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Connections of Differential Operators

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Introduction

A connection θ of a vector bundle F may be regarded to be the lower order term of a differential operator $D: C^\infty(M, \Lambda^p T^*(M)\otimes F)\to C^\infty(M, \Lambda^{p+1}T^*(M)\otimes F)$ with the symbol $\sigma(d)\otimes id_F$ (cf. [1]). Similarly, for an arbitrary differential operator $D: C^\infty(M, E_1)\to C^\infty(M, E_2)$, E_1 and E_2 being vector bundles over M, we may consider the lower order term of a differential operator $\widetilde{D}: C^\infty(M, E_1\otimes F)\to C^\infty(M, E_2 \otimes F)$ with $\sigma(\widetilde{D})=\sigma(D)\otimes id_F$, ($\sigma(D)$, *etc.*, mean the symbols of D, *etc.*), to be a connection of D with respect to F. This connection has many (formally) similar properties as usual connection. For example, the action of the group of automorphisms of F to the set of all connections of D with respect to F is formally same as usual case (cf. [9]), and the obstruction class $o(D, F)$ which has similar properties as curvature or characteristic classes, can be defined by the help of connection.

The outline of this paper is as follows : In §1, we define the connection of D with respect to a vector bundle F. After showing the existence of connection, the action of the automorphism of F to the connection is calculated in §1. In §2, we define the obstruction class $o(D, F)$ and show D has a connection with respect to F with the degree at most $\deg D-2$ if and only if $o(D, F)=0$. The higher obstructions $o^j(D, F)$ are also defined under the assumption $o^{j-1}(D, F)=0$. It is shown that D has a connection with the degree at most $\deg D-j-1$ if and only if $o^j(D, F)=0$. If F is a complex line bundle, $o(d, F)\in H^1(M, \mathscr{B}^1)$, $\mathscr{B}^1$ is the sheaf of germs of closed 1–forms on M, and its de Rham image in $H^2(M, \mathrm{C})$ is the 1–st Chern class of F, the closed 2–form on M whose de Rham image $o(d. F)$, is the curvature form of F. For this reason, we may define $ch(D, F)$ and $ch^j(D, F)$ using non–abelian cohomology theory ([6], [8]). In §3, we consider the extension of differential operator D on the base space M to the tatal space M_F of a fibre bundle F and show this problem is also treated by the same way as the connection of D defined in §1. For this reason, to fix a connection $\theta(F)$ of F, we call the lower order term of the differential operator $\widetilde{D}: C^\infty(M_F,\ \pi_F{}^*(E_1))\rightarrow C^\infty(M_F,\ \pi_F{}^*(E_2))$ with $\sigma(\widetilde{D})=\pi_{\theta(F)}{}^*[\pi_F{}^*(\sigma(D))]$ is called the connection of D with respect to F and $\theta(F)$. Here $\pi_{\theta(F)}{}^*: T^*(M_F)\rightarrow\pi_F{}^*(T^*(M))$ is the map defined by $\theta(F)$. It is shown that if F is an $SO(n)$-bundle or $SU(n)$-bundle with the fibre $\mathbf{R}^n$ or $\mathbf{C}^n$, D has a connection such that decomposed as the sum of connections of D with respect to $\chi_p(F)$ or $\chi_{C,p}(F)$, $q\geqq 0$. Here $\chi_p(F)$, or $\chi_{C,p}(F)$, is the associate p-th degree harmonic polynomials bundle, or $(p,\ p)$-type harmonic polynomial bundle, of F.

§ 1. Definition of connections

1. Let M be a connected n-dimensional smooth manifold, E_1, E_2 and F are complex (or real) vector bundles over M. The dimensions of the fibres of E_1 and E_2 are assumed to be finite, but the dimension of the fibre of F need not be finite (cf. § 3). We fix a common (locally finite) coordinate neighborhood $\{U\}$ of E_1, E_2 and F. The (fixed) transition functions of E_1, E_2 and F defined by $\{U\}$ are denoted by $\{g_{1,UV}(x)\}$, $\{g_{2,UV}(x)\}$ and $\{g_{UV}(x)\}$. We denote by $C^\infty(M,\ E_1)$, *etc.*, the space of C^∞-cross-sections of E_1 over M, *etc.*. Under these notations, a differential operator $D: C^\infty(M,\ E_1)\rightarrow C^\infty(M,\ E_2)$ is a collection of differential operators $D_U: C^\infty(U,\ E_1)\rightarrow C^\infty(U,\ E_2)$ such that

$$D_U g_{1,UV}(x)=g_{2,UV}(x)D_V,\quad x\in U\cap V.$$

We set $\deg D=k$. Then D_U is written

$$D_U=\sum_{|\mathbf{I}|\leqq k}A_{\mathbf{I},U}(x)\left(\frac{\partial}{\partial x_U}\right)^{\mathbf{I}},\quad \mathbf{I}=(i_1,\dots,i_n),\quad |\mathbf{I}|=i_1+\dots+i_n, \tag{1}$$

$$\left(\frac{\partial}{\partial x_U}\right)^{\mathbf{I}}=\frac{\partial^{|\mathbf{I}|}}{\partial x_{U,1}{}^{i_1}\dots\partial x_{U,n}{}^{i_n}},$$

where $(x_{U,1},\dots,x_{U,n})$ is the local coordinate on U. We set

$$D_U\otimes 1_F=\sum_{|\mathbf{I}|\leqq k}A_{\mathbf{I},U}(x)\otimes 1_F\left(\frac{\partial}{\partial x_U}\right)^{\mathbf{I}},\ 1_F \text{ is the identity map of the fibre of } F,$$

Then $D_U\otimes 1_F: C^\infty(U,E_1\otimes F)\rightarrow C^\infty(U,E_2\otimes F)$ is a differential operator on U.

Definition. *A collection $\{\theta_U\}$ of differential operators $\theta_U: C^\infty(U,\ E_1\otimes F)\rightarrow C^\infty(U,\ E_2\otimes F)$, is called a connection of D (with respect to F) if it satisfies*

(i) $\quad g_{2,UV}(x)\otimes g_{UV}(x)(D_V\otimes 1_F+\theta_V)=(D_U\otimes 1_F+\theta_U)g_{1,UV}(x)\otimes g_{UV}(x),$

(ii) $\quad \deg\theta_U\leqq k-1,\ (k=\deg D).$

Proposition 1. *For any D and F connection exists.*

Proof. Let $\{e_U(x)\}$ be a C^∞-partition of unity subordinate to $\{U\}$. Then to set

$$(2) \qquad \theta_U(x) = \sum_{U \cap W \neq \phi} e_W(x) g_{2,UW}(x) \otimes g_{UW}(x) \{D_W \otimes 1_F(g_{1,WU}(x) \otimes g_{WU}(x)) - (g_{2,WU}(x) \otimes g_{WU}(x)) D_U \otimes 1_F\},$$

$\{\theta_U(x)\}$ satisfies (i), (ii).

