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Indexes of Some Degenerate Operators

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§ 5. Fundamental solutions on the cylinder.

Let $A : C^\infty(Y, E) \to C^\infty(Y, E)$ be a 1-st order selfadjoint elliptic operator, where Y is a compact oriented Riemannian manifold and E is a (complex) vector bundle over Y. On $Y \times \mathbf{R}^+$, We consider differential operators $D_+ = D_{+,k}$ and $D_- = D_{-,k}$ given by

$$D_{+,k} = \frac{\partial}{\partial u} + u^k A,\ k > 0, \tag{15}_{+,k}$$

$$D_{-,k} = u^k \frac{\partial}{\partial u} + A,\ k > 0. \tag{15}_{-,k}$$

By definitions, $D = D_{+,k}$, or $D_{-,k}$, is a differential operator from $C^\infty(Y \times \mathbf{R}^+, E)$ into itself. Here $E = \pi^*(E)$ is the induced bundle of E on $Y \times \mathbf{R}^+$ by the projection $\pi : Y \times \mathbf{R}^+ \to Y$.

Since $u^k \neq 0$ on $Y \times (\mathbf{R}^+ - \{0\})$, D_+ $(= D_{+,k})$ and D_- $(= D_{-,k})$ are both elliptic on $Y \times (\mathbf{R}^+ - \{0\})$ and their formal adjoints are given by

$$D_{+,k}^* = -\frac{\partial}{\partial u} + u^k A, \tag{15}_{+,k}^*$$

$$D_{-,k}^* = -u^k \frac{\partial}{\partial u} - k u^{k-1} + A. \tag{15}_{-,k}^*$$

The eigenvalues and eigenfunctions of A are denoted by λ and ϕ_λ. The projection from $C^\infty(Y, E)$ onto the space spanned by $\{\phi_\lambda | \lambda \geqq 0\}$ is denoted by P. We set $C^\infty(Y \times \mathbf{R}^+, E ; P) = \{f(y,u) | (Pf)(y,0) = 0.\}$ $C^r(Y \times \mathbf{R}^+, E ; P)$, *etc.*, are similarly defined. The adjoint condition of $(Pf)(y,0) = 0$ is $((I-P)f)(y,0) = 0$. The space $C^\infty(Y \times R^+, E ; I-P)$, *etc.*, are similarly defined,

As in § 2, $L^2 = L^2(Y \times \mathbf{R}^+)$, $H^t = H^t(Y \times \mathbf{R}^+)$, *etc.*, mean the Hilbert space, t-th Sobolev space, *etc.*, on $Y \times \mathbf{R}^+$, *etc.*. C_c^∞.. means the space of compact support C^∞-functions (or cross-sections) and set $(n = \dim Y + 1)$

$$H^t{}_{(s)} = \{f \mid f \in H^t,\ f(y,0) = \frac{\partial f}{\partial u}(y,0) = \cdots = \frac{\partial^s f}{\partial u^s}(y,0) = 0\},\ s \leqq t - \left[\frac{n}{2}\right],$$

$$C_{[s]}{}^\infty = \{f \mid f \in C^\infty,\ f(y,0) = \frac{\partial f}{\partial u}(y,0) = \cdots = \frac{\partial^s f}{\partial u^s}(y,0) = 0\}, C_{c,[s]}{}^\infty = C_c{}^\infty \cap C_{[s]}{}^\infty.$$

Definition. *Let $g(y,u)$ be a cross-section of E such that $g(y,u) = \sum g_\lambda(u)\phi_\lambda(y)$ Then we define operators $Q_{+,k}$ and $Q_{-,k}$ by*

(16) $$Q_{+,k}(g) = \sum_\lambda Q_{\lambda,k}(g_\lambda)(u)\phi_\lambda(y),\quad Q_{-,k}(g) = \sum_\lambda Q_{\lambda,-k}(g_\lambda)(u)\phi_\lambda(y),\quad 0 < k < 1,$$
$$g_\lambda \in C_c{}^\infty(y \times \mathbf{R}^+, E),$$

(16)$_-$ $$Q_{-,k}(g) = \sum_\lambda Q_{\lambda,-k}(g_\lambda)(u)\phi_\lambda(y),\quad k \geqq 1,\quad g_\lambda \in C_{c,[[k']-1]}{}^\infty(Y \times \mathbf{R}^+, E).$$

Proposition 1. *$Q_{\pm,k}$ are the fundamental solutions of $D_{\pm,k}$ with the following properties.*

(a). *The kernels $Q_{\pm,k}(y, u\,;\, z, v)$ of $Q_{\pm,k}$ are C^∞ for $u \neq v$, $u \neq 0$, $v \neq 0$.*

(b). (i). *$Q_{+,k}$ and $Q_{-,k}$, $0 < k < 1$, are defined on $C_c{}^\infty(Y \times \mathbf{R}^+, E)$ and map it into $C^\infty(Y \times (\mathbf{R}^+ - \{0\}), E)$.*

(ii). *$Q^-{}_{,k}$, $k \geqq 1$, is defined on $C_{c,[[k]'-1]}{}^\infty(Y \times \mathbf{R}_+, E)$ and maps it into $C^\infty(Y \times (\mathbf{R}^+ - \{0\}), E) \cap C^0(Y \times \mathbf{R}^+, E\,;\, P)$.*

(c). *For any $0 < m < M$, $Q_{\pm,k}$ is extended to a continuous map $L^2(Y \times [m, M] \to L^2{}_{loc.}$. More precisely, we have*

(i). *$Q_{+,k}$ is extended to a continuous map $Q_{+k} : L^2 \to L^2{}_{loc.}$.*

(ii). *$Q_{-,k}$, $k \geqq 1$, is extended to a continuous map $Q_{-k} : H_{([k]'-1)}{}^{[k]'+1}[0, L] \to L^2{}_{loc.}$, for any $L > 0$.*

Proof. Except (a), the proposition follows from lemma 2 and lemma 6. To show (a), as [4], we set

$$K_A(u) = Y(u)e^{-u|A|}P - Y(-u)e^{u|A|}(I - P),$$

$Y(u)$ is the characteristic function of $\mathbf{R}^+$, $|A| = AP - A(I - P)$.

Then, it is known that the kernel $E_A(y, z, u)$ of K_A is a C^∞ *function on* $Y \times Y \times \mathbf{R}^+$ ([4], I). Then, since

$$Q_{+,k}(y, u\,;\, z, v) = E_{(A/(k+1))}(y, z, u,\ {}^{k+1} - v^{k+1}),$$

$$Q_{-,k}(y, u\,;\, z, v) = v^{-k}E_{(A/(1-k))}(y, z, u^{1-k} - v^{1-k}),\quad k \neq 1,$$

$$Q_{-,1}(y, u\,;\, z, v) = v^{-1}E_A(y, z, \log u - \log v),$$

we have the proposition.

