Explicit Isomorphism between $SU(4)$ and $Spin(6)$

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It is well known that the special unitary group $SU(4)$ and the spinor group $Spin(6)$ are isomorphic. To prove this it is usually used that their Lie algebras are isomorphic. In this paper, we shall prove it by giving a homomorphism $\varphi : SU(4) \rightarrow SO(6)$ explicitly.

1. Preliminaries.

(1) Let $C$ and $H=C\oplus jC$ be the complex and the quaternion fields respectively. $H$ is isomorphic to the space $\mathfrak{h}=(x \in M(2, C) | xj = jx)$, where $j=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, as algebra, by the correspondence $k : H \rightarrow \mathfrak{h}$,

$$k(a+jb)=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a, b \in C,$$

and $k$ has the following properties:

$$k(x)=x^*, \quad \frac{1}{2}(xy^*+yx^*)=(x, y)E, \quad xx^* = x^*x = |x|^2E$$

where $x=k(x)$, $y=k(y)$ and $E$ is the unit matrix. This mapping $k$ is naturally extended to the spaces of matrices:

$$k : M(2, H) \rightarrow M(4, C), \quad k\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} k(x_{11}) & k(x_{12}) \\ k(x_{21}) & k(x_{22}) \end{pmatrix}.$$

(2) Let $\mathfrak{h}(2, H)$ be the vector space of all $2 \times 2$ quaternion Hermitian matrices:

$$\mathfrak{h}(2, H)=(X \in M(2, H) | X^* = X).$$

In $\mathfrak{h}(2, H)$, we define the inner product $(X, Y)$ by

$$(X, Y)=\frac{1}{2} \text{tr}(XY+YX).$$

Let $\mathfrak{h}(2, H)^C=(X=X_1+iX_3 | X_1, X_3 \in \mathfrak{h}(2, H))$ be the complexification of $\mathfrak{h}(2, H)$. 
In $\mathfrak{S}(2, H)^C$ we define the Hermitian inner product $\langle X, Y \rangle$ by
$$\langle X_1+iX_2, Y_1+iY_2 \rangle = (X_1, Y_1) + (X_2, Y_2) + i((X_1, Y_2) - (X_2, Y_1)).$$
Furthermore let $\mathfrak{S}(4, C)$ be the vector space of all $4 \times 4$ complex skew-symmetric matrices:
$$\mathfrak{S}(4, C) = \{ P \in M(4, C) | P = -P \}.$$  
In $\mathfrak{S}(4, C)$ we define the Hermitian inner product $\langle P, Q \rangle$ by
$$\langle P, Q \rangle = -\frac{1}{4} \operatorname{tr}(PQ + QP).$$
Then the space $\mathfrak{S}(2, H)^C$ is isomorphic to the space $\mathfrak{S}(4, C)$ by the correspondence $h : \mathfrak{S}(2, H)^C \rightarrow \mathfrak{S}(4, C),$
$$h(X_1+iX_2) = (k(X_1)+ik(X_2))J,$$ where
$$J = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}.$$

(3) Let $\mathfrak{c}_2$ be the Lie algebra of all $2 \times 2$ quaternion skew-Hermitian matrices:
$$\mathfrak{c}_2 = \{ D \in M(2, H) | D^* = -D \}$$
and $\mathfrak{a}_3$ the Lie algebra of all $4 \times 4$ complex skew-Hermitian matrices with zero trace:
$$\mathfrak{a}_3 = \{ S \in M(4, C) | S^* = -S, \operatorname{tr}(S) = 0 \}.$$  
Any element $S$ of $\mathfrak{a}_3$ can be represented by the form
$$S = k(D) + ik(T), \quad D \in \mathfrak{c}_2, \quad T \in \mathfrak{S}(2, H), \quad \operatorname{tr}(T) = 0$$
where $F(a) = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$, $a \in H$, $E_1-E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $t \in \mathbb{R}$ ($\mathbb{R}$ is the field of real numbers). In fact, for $S \in \mathfrak{a}_3$, put $D_1 = -\frac{1}{2}(S-JSJ)$ and $T_1 = -\frac{i}{2}(S+JSJ)$, then we have $S = D_1 + iT_1$, $D_1^* = -D_1$, $D_1J = JD_1$ and $T_1^* = T_1$, $T_1J = JT_1$, $\operatorname{tr}(T_1) = 0$.  
So $D = k^{-1}(D_1)$ and $T = k^{-1}(T_1)$ satisfy the required conditions.

