) = 1$) Lie group of type $F_4$.

**Proof.** We define the group $F_{4(-2e)}$ by
\[ F_{4(-2e)} = \{ \alpha \in \text{IsoR}(\mathbb{R}_+, \mathbb{R}_-) | \alpha(X, Y) = (\alpha X, \alpha Y) \} \]
\[ = \{ \alpha \in E_{4(-2e)} | \alpha E = E \} \]
where $E_{4(-2e)} = \{ \alpha \in \text{IsoR}(\mathbb{R}_+, \mathbb{R}_-) | \det \alpha X = \det X \}$. Then the argument used in the proof of Proposition 1 of [8] shows that $F_{4(-20)}$ is isomorphic to $F_{4,41}$ by the complexification $\alpha \mapsto \alpha^c$ (which means $\alpha^c(X_1 + iX_2) = \alpha X_1 + i\alpha X_2$, $X_1, X_2 \in \mathbb{R}_+$). Recall now that $F_{4(-20)}$ is homeomorphic to $\text{Spin}(9) \times \mathbb{R}^{16}$ and a simple (in the sense of the center $z(F_{4,-20}) = 1$) Lie group of type $F_4$ (Theorem 8 and 11 [6]), then results follow.

**Proposition 3.** The group $E_{6,4}$ is isomorphic to the group $E_{6,41}$ and also $F_{4,4}$ to $F_{4,41}$. In particular, $F_{4,4}$ is homeomorphic to $\text{Spin}(9) \times \mathbb{R}^{16}$ and a simple (in the sense of the center $z(F_{4,4}) = 1$) Lie group of type $F_4$.

**Proof.** By using the isomorphism $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ in Proposition 1, we define a mapping $\phi: E_{6,4} \rightarrow E_{6,41}$ by
\[ \phi(\alpha)X = f^{-1}(\alpha fX), \quad X \in \mathbb{R}_+. \]
Then from Proposition 1 it is easily obtained that $\phi$ gives an isomorphism between $E_{6,4}$ and $E_{6,41}$. Furthermore we can readily show that the restriction $\phi|F_{4,4}$ gives an isomorphism between $F_{4,4}$ and $F_{4,41}$.

**Remark.** Let the group $E_{6(-20)}$ and its subgroup $F_{4(-20)}$ be defined respectively by
\[ E_{6(-20)} = \{ \alpha \in \text{IsoR}(\mathbb{R}_+, \mathbb{R}_-) | \det \alpha X = \det X \}, \]
\[ F_{4(-20)} = \{ \alpha \in E_{6(-20)} | \alpha(X, \alpha Y) = (X, Y) \} \]
\[ = \{ \alpha \in E_{6(-20)} | \alpha E = E \} \]
where $(X, Y)_\sigma = (\sigma X, Y)$. Then we have already known that
\[ E_{6(-20)} \cong F_4 \times \mathbb{R}^{20}, \quad z(E_{6(-20)}) = 1 \quad ([1], \quad [3]), \]
\[ F_{4(-20)} \cong \text{Spin}(9) \times \mathbb{R}^{16}, \quad z(F_{4(-20)}) = 1 \quad ([6]). \]
Now, define a mapping $g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by
then $g$ is a linear isomorphism over $\mathbb{R}$ and satisfies the properties $\det X = -\det gX$ and $(X, Y)_A = (gX, gY)_A$. We see therefore that the mapping $\varphi' : E_6(-26) \rightarrow E_1^4(-26)$ defined by

$$
\varphi'(\alpha)X = g^{-1}a gX, \quad X \in \mathbb{R}
$$
gives an isomorphism between $E_6(-26)$ and $E_1^4(-26)$ and that the restriction $\varphi'|E_4(-26)$ gives one between $F_4(-26)$ and $F_4(-26)$.