Corollary. *$D_{\pm,k}$ and $D_{\pm,k}^*$ have closed extensions $\mathscr{D}_{\pm,k}$ and $\mathscr{D}_{\pm,k}^*$. $\mathscr{D}_{+,k}$ and $\mathscr{D}_{+,k}^*$, $\mathscr{D}_{-,k}$ and $\mathscr{D}_{-,k}^*$ are adjoints in L^2 each others.*

§ 6. A lemma on Volterra's integral equation.

It is known (cf. [5]) that if the fundamental solution of the heat equation on $\mathbf{R}^+ \times D$ with time variable t and space variables x given by

(17) $\frac{\partial f}{\partial t} + Lf = 0$, *$L$ is an operator on D, the condition is given at $t=0$*, is given

by G, $G\varphi = \int_D G(t, x, \xi)\varphi(\xi)d\xi$, then the fundamental solution E of the equation

(18) $\frac{\partial f}{\partial t} + (L+K)f = 0$, *$K$ is an operator on D,*

is obtained in the form

$$E = G + G^*H,\ G^*H = \int_0^t \int_D G(t-s, x, \eta)H(s, \eta, \xi)d\eta ds. \tag{19}$$

Here, H is the solution of the following Volterra type integral equation

$$H + K_xG + K_x(G^*H) = 0. \tag{20}$$

Lemma 8. *In* (20), *if G satisfies*

$$\lim_{t\to 0} \frac{\partial^n}{\partial^n t^n} G(t, x, \xi) = 0,\ x \neq \xi,\ n \leqq N, \tag*{$(21)_N$}$$

and assume H satisfies the condition

$$\lim_{t\to 0} (1+K_x)\left(\frac{\partial^n H}{\partial t^n}(t, x, \xi)\right) = 0 \text{ implies } \lim_{t\to 0} \frac{\partial^n H}{\partial t^n}(t, x, \xi) = 0,\ n \leqq N,\ x \neq \xi, \tag*{$(c)_N$}$$

then

$$\lim_{t\to 0} \frac{\partial^n H}{\partial t^n}(t, x, \xi) = 0,\ n \leqq N,\ x \neq \xi. \tag*{$(22)_N$}$$

Proof. Since $lim_{t\to 0}(H+KG+K(G^*H)) = lim_{t\to 0}(H+KH) = 0$, we have $lim_{t\to 0}\ H(t, x, \xi) = 0$, $x \neq \xi$ by $(c)_0$. Then we get

$$G_t{}^*H = -G_s{}^*H = [-G(t-s)H(s)]_{s=0}^{t} + G^*H_s = G^*H_s.$$

Hence we obtain

$$H_t = -(KG_t + KH + K(G^*H_s)). \tag*{$(23)_1$}$$

$(23)_1$ shows $lim_{t\to 0}H_t(t, x, \xi) = 0$, $x \neq \xi$, by $(c)_1$. In general, we assume that we have

$$(22)_n \qquad \lim_{t\to 0} \frac{\partial^m}{\partial t^m} H(t, x, \xi) = 0, \ x \neq \xi, \ m \leqq n,$$

$$(23)_n \qquad H_{(n)} = -(KG_{(n)} + nKH_{(n-1)} + K(G^*H_{(n)})), \ F_{(n)} \text{ means } \frac{\partial^n F}{\partial t^n},$$

then, since $G_t{}^*H_{(n)} = -G_s{}^*H_{(n)} = [-G^*H_{(n)}]_0^t + G^*H_{(n+1)}$, we get

$$\begin{aligned} H_{(n+1)} &= -(KG_{(n+1)} + nKH_{(n)} + KH_{(n)} + K(G^*H_{(n+1)})) \\ &= -(KG_{(n+1)} + (n+1)KH_{(n)} + K(G^*H_{(n+1)})). \end{aligned}$$

Hence we obtain $(23)_{n+1}$ and therefore we have $(22)_{n+1}$ if $n+1 \leqq N$ by assumption.

Note. If $(1 + K_x)f = 0$ implies $f = 0$, for example, K_x is an operator given by multiplying a function, then (22) holds under the assumption (21).

Lemma 9. *If G satisfies $(21)_N$ and $F = F(t, x, \xi)$ is a solution of the equation*

$$(24) \qquad (1 + tK_x)F(t, x, \xi) = -K_xG(t, x, \xi),$$

then F also satisfies $(21)_N$ if F is continuous on $t \geqq 0$.

Proof. Since

$$(25) \qquad \frac{\partial^n}{\partial t^n}(1 + tK_x)F = (1 + tK_x)\frac{\partial^n F}{\partial t^n} + nK_x\frac{\partial^{n-1}F}{\partial t^{n-1}},$$

we have the lemma by induction.

Lemma 10. *If G is real analytic in t, $t > 0$, and satisfies $(21)_\infty$, F, a continuous solution of (24) on $t \geqq 0$, is also real analytic in t, $t > 0$, and G^*F exists, then F is a solution of (20) if F satisfies $(c)_\infty$. Conversely, under the same assumptions on G, if a solution H of (20) is real analytic in t, $t > 0$, and satisfies $(c)_\infty$, then H is a solution of (24).*

Proof. To show the first assertion, it is sufficient to show

$$(26) \qquad tK_xF(t, x, \xi) = K_x(G^*F(t, x, \xi)).$$

But since $F(t, x, \xi) = lim_{h\to 0}\int_D G(h - s, x, \eta)F(s + t, \eta, \xi)d\eta$, to set

$$tK_xF(r + t, x, \xi) = \int_r^{r+t}\int_D G(r + t - s, x, \eta)F(s + r + t, \eta, \xi)d\eta ds + f(r, t),$$

$f(r, t) = O(r)$, $r \to 0$, and $O(t)$, $t \to 0$, and for $r > 0$, $f(r,t)$ is real analytic in t although at $t = 0$.

On the other hand, since

$$\int_r^{r+t} \int_D \frac{\partial^n G}{\partial t^n}(r+t-s, x, \eta) F(s+r+t, \eta, \xi) d\eta ds$$

$$= \int_r^{r+t} \int_D G(r+t-s, x, \eta) \frac{\partial^n F}{\partial s^n}(s+r+t, \eta, \xi) d\eta ds + O(t),$$

for any n by lemma 9, we get by the same reason an in the proof of lemma 8

$$\frac{\partial^n F}{\partial t^n}(r+t, x, \xi) + K_x \frac{\partial^n G}{\partial t^n}(r+t, x, \xi) + nK_x \frac{\partial^n G}{\partial t^n}(r+t, x, \xi)$$

$$+ K_x(\int_r^{r+t} \int_D G(r+t-s, x, \eta) \frac{\partial^n F}{\partial s^n}(s+r+t, \eta, \xi) d\eta ds + \frac{\partial^n}{\partial t^n} f(r,t) + o(t) = 0,$$

because F satisfies $(c)_\infty$. But by (26), we also obtain

$$\frac{\partial^n F}{\partial t^n}(r+t, x, \xi) + K_x \frac{\partial^n G}{\partial t^n}(r+t, x, \xi) + nK_x \frac{\partial^n F}{\partial t^n}(r+t, x, \xi)$$

$$+ K_x(\int_r^{r+t} \int_D G(r+t-s, x, \eta) \frac{\partial^n F}{\partial s^n}(s+r+t, \eta, \xi) d\eta ds + o(t) = 0.$$

Hence for any n, $(\partial^n/\partial t^n) f(r, \mathrm{t}) = o(t)$. This shows $f(r, t) = 0$, $r > 0$, because $f(r, t)$ is real analytic in t although at $t = 0$, for $r > 0$. Therefore we get

$$(26)' \qquad TK_x F(r+t, x, \xi) = \int_r^{r+t} \int_D G(r+t-s, x, \eta) F(s+r+t, \eta, \xi) d\eta ds.$$

Tends r to 0 in (26)$'$, we obtain (26) which shows the first assertion.