2. **Low dimensional spinor groups.**

We define the low dimensional symplectic groups, the special unitary group and the orthogonal groups by
$$\mathfrak{sp}(1) = \{ a \in H \mid |a| = 1 \},$$
$$\mathfrak{sp}(2) = \{ A \in M(2, H) \mid A^*A = E \},$$
$$\mathfrak{SU}(4) = \{ A \in M(4, C) \mid A^*A = E, \operatorname{det}A = 1 \},$$
$$\mathfrak{SO}(3) = \mathfrak{SO}(H_0) = \{ \alpha \in \operatorname{Isom}(H_0, H_0) \mid (ax, ay) = (x, y), \operatorname{det} \alpha = 1 \}.$$
where $H_6 = \{ x \in H \mid \bar{x} = -x \}$,
\[
SO(4) = SO(H) = \{ \alpha \in Iso_R(H, H) \mid (\alpha x, \alpha y) = (x, y), \det \alpha = 1 \},
\]
\[
SO(5) = SO(\mathcal{Z}_0) = \{ \alpha \in Iso_R(\mathcal{Z}_0, \mathcal{Z}_0) \mid (\alpha X, \alpha Y) = (X, Y), \det \alpha = 1 \}
\]
where $\mathcal{Z}_0 = \mathbb{H}(2, H)_0 = \{ X \in \mathbb{H}(2, H) \mid \text{tr}(X) = 0 \}$ and
\[
SO(6) = SO(V) = \{ \alpha \in Iso_R(V, V) \mid \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle, \det \alpha = 1 \}
\]
where $V = \left\{ \begin{pmatrix} \xi & x \\ \bar{x} & \bar{\xi} \end{pmatrix} \mid \xi \in \mathbb{C}, x \in \mathbb{H} \right\} \subset \mathbb{H}(2, H)^c$.

We note that the restriction of the mapping $h$ of the section 1 on $V$ is an isometry:
\[
\langle h(X), h(Y) \rangle = \langle X, Y \rangle, \quad X, Y \in V
\]
and the group $Sp(2)$ acts on the space $\mathbb{H}(2, H)$ by $\mu : Sp(2) \times \mathbb{H}(2, H) \to \mathbb{H}(2, H)$, $\mu(A, X) = AXA^*$ and it holds that
\[
\langle AXA^*, AYA^* \rangle = \langle X, Y \rangle, \quad \text{tr}(AXA^*) = \text{tr}(X).
\]
On the other hand, the group $SU(4)$ acts on the space $\mathbb{H}(4, C)$ by $\mu : SU(4) \times \mathbb{H}(4, C) \to \mathbb{H}(4, C)$, $\mu(A, P) = AP^*A$ and it holds that
\[
\langle AP^*A, AQ^*A \rangle = \langle P, Q \rangle.
\]

Now we define the following homomorphisms.
\[
p_1 : Sp(1) \to SO(3), \quad p_1(a)x = ax\bar{a}, \quad a \in H_6,
\]
\[
p_2 : Sp(1) \times Sp(1) \to SO(4), \quad p_2(a, b)x = axb, \quad a, b \in H,
\]
\[
p_3 : Sp(2) \to SO(5), \quad p_3(A)X = AXA^*, \quad X \in \mathcal{Z}_0,
\]
\[
p_4 = p_4 : SU(4) \to SO(6), \quad p_4(A)X = h^{-1}(Ah(X)h^A), \quad X \in V.
\]

Then we have

**Theorem 1.** The following diagram is commutative

\[
\begin{array}{ccc}
Sp(1) & \xrightarrow{k_1} & Sp(1) \times Sp(1) & \xrightarrow{k_2} & Sp(2) & \xrightarrow{k} & SU(4) \\
\downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 & & \downarrow p \\
SO(3) & \xrightarrow{j_1} & SO(4) & \xrightarrow{j_2} & SO(5) & \xrightarrow{j} & SO(6)
\end{array}
\]

where $k_1$ is the diagonal mapping and $k_2$, $j_1$, $j_2$, $j$ are natural inclusions. And each mapping $p_i$ is the universal covering homomorphism. In particular, we have the following isomorphisms.
\[
Sp(1) \cong Spin(3), \quad Sp(1) \times Sp(1) \cong Spin(4),
\]
\[
Sp(2) \cong Spin(5), \quad SU(4) \cong Spin(6).
\]
Proof. As for the mapping \( p_1, p_2 \), they are well known (Chap. I [1]). The mapping \( p_3 \) is also well known, however we will give a proof that \( p_3 \) is onto by using the following