Eefinition. *Let $\theta = \{\theta_U(x)\}$ be a connection of D with respect to F. Then the collection $\{D_U \otimes 1_F + \theta_U\}$ is denoted by D_θ.*

By definition, $D_\theta : C^\infty(M,\ E_1 \otimes F) \to C^\infty(M,\ E_2 \otimes F)$ is a differential operator on M and $\deg D_\theta = k$. Hence we have

$$(3) \qquad \sigma(D_\theta) = \sigma(D) \otimes id_F,$$

where $\sigma(D)$, *etc.*, are the symbols of D, *etc.*, and id_F is the identity map of $\pi^*(F)$, π is the projection of $T^*(M)$, the cotangent bundle of M.

Note 1. If $E_1 = \Lambda^p T^*$, $E_2 = \Lambda^{p+1} T^*$ and $D = d$, the exterior differential, then a connection of d with respect to F is a linear connection of F.

Note 2. For a differential complex

$$(D) : C^\infty(M,\ E_1) \xrightarrow{D_1} C^\infty(M,\ E_2) \xrightarrow{D_2} \cdots,$$

Connection (with respect to F) is also defined. But the lifted sequence

$$(D_\theta) : C^\infty(M,\ E_1 \otimes F) \xrightarrow{D_1, \theta_1} C^\infty(M, E_2 \otimes F) \xrightarrow{D_2, \theta_2} \cdots,$$

is not a differential complex in general, although its symbol sequence is exact (cf. [2]).

2. Let $\varphi : E_1 \to E_1$ be a bundle map, then we set

$$(4) \qquad {}_\varphi D_U = \sum_{|I| \leqq k} A_{I,U}(x) \varphi(x) \left(\frac{\partial}{\partial x_U}\right)^I,$$

and set

$$(5) \qquad D_U \varphi = {}_\varphi D_U + D_{U,\varphi}.$$

By definition, $\deg D_{U,\varphi} \leqq k-1$.

Lemma 1. *If φ is an automophism, then*

$$(6) \qquad D_{U,\varphi^{-1}} = -{}_{\varphi^{-1}} D_{U,\varphi} \varphi^{-1}.$$

Proof. Since we have ${}_{\varphi_{-1}}({}_\varphi D) = D$, we get

$$D_U = D_U \varphi^{-1} \varphi = {}_{\varphi^{-1}} D_U \varphi + D_{U,\varphi^{-1}} = D_U + {}_{\varphi^{-1}} D_{U,\varphi} + D_{U,\varphi^{-1}},$$

we obtain (6).

If $\{\theta_U\}$ is a connection of D with respect to F, we have

$$g_{2,UV}\otimes g_{UV}D_V\otimes 1_F - D_U\otimes 1_F g_{1,UV}\otimes g_{UV} = g_{2,UV}\otimes g_{UV}\theta_V - \theta_U g_{1,UV}\otimes g_{UV}.$$

Hence we get

$$(7)\qquad g_{2,UV}\otimes h_U g_{UV} h_V^{-1}\{(1_{E_2}\otimes h_V)D_V\otimes 1_F\ (1_{E_1}\otimes h_V^{-1}) - (1_{E_2}\otimes h_U)\ D_U\otimes 1_F\ (1_{E_1}\otimes h_U^{-1})\}\ g_{1,UV}\otimes g_{UV}$$
$$=(1_{E_2}\otimes h_U)\{g_{2,UV}\otimes g_{UV}\theta_V - \theta_U g_{1,UV}\otimes g_{UV}\}(1_{E_1}\otimes h_V^{-1}),$$

where 1_{E_1} and 1_{E_2} are the identity maps of the fibres of E_1 and E_2. But since we get by (6)

$$(1_{E_2}\otimes h_U)D_U\otimes 1_F(1_{E_1}\otimes h_U^{-1})$$
$$=(1_{E_2}\otimes h_U)\{{}_{(1_{E_1}\otimes h_U^{-1})}D_U\otimes 1_F - {}_{(1_{E_1}\otimes h_U^{-1})}(D_U\otimes 1_F)_{1_{E_1}\otimes h_U}(1_{E_1}\otimes h_U^{-1})\},$$

and since

$${}_{(1_{E_1}\otimes h_U^{-1})}D_U\otimes 1_F = (1_{E_2}\otimes h_U^{-1})D_U\otimes 1_F,$$

we obtain

$$(8)\qquad (1_{E_2}\otimes h_U)D_U\otimes 1_F(1_{E_1}\otimes h_U^{-1}) = D_U\otimes 1_F - (D_U\otimes 1_F)_{1_{E_1}\otimes h_U}(1_{E_1}\otimes h_U^{-1}).$$

By (7), (8), we have

Lemma 2. *If $\{\theta_U\}$ is a connection of D with respect to F, where $\{g_{UV}\}$, a transition function of F, is fixed, then by the change of transition function of F by $\{h_U\}$, $\{\theta_U\}$ is changed to $\{\theta_U'\}$ given by*

$$(9)\qquad \theta_U' = (1_{E_2}\otimes h_U)\{\theta_U - (1_{E_2}\otimes h_U^{-1})(D_U\otimes 1_F)_{1_{E_1}\otimes h_U}\}(1_{E_1}\otimes h_U^{-1}).$$

Note. Since $d_f = df$, the action of the automorphism of F to the connection of D is formally similar as the usual connection (cf. [1], [9]).

Definition. *If $\{\theta_{1,U_1}\}$ and $\{\theta_{2,U_2}\}$ are the connections of D with respect to F, we call $\{\theta_{1,U_1}\}$ and $\{\theta_{2,U_2}\}$ to be equivalent if there exists a common locally finite refinement $\{U\}$ of $\{U_1\}$, $\{U_2\}$ and a collection of bundle automorphisms $\{h_U\}$ of F, each h_U is defined on U, such that*

$$\theta_{1,U_1}|U = (1_{E_2}\otimes h_U)\{\theta_{2,U_2}\ |U - (1_{E_2}\otimes h_U^{-1})(D_U\otimes 1_F)_{1_{E_1}\otimes h_U}\}(1_{E_1}\otimes h_U^{-1}),$$
$$U\subset U_1\cap U_2,$$

for each U.

Note. If $E_1 = E_2 = E$, E and F both have unitary structures and D is formally selfadjoint, that is, $\{g_{1,UV}\}$ $(=\{g_{2,UV}\})$ and $\{g_{UV}\}$ both take the values in unitary group, then to denote inner product on $C^\infty(M, E\otimes F)$ defined from the inner products of E and F by $<\varphi, \phi>$, we get

$$<g_{1,UV}{}^{-1}\otimes g_{UV}{}^{-1}D_U g_{1,UV}\otimes g_{UV}\varphi, \phi> = <\varphi, g_{1,UV}{}^{-1}\otimes g_{UV}{}^{-1}Dg_{1,UV}\otimes g_{UV}>,$$

if *Supp.* φ and *Supp.* ϕ both contained in $U\cap V$. Hence D has a connection θ such that D_θ is formally selfadjoint. In this case, if $\{h_U\}$ take values in unitary group, the change $\{\theta_U'\}$ of $\{\theta_U\}$ by $\{h_U\}$ given by (9) also gives a formally selfadjoint operator $D_{\theta'}$.