To show the second assertion, we note that we obtain

$$\frac{H_{(n)}}{n!} r^n = -(K_x(G_{(n)} \frac{r^n}{n!}) + K_x(H_{(n-1)} \frac{r^n}{(n-1)!} + K_x(G^* H_{(n)} \frac{r^n}{n!})),$$

by lemma 8 (and $(23)_n$). Hence we get

$$\sum_{n=0}^\infty \frac{H_{(n)}}{n!} r^n$$

$$= -(K_x(\sum_{n=0}^\infty \frac{G_{(n)}}{n!} r^n) + rK_x(\sum_{n-1=0}^\infty \frac{H_{(n-1)}}{(n-1)!} r^{n-1}) + K_x(G^*(\sum_{n=0}^\infty \frac{H_{(n)}}{n!} r^n)),$$

because G and H are both real analytic in t, $t > 0$, by assumption. Therefore we have

$$H(t+r,x,\xi) = -K_xG(t+r,x,\xi)+rK_xH(t+r,x,\xi)+K_x(G^*(H|_{t=s+r})). \tag{27}$$

In (27), tends t to 0 and change r to t, H satisfies (24). Hence we obtain the lemma.

Corollary. *If G is real analytic in t, $t>0$, and satisfies* $(21)_\infty$, *K_x is a real analytic coefficients differential operator (may be degree is* 0*), then* (20) *has a solution H which satisfies* $(22)_\infty$ *if* (24) *has a solution which satisfies* $(c)_\infty$. *Especially, if* deg. $K=0$, *then* (20) *has a solution H which satisfies* $(22)_\infty$.

§ 7. Construction of the kernels of $e^{-t\triangle_{i,+,k}}$, $i=1, 2$, on the cylinder.

As in § 3, we set $\Delta_{1,\pm,k} = \mathscr{D}_{\pm,k}{}^*\mathscr{D}_{\pm,k}$ and $\Delta_{2,\pm,k} = \mathscr{D}_{\pm,k}\mathscr{D}_{\pm,k}{}^*$.

Definition. *For any $\varepsilon>0$, we set on $Y\times R^+$*

$$D_{+,k,\varepsilon} = \frac{\partial}{\partial u} + (u^k+\varepsilon)A, \tag{28$_+$}$$

$$D_{-,k,\varepsilon} = (u^k+\varepsilon)\frac{\partial}{\partial u} + A. \tag{28$_-$}$$

By definitions, $D_{\pm,k,\varepsilon}$ are elliptic on $Y\times R^+$. Their closures (in L^2) $\mathscr{D}_{\pm,k,\varepsilon}$ and their adjoints $\mathscr{D}_{\pm,k,\varepsilon}{}^*$ are defined and we set

$$\Delta_{1,\pm,k,\varepsilon} = \mathscr{D}_{\pm,k,\varepsilon}{}^*\mathscr{D}_{\pm,k,\varepsilon},\quad \Delta_{2,\pm,k,\varepsilon} = \mathscr{D}_{\pm,k,\varepsilon}\,\mathscr{D}_{\pm,k,\varepsilon}{}^*.$$

Similarly, $D_{\pm,k,\lambda,\varepsilon}$, $\Delta_{i,\pm,k,\lambda,\varepsilon}$, $i=1,2$, *etc.*, are defined. Explicitly, they take the forms

$$\Delta_{i,+,k,\lambda,\varepsilon} = -\frac{\partial^2}{\partial u^2} + (-1)^i k\,u^{k-1} + \lambda^2(u^k+\varepsilon)^2,\quad i=1,\ 2, \tag{29$_+$}$$

$$\Delta_{1,-,k,\lambda,\varepsilon} = -(u^k+\varepsilon)^2\frac{\partial^2}{\partial u^2} - 2ku^{k-1}(u^k+\varepsilon)\frac{\partial}{\partial u} - \lambda\,ku^{k-1} + \lambda^2, \tag{29$_{-,1}$}$$

$$\Delta_{2,-,k,\lambda,\varepsilon} = -(u^k+\varepsilon)^2\frac{\partial^2}{\partial u^2} - 2ku^{k-1}(u^k+\varepsilon)\frac{\partial}{\partial u} - k(k-1)n^{k-2}(u^k+\varepsilon) - \lambda ku^{k-1} + \lambda^2. \tag{29$_{-,2}$}$$

The boundary conditions for these operators are

$$f_\lambda(0)=0,\ \lambda\geq 0,\ (\frac{df}{du}+\varepsilon\lambda f)|_{u=0}=0,\ \lambda<0,\ \text{for } \Delta_{1,+k,\lambda,\varepsilon}, \tag{30$_{1,+,\varepsilon}$}$$

$$f_\lambda(0)=0,\ \lambda\leq 0,\ (\frac{df}{du}+\varepsilon\lambda f)|_{u=0}=0,\ \lambda>0,\ \text{for } \Delta_{2,+,k,\lambda,\varepsilon}, \tag{30$_{2,+,\varepsilon}$}$$

$(30)_{1,-,\varepsilon}$ $\quad f_\lambda(0)=0,\ \lambda\geqq 0,\ (\varepsilon\frac{df}{du}+\lambda f)|_{u=0}=0,\ \lambda<0,\ \mathit{for}\ \Delta_{1,-,k,\lambda,\varepsilon},\ k<1,$

$\mathit{for}\ \Delta_{2,-,k,\lambda,\varepsilon},\ k\geqq 1,$

$(30)_{2,-,\varepsilon}$ $\quad f_\lambda(0)=0,\ \lambda\leqq 0,\ (\varepsilon\frac{df}{du}+\lambda f)|_{u=0}=0,\ \lambda>0,\ \mathit{for}\ {}_2\Delta_{,-,k,\lambda,\varepsilon},\ k<1,$

$\mathit{for}\ \Delta_{1,-,k,\lambda,\varepsilon},\ k\geqq 1.$

To construct the elementary solutions of the heat equations associated to $\Delta_{i,\pm,k,\lambda,\varepsilon}$, we set

$(31)_+$
$$\Delta_{i,+,k,\lambda,\varepsilon}=-\frac{\partial^2}{\partial u^2}+\varepsilon^2\lambda^2+K,$$

$$K=K_{i,+,k,\lambda,\varepsilon}=(-1)^i k\lambda u^{k-1}+\lambda^2(u^{2k}+2\varepsilon u^k),\ i=1,2,$$