Lemma 2. Let \( G, G' \) be groups, \( H, H' \) subgroups of \( G, G' \) respectively and \( p : G \rightarrow G' \) a homomorphism satisfying \( p(H) \subseteq H' \). If \( p' = p|H : H \rightarrow H' \) and \( \tilde{p} : G/H \rightarrow G'/H' \) (the induced mapping of \( p \)) are both onto, then \( p : G \rightarrow G' \) is also onto.

\[
\begin{array}{ccccccc}
1 & \rightarrow & H & \rightarrow & G & \rightarrow & G/H & \rightarrow & * \\
\downarrow{p'} & & \downarrow{p} & & \downarrow{\tilde{p}} & & \\
1 & \rightarrow & H' & \rightarrow & G' & \rightarrow & G'/H' & \rightarrow & *
\end{array}
\]

Proof of Lemma 2 is easy (Lemma 1.50 [2]).

Let \( S^4 \) be the unit sphere in \( \mathbb{S}(2, H) \):

\[
S^4 = \{X \in \mathbb{S}(2, H) \mid \langle X, X \rangle = 2\}.
\]

By using that any element of \( \mathbb{S}(2, H) \) can be transformed in a diagonal form by the action \( \mu \) of \( Sp(2) \), we see that any element \( X \) of \( S^4 \) can be transformed to \( E_1 - E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) by \( Sp(2) \). This shows that the group \( Sp(2) \) acts transitively on \( S^4 \). Since the isotropy subgroup of \( Sp(2) \) at \( E_1 - E_2 \) is \( k_5(\pm 1) \times \pm 1 \), we have the following homeomorphism

\[
Sp(2)/k_5(\pm 1) \times \pm 1 = S^4.
\]

Thus we have the following diagram

\[
\begin{array}{ccccccc}
1 & \rightarrow & Sp(1) \times Sp(1) & \xrightarrow{k_5} & Sp(2) & \rightarrow & S^4 & \rightarrow & * \\
\downarrow{p_2} & & \downarrow{p} & & \downarrow{\tilde{p}} & & \| & & \\
1 & \rightarrow & SO(4) & \xrightarrow{j_5} & SO(5) & \rightarrow & S^4 & \rightarrow & *
\end{array}
\]

Therefore, from Lemma 2, we see that \( p_3 \) is onto. \( \text{Ker}p_3 = Z_2 = \{E, -E\} \) is easily obtained.

Now, we consider the mapping \( p : SU(4) \rightarrow SO(6) \). In order to prove that the mapping \( p \) is well-defined, first we have to show that, for \( A \in SU(4) \) and \( X \in V \), we have

\[
p(A)X = h^{-1}(Ah(X)^fA) \in V.
\]

Since any element \( S \) of the Lie algebra \( a_8 \) of \( SU(4) \) is represented by the form

\[
S = k(D) + ik(F(a)) + itk(E_1 - E_2)
\]

as §1 (3), the group \( SU(4) \) is generated by the elements such as \( \exp k(D) \), \( \exp ik(F(a)) \) and \( \exp itk(E_1 - E_2) \). For \( A = h(A_1) \) where \( A_1 = \exp D \in Sp(2), X \in V \), we have
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\[ h^{-1}(Ah(X)^tA) = h^{-1}(k(A_1^t)k(X)^tA_1) = h^{-1}(k(A_1)k(X)k(A_1)^*) = h^{-1}(k(A_1XA_1^*)) = A_1XA_1^* ∈ V. \]