3. We set

$$\mathrm{Con}(D,\ F)=\{\theta|\theta \textit{ is a connection of } D \textit{ with respect to } F\},$$

$$\mathscr{D}^j(E_1\otimes F,\ E_2\otimes F)=\{\eta:C^\infty(M,\ E_1\otimes F)\to C^\infty(M,\ E_2\otimes F)|\eta \textit{ is a differential operator with degree at most } j\}.$$

Then by definition, to fix $\theta_0=\{\theta_{0,U}\}\in\mathrm{Con}(D,\ F)$ and define $i_{\theta_0}(\theta)=\{\theta_U-\theta_{0,U}\}=D_\theta-D_{\theta_0}$, $\theta=\{\theta_U\}\in\mathrm{Con}(D,\ F)$, we have a bijection

$$(10)\qquad i_{\theta_0}:\mathrm{Con}(D,\ F)\to\mathscr{D}^{k-1}(E_1\otimes F,\ E_2\otimes F).$$

Since $\mathscr{D}^{k-1}(E_1\otimes F,\ E_2\otimes F)=C^\infty(M,\ Hom(E_1\otimes F,\ E_2\otimes F)\otimes J_{k-1}(M))$ is a topological space by C^∞—topology, Con$(D,\ F)$ becomes a topological space by (10) and this topology does not depend on the choice of θ_0.

Denote $\mathfrak{G}(F)$ the group of bundle automorphisms of F, $\mathfrak{G}(F)$ acts on Con $(D,\ F)$ by lemma 2. To copy this action to $\mathscr{D}^{k-1}(E_1\otimes F,\ E_2\otimes F)$ by i_{θ_0}, we can define an action of $\mathfrak{G}(F)$ on $\mathscr{D}^{k-1}(E_1\otimes F,\ E_2\otimes F)$ which may different from usual action. By (9), the isotropy group $\mathfrak{G}(F)_\theta$ of $\mathfrak{G}(F)$ at θ is given by

$$(11)\qquad \mathfrak{G}(F)_\theta=\{\{h_U\}|(1_{E_2}\otimes h_U)D_{U,\theta_U}=D_{U,\theta_U}(1_{E_1}\otimes h_U)\},$$

$$D_{U,\theta_U}=D_U\otimes 1_F+\theta_U.$$

By (10), Con$(D,\ F)$ is imbedded in $\mathscr{D}^{k-1,s}(E_1\otimes F,\ E_2\otimes F)=\mathscr{H}^s(M,\ Hom(E_1\otimes F,\ E_2\otimes F)\otimes J_{k-1}(M))$, where $\mathscr{H}^s$ means s—th Sobolev space. Hence, if F has a fixed unitary structure, $\mathfrak{G}(F)$ is the group of bundle automorphisms of F with the unitary structure and M is compact, local slice theorem is valid ([4], [7], [9]).

In the case D is formally selfadjoint, we set

$$\mathrm{Con}_s(D,\ F)=\{\theta|\theta \textit{ is a formally selfadjoint connection of } D \textit{ with respect to } F\},$$

$$\mathscr{D}^j{}_s(E_1\otimes F,\ E_2\otimes F)=\{\eta:C^\infty(M,\ E_1\ F)\to C^\infty(M,\ E_2F)\ E\ \eta \textit{ is a formally selfadjoint differential operator with degree at most } j\}.$$

Then to fix a formally selfadjoint connection θ_0 of D with respect to F, we get

$$(10)'\qquad i_{\theta_0}:\mathrm{Con}_s(D,\ F)\to\mathscr{D}^{k-1}{}_s(E_1\otimes F,\ E_2\otimes F).$$

If M is compact, by the action of $\mathfrak{G}(F)$, $\mathrm{Con}_s(D, F)$ has local slice.

§ 2. The obstruction class.

4. For the index set $\mathbf{I}=(i_1, \dots, i_n)$, we set

$$\mathbf{I}+1_j=(i_1, \dots, i_{j-1},\ i_j+1,\ i_{j+1}, \dots, i_n)$$

(cf. [3]). Using this notation, we set

$$D_U = \sum_{|\mathbf{I}|=k-1} [\sum_{j=1}^{n} A_{U,\mathbf{I}+1_j}(x)\,(\frac{\partial}{\partial x_U})^{\mathbf{I}} \frac{\partial}{\partial x_{u,j}}]$$

$$+\sum_{|\mathbf{I}|=k-1} B_{U,\mathbf{I}}(x)(\frac{\partial}{\partial x_U})^{\mathbf{I}} + \textit{lower order terms},$$

$$A_{U,\mathbf{I}+1_j}(x)=A_{U,\mathbf{I}'+1_k}(x) \ \textit{if}\ \mathbf{I}+1_j=\mathbf{I}'+1_k.$$

Then, since $g_{2,UV}D_V=D_U g_{1,UV}$, we get

$$g_{2,UV}\otimes g_{UV}D_V\otimes 1_F-D_U\otimes 1_F g_{1,UV}\otimes g_{UV}$$

$$=-\sum_{\mathbf{I}}[\sum_{j} A_{U,\mathbf{I}+1_j}(\frac{\partial (x_V)}{\partial (x_U)})^{\mathbf{I}+1_j}\otimes 1_F (g_{1,UV}\otimes\frac{\partial g_{UV}}{\partial x_{V,j}})](\frac{\partial}{\partial x_V})^{\mathbf{I}}$$

$$+\textit{lower order terms}.$$

Therefore, to set a connection $\{\theta_U\}$ of D with respect to F by

$$\theta_U=\sum_{|\mathbf{I}|=k-1} \theta_{U,\mathbf{I}}(x)\,(\frac{\partial}{\partial x_U})^{\mathbf{I}}+\textit{lower order terms},$$

we obtain

$$g_{2,UV}\otimes g_{UV}\theta_{V,\mathbf{I}}-\theta_{U,\mathbf{I}}(\frac{\partial (x_V)}{\partial (x_U)})^{\mathbf{I}} g_{1,UV}\otimes g_{UV}$$

$$=-\sum_{j=1}^{n} A_{U,\mathbf{I}+1_j}(\frac{\partial (x_V)}{\partial (x_U)})^{\mathbf{I}+1_j} g_{1,UV}\otimes\frac{\partial g_{UV}}{\partial x_{V,j}},$$

for each $\mathbf{I}$, $|\mathbf{I}|=k-1$. But since $g_{2,UV}D_V=D_U g_{1,UV}$, we get

$$\sum_{j=1}^{n} A_{U,\mathbf{I}+1_j}(\frac{\partial (x_V)}{\partial (x_U)})^{\mathbf{I}+1_j} g_{1,UV}\otimes\frac{\partial g_{UV}}{\partial x_{V,j}}=\sum_{j=1}^{n} g_{2,UV}A_{V,\mathbf{I}+1_j}\otimes\frac{\partial g_{UV}}{\partial x_{V,j}}.$$