$(31)_-$
$$\Delta_{i,-,k,\lambda,\varepsilon}=-\varepsilon^2\frac{\partial^2}{\partial u^2}+\lambda^2+K,\ \ K=K_{i,-,k,\lambda,\varepsilon},\ i=1,2,$$

$$K_{1,-,k,\lambda,\varepsilon}=-(u^{2k}+2\varepsilon u^k)\frac{\partial^2}{\partial u^2}-2ku^{k-1}(u^k+\varepsilon)\frac{\partial}{\partial u}-\lambda ku^{k-1},$$

$$K_{2,-,k,\lambda,\varepsilon}=-(u^{2k}+2\varepsilon u^k)\frac{\partial^2}{\partial u^2}-2ku^{k-1}(u^k+\varepsilon)\frac{\partial}{\partial u}$$
$$-k(k-1)u^{k-2}(u^k+\varepsilon)-\lambda ku^{k-1}.$$

The fundamental solutions of $\partial/\partial t-\partial^2/\partial u^2+\varepsilon^2\lambda^2$ (and $\partial/\partial t-\varepsilon^2\partial^2/\partial u^2+\lambda^2$) with the boundary conditions $(30)_{i,+,\varepsilon}$, $i=1,2$, (and $(30)_{i,-,\varepsilon}$, $i=1,2$,) are given in [4] and they satisfy the assumptions of lemma 10. Hence we may construct the fundamental solutions of $\partial/\partial t-\Delta_{i,+,k,\lambda,\varepsilon}$, $i=1,2$, (and $\partial/\partial t-\Delta_{i,-,k,\lambda,\varepsilon}$, $i=1,2$,) with the boundary conditions $(30)_{i,+,\varepsilon}$, $i=1,2$, (and $(30)_{i,-,\varepsilon}$, $i=1,2$,) by lemma 10. In this §, we treat $\partial/\partial t-\Delta_{i,+,k,\lambda,\varepsilon}$, $i=1,2$.

Since $K_{i,+,k,\lambda,\varepsilon}$, $i=1,2$, are the operators of order 0, the solutions of the equation (24) is given by

$$F(t,u,v)=F_{i,+,k,\lambda,\varepsilon}(t,u,v)$$
$$=-\frac{\{(-1)^i k\lambda u^{k-1}+\lambda^2(u^{2k}+2\varepsilon u^k)\}G(t,u,v)}{1+\{(-1)^i k\lambda u^{k-1}+\lambda^2(u^{2k}+2\varepsilon u^k)\}t},\ i=1,2.$$

Here, $G=G_{i,+,\lambda,\varepsilon}(t,u,v)$, $i=1,2$, are the kernels of the fundamental solutions of $\partial/\partial t-\partial^2/\partial u^2+\varepsilon^2\lambda^2$ with the boundary conditions $(30)_{i,+,\varepsilon}, i=1,2$, given by ([4])

$$G_{i,+,\lambda,\varepsilon}(t,u,v)=\frac{e^{-\varepsilon^2\lambda^2 t}}{\sqrt{4\pi t}}\Big\{\exp\Big(\frac{-(u-v)^2}{4t}\Big)-\exp\Big(\frac{-(u+v)^2}{4t}\Big)\Big\},$$

$$\lambda\geqq 0,\ \mathit{for}\ i=1,\ \lambda\leqq 0,\ \mathit{for}\ i=2,$$

$$G_{i,+,\lambda,\varepsilon}(t,u,v)=\frac{e^{-\varepsilon^2\lambda^2 t}}{\sqrt{4\pi t}}\Big\{\exp\Big(\frac{-(u-v)^2}{4t}\Big)+\exp\Big(\frac{-(u+v)^2}{4t}\Big)\Big\}$$

$$-\varepsilon|\lambda|e^{\varepsilon|\lambda|(u+v)}\,\mathrm{erfc.}\Big\{\frac{u+v}{2\sqrt{t}}+\varepsilon|\lambda|\sqrt{t}\Big\},$$

$$\mathrm{erfc}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\xi^2}d\xi,\ \ \lambda<0,\ \ for\ i=1,\ \ \lambda>0,\ \ for\ i=2.$$

Hence $lim_{\varepsilon\to 0}G_{i,+,\lambda,\varepsilon}=G_{i,+,0}$ in H^s for any $s\geq 0$, where $G_{i,+,0}$ means the fundamental solution of $\partial/\partial t-\partial^2/\partial u^2$ with the boundary condition $(14)_+$. Therefore we have

$$\lim_{\varepsilon\to 0}F_{i,+,k,\lambda,\varepsilon}(t,u,v)=-\frac{\{(-1)^i k\lambda u^{k-1}+\lambda^2u^{2k}\}G_{i,+,0}(t,u,v)}{1+\{(-1)^i k\lambda u^{k-1}+\lambda^2u^{2k}\}t},\ \ i=1,\ 2,$$

where the right hand side is the solution of (24) with $G=G_{i,+,0}$ and $K=K_{i,+,k,\lambda}=(-1)^i k\lambda u^{k-1}+\lambda^2u^{2k}$, $i=1,2$. Then, since

$$(32)_{\mathrm{i}}\qquad 1+\{(-1)^i k\lambda u^{k-1}+\lambda^2u^{2k}\}t\leqq 1+\{(-1)^i k\lambda u^{k-1}+\lambda^2(u^{2k}+2\varepsilon u^k)\}t,$$

$$(32)_{\mathrm{ii}}\qquad 1+\{(-1)^i k\lambda u^{k-1}+\lambda^2u^{2k}\}t\geqq 1,\ \ k\leqq 1,\ \ or\ \ k>1\ \ and\ \ (-1)^i\lambda\geqq 0,$$

$$(32)_{\mathrm{iii}}\qquad 1+\{(-1)^i k\lambda u^{k-1}+\lambda^2u^{2k}\}t\geqq 1-\frac{t(k+1)}{2}\Big(\frac{k-1}{2}\Big)^{(k-1)/(k+1)}|\lambda|^{2/(k+1)},$$

$$(-1)^i\lambda<0,\ k>1,$$

we obtain

Lemma 11. (i). *If* $k\leqq 1$, *or* $k>1$ *and* $(-1)^i\lambda\geqq 0$, *the fundamental solutions of the equations* $\partial/\partial t-\Delta_{i,+,k,\lambda,\varepsilon}$, $i=1,2$, *with the boundary conditions* $(30)_{i,+,\varepsilon}$, $i=1,2$, *tend to the fundamental solutions of the equations* $\partial/\partial t-\Delta_{i,+,k,\lambda}$, $i=1,2$, *with the boundary conditions* $(14)_+$ *in* H^s *for any* $s\geqq 0$ *and this convergence is uniform in* λ. (ii). *If* $k>1$ *and* $(-1)^i\lambda<0$, *we set* $L^2{}_C$ *(or* $H^s{}_C$*) the subspace of* L^2 *(or* H^s*) spanned by* $\{\phi_\lambda\,|\,|\lambda|<C\}$, *then the fundamental solution of* $\partial/\partial t-\Delta_{i,k,\lambda,\varepsilon}$ *with the boundary condition* $(30)_{i,+,\varepsilon}$ *tends to the fundamental solution of the equation* $\partial/\partial t-\Delta_{i,+,k,\lambda}$ *with the boundary condition* $(14)_+$ *in* $L^2{}_C$ *(or* $H^s{}_C$*) on the interval*