3. Lie algebra $\mathfrak{e}_{6,\sigma}$ of $E_6,\sigma$.

We consider the Lie algebra $\mathfrak{e}_{6,\sigma}$ of $E_6,\sigma$:

$$
\mathfrak{e}_{6,\sigma} = \{ \xi \in \text{Hom}(\mathfrak{g}, \mathfrak{g}) \mid (\xi X, X, X) = 0, <\xi X, Y>_\sigma = -<X, \xi Y>_\sigma \}.
$$

Theorem 4. Any element $\xi$ of the Lie algebra $\mathfrak{e}_{6,\sigma}$ of the group $E_6,\sigma$ is uniquely represented by the form

$$
\xi = \delta + \tilde{S}, \quad \delta \in \mathfrak{e}_{6,\sigma}, \quad S = \begin{pmatrix}
0 & s_3 & \bar{s}_2 \\
\bar{s}_2 & 0 & 0 \\
s_2 & 0 & 0
\end{pmatrix}
$$

where $\sum \sigma_i = 0$, $\sigma_i \in \mathbb{R}$, $s_i \in \mathbb{C}$ and $\mathfrak{e}_{6,\sigma} = \{ \delta \in \mathfrak{e}_{6,\sigma} \mid (\delta X, Y) = -<X, \delta Y> = -<\delta E = 0 \}$
is the Lie algebra of the group $F_{4,\sigma}$ and, for $S$, $\tilde{S} \in \text{Hom}(\mathfrak{g}, \mathfrak{g})$ is defined by $\tilde{S}X = S\circ X$. In particular, the type of the Lie group $E_6,\sigma$ is $E_6$.

Proof. It is easily seen by the analogous argument as in the proof of Theorem 2 of [8].

4. Compact subgroup $(E_6,\sigma)_K$ of $E_6,\sigma$.

We shall consider the following subgroup $(E_6,\sigma)_K$ of $E_6,\sigma$:

$$(E_6,\sigma)_K \equiv \{ \alpha \in E_6,\sigma \mid <\alpha X, \alpha Y> = <X, Y>_\sigma \}$$

$$= \{ \alpha \in E_6 \mid <\alpha X, \alpha Y>_\sigma = <X, Y>_\sigma \}.$$

To do this, we need some preparations. Following [8], we first define the subgroups $E_6$ of $E_6$ and $E_{6,1}$ of $E_6$ by

$$E_6 = \{ \alpha \in E_6 \mid \sigma \alpha = \alpha \}$$

$$E_{6,1} = \{ \alpha \in E_6 \mid \alpha E_1 = E_1 \}$$

where $E_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$. Then we have already known the following
Lemma 5. (Proposition 11 [8]). The group $E_{a,1}$ is isomorphic to the spinor group $Spin(10)$.

From now on, we identify the group $E_{a,1}$ with the group $Spin(10)$.

We next define the subgroup $U(1)$ of $E_{a,1}$ by

$$U(1) = \{ \phi(\theta) | \phi(\theta) X(\xi, x) = \begin{pmatrix} \theta \xi_1 & \theta x_3 & \theta x_2 \\ \theta x_3 & \theta^{-2} \xi_2 & \theta^{-2} x_1 \\ \theta x_2 & \theta^{-2} x_1 & \theta^{-2} \xi_3 \end{pmatrix}, \theta \in \mathbb{C}, |\theta| = 1 \}.$$  

It is obvious that the group $U(1)$ is isomorphic to the usual unitary group $U(1) = \{ \theta \in \mathbb{C} | |\theta| = 1 \}$. Furthermore we have known that the subgroups $U(1)$ and $Spin(10)$ of $E_6$ commute elementwisely (Lemma 12 [8]).

Finally we denote by $\alpha^*$ and $\hat{\alpha}$ the transpose of $\alpha \in \text{Isoc}(\mathfrak{z}^C, \mathfrak{z}^C)$ relative to $\langle X, Y \rangle$ and $\langle X, Y \rangle_\sigma$ respectively:

$$\langle \alpha X, Y \rangle = \langle X, \alpha^* Y \rangle, \quad \langle \alpha X, Y \rangle_\sigma = \langle X, \hat{\alpha} Y \rangle_\sigma.$$

Then it holds generally

$$\hat{\alpha} = \sigma \alpha^* \sigma, \quad \sigma \in \text{Isoc}(\mathfrak{z}^C, \mathfrak{z}^C),$$

since we have $\langle X, \hat{\alpha} Y \rangle = \langle \alpha X, \hat{\alpha} Y \rangle_\sigma = \langle \sigma \alpha X, Y \rangle_\sigma = \langle X, \alpha^* \sigma Y \rangle = \langle X, \alpha^* \sigma Y \rangle$, noting that $\sigma = \sigma^* = \hat{\sigma}$.