Hence we have

$$(12)\qquad g_{2,UV}\otimes g_{UV}\ \theta_{V,\mathbf{I}}-\theta_{U,\mathbf{I}}(\frac{\partial (x_V)}{\partial (x_U)})^{\mathbf{I}} g_{1,UV}\otimes g_{UV}$$

$$=-\sum_{j=1}^{n} g_{2,UV} A_{V,\mathrm{I}+1j} \otimes \frac{\partial g_{UV}}{\partial x_{V,j}}, \quad |\mathrm{I}|=k-1.$$

5. We set by $\xi_U^1, \dots, \xi_U^n$, the dual basis of $\partial/\partial x_{U,1}, \dots, \partial/\partial x_{U,n}$ in the cotangent space. Then to set

$$\mathscr{A}_U = \sum_{|\mathrm{I}|=k-1} \left(\sum_{j=1}^{n} A_{U,\mathrm{I}+1j} \xi_U^{I+1j} \otimes \frac{\partial}{\partial x_{U,j}}\right),$$

$\mathscr{A}_U$ is a cross-section of $Hom(E_1, E_2) \otimes S^k(T^*(M)) \otimes T(M)$ over U. Here, $S^k(T^*(M)) = J_k(M)/J_{k-1}(M)$ is the k-th symmetric product of $T^*(M)$. By (12), to set

$$\sigma(\theta_U) = \sum_{|\mathrm{I}|=k-1} \theta_{U,\mathrm{I}}\ \xi_U^{\mathrm{I}}, \quad \sigma(\theta_V)^{\mathfrak{g}_{VU}} = g_{2,UV} \otimes g_{UV}\ \sigma(\theta_V) g_{1,VU} \otimes g_{VU},$$

we get

$$\begin{aligned}\sigma(\theta_U) - \sigma(\theta_V)^{\mathfrak{g}}{}_{VU} &= g_{2,UV} [\mathscr{A}_V(g_{UV})] g_{1,VU} \otimes g_{VU} \\ &= \mathscr{A}_U(g_{UV})(1_{E_1} \otimes g_{VU}),\end{aligned}$$

because $g_{2,UV}\mathscr{A}_V = \mathscr{A}_U g_{1,UV}$. Since this right hand side does not depend on th choice of $\{\theta_U\}$, we have

Lemma 3. *To set*

$$\mathscr{A}_{UV} = \sigma(\theta_U) - \sigma(\theta_V)^{\mathfrak{g}_{VU}} = \mathscr{A}_U(g_{UV})\,(1_{E_1} \otimes g_{VU}),$$

$\mathscr{A}_{UV}$ is a cross-section of $Hom(E_1, E_2) \otimes S^{k-1}(T^*(M)) \otimes Hom(F, F)$ over $U \cap V$ and does not depend on the chice of $\{\theta_U\}$.

By definition, the collection $\{\mathscr{A}_{UV}\}$ satisfies cochain condition

(13) $$\mathscr{A}_{UV} + \mathscr{A}_{VW}{}^{\mathfrak{g}_{VU}} + \mathscr{A}_{WU}{}^{\mathfrak{g}_{WU}} = 0.$$

If $\theta' = \{\theta'_U\}$ is equivalent to θ, we have by (9)

(14) $$\theta'_{U,\mathrm{I}} = \theta_{U,\mathrm{I}} - \sum_{j=1}^{n} A_{U,\mathrm{I}+1} \otimes h_U^{-1} \frac{\partial h_U}{\sigma x_{U,j}},$$

for any I, $|\mathrm{I}|=k-1$. Conversely, if θ' satisfies (14), there exists $\eta \in \mathscr{D}^{k-2}(E_1 \otimes F, E_2 \otimes F)$ such that $\theta' + \eta$ is equivalent to θ. If θ satisfies (14), then

(15) $$\sigma(\theta'_U) = \sigma(\theta_U) - (1_{E_2} \otimes h_U^{-1})\ \mathscr{A}_U\ (h_U).$$

But, since $\mathscr{A}_U(h_U^{-1}) = -(1_{E_2} \otimes h_U^{-1})[\mathscr{A}_U(h_U)](1_{E_1} \otimes h_U^{-1})$, (15) is rewritten

(15)′ $$\sigma(\theta'_U) = \sigma(\theta_U) + \mathscr{A}_U(h_U^{-1})\ (1_{E_1} \otimes h_U).$$

Therefore, if $\{\theta_U\}$ and $\{\theta'_U\}$ are equivalent each other, then

$$\{\sigma(\theta'_U)-\sigma(\theta'_V)^{\mathfrak{g}VU}\}-\{\sigma(\theta_U)-\sigma(\theta_V)^{\mathfrak{g}VU}\}$$
$$=\mathscr{A}_U(f_U)(1_{E_1}\otimes f_U^{-1})-[\mathscr{A}_V(f_V)(1_{E_1}\otimes f_V^{-1})]^{\mathfrak{g}VU}, \tag{16}$$

with suitable $\{f_U\}$, $f_U:U\to Hom(F, F)$. Conversely, if (16) is satisfied, then with suitable $\eta\in\mathscr{D}^{k-2}(E_1\otimes F, E_1\otimes F)$, $\theta'+\eta$ is equivalent to θ. Especially, we have

Lemma 4. *(i). The symbol* $\sigma(\theta)$ *of* θ *is defined if and only if there exist* f_U, $f_U:U\to Hom(F, F)$, *such that*

$$\mathscr{A}_{UV}=\mathscr{A}_U(f_U)(1_{E_1}\otimes f_U^{-1})-[\mathscr{A}_V(f_V)(1_{E_1}\otimes_V^{-1})]^{\mathfrak{g}VU}, \tag{17}$$

for each $U\cap V$.

(ii). If D *has a connection* θ *with respect to* F *such that* $\sigma(\theta)$ *is defined, then there exists a connection* θ_0 *of* D *with respect to* F *such that* deg $\theta_0\leq k-2$.

Proof. We only need to show (ii). But since $\sigma(\theta)$ is defined, $g_{2,UV}\otimes g_{UV}\theta_V-\theta_U g_{1,UV}\otimes g_{UV}$ is a differential operator of degree at most $k-2$. Hence there exists $\eta_0\in\mathscr{D}^{k-1}(E_1\otimes F, E_1\otimes F)$ such that $\sigma(\theta)=\sigma(\eta_0)$ (cf. the proof of proposition 1). Then, since $\sigma(\theta+\eta)=\sigma(\theta)+\sigma(\eta)$ for $\eta\in\mathscr{D}^{k-1}(E_1\otimes F, E_2\otimes F)$, we have the lemma.

6. We denote by $\mathfrak{P}(F)$ the associate principal bundle of F and define a differential operator $\mathscr{D}:C^\infty(U,\mathfrak{P}(F))\to C^\infty(U, Hom(E_1, E_2)\otimes S^{k-1}(T^*(M))\otimes Hom(F, F))$ by

$$\mathscr{D}f_U=\mathscr{A}_U(f_U)(1_{E_1}\otimes f_U^{-1}).$$

The sheaf of germs of images of $\mathscr{D}$ is denoted by $\mathrm{R}(\mathscr{D})$. Then (12) and (13) show that $\{\sigma(\theta_U)-\sigma(\theta_V)^{\mathfrak{g}VU}\}=\{\mathscr{A}_{UV}\}$ defines a cohomo logy class in $H^1(M, \mathrm{R}(\mathscr{D}))$. By (16), this class is same if θ and θ' are equivalent. By lemma 3, this class does not depend on the choice of θ.