$$(33)\qquad 2\Big(\frac{2}{k-1}\Big)^{(k-1)/(k+1)}C^{-2/(k+2)}>t\geqq 0,$$

and this convergence is uniform in λ*(if* $|\lambda|<C$*).*

Proof. By $(32)_{\mathrm{i}}$ and $(32)_{\mathrm{ii}}$, to show (i), we only need to show the uniformity of the convergence in λ, But, since $G_{i,+,\lambda,\varepsilon}$ tends to 0 at least in the order of exp $(-\varepsilon^2\lambda^2)$ becouse erfc $(x)=O(\exp(-x^2))$, we have the uniformity in λ.

On the other hand, by $(32)_{\mathrm{i}}$ and $(32)_{\mathrm{iii}}$, $F_{i,+,k,\lambda,\varepsilon}$ is continuous on $0 \leqq t < 2(2/(k-1))^{(k-1)/(k+1)}|\lambda|^{-2/(k+2)}$, we get (ii).

Corollary. (i). *If* $k \leqq 1$, *denote the kernels of* $\exp(-t\Delta_{1,+,k,\varepsilon}) - \exp(-t\Delta_{2,+k,\varepsilon})$ *and* $\exp(-t\Delta_{1,+,k}) - \exp(-t\Delta_{2,+,k})$ *in* L^2 *by* $F_{+,k,\varepsilon}(t,y,u)$ *and* $F_{+,k}(t,y,u)$, *we have in* L^2

$$(34)_{\mathrm{i}} \qquad \lim_{\varepsilon \to 0} F_{+,k,\varepsilon}(t,y,u) = F_{+,k}(t,y,u), \quad 0 \leqq t < \infty.$$

(ii). *If* $k > 1$, *denote the kernels of* $\exp(-t\Delta_{1,+k,\varepsilon}) - \exp(-t\Delta_{2,+,k,\varepsilon})$ *and* $\exp(-t\Delta_{1,+,k}) - \exp(-t\Delta_{2,+,k})$ *in* $L^2{}_C$ *by* $F_{+,k,\varepsilon,C}(t,y,u)$ *and* $F_{+,k,C}(t,y,u)$, we have in $L^2{}_C$

$$(34)_{\mathrm{ii}} \qquad \lim_{\varepsilon \to 0} F_{+,k,\varepsilon,C}(t,y,u) = F_{+,k,C}(t,y,u), \quad 0 \leqq t < 2\left(\frac{2}{k-1}\right)^{(k-1)/(k+1)} C^{-2}/(k+2).$$

On the other hand, since $\lim_{t\to 0} F_{i,+,\lambda,\varepsilon}(t,u,v) = 0$, $u \neq v$, *for* $\varepsilon \geqq 0$, to set

$$\int_0^\infty \int_Y F_{+,k,\varepsilon}(t,y,u)\,dydu = F_{+,k,\varepsilon}(t), \quad k \leqq 1,$$

$$\int_0^\infty \int_Y F_{+,k,\varepsilon,C}(t,y,u)\,dydu = F_{+,k,\varepsilon,C}(t), \quad k > 1,$$

we obtain

$$(35)_{\mathrm{i}} \qquad \lim_{t\to 0} F_{+,k,\varepsilon}(t) = \lim_{t\to 0} \int_0^\infty \int_Y \{G_{1,+,\varepsilon}(t,y,u) - G_{2,+,\varepsilon}(t,y,u)\}\,dydu, \quad k \leqq 1,$$

$$(35)_{\mathrm{ii}} \qquad \lim_{t\to 0} F_{+,k,\varepsilon,C}(t) = \lim_{t\to 0} \int_0^\infty \int_Y \{G_{1,+,\varepsilon,C}(t,y,u) - G_{2,+,\varepsilon,C}(t,y,u)\}\,dydu, \quad k > 1.$$

Here, $G_{i,+,\varepsilon}$ and $G_{i,+,\varepsilon,C}$ are the kernels of $\exp(-t\Delta_{i,+,\varepsilon})$ on L^2 or on $L^2{}_C$, where $\Delta_{i,+,\varepsilon}$ mean $-\partial^2/\partial u^2 + \varepsilon^2 A^2$ with the boundary conditions

$$(36)_{+,1} \qquad (Pf)(y,0) = 0, \quad (I-P)\left\{\left(\frac{\partial}{\partial u} + A\right)f\right\}(y,0) = 0, \quad i = 1,$$

$$(36)_{+,2} \qquad ((I-P)f)(y,0) = 0, P\left\{\frac{\partial}{\partial u} + A\right)f\right\}(y,0) = 0, \quad i = 2.$$

Then by [4], to set $G_{+,\varepsilon}(t) = \int_0^\infty \int_Y G_{1,+,\varepsilon}(t,y,u) - G_{2,+,\varepsilon}(t,y,u)\}\,dydu$ and $G_{+,\varepsilon,C}(t) = \int_0^\infty \int_Y G_{1,+,\varepsilon,C}(t,y,u) - G_{2,+,\varepsilon,C}(t,y,u)\}\,dydu$, they are both defined on $0 \leqq t < \infty$ and we get

(37)i $$\int_0^\infty \{G_{+,\varepsilon}(t)+\frac{h}{2}\}t^{s-1}dt=-\frac{1}{2s\sqrt{\pi}}\Gamma(s+\frac{1}{2})\varepsilon^{-2s}\eta(2s),$$

(37)ii $$\int_0^\infty \left\{G_{+,\varepsilon,C}(t)+\frac{h}{2}\right\}t^{s-1}dt=-\frac{1}{2s\sqrt{\pi}}\Gamma(s+\frac{1}{2})\varepsilon^{-2s}\eta_C(2s),$$

where $h=\dim.\ker. A$, $\eta(s)$ is the η-function of A given by $\sum_{\lambda\neq 0,\ \lambda\in\text{spec}.A}\operatorname{sign}\lambda|\lambda|^{-s}$ and $\eta_C(s)=\sum_{\lambda\neq 0,\ \lambda\in\text{spec}.A,\ |\lambda|<C}\operatorname{sign}\lambda|\lambda|^{-s}$. Hence, if $G_{+,\varepsilon}$, $G_{+,\varepsilon,C}$, $F_{+,k,\varepsilon}$ and $F_{+,k,\varepsilon,C}$ have asymptotic expansions at $t\to 0$ of the forms

(38)i $$G_{+,\varepsilon}(t)=\sum_{m\geqq -n}a_{+,m,\varepsilon}t^{m/2},\quad F_{+,k,\varepsilon}(t)=\sum_{m\geqq -n}b_{+,m,k,\varepsilon}t^{m/2},\quad k\leqq 1,$$