Proposition 6. The group $(E_{a,a})_K$ is isomorphic to the group $(U(1) \times Spin(10))/\mathbb{Z}_4$ where $\mathbb{Z}_4 = \{(1, 1), (-1, 1), (i, 1), (-i, 1)\}$.

Proof. First we shall show that $(E_{a,a})_K = E_a$. Let $\alpha$ be an element of $(E_{a,a})_K$, that is, $\alpha a = \alpha \sigma = 1$, then from $\hat{\alpha} = \sigma \alpha^* \sigma$ we have $\sigma \alpha \sigma = \alpha$, that is, $\alpha \in E_a$. Conversely, let $\alpha$ be an element of $E_a$, then we have $\alpha \hat{\alpha} = \alpha \sigma \sigma = \alpha \sigma \sigma = \sigma = 1$, that is, $\alpha \in (E_{a,a})_K$. Now, we have already known that a homomorphism $\varphi: U(1) \times Spin(10) \to E_a = (E_{a,a})_K$ defined by $\varphi(\theta, \beta) = \phi(\theta) \beta$ induces an isomorphism $(E_{a,a})_K \cong (U(1) \times Spin(10))/\mathbb{Z}_4$ (Theorem 13 [8]). Thus Proposition 6 is proved.

5. Polar decomposition of $E_{a,a}$.

To give a polar decomposition of $E_{a,a}$, we use the following

Lemma 7 ([2] pp. 345). Let $G$ be a pseudoalgebraic subgroup of the general linear group $GL(n, \mathbb{C})$ such that the condition $A \in G$ implies $A^* \in G$. Then $G$ is homeomorphic to the topological product of $G \cap U(n)$ (which is a maximal compact subgroup of $G$) and a Euclidean space $\mathbb{R}^d$:

$$G \cong (G \cap U(n)) \times \mathbb{R}^d, \quad d = \text{dim} G - \text{dim}(G \cap U(n))$$

where $U(n)$ is the unitary subgroup of $GL(n, \mathbb{C})$.

To use the above Lemma, first of all we show the following

Lemma 8. $E_{a,a}$ is a pseudoalgebraic subgroup of the general linear group $GL(27, \mathbb{C}) = \text{Isoc}(\mathfrak{z}^C, \mathfrak{z}^C)$ and satisfies the condition $\alpha \in E_{a,a}$ implies $\alpha^* \in E_{a,a}$.

Proof. Since $\hat{\alpha} = \alpha \sigma^* \sigma, \alpha \hat{\alpha} = 1$ for $\alpha \in E_{a,a}$, we have $\alpha^* = \sigma \alpha^* \sigma \in E_{a,a}$. It is obvious
that \( E_{6,\sigma} \) is pseudoalgebraic, because \( E_{6,\sigma} \) is defined by the pseudoalgebraic relations \( \det \alpha X = \det X \) and \( \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle \).

Let \( U(3^C) \) be the unitary subgroup of \( \text{Isoc}(3^C, 3^C) \):

\[
U(2^7) = U(3^C) = \{ \alpha \in \text{Isoc}(3^C, 3^C) | \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle \}.
\]

Then we have

\[
E_{6,\sigma} \cap U(3^C) = (E_{6,\sigma})_\text{R} \cong (U(1) \times \text{Spin}(10))/\mathbb{Z}_4
\]

by Proposition 6. Finally we shall determine the dimension of the Euclidean part of \( E_{6,\sigma} \). Since \( E_{6,\sigma} \) is a simple Lie group of type \( E_6 \) by Theorem 4, the dimension \( d \) is obtained by

\[
d = \dim E_{6,\sigma} - \dim (U(1) \times \text{Spin}(10)) = 78 - 46 = 32.
\]

Thus we get the following

**Theorem 9.** The group \( E_{6,\sigma} \) is homeomorphic to the topological product of the group \( (U(1) \times \text{Spin}(10))/\mathbb{Z}_4 \) and a 32-dim. Euclidean space \( \mathbb{R}^{32} \):

\[
E_{6,\sigma} \cong (U(1) \times \text{Spin}(10))/\mathbb{Z}_4 \times \mathbb{R}^{32}.
\]

In particular, \( E_{6,\sigma} \) is a connected (but not simply connected) Lie group.