Definition. *The cohomology class of* $\{\mathscr{A}_{UV}\}$ *in* $H^1(M, \mathrm{R}(\mathscr{D}))$ *is denoted by* $o(D, F)$.

By this definition, lemma 4 is restated as follows:

Theorem 1. *D has a connection* θ *with respect to* F *such that* deg$\theta\leq k-2$ *if and only if* $o(D, F)=0$.

Example 1. If $D=d$, the exterior differential, $\mathscr{A}(f)$ is equal to df. Hence to define $D_2:C^\infty(M, Hom(F, F)\otimes T^*(M))\to C^\infty(M, Hom(F, F)\otimes\Lambda^2T^*(M))$ by $D_2F=dF+F$, we get an exact sequence of sheaves

$$0\to\mathrm{R}(\mathscr{D})\to\mathscr{B}^1(M)\otimes Hom(F, F)\xrightarrow{D_2}D_2(\mathscr{B}^1(M)\otimes Hom(F, F))\to 0,$$

where $\mathscr{B}^1(M)$ is the sheaf of germs of closed 1-forms on M. Therefore, there exists a $Hom(F, F)$-valued 2-form Θ on M such that whose de Rham image by this exact sequence just covers the representative of $o(D, F)$ defined by θ, a connection of F. This Θ is the curvature form of θ. On the other hand, if F is a complex line bundle, then the kernel sheaf of $\mathscr{D}$ is the constant sheaf of complex numbers over M and we have the exact sequence

$$0\rightarrow\mathbf{C}\rightarrow\mathbf{C}^*{}_d\rightarrow\mathbf{R}(\mathscr{D})\rightarrow 0.$$

Hence we can define $\delta(o(D, F))\in H^2(M,\mathbf{C})$. It is the 1-st Chern class of F.

Example 2. If $k=2$, $E_1=E_2=1$, the 1-dimensional trivial bundle, then set $D=\sum_{i,j}A_{i,j}(x)\partial^2/\partial x_i\partial x_j+$*lower order terms*, $A_{i,j}=A_{j,i}$, $\mathscr{D}(f)$ is given

$$\mathscr{D}(f)\sum_{i=1}^{n}(\sum_{j=1}^{n}A_{i,j}(x)\frac{\partial f}{\partial x_j}f^{-1})\,dx_i.$$

Hence if F is a complex line bundle and the matrix $(A_{i,j}(x))$ is regular at any point of M, the kernel shdaf of $\mathscr{D}$ is the constant sheaf of complex numbers over M.

We denote the kernel sheaf of $\mathscr{D}$ by $\mathfrak{ker}(\mathscr{D})$. The sheaf of germs of smooth sections of $\mathfrak{P}(F)$ is denoted by $\mathfrak{P}(F)$. Then we have the exact sequence of sheaves

$$0\rightarrow\mathfrak{ker}(\mathscr{D})\rightarrow\mathfrak{P}(F)\rightarrow\mathbf{R}(D)\rightarrow 0$$

Then to set $\Phi_{\mathscr{D}}$ the sheaf of germs of those automorphisms of $\mathfrak{ker}(D)$ that can be extended to automorphisms of $\mathfrak{P}(F)$, there exists 2-dimensional cohomology set $H^2(M,\Phi_{\mathscr{D}})$ and map $\delta: H^1(M, \mathbf{R}(\mathscr{D}))\rightarrow H^2(M, \Phi_{\mathscr{D}})$ ([6], [8]).

Definition. *We denote* $\delta(o(D, F))$ *by* $ch(D, F)$.

On the other hand, if there is an operator $\mathscr{D}_2=\{\mathscr{D}_{2,U}\}$ such that the local integrabillity condition for the equation $g=\mathscr{D}(f)$ is given by $\mathscr{D}_2(g)=0$, then we define the curvature $\Theta=\Theta(\theta, D, F)$ of a connection θ of D with respect so F by

$$(18)\qquad \Theta_U=\mathscr{D}_{2,U}(\theta_U),\quad \Theta=\{\Theta_U\}.$$

7. We assume there is a transition function $\{g_{UV}\}$ of F such that

$$(19)\qquad \deg[g_{2,UV}\otimes g_{UV}D_V\otimes 1F-D_U\otimes 1_F g_{1,UV}\otimes g_{UV}]=k-j,\quad j\geqq 2.$$

We note that under thi assumption, D has a connection θ with respect to F such that deg $\theta\leqq k-j$ (cf. the proof of proposition 1).

Under the assumption (19), we set

$$D_U=\sum_{|\mathbf{I}|=k-j,\ |j|\leqq j}A_{U,\mathbf{I}+\mathbf{J}}(x)(\frac{\partial}{\partial x_U})^{\mathbf{I}+j}+\textit{lower order terms},$$

$$A_{U,\mathbf{I}+\mathbf{J}}=A_{U,\mathbf{I}'+\mathbf{J}'},\ \textit{if}\ \mathbf{I}+\mathbf{J}=\mathbf{I}'+\mathbf{J}'.$$

Then we have

$$g_{2,UV}\otimes g_{UV}\otimes 1_F - D_U\otimes 1_F g_{1,UV}\otimes g_{UV}$$

$$= -\sum_{|I|=k-j}[\sum_{1\leq|J|\leq j} A_{U,I+J}\otimes 1_F\frac{|J|!}{J!}[\sum_{J\geq K, |K|\geq 1}\frac{|J|!}{K!(J-K)!}\{(\frac{\partial}{\partial x_U})^{J-K}g_{1,UV}\}\otimes$$

$$\{(\frac{\partial}{\partial x_U})^K g_{UV}\}](\frac{\partial}{\partial x_U})^I + \textit{lower order terms.}$$

Let $\{\theta_U\}$ be a connection of D with respect to F such that deg $\theta_U\leq k-j$, then to set $\theta_U=\sum_{|I|=k-j}\theta_{U,I}(\partial/\partial x_U)^I+\textit{loder terms}$, we have

$$g_{2,UV}\otimes g_{UV}\theta_{V,I}(\frac{\partial(x_U)}{\partial(x_V)})^I-\theta_{U,I}g_{1,UV}\otimes g_{UV}$$

$$= -\sum_{1\leq|J|\leq j}A_{U,I+J}\otimes 1_F\frac{|J|!}{J!}[\sum_{J\geq K,|K|\geq 1}\frac{|J|!}{K!(J-K)!}\{(\frac{\partial}{\partial x_U})^{J-K}g_{1,UV}\{\otimes\}(\frac{\partial}{\partial x_U})^K g_{UV}\}].$$