(38)ii $$G_{+,\varepsilon,C}(t)=\sum_{m\geqq -n}a_{+,m,\varepsilon,C}t^{m/2},\quad F_{+,k,\varepsilon,C}(t)=\sum_{m\geqq -n}b_{+,m,k,\varepsilon,C}t^{m/2},\quad k>1,$$

we have by (35) and (37)

(39)i $$\eta(0)=-2(2a_{+,0,\varepsilon}+h)=-(2b_{+,0,k,\varepsilon}+h),\quad k\leqq 1,$$

(39)ii $$\eta_C(0)=-(2a_{+,0,\varepsilon,C}+h)=-(2b_{+,0,k,\varepsilon,C}+h),\quad k>1.$$

Therefore we obtain

Proposition 2. (i). *If $k\leqq 1$ and $G_{+,\varepsilon}$, $F_{+,k,\varepsilon}$ have asymptotic expansions of the form* (38)i, *then their cofficients of the terms of order* 0 *do not depend on ε and k.*

(ii). *If $k>1$ and $G_{+,\varepsilon,C}$, $F_{+,k,\varepsilon,C}$ have asymptotic expansions of the form* (38)ii, *then their coefficients of the terms of degree* 0 *do not depend on ε and k and their limits at $C\to\infty$ exist.*

In the rest, we set these constants by $a_{+,0}$ and $a_{+,0,C}$. Hence we get

(39)i′ $$\eta(0)=-(2a_{+,0}+h),$$

(39)ii′ $$\eta_C(0)=-(2a_{+,0,C}+h),\quad \lim_{C\to\infty}a_{+,0,C}=a_{+,0}.$$

§ 8. Construction of the kernels of $e^{-t\Delta_{i,-,k}}$ $i=1,2$, on the cylinder, I.

Lemma 12. *The fundamental solutions of $\partial/\partial t-\varepsilon^2\partial^2/\partial u^2+\lambda^2$ with the boundary conditions $(30)_{i,-,\varepsilon}$, $i=1,2$, tends to the fundamental solution of $\partial/\partial t+\gamma^2$ on (t,u) — space if $u>0$.*

Proof. Since the fundamental solutions are given

$$G_{i,-,\lambda,\varepsilon}(t,u,v)=\frac{1}{\varepsilon}\frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}}\left[\exp\left\{\frac{-(u-v)^2}{4\varepsilon^2 t}\right\}-\exp\left\{\frac{-(u+v)^2}{4\varepsilon^2 t}\right\}\right],$$

$$\lambda\geqq 0 \text{ for } i=1,\quad \lambda\leqq 0 \text{ for } i=2,$$

$$G_{i,-,\lambda,\varepsilon}(t,u,v)=\frac{1}{\varepsilon}\left[\frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}}\left[\exp\left\{\frac{-(u-v)^2}{4\varepsilon^2 t}\right\}+\exp\left\{\frac{-(u+v)^2}{4\varepsilon^2 t}\right\}\right]\right.$$

$$\left.-|\lambda|e^{|\lambda|/\varepsilon(u+v)}\,\text{erfc.}\left\{\frac{u+v}{2\varepsilon\sqrt{t}}+|\lambda|\sqrt{t}\right\}\right],$$

$$\lambda<0 \textit{ for } i=1,\ \lambda>0 \textit{ for } i=2,$$

and

$$\frac{|\lambda|}{\varepsilon}\int_0^\infty e^{|\lambda|/\varepsilon(u+v)}\,\text{erfc.}\left\{\frac{u+v}{2\varepsilon\sqrt{t}}+|\lambda|\sqrt{t}\right\}f(v)dv$$

$$=2|\lambda|\sqrt{t}\int_{u/2\varepsilon\sqrt{t}}^\infty e^{2|\lambda|\sqrt{t}\,w}\,\text{erfc.}\,\{w+|\lambda|\sqrt{t}\}f(2\varepsilon\sqrt{t}\,w-u)dw,$$

$$\int_0^\infty e^{2|\lambda|\sqrt{t}\,w}\,\text{erfc.}\,\{w+|\lambda|\sqrt{t}\}dw=\frac{1}{2|\lambda|\sqrt{t}}(e^{-|\lambda|^2 t}-\text{erfc.}\,(|\lambda|\sqrt{t}),$$

we have

(40)i $$\lim_{\varepsilon\to 0} G_{i,-,\lambda,\varepsilon}(t,u,v)=e^{-2t}\delta_u,\ u\neq 0,$$

$$\lim_{\varepsilon\to 0} G_{i,-,\lambda,\varepsilon}(t,0,v)=0,\quad \lambda\geqq 0 \textit{ for } i=1,\ \lambda\leqq 0 \textit{ for } i=2,$$

(40)ii $$\lim_{\varepsilon\to 0} G_{i,-,\lambda,\varepsilon}(t,u,v)=e^{-\lambda^2 t}\delta_u,\ u\neq 0,$$

$$\lim_{\varepsilon\to 0} G_{i,-,\lambda,\varepsilon}(t,0,v)=\{e^{-\lambda^2 t}+\text{erfc.}\,(|\lambda|\sqrt{t})\}\delta_0,$$

$$\lambda<0 \textit{ for } i=1,\ \lambda>0 \textit{ for } i=2,$$

in $(C_C{}^1)^*$, the dual space of compact support C^1-class functions. Here, δ_u means the Dirac measure concentrated at $\{u\}$. Because we get

$$\lim_{\varepsilon\to 0}\int_{u/2\varepsilon\sqrt{t}}^\infty e^{-w^2}f(2\varepsilon\sqrt{t}\,w-u)dw$$

$$=\lim_{\varepsilon\to 0}\int_{u/2\varepsilon\sqrt{t}}^\infty e^{2|\lambda|\sqrt{t}\,w}\,\text{erfc.}\,\{w+|\lambda|\sqrt{t}\}f(2\varepsilon\sqrt{t}\,w-u)dw=0,$$

$$u\neq 0,\ f\in C_C{}^1.$$

Hence we obtain the lemma.

Corrolary. *Let* $H_\varepsilon=H_{\lambda,\varepsilon}=H_{i,\lambda,\varepsilon}$ *be a solution of the equation*

(41) $$\frac{\partial^2}{\partial u^2}H_{i,\lambda,k,\varepsilon}(t,u,v)-\frac{1}{tu^{2k}}H_{i,\lambda,k,\varepsilon}(t,u,v)=-\frac{1}{t}\frac{\partial^2}{\partial u^2}G_{i,-,\lambda,\varepsilon}(t,u,v),\ i=1,2,$$

and assume $G_\varepsilon+G_\varepsilon{}^*H_\varepsilon$ *and* $\lim_{\varepsilon\to 0}G_\varepsilon+G_\varepsilon{}^*H_\varepsilon$ *both exist and* H_ε *tends to a solution* *of* $\partial^2 H_\lambda/\partial u^2 - H_\lambda/tu^{2k} = -(e^{-\lambda^2 t}/t)\delta_u^{(2)}$. *Then the fundamental solution of* $\partial/\partial t - (u^{2k}+\varepsilon^2)\,\varrho^2/\varrho u^2 + \lambda^2$ *given by* $G_\varepsilon + G_\varepsilon * H_\varepsilon$ *tends to a fundamental solution of* $\partial/\partial t - u^{2k}\partial^2/\partial u^2 + \lambda^2$ *on* $u>0$ *if* $\lim_{t\to 0} H_\varepsilon(t,u,v) = u^{2k}\delta_u^{(2)}$. *Here,* $G_\varepsilon = G_{\lambda,\varepsilon}$ *means* $G_{i,^-,\lambda,\varepsilon}$.