**6. Center** \( z(E_{6,\sigma}) \) of \( E_{6,\sigma} \).

**Lemma 10.** For \( a \in \mathbb{C} \), \( a \neq 0 \), the mapping \( \alpha(a) : \mathbb{C}^3 \to \mathbb{C}^3 \) defined by \( \alpha(a)X(\xi, x) = Y(\eta, y) \) belongs to \( E_{6,\sigma} \), where

\[
\begin{align*}
\eta_1 &= \frac{\xi_1 - \xi_2}{2} + \frac{\xi_1 + \xi_3}{2} \cosh |a| + \frac{(a, x_3)}{|a|} \sinh |a|, \\
\eta_2 &= \xi_2, \\
\eta_3 &= -\frac{\xi_1 - \xi_3}{2} + \frac{\xi_1 + \xi_3}{2} \cosh |a| + \frac{(a, x_2)}{|a|} \sinh |a|, \\
\eta_4 &= x_1 \cosh \frac{|a|}{2} + \frac{a \bar{x}_3}{|a|} \sinh \frac{|a|}{2}, \\
\eta_5 &= x_2 + \frac{2(a, x_2) a}{|a|^2} \sinh^2 \frac{|a|}{2} + \frac{\xi_1 + \xi_3}{2 |a|} \sinh |a|, \\
\eta_6 &= x_3 \cosh \frac{|a|}{2} + \frac{\bar{x}_3 a}{|a|} \sinh \frac{|a|}{2}.
\end{align*}
\]

**Proof.** Since, for \( F_2(a) = \begin{pmatrix} 0 & 0 & \bar{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix} \), \( \tilde{F}_3(a) \) is an element of \( E_{6,\sigma} \) by Theorem 4, it follows \( \alpha(a) = \exp \tilde{F}_3(a) \in E_{6,\sigma} \).
Theorem 11. The center \( z(E_{6, \omega}) \) of the group \( E_{6, \omega} \) is isomorphic to the cyclic group \( \mathbb{Z}_3 \) of order 3:

\[
z(E_{6, \omega}) = \mathbb{Z}_3 = \{1, \omega 1, \omega 2 \}, \quad \omega \in C, \quad \omega^3 = 1, \quad \omega \neq 1.
\]

Proof. Let \( \alpha \in z(E_{6, \omega}) \). From the commutativity with \( \sigma \in E_{6, \omega} \), we have \( \sigma \alpha = \alpha \sigma \), that is, \( \alpha \in (E_{6, \omega})_R \). Hence there exists an element \( (\theta, \beta) \in U(1) \times Spin(10) \) such that \( \alpha = \phi(\theta, \beta) = \phi(\theta) \beta \) by Proposition 6. Moreover we see that \( \beta \) is an element of the center \( z(Spin(10)) \), noting that the groups \( U(1) \) and \( Spin(10) \) commute elementwisely.

In fact, it holds \( \phi(\theta) \beta \beta' = \beta' \phi(\theta) \beta = \phi(\theta) \beta' \beta \), hence \( \beta \beta' = \beta' \beta \) for all \( \beta' \in Spin(10) \). Now, as is well known, the order of \( z(Spin(10)) \) is 4 and obviously \( \phi(\varepsilon) \in z(Spin(10)) \) for \( \varepsilon = \pm 1, \pm i \), therefore we have

\[
z(Spin(10)) = \{ \phi(1), \phi(-1), \phi(i), \phi(-i) \} \subset U(1).
\]