Hence to set

$$\mathscr{A}^j{}_{UV}(f)=\sum_{|I|=k-j}\xi_U{}^I[\sum_{1\leq|J|\leq j}A_{U,I+J}\otimes 1_F\xi_U\frac{J|J|!}{J!}[\sum_{J\geq K,|K|\geq 1}\frac{|J|!}{K!(J-K)!}$$

$$\{(\frac{\partial}{\partial x_U})^{J-K}g_{1,UV}\{\otimes\}(\frac{\partial}{\partial x_U})^K f\}]],$$

we obtain

$(12)_j$ $\qquad \sigma(\theta_U)-\sigma(\theta_V)^{g_{VU}}=\mathscr{A}^j{}_{UV}(g_{UV})g_{1,VU}\otimes g_{VU}.$

By $(12)_j$, $\sigma(\theta_U)-\sigma(\theta_V)^{g_{VU}}=\mathscr{A}^j{}_{UV}$ does not depend on the choice of θ if deg $\theta\leq k-j$. On the other hand, if h_U satisfies

$(20)_j$ $\qquad \deg(D_U\otimes 1_F)_{1_{E_1}\otimes h_U}\leq k-j,$

and $\theta'_U=1_{E_2}\otimes h_U(\theta_U-1_{E_2}\otimes h_U{}^{-1}D_{U,h_U})1_{E_1}\otimes h_U{}^{-1}$, then

$$\{\sigma(\theta'_U)-\sigma(\theta'_V)^{g_{VU}}\}-\{\sigma(\theta_U)-\sigma(\theta_V)^{g_{VU}}\}$$
$$=\mathscr{A}^j{}_{UV}(h_U{}^{-1})(1_{E_1}\otimes h_U)-[\mathscr{A}^j{}_{UV}(h_V{}^{-1})(1_{E_1}\otimes h_V)]^{g_{VU}}.$$

We set $C_j{}^\infty(U,\mathfrak{P}(F))=\{f\,|\,f\in C^\infty(U,\ \mathfrak{P}(F)),\ \deg(D_U\otimes 1_F)_{1_{E_1}\otimes f}\leq k-j\}$, $j\leq k$. Since constant section belongs in $C_j{}^\infty(U,\ \mathfrak{P}(F))$, $C_j{}^\infty(U,\ \mathfrak{P}(F))\neq\emptyset$ for all j. We define $\mathscr{D}^j$ $:C_j{}^\infty(U,\ \mathfrak{P}(F))\rightarrow C^\infty(U,\ Hom(E_1,\ E_2))\otimes S^{k-j}(T^*(M))\otimes Hom(F,\ F))$ by

$$\mathscr{D}^j(f)=\mathscr{A}^j(f)f^{-1}.$$

Then, to set $\mathbf{R}(\mathscr{D}^j)$ the image sheaf of $\mathscr{D}^j$, $\{\sigma(\theta_U)-\sigma(\theta_V)^{g_{VU}}\}$ defines an element of $H^1(M,\mathbf{R}(\mathscr{D}^j))$.

Definition. *Under the above assumptions, the class* $\sigma(\theta_U)-\sigma(\theta_V)^{g_{VU}}$ in $H^1(M, \mathbf{R}(\mathscr{D}^j))$ *is denoted by* $o^j(D, F)$.

By definition, we have

Theorem 1'. *D has a connection* $\{\theta_U\}$ *with respect to F such that* deg $\theta_U \leqq k-j-1$ *if and only if* $o^j(D, F)=0$.

Corollary. $o^j(D, F)$ *is defined if and only if* $o(D, F)$ (D, F) $(=o^1(D, F))=o^2(D. F)=\cdots=o^{j-1}(D, F)=0$.

As in n°6, we can define $ch^j(D, F)$ and $\Theta^j(\theta)$ under the assumption $o^{j-1}(D, F)=0$.

§ 3. Extension of differential operators

8. Let $\xi=\xi(F)=\{M_F, M, \pi_F, F\}$ be a G-bundle, G is a Lie group, over M with the coordinate neighborhood system $\{U\}$ and transition function $\{g_{UV}(x)$. Let E_1 and E_2 are the vector bundles over M such that trivial on each U. Then $f\in C^\infty(M_F, \pi_F{}^*(E_i))$, $i=1,2$, can be written

$$f=\{f_U(x, y)\},\ f_U\in C^\infty(U\times F, \pi_F{}^*(E_i)),\ x\in U,\ y\in F,$$
$$f_U(x, y)=f_V(x, g_{UV}(x)y),\ (x,y)\in(U\cap V)\times F.$$

We set

$$(21)\qquad g_{UV}(x)^\# f_V(x, y)=f_V(x, g_{UV}(x)y).$$

Let $D:C^\infty(M, E_1)\to C^\infty(M, E_2)$ be adifferential operator of degree $k$on M. Then to fix a connection $\theta=\theta(F)$ of F, $\pi_F{}^*(D)=\pi_F{}^*(D)=\pi_F{}^*(D_U)$ is defined on each $C^\infty(U\times F, \pi_F{}^*(E_1))$.

Definition. *A collection of differential operators* $\{\theta_U\}$, $\theta_U:C^\infty(U\times F, \pi_F{}^*(E_1))\to C^\infty(U\times F, \pi_F{}^*(E_2))$ *is called a connection of D with respect to* $\xi(F)$ *and* $\theta(F))$ *if it satisfies*

(i) $\quad g_{UV}(x)^\#(\pi_F{}^*(D_V)+\theta_V)=(\pi_F{}^*(D_U)+\theta_U)g_{UV}(x)^\#$,

(ii) $\quad \deg\theta_U\leqq k-1$,

(iii) $\quad \theta_U\pi_F{}^*f=0,\ f\in C^\infty(U,E_1)$.

proposition 1'. *For any D and* ξ *(and* $\theta(F)$*), connection exists.*

Proof. Under the same notations as in proposition 1, it is sufficient to set

$$(1)'\qquad \theta_{U,\xi}(x)=\sum_{W\cap U\neq\phi} e_W(x)g_{UW}(x)^\#\{\pi_F{}^*(D_W)g_{WU})x)^\# - g_{WU}(x)^\#\pi_F{}^*(D_U)\}.$$

Definition. *We define a differential operator* $D_\theta:C^\infty(M_F, \pi_F{}^*(E_1))\to C^\infty(M_F, \pi_F{}^*$

$(E_2))$ *by*

$$D\theta f_U=(\pi_F^*(D_U)+\theta_U)f_U,\ \ f=\{f_U\}.$$

By definition, denote the projection from $T^*(M_F)$ to $\pi_F^*(T^*(M))$ defined by $\theta(F)$, by $\pi_{\theta(F)}$, we get

$$\sigma(D_\theta)=\pi_{\theta(F)}^*(\pi_F^*(\pi_F^*(\sigma(D)))). \tag{22}$$

Proposition 2. *If F has a G-invariant measure μ, M is a Rimannian manfold with the volume element dv, and $E=E_1=Ei_2$ has a (fixed) unitary structure, then the formal adjoint $\theta_{\xi'}=\theta_{U,\xi'}$ of θ_ξ is a connection of D', the formal adjoint of D. Especially, if D is formally selfadjoint, then D has a formally selfadjoint connection.*

Proof. By assumption, $g_{UV}(x)^\#$ is extended to a unitary operator of $L^2((U\cap V)\times F,\ dv\otimes\mu)\otimes E_x$, E_x is the fibre of E at x. Hence we have the proposition.