Proof. First we note that H_λ is given by $e^{-\lambda^2 t}H_0\ H_0$ if H_0 exists. Then, since $tu^{2k}H_{0,uu} - H_0 = -u^{2k}\delta_u^{(2)}$, we have $u^{2k}H_{0,uu} = H_{0,t}$ and therefore

$$u^{2k}H_{\lambda,uu} - H_{\lambda,\varepsilon} = \lambda^2 H_\lambda. \tag{42}$$

On the other hand, since we get by (40)

$$\lim_{\varepsilon\to 0} G_{\lambda,\varepsilon} + G_{\lambda,\varepsilon}*H_{\lambda,\varepsilon} = e^{-\lambda^2 t}\delta_u + \int_0^t e^{-\lambda^2(t-s)}H_\lambda(s,u,v)ds,$$

we have

$$\Big(\frac{\partial}{\partial t} - u^{2k}\frac{\varrho^2}{\partial u^2} + \lambda^2\Big)\big(\lim_{\varepsilon\to 0} G_{\lambda,\varepsilon} + G_{\lambda,\varepsilon}*H_{\lambda,\varepsilon}\big)$$

$$= H_\lambda(t,u,v) - u^{2k}e^{-\lambda^2 t}\delta_u^{(2)} - u^{2k}e^{-\lambda^2 t}\int_0^t e^{\lambda^2 s}H_{\lambda,uu}(s,u,v)ds.$$

Hence $\partial/\partial t\{e^{\lambda^2 t}(\partial/\partial t - u^{2k}\partial^2/\partial u^2 + \lambda^2)(\lim_{\varepsilon\to 0}G_{\lambda,\varepsilon} + G_{\lambda,\varepsilon}*H_\lambda{}'_\varepsilon)\} = 0$ by (42). Then, Since to set

$$\Big(\frac{\partial}{\partial t} - u^{2k}\frac{\partial^2}{\partial u^2} + \lambda^2\Big)\big(\lim_{\varepsilon\to 0} G_{\lambda,\varepsilon} + G_{\lambda,\varepsilon}*H_{\lambda,\varepsilon}\big) = e^{-\lambda^2 t}\cdot C,$$

C is given by $\lim_{t\to 0}\{H_\lambda(t,u,v) - u^{2k}e^{-\lambda^2 t}\delta_u^{(2)} - u^{2k}e^{-\lambda^2 t}\int_0^t e^{\lambda^2 s}H_{\lambda,uu}(s,u,v)ds\} = \lim_{t\to 0}\{H_\lambda(t,u,v) - u^{2k}\delta_u^{(2)}\}$, we obtain the corollary.

To solve the equation (41), we set

$$y_1(t,u) = \sqrt{u}\,J_{1/|2-2k|}\Big(\sqrt{\frac{-1}{t}}\frac{u^{1-k}}{1-k}\Big),\quad y_2(t,u) = \sqrt{u}\,Y_{1/|2-2k|}\Big(\sqrt{\frac{-1}{t}}\frac{u^{1-k}}{1-k}\Big),\ k\neq 1,$$

$$y_1(t,u) = \sqrt{u}\cdot u^{-\sqrt{1/t+1/4}},\quad y_2(t,u) = \sqrt{u}\cdot u^{\sqrt{1/t+1/4}},\quad k=1,$$

where J_α and Y_β are α-th Bessel function and β-th Bessel function of the second kind. Then y_1 and y_2 are the solutions of the equation $d^2y/du^2 - y/tu^{2k} = 0$ ([10], [18]) and their Wronskians $W(y_1,y_2)$ are given by

$$W(y_1,y_2) = \frac{2(1-k)\sqrt{-1}}{\pi\sqrt{t}},\quad k\neq 1,\quad W(y_1,y_2) = \sqrt{1+\frac{4}{t}},\quad k=1.$$

Hence a solution of (41) is given by

(43) $$H_{\varepsilon,c}(t,u,v)=-\frac{1}{t}G_\varepsilon(t,u,v)-\frac{1}{t^2}\int_c^u\frac{1}{w^{2k}}g(t,u,w)G_\varepsilon(t,w,v)dw,$$

$$g(t,u,v)=\frac{1}{W(y_1,y_2)}\{y_1(t,v)y_2(t,u)-y_1(t,u)y_2(t,v)\}.$$

By (43) and (40), we have

(44) $$\lim_{\varepsilon\to 0}H_{\varepsilon,\varepsilon}(t,u,v)=-e^{-\lambda^2 t}\left\{\frac{\delta_u}{t}+\frac{Y(u-v)}{t^2v^{2k}}g(t,u,v)\right\},\quad u>0,\ v>0,\ in\ (C_c{}^1)^*.$$

In other word, $lim_{\varepsilon\to 0}H_{\varepsilon,\varepsilon}(t,u,v)$ tends to a solution of the equation $\partial^2H/\partial u^2-H/tu^{2k}=-(e^{-\lambda^2t}/t)\delta_u{}^{(2)}$ in $(C_c{}^2)^*$ if $u>0$.

Definition. *For $k>0$, we set*

$$(T_kf)(w)=f(\{1-k)v\}^{1/(1-k)}),\quad v=\frac{w^{1-k}}{1-k},\quad k\neq 1,$$

$$(T_1f)(w)=f(e^v),\quad v=\log.w,\quad k=1,$$

and define the subspaces $\mathscr{H}_k$ of the space of cotinuous funnctinons in v–space by

$$\mathscr{H}_k=\{f\,|\,T_k\Big(\frac{f}{\sqrt{v}}\Big)(w)\ \textit{is continuous on}\ 0\leqq \arg.w\leqq\frac{\pi}{2},\ \textit{holomorphic on}$$

$$0<arg.\ w<\frac{\pi}{2},\ w\neq 0,\ |T_k\Big(\frac{f}{\sqrt{v}}\Big)(re^{\sqrt{-1}\theta})|=O(e^{-r^{1+\varepsilon}}),\ r\to\infty,\ for$$

$$some\ \varepsilon>0,\ w=re^{\sqrt{-1}\theta},\quad 0\leqq\theta\leqq\frac{\pi}{2}\},\quad k\neq 1,$$

$$\mathscr{H}_1=\{f\,|\,T_1\Big(\frac{f}{\sqrt{v}}\Big)(w)\ \textit{is continuous on}\ 0\leqq arg.\ w\leqq\pi,\ \textit{holomorphic on}$$

$$0<arg.\ w<\pi,\quad |T_1\Big(\frac{f}{v}\Big)(re^{\sqrt{-1}\theta})|\in L^1(0,\infty)\}.$$

We denote by $\mathscr{L}$ the Laplace transform and by H_ν, the ν–th Hankel transform given by $H_\nu(f)(x)=\int_0^\infty x^{-\nu}y^{\nu+1}J_\nu(xy)f(y)dy$, Re. $\nu\geqq -1/2$. *We know that* $H_\nu(H_\nu(f))=f$ if f is C^∞ and rapidly decreasing at ∞ ([2], [6], [18]).