Hence \( \alpha = \phi(\theta') \in U(1) \) for some \( \theta' \in C, \ |	heta'| = 1 \). Next, from the commutativity with \( \sigma(a) \in E_{6, \omega} \) as in Lemma 10, we have \( \sigma \alpha(a) E = \alpha(a) \alpha E \), that is,

\[
\begin{pmatrix}
\lambda \cosh |a| & 0 & 0 \\
0 & \mu & 0 \\
0 & 0 & \mu \cosh |a|
\end{pmatrix} =
\begin{pmatrix}
\frac{\lambda - \mu + \lambda + \mu}{2} \cosh |a| & 0 & 0 \\
0 & \mu & 0 \\
0 & 0 & \frac{\lambda - \mu + \lambda + \mu}{2} \cosh |a|
\end{pmatrix}
\]

where we denote \( \theta' \) by \( \lambda \) and \( \theta'^{-2} \) by \( \mu \). Hence we have \( \lambda = \mu (= \omega) \), that is,

\[
aE = \omega E
\]

where \( \omega \in C \) and \( \omega^3 = \text{det} \alpha E = \text{det} E = 1 \). Since \( \omega 1 \in z(E_{6, \omega}) \), we have \( \omega^{-1} \alpha \in z(E_{6, \omega}) \) and \( \omega^{-1} \alpha E = E \), hence \( \omega^{-1} \alpha \in z(F_{4, \omega}) \). Therefore it follows that \( \omega^{-1} \alpha = 1 \), that is, \( \alpha = \omega 1 \), since \( z(F_{4, \omega}) = 1 \) by proposition 3. Thus the proof of Theorem 11 is completed.

II. Non-compact simple Lie group \( E_{6, \tau} \) of type \( E_6 \)

7. Split Jordan algebra \( \mathfrak{S}_E \).

Let \( \mathfrak{C} \) be the split Cayley algebra over \( \mathbb{R} \). In \( \mathfrak{C} = H \oplus He' \), the multiplication \( xy \), the conjugate \( \bar{x} \), the scalar part \( t(x) \) and the inner product \( (x, y)' \) are defined respectively by

\[
(a + be')(c + de') = (ac + db) + (bc + da)e',
\]

\[
\bar{a + be'} = \bar{a} - be', \quad t(x) = x + \bar{x},
\]

\[
(a + be', c + de')' = (a, c) - (b, d).
\]
Let \( \mathfrak{C}' \) be the complexification algebra of \( \mathfrak{C}' \). In \( \mathfrak{C}' \), the conjugate \( \bar{x} \), the scalar part \( t(x) \) and the inner product \( (x, y)' \) are also defined naturally. The mapping \( k : \mathfrak{C}' \to \mathfrak{C}' \) defined by
\[
k(a+be') + i(c+de') = (a+de) + i(c-be)
\]
gives an isomorphism as algebra over \( \mathbb{C} \) and satisfies
\[
k(x) = k(x), \quad (x, y)' = (k(x), k(y)).
\]

Let \( \mathfrak{S}_3 = \mathfrak{S}(3, \mathfrak{C}') \) be the Jordan algebra consisting of all \( 3 \times 3 \) Hermitian matrices with entries in \( \mathfrak{C}' \)
\[
X = X(\xi, x) = \begin{pmatrix}
x_1 & x_3 & \bar{x}_2 \\
\bar{x}_3 & x_2 & x_1 \\
\bar{x}_2 & x_1 & x_3
\end{pmatrix}, \quad \xi \in \mathbb{R}, x_i \in \mathfrak{C}'
\]
with respect to the multiplication \( X \cdot Y = \frac{1}{2} (XY + YX) \). In \( \mathfrak{S}_3 \) also, the inner product \( \langle X, Y \rangle \), the crossed product \( X \times Y \), the cubic form \( \langle X, Y, Z \rangle \) and the determinant \( \det X \) are defined by the quite same formulae in \( \mathfrak{S} \).

Furthermore the complexification \( \mathfrak{S}_3^\mathbb{C} \) of \( \mathfrak{S}_3 \) and the several operations in \( \mathfrak{S}_3^\mathbb{C} \) are also similar to the definitions in the section 1.

From now on, we will use the same notations for the same operations in \( \mathfrak{S} \) and \( \mathfrak{S}_3 \), but as occasion demands the notations in \( \mathfrak{S}_3 \) will be indexed by the figure 2.