As in n^o3, we denote the set of all connections (respectively, all formally selfadjoint connections) of D with respect to ξ and $\theta(F)$ by $\mathrm{Con}_{\theta(F)}(D,\ \xi)$ and $\mathrm{Con}_{\theta(F),S}(D,\ \xi)$. Then we have

$$(10)'\qquad \mathrm{Con}_{\theta(F)}(D,\ \xi)\cong\mathscr{D}^{k-1}(\pi_F^*(E_1),\ \pi_F^*(E_2)),$$
$$\mathrm{Con}_{\theta(F),S}(D,\ \xi)\cong\mathscr{D}^{k-1}{}_S(\pi_F^*(E_1),\ \pi_F^*(E_2)).$$

9. Let $\mathscr{F}(F)$ be a function space on F such that G acts on $\mathscr{F}(F)$ by the action $\tau^\# f(y)=f(\tau y)$, $\tau\ni G$, $y\in F$, $f\in\mathscr{F}(F)$. Then we can construct associate $\mathscr{F}(F)$-bundle $\mathscr{F}(\xi)$ of ξ. The associate $C^\infty(F)$-bundle of ξ is denoted by $C^\infty(\xi)$.

Lemma 5. (i). *Let E be a finite dimensional vector bundle, then there is an isomorphism ι* such that

$$(23)_i\qquad \iota: C^\infty(M_F,\ \pi_F^*(E))\cong C^\infty(M,\ E\otimes C^\infty(\xi)).$$

(ii). *To fix a connection $\theta(F)$ of F, there is an isomorphism $\iota_{\theta(F)}=\iota_{\theta(F),E_1,E_2}$ such that*

$$(23)_{ii}\qquad \iota_{\theta(F)}: \mathscr{D}^j(\pi_F^*(E_1),\ \pi_F^*(E_2))\cong\mathscr{D}^j(E_1\otimes C^\infty(\xi),\ E_2\otimes C^\infty(\xi)),\ \ j\geqq 1.$$

Proof. Since $f_U(x,\ y)\in E_x\otimes C^\infty(F)$ *if* $\{f_U\}\in C^\infty(M_F,\ \pi_F^*(E))$, $\{f_U\}$ defines a C^∞-cross-section of $E\otimes C^\infty(\xi)$. Conversely, a C^∞-cross-section of $E\otimes C^\infty(\xi)$ satisfies $f=\{f_U\}$, $f_U(x)\in Ex\otimes C^\infty(F)$, $g_{UV}(x)^\# f_V(x)=f_U(x)$, we have (i).

Since a splitting of tangent bundle of M_F induces (local) tensor product decomposition of differential operator on M_F, we have (ii) by (i).

Lemma 6. (i). *Let $\theta_\xi=\{\theta_{U,\xi}\}$ be a connection of D with respect to $\theta(F)$, then*

$$\{\iota^*(\theta)_\xi\}=\{\iota_2\theta_{U,\xi}\iota_1^{-1}\},$$

is a connection of D with respect to $C^{\infty}(\xi)$ *and satisfies*

$$D_{\iota*(\theta_{\xi})}|C^{\infty}(M,\ E_1\otimes\mathbf{K})=D,\ \ \mathbf{K}=\mathbf{R}\ \text{or}\ \mathbf{C}, \tag{24}$$

where K is the space of constant functions on F and D in the right hand side means $D(f\otimes c)=Df\otimes c$.

(ii). *Let* $\theta=\{\theta_U\}$ *be a connection of D with respect to* $C^{\infty}(\xi)$ *and satisfies* (24), *then*

$$(\iota_{\theta(F)}{}^{-1})^*\theta=\{\iota_{\theta(F)}{}^{-1}\theta_U\},$$

is a connection of D with respect to ξ (F) *and* θ (F).

Corollary. *To fix a subbundle* $C^{*\infty}(\xi)$ *of* $C^{\infty}(\xi)$ *such that* $\mathbf{K}\otimes C_*{}^{\infty}(\xi)=C^{\infty}(\xi)$, *we have*

$$\iota^*\mathrm{Con}_{\theta(F)}\,(D,\ \xi)\cong\mathrm{Con}(D,\ C_*{}^{\infty}(\xi)). \tag{24}$$

By this corollary, we can define the action of $\mathfrak{G}(F)$ on $\mathrm{Con}_{\theta(F)}\,(D,\ \xi)$, the obstruction class $o(D,\ \xi)$, characteristic class $ch\ (D,\ \xi)$, *etc.*. Especially, $ch(D,\ \xi)$ belongs in $H^?\ (M,\ \Phi)$, where Φ is a subsheaf of the sheaf of germs of automorphisms of $\mathfrak{P}(F)$.

Lemma 5'. *Denote* $C_0{}^{\infty}(F)$ and $C_0{}^{\infty}(M_F,\ F)$ *be the spaces of compact support smooth functions on F and compact support smooth cross-sections of* $C_0{}^{\infty}(\xi)$, *the associate* $C_0{}^{\infty}(F)$*-bundle of* ξ, *over* M_F, *we have*

$$\iota{:}C_0{}^{\infty}(M_F,\ \pi_F{}^*(E))\cong C_0{}^{\infty}(M,\ E\otimes C_0{}^{\infty}(\xi)). \tag{23$_i'$}$$

By $(23)_i'$, we obtain

Lemma 6'. *We have the isomorphism*

$$\iota^*{:}\mathrm{Con}_{\theta(F)}\,(D,\ \xi)\cong\mathrm{Con}(D,\ E\otimes C_0{}^{\infty}(\xi)) \tag{24$'$}$$

10. Definition. *Let M be a Riemannian manifold with the volume element dv, F has a G-invariant measure* μ *such that* $L^2(F,\ \mu)$ *containes* $C_0{}^{\infty}\ (F)$ *as a dense subspace and E is an Hermitian vector bundle over M, then we define* $n_{\xi}{:}\Gamma[(M,\ E\otimes L^?(\xi))\to\Gamma(M)$ *by*

$$n_{\xi}(f)\,(x)=||f(x)||_{E_X\otimes L^2(F,\mu)}, \tag{25}$$

where $\Gamma(M,\ E\otimes L^?(\xi))$ *is the space of (not necessarily continuous) cross-sections of* $E\otimes L^2(\xi)$ *over M,* $\Gamma(M)$ *is the space of (dv-measurable) functions on M and* $||f(x)||_{E_X\otimes L^2(F,\mu)}$ *is the norm of* $f(x)$ *in* $E_X\otimes L^2(F,\ \mu)$.

Definition. *Under the same assumptions as above, we set*

(26) $$L^2(M,\ E\otimes L^2(\mu))=\{f\,|\,f\in\Gamma(M,\ E\otimes L^2(\mu)),\ n_\xi(f)\in L^2(M,\ dv)\},$$
$$||f||=||n_\xi(f)||\ \text{in}\ L^2(M,\ dv),\ \text{if}\ f\in L^2(M,\ E\otimes L^2(\xi)).$$

Lemma 7. *Under the same assumptions as above, we have*

$(23)_{i,L^2}$ $$\iota_{L^2}: L^2(M_F, \pi_F^*(E))\cong L^2(M,\ E\otimes L^2(\xi)),$$
and ι_{L^2} is a unitary transformation. Here the measure on M_F is given by $dv\otimes\mu$.