For $A = exp i(k(F(a)))$, $X ∈ V$, we have

\[ h^{-1}(Ah(X)^tA) = h^{-1}((exp i(k(F(a))))k(X)^t(exp i(k(F(a)))))) \]
\[ = h^{-1}((exp i(k(F(a))))k(X)(exp i(k(F(a))))J) \]
\[ = h^{-1}(k((exp i(F(a)))X(exp i(F(a))))J) \]
\[ = (exp i(F(a)))X(exp i(F(a))) \]
\[ = \left( \begin{array}{c}
\cos |a| & i\frac{a}{|a|}\sin |a| \\
\frac{a}{|a|}\sin |a| & \cos |a|
\end{array} \right) \left( \begin{array}{c}
x \\
-\xi
\end{array} \right) \left( \begin{array}{c}
\cos |a| & i\frac{a}{|a|}\sin |a| \\
\frac{a}{|a|}\sin |a| & \cos |a|
\end{array} \right) \left( \begin{array}{c}
x \\
-\xi
\end{array} \right) \]
\[ = \left( \begin{array}{c}
\eta \\
y
\end{array} \right) \in V, \]

where

\[ \eta = \xi\cos |a| + \xi\sin |a| + i\frac{2(a, x)}{|a|}\sin |a|\cos |a|, \]
\[ y = x + \frac{2(a, x)}{|a|}\sin |a| + i\frac{(\xi - \bar{\xi})a}{|a|}\sin |a|\cos |a|. \]

For $A(t) = exp \frac{it}{2} k(E_1 - E_2)$, $X = \left( \begin{array}{c}
x \\
-x
\end{array} \right) ∈ V$, it is easy to verify that

\[ h^{-1}(A(t)h(X)^tA(t)) = \left( \begin{array}{c}
\cos |a| & i\frac{a}{|a|}\sin |a| \\
\frac{a}{|a|}\sin |a| & \cos |a|
\end{array} \right) \left( \begin{array}{c}
x \\
-x
\end{array} \right) \left( \begin{array}{c}
\cos |a| & i\frac{a}{|a|}\sin |a| \\
\frac{a}{|a|}\sin |a| & \cos |a|
\end{array} \right) \left( \begin{array}{c}
x \\
-x
\end{array} \right) \]
\[ = \left( \begin{array}{c}
\eta \\
y
\end{array} \right) \in V. \]

Thus $p(A)X ∈ V$ is proved. For $A ∈ SU(4)$, we see that $p(A) ∈ O(8) = O(V) = \{ a ∈ Isom_R (V, V) | \langle aX, aY \rangle = \langle X, Y \rangle \}$, because

\[ \langle p(A)X, p(A)Y \rangle = \langle h(p(A)X), h(p(A)Y) \rangle \]
\[ = \langle h(Ah(X)^tA), h(Ah(Y)^tA) \rangle = \langle h(X), h(Y) \rangle = \langle X, Y \rangle. \]

Since $SU(4)$ is connected, $p(SU(4))$ is contained in the connected component $SO(6)$ of identity $E$ in $O(V)$, i.e. $p(SU(4)) ⊂ SO(6)$. Thus we see that the mapping $p$ is well-defined.

Let $S^5$ be the unit sphere in $V$:

\[ S^5 = \{ X ∈ V | \langle X, X \rangle = 2 \} \]

We shall prove that the group $SU(4)$ acts transitively on $S^5$. To prove this, it
is sufficient to show that any element $X$ of $S^4$ can be transformed to $i(E_1 + E_2) = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$. For a given $X \in S^4$, operate some element $A(t_0) = \exp \frac{it}{2}(E_1 - E_2)$, then we have

$$p(A(t_0))X \in S^4.$$ 

Since $Sp(2)$ acts transitively on $S^4$, there exists $A \in Sp(2)$ such that

$$p(k(A)A(t_0))X = E_1 - E_2,$$

and then operate $A\left(\frac{\pi}{2}\right) = \exp \frac{i\pi}{4}(E_1 - E_2)$ on it, then we have

$$p(A\left(\frac{\pi}{2}\right)k(A)A(t_0))X = i(E_1 + E_2).$$

This implies the transitivity of $SU(4)$. Since the isotropy subgroup of $SU(4)$ at $i(E_1 + E_2)$ is $k(Sp(2))$, we have the following homeomorphism

$$SU(4)/k(Sp(2)) \cong S^5.$$

Thus we have the following commutative diagram

$$1 \rightarrow Sp(2) \xrightarrow{k} SU(4) \rightarrow S^4 \rightarrow *$$

$$1 \rightarrow SO(5) \xrightarrow{j} SO(6) \rightarrow S^5 \rightarrow *$$

Therefore, from Lemma 2, we see that $p$ is onto. $\text{Ker}p = \{E, -E\}$ is easily obtained. Thus the proof of Theorem 1 is completed.

References