**Proposition 12.** \( \mathfrak{S}_3^\mathbb{C} \) is isomorphic to \( \mathfrak{S}' \) as Jordan algebra over \( \mathbb{C} \) by an isomorphism \( h : \mathfrak{S}_3^\mathbb{C} \to \mathfrak{S}' \) defined as follows:
\[
hX(\xi, x) = X(\xi, h(x)).
\]
And \( h \) satisfies the following properties.

(i) \( \langle X, Y \rangle_3 = \langle hX, hY \rangle \),
(ii) \( \det X = \det hX \),
(iii) \( \langle X, Y \rangle_3 = \langle hX, hY \rangle_\gamma \)
where \( \gamma : \mathfrak{S}' \to \mathfrak{S}' \) is the linear involution defined by
\[
\gamma X(\xi, a+be) = X(\xi, a-be)
\]
where \( \xi \in \mathfrak{C}, a, b \in \mathfrak{H}' \) and the inner product \( \langle X, Y \rangle_\gamma \) in \( \mathfrak{S}' \) is defined by
\[
\langle X, Y \rangle_\gamma = \langle \gamma X, Y \rangle.
\]

**Proof.** It is easy to see that \( h \) is a linear isomorphism over \( \mathbb{C} \) and satisfies \( h(X \cdot Y) = hX \cdot hY \). The properties (i), (ii) and (iii) are shown similarly in the proof of Proposition 1.
Non-compact Simple Lie Groups $E_6(\mathbb{C})$ and $E_6(\mathbb{C}_2)$ of Type $E_6$


The group $E_6(\mathbb{C})$ is defined to be the group of linear isomorphisms of $\mathbb{C}^6$ leaving the determinant $\det X$ and the Hermitian inner product $\langle X, Y \rangle$ invariant:

$$E_6(\mathbb{C}) = \{ \alpha \in \text{Isoc}(\mathbb{C}^6, \mathbb{C}^6) | \det \alpha X = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle \}
= \{ \alpha \in \text{Isoc}(\mathbb{C}^6, \mathbb{C}^6) | (\alpha X, \alpha Y, \alpha Z) = (X, Y, Z), \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle \}$$

and $F_4(\mathbb{C})$ the subgroup of $E_6(\mathbb{C})$ preserving the inner product $(X, Y)$:

$$F_4(\mathbb{C}) = \{ \alpha \in E_6(\mathbb{C}) | \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle \}
= \{ \alpha \in E_6(\mathbb{C}) | \alpha E = E \}.$$ 

Next, to consider the group $E_6(\mathbb{C})$ we need to define the group $E_6(\mathbb{C}_2)$ and the subgroup $F_4(\mathbb{C}_2)$ of $E_6(\mathbb{C}_2)$:

$$E_6(\mathbb{C}_2) = \{ \alpha \in \text{Isoc}(\mathbb{C}_2, \mathbb{C}_2) | \det \alpha X = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle \}
= \{ \alpha \in \text{Isoc}(\mathbb{C}_2, \mathbb{C}_2) | (\alpha X, \alpha Y) = (X, Y), \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle \}.$$ 

$$F_4(\mathbb{C}_2) = \{ \alpha \in E_6(\mathbb{C}_2) | \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle \}
= \{ \alpha \in E_6(\mathbb{C}_2) | \alpha E = E \}.$$ 

Lemma 13. The group $F_4(\mathbb{C}_2)$ is homeomorphic to $(Sp(1) \times Sp(3))/Z_2 \times \mathbb{R}^{28}$ and a simple (in the sense of the center $z(F_4(\mathbb{C}_2)) = 1$) Lie group of type $F_4$.

Proof. We define the group $F'_4(\mathbb{C})$ by

$$F'_4(\mathbb{C}) = \{ \alpha \in \text{Isoc}(\mathbb{C}_2, \mathbb{C}_2) | \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle \}
= \{ \alpha \in E_6(\mathbb{C}_2) | \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle \}
= \{ \alpha \in E_6(\mathbb{C}_2) | \alpha E = E \}.$$ 

where $E_6(\mathbb{C}_2) = \{ \alpha \in \text{Isoc}(\mathbb{C}_2, \mathbb{C}_2) | \det \alpha X = \det X \}$. Then the argument used in the proof of Proposition 1 of [8] shows that $F'_4(\mathbb{C})$ is isomorphic to $F_4(\mathbb{C}_2)$ by the complexification $a \mapsto \alpha a$. Recall now that $F'_4(\mathbb{C})$ is homeomorphic to $(Sp(1) \times Sp(3))/Z_2 \times \mathbb{R}^{28}$ and a simple (in the sense of the center $z(F'_4(\mathbb{C})) = 1$) Lie group of type $F_4$ [7], then the results follow.