Proof. To triangulate M sufficiently fine such that on each simplex σ_i of M by triangulation, ξ and E are both trivial. Then to denote the characteristic function of $\sigma_i\times F$ by χ_i, we have

(27) $$||f||=\sum_i||\chi_i f||,\ f\in L^2(M_F,\ \pi_F^*(E))=L^2(M_F\ dv\otimes\mu)\otimes E.$$

Then by Fubini's theorem, $f(x)$ belongs in $E_x\otimes L^2(F,\ \mu)$ almost everywhere on each $\sigma_i\times F$ and

$$||\chi_i f||^2=\int_{\sigma i}[\int_F\{\chi_i(x)f(x,\ y)\}^2 d\mu]dv$$
$$=\int_{\sigma i}||n_\xi(\chi_i f)||^2(x)dv=\int_M||n_\xi(\chi_i f)||^2(x)dv.$$

Hence we have (27) by (26). Then, since $C_0^\infty(M_F,\ \pi_F^*(E))$ is dense in $L^2(M_F,\ \pi_F^*(E))$, ι_{L^2} is defined on $L^2(M_F,\ \pi_F^*(E))$ and we have the lemma.

In the rest, we assume $G=\mathrm{SO}(n)$ or $\mathrm{SU}(n)$ and $F=\mathbf{R}^n$ or $\mathbf{C}^n$. First we note

$(28)_{\mathrm{R}}$ $$L^2(\mathbf{R}^n)=L^2(\mathbf{R}^+,\ r^{n-1}dr)\otimes L^2(\mathrm{S}^{n-1},\ d\Omega),$$
$(28)_{\mathrm{C}}$ $$L^2(\mathbf{C}^n)=L^2(\mathbf{C}^*,\ r^{n-1}drd\theta)\otimes L^2(CP^{n-1},\ d\omega),$$

and the actions of G on $L^2(\mathbf{R}^+,\ r^{n-1}dr)$ or on $L^2(\mathbf{C}^*,\ r^{n-1}brd\theta)$ are trivial. Here $d\Omega$ and $d\omega$ are the standard volume elements on S^{n-1} and CP^{n-1}. It is known that the o. n.-basis of $L^2(S^{n-1},\ d\Omega)$ and $L^2(CP^{n-1},\ b\omega)$ are taken by harmonic polynomials of homogeneous degree p and type $(p,\ p)$, $p=0,\ 1,\ 2,\dots$ We set the space of harmonic polynomials of homogeneous degree p (with n-variables) and type $(p,\ p)$ by $\mathfrak{H}^q{}_n$ and $\mathfrak{H}^{p,p}{}_n$ ([5], [10]). Then each $\mathfrak{H}^p{}_n$ or $\mathfrak{H}^{p,p}{}_n$ is the representation spaces of SO (n) or SU (n) Denoting their representations by χ_p and $\chi_{\mathrm{C},p}$, the representations of SO (n) and SU (n) in $L^2(R^n)$ and in $L^2(\mathrm{C}^n)$ (equivalently, in $L^2(\mathrm{S}^{n-1},\ b\Omega)$ and in $L^2(\mathrm{CP}^{n-1},\ d\omega)$, denoted by χ and χ_{C}, are decomposed as

(29) $$\chi=\sum_{p=0}^{\infty}\chi_p,\ \chi_{\mathrm{C}}=\sum_{p=0}^{\infty}\chi_{\mathrm{C},p}.$$

We denote the associate $L^2(S^{n-1}, d\Omega)$-bundle or L^2 $(\mathrm{CP}^{n-1}, d\omega)$-bundle of ξ by $\chi(\xi)$ or $\chi_{\mathrm{C}}(\xi)$, and the associate $\mathfrak{H}^p{}_n$-bundle or $\mathfrak{H}^{p,p}{}_n$-bundle by $\chi_p(\xi)$ or $\chi_{\mathrm{C},p}(\xi)$.

Proposition 3. *There exists a connection θ_ξ of D with respect to ξ which induces connection $\chi_p(\theta_\xi)$ or $\chi_{\mathrm{C},p}(\theta_\xi)$ of D with respect to $\chi_p(\xi)$ or $\chi_{\mathrm{C},p}(\xi)$ for each p and satisfies*

$$\chi_0(\theta_\xi)=0, \quad or \quad \chi_{\mathrm{C},0}(\theta_\xi)=0.$$

Proof. The connection θ_ξ constructed in the proof of proposition 1′ satisfies the requirements of the proposition.

We denote the sheaves defined for D and $\chi(\xi)$ or $\chi_{\mathrm{C}}(\xi)$, $\chi_p(\xi)$ or $\chi_{\mathrm{C},p}(\xi)$ by $\mathrm{R}(\mathscr{D}\chi)$, $\mathrm{R}(\mathscr{D}\chi_{\mathrm{C}})$, $\mathrm{R}(\mathscr{D}\chi)$ or $\mathrm{R}(\mathscr{D}\chi_{\mathrm{C},p})$ (cf. n°6). Then there are maps $\iota_p: H^1(M, \mathrm{R}(\mathscr{D}\chi))\to H^1(M, \mathrm{R}(\mathscr{D}\chi))$ and $\chi_{\mathrm{C},p}: H^1(M, \mathrm{R}(\mathscr{D}\chi_{\mathrm{C}}))\to H^1(M, \mathrm{R}(\mathscr{D}\chi_{\mathrm{C},p}))$ induced by the inclusions $\mathfrak{H}^p{}_n\to L^2(S^{n-1}, d\Omega)$ and $\mathfrak{H}^{p,p}{}_n\to L^2(CP^{n-1}, d\omega)$. Then we have by proposition 3 (and lemma 3)

Theorem 2. *We have*

$$\begin{aligned}(30)\qquad &\iota_p(o(D, \chi(\xi)))=o(D, \chi_p(\xi)),\\ &\iota_{\mathrm{C},p}(o(D, \chi_{\mathrm{C}}(\xi)))=o(D, \chi_{\mathrm{C},p}(\xi)),\end{aligned}$$

and $o(D, \chi(\xi))$ (respectively, $o(D, \chi_{\mathrm{C}}(\xi))$ vanishes if $o(D, \chi_p(\xi))=0$ (respectively $o(D, \chi_{\mathrm{C},p}(\xi))=0$) for all p.

Proof. We only need to show the second assertion. But since $L^2(S^{n-1}, d\Omega)$ (respectively $L^2(CP^{n-1}, d\omega)$) is the direct sum of $\mathfrak{H}^p{}_n$ (respectively $\mathfrak{H}^{p,p}{}_n$), we have the second assertion.

Same theorems hold for *ch*$(D, \chi(\xi))$, *etc.*, and higher obstruction classes.

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