Proposition 14. The group $E_6(\mathbb{C})$ is isomorphic to the group $E_6(\mathbb{C}_2)$ and also $F_4(\mathbb{C})$ to $F_4(\mathbb{C}_2)$. In particular, $F_4(\mathbb{C})$ is homeomorphic to $(Sp(1) \times Sp(3))/Z_2 \times \mathbb{R}^{28}$ and a simple (in the sense of the center $z(F_4(\mathbb{C})) = 1$) Lie group of type $F_4$.

Proof. By using the isomorphism $h : \mathbb{C}^6 \to \mathbb{C}^6$ in Proposition 12, we define a mapping $\phi : E_6(\mathbb{C}) \to E_6(\mathbb{C}_2)$ by

$$\phi(a)X = h^{-1}ahX, \quad X \in \mathbb{C}^6.$$ 

Then from proposition 12 it is easily obtained that $\phi$ gives an isomorphism between
9. Lie algebra $\mathfrak{e}_{6,7}$ of $E_{6,7}$.

We consider the Lie algebra $\mathfrak{e}_{6,7}$ of $E_{6,7}$:

$$\mathfrak{e}_{6,7} = \{ g \in \text{Hom}_c(\mathbb{R}^6, \mathbb{R}^6) \mid \langle \delta X, Y \rangle = \langle X, \delta Y \rangle \}.$$ 

Theorem 15. Any element $\zeta$ of the Lie algebra $\mathfrak{e}_{6,7}$ of the group $E_{6,7}$ is uniquely represented by the form

$$\zeta = \delta + \tilde{S},$$

where $\sum \tau_i = 0$, $\tau_i \in \mathbb{R}$, $t_i \in \mathbb{R}$ and $\{ \tau \in \mathfrak{e}_{6,7} \mid \langle \delta X, Y \rangle = \langle X, \delta Y \rangle = 0 \}$ is the Lie algebra of the group $F_{4,7}$ and, for $S, \tilde{S} \in \text{Hom}_c(\mathbb{R}^6, \mathbb{R}^6)$ is defined by $\tilde{S}X = S\cdot X$. In particular, the type of the Lie group $E_{6,7}$ is $E_6$.

Proof. It is easily seen by the analogous argument as in the proof of Theorem 2 of [8].

10. Compact subgroup $(E_{6,7})_K$ of $E_{6,7}$.

We shall consider the following subgroup $(E_{6,7})_K$ of $E_{6,7}$:

$$(E_{6,7})_K = \{ \alpha \in E_6 \mid \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle \} = \{ \alpha \in E_6 \mid \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle \}.$$

To do this, we need some preparations. Following [8], we first define the subgroup $E_r$ of $E_6$ by

$$E_r = \{ \alpha \in E_6 \mid r \alpha = \alpha \}.$$

Next we denote by $'a$ the transpose of $\alpha \in \text{Isoc}(\mathbb{R}^6, \mathbb{R}^6)$ relative to $\langle X, Y \rangle = \langle \alpha X, Y \rangle = \langle X, \alpha Y \rangle$. Then it holds similarly in the section 4,

$$'a = r^{-1} \alpha r, \quad \alpha \in \text{Isoc}(\mathbb{R}^6, \mathbb{R}^6),$$

noting that $r = r^* = r^{-1}$.

Proposition 16. The group $(E_{6,7})_K$ is isomorphic to the group $(Sp(1) \times SU(6))/Z_2$ where $Z_2 = \{ (1, E), (-1, -E) \}$.

Proof. By the proof similar to that of Proposition 6, it follows that $(E_{6,7})_K = E_r$. On the other hand, we have already known that $E_r$ is isomorphic to the group $(Sp(1) \times SU(6))/Z_2$ (Theorem 16 [8]). Thus Proposition 16 is proved.

11. Polar decomposition of $E_{6,7}$.

To use Lemma 7, first of all we show the following
Lemma 17. $E_{6, r}$ is a pseudoalgebraic subgroup of the general linear group $GL(n, \mathbb{C}) = IsoC(\mathbb{C}, \mathbb{C})$ and satisfies the condition $a \in E_{6, r}$ implies $a^* \in E_{6, r}$.

Proof. Since $a = a^{-1} a$, $a \in E_{6, r}$, we have $a^* = a^{-1} a \in E_{6, r}$. It is obvious that $E_{6, r}$ is pseudoalgebraic, because $E_{6, r}$ is defined by the pseudoalgebraic relations $\det X = \det X$ and $\langle aX, aY \rangle = \langle X, Y \rangle$.

Next, let $U(\mathbb{C})$ be the unitary subgroup of $IsoC(\mathbb{C}, \mathbb{C})$ as in the section 5, then we have

$$E_{6, r} \cap U(\mathbb{C}) = (E_{6, r})_{K} \cong (Sp(1) \times SU(6))/\mathbb{Z}_2$$

by Proposition 16. Finally we shall determine the dimension of the Euclidean part of $E_{6, r}$. Since $E_{6, r}$ is a simple Lie group of type $E_6$ by Theorem 15, the dimension $d$ is obtained by

$$d = \dim E_{6, r} - \dim (Sp(1) \times SU(6)) = 78 - 38 = 40.$$ 

Thus we get the following

**Theorem 18.** The group $E_{6, r}$ is homeomorphic to the topological product of the group $(Sp(1) \times SU(6))/\mathbb{Z}_2$ and a 40-dim. Euclidean space $\mathbb{R}^{40}$:

$$E_{6, r} \cong (Sp(1) \times SU(6))/\mathbb{Z}_2 \times \mathbb{R}^{40}.$$ 

In particular, $E_{6, r}$ is a connected (but not simply connected) Lie group.

12. Center $z(E_{6, r})$ of $E_{6, r}$.

**Theorem 19.** The center $z(E_{6, r})$ of the group $E_{6, r}$ is isomorphic to the cyclic group $\mathbb{Z}_3$ of order 3:

$$z(E_{6, r}) = \mathbb{Z}_3 = \{1, \omega, \omega^2\}, \quad \omega \in \mathbb{C}, \quad \omega^3 = 1, \quad \omega \neq 1.$$ 

Proof. We define the linear transformations $\beta_i, i = 1, 2, 3$ of $\mathbb{C}$ by

$$\beta_1 X = \begin{pmatrix} \xi_1 - x_3 - x_2 \\ -x_3 \xi_2 - x_1 \\ -x_2 x_1 - x_1 \end{pmatrix}, \quad \beta_2 X = \begin{pmatrix} \xi_1 - x_3 - x_2 \\ -x_3 \xi_2 - x_1 \\ -x_2 x_1 - x_1 \end{pmatrix}, \quad \beta_3 X = \begin{pmatrix} \xi_2 & x_1 & x_3 \\ x_1 & \xi_3 & x_2 \\ x_2 & x_1 & \xi_3 \end{pmatrix}$$

for $X = X(\xi, x) \in \mathbb{C}$. Then as readily seen they are elements of $E_{6, r}$. Now, let $\alpha \in z(E_{6, r})$. From the commutativity with the above $\beta_i, i = 1, 2, 3$, that is, $\beta_i \alpha E = \alpha \beta_i E = \alpha E$, we have

$$\alpha E = \omega E, \quad \omega \in \mathbb{C}, \quad \omega^3 = 1.$$ 

Thus, since $z(F_4, r) = 1$ by Proposition 14, the result follows similarly in the proof of Theorem 11.

Since the fundamental group of $E_{6, r}$ is $\mathbb{Z}_2$ from Theorem 18 and the center $z(E_{6, r})$ of $E_{6, r}$ is $\mathbb{Z}_3$, we have the following
Theorem 20. The center $z(\tilde{E}_{6,7})$ of the simply connected non-compact Lie group $	ilde{E}_{6,7} = E_{6(2)}$ is isomorphic to the cyclic group $\mathbb{Z}_6$ of order 6.

References