Definition. *We define the subspaces $\mathscr{G}_k$ of the space of conitinuous functions in t–space by*

$$\mathscr{G}_k=\{\varphi\,|\,\varphi=H_{1/|2-k|}(f)\Big(\sqrt{\frac{-1}{t}}\Big),\ f\in\mathscr{H}_k\},\ k\neq 1,$$

$$\mathscr{G}_1=\{\varphi|\varphi=\mathscr{L}^{-1}(f)\left(\sqrt{\frac{-1}{t}}\right),\ f\in\mathscr{H}_1\}.$$

$\mathscr{H}_k$ and $\mathscr{G}_k$ are both considered to be topological vector spaces (not complete) with the uniform convergence topology. Then we obtain by (44) and the corollary of lemma 12

Lemma 13. *There exist fundamental solutions* E_ε *of the equation* $\frac{\partial}{\partial t}-(u^{k2}+\varepsilon^2)\partial^2/\partial u^2+\lambda^2$ *(with the boundary conditions* $(30)_{i,-}$*) which tend to a fundamental solution* E *of* $\partial/\partial t-u^{2k}\partial^2/\partial u^2+\lambda^2$ *in* $(\mathscr{G}_k)^*\otimes(\mathscr{H}_k)^*$*, and this* E *satisfies*

(45) $$Supp.\ E\subset\mathbf{R}^1\times\{(u,v)|u\geqq v\}.$$

Proof. By Hankel's inversion theorem ([2], [6], [18]), δ_u is approximated by the solutions of $\partial^2H/\partial u^2-H/tu^{2k}=0$ in $(\mathscr{F}_k)^*\otimes(\mathscr{H}_k)^*$. On the other hand, $(\mathrm{e}^{-\lambda^2}/t^2v^{2k})\,g(t,u,v)$ is a solution of $\partial^2H/\partial u^2-H/tu^{2k}=0$. Then, since

$$g(t,u,v)\sim\frac{2(k-1)}{\pi\sqrt{-1}}\sqrt{tu^kv^k}\,[\sin\left\{\sqrt{\frac{-1}{t}}\,\frac{1}{1-k}(u^{1-k}-v^{1-k})\right\}$$

$$+\frac{|1-k|}{2\sqrt{-1}}\sqrt{t}\,(u^{k-1}-v^{k-1})\cos\left\{\sqrt{\frac{-1}{t}}\,\frac{1}{1-k}(u^{1-k}-v^{1-k})\right\}+\cdots],\ k\neq1,$$

$$g(t,u,v)=\sqrt{\frac{tuv}{t+4}}\left[\left(\frac{u}{v}\right)^{\sqrt{1/t+1/4}}-\left(\frac{v}{u}\right)^{\sqrt{1/t+1/4}}\right],\ k=1,$$

$lim_{\varepsilon\to0}(G_\varepsilon+G_\varepsilon*[(-\mathrm{e}^{-\lambda^2t}/t^2v^{2k})\{1-Y(u-v)\}g(t,u,v)])$ is defined as an element of $(\mathscr{F}_k)^*\otimes(\mathscr{H}_k)^*$ if $u>0$. Hence we obtain the lemma.

Corollary 1. *There exist fundamental solutions* $E_{1,\varepsilon}$ *of the equations* $\partial/\partial t-(u^k+\varepsilon)^2\partial^2/\partial u^2+\lambda^2$ *which tend to a fundamental solution* E *of* $\partial/\partial t-u^{2k}\partial^2/\partial u^2+\lambda^2$ *in* $(\mathscr{F}_k)^*\otimes(\mathscr{H}_k)^*$*, which satisfies* (45).

Proof. By lemma 10, a fundamental solution of $\partial/\partial t-(u^k+\varepsilon)^2\partial^2/\partial u^2+\lambda^2$ is obtained from E_ε by solving the equation

$$\frac{\partial^2H_{1,\varepsilon}}{\partial u^2}-\frac{H_{1,\varepsilon}}{2tu^k}=-\frac{1}{t}\frac{\partial^2E_\varepsilon}{\partial u^2}.$$

The solution $H_{1,\varepsilon}$ of this equation is given by

$$H_{1,\varepsilon,c}(t,u,v,)=-\frac{1}{t}\int_c^u g_\varepsilon(t,u,w)E_{\varepsilon,ww}(t,w,v)dw,$$

$$g_\varepsilon(t,u,v)=\frac{\pi\sqrt{\varepsilon t}}{(2-k)\sqrt{-1}}J_{1/|2-k|}\left(\sqrt{\frac{-1}{2\varepsilon t}}\,\frac{v^{1-k/2}}{2-k}\right)Y_{1/|2-k|}\left(\sqrt{\frac{-1}{2\varepsilon t}}\,\frac{u^{1-k/2}}{2-k}\right)$$

$$-J_{1/|2-k|}\left(\sqrt{\frac{-1}{2\varepsilon t}}\,\frac{u^{1-k/2}}{2-k}\right)Y_{1/|2-k|}\left(\sqrt{\frac{-1}{2\varepsilon t}}\,\frac{v^{1-k/2}}{2-k}\right),\quad k\neq 2,$$

$$g_\varepsilon(t,u,v)=\sqrt{\frac{2\varepsilon tuv}{1+2\varepsilon t}}\left\{\left(\frac{u}{v}\right)^{\sqrt{1/4+1/8\varepsilon t}}-\left(\frac{v}{u}\right)^{\sqrt{1/4+1/8\varepsilon t}}\right\},\quad k=2.$$

Hence $lim_{\varepsilon\to 0}H_{1,\varepsilon,\varepsilon}{}^*E_\varepsilon=0$ by (45) because $lim_{\varepsilon\to 0}g_\varepsilon(t,u,u)=0$.

Corollary 2. *There exist fundamental solutions* $E_{2,\varepsilon}$ *of the equation* $\partial/\partial t-\Delta_{i,-,k,\lambda,\varepsilon}$ *on* (t,u)*-space which tend to a fundamental solution* E *of the equation* $\partial/\partial t-u^{2k}\partial^2/\partial u^2+\lambda^2$ *in* $(\mathscr{F}_k)^*\otimes(\mathscr{H}_k)^*$ *and this* E *satisfies* (45).

Proof. By lemma 10, a fundamental solution of $\partial/\partial t-\Delta_{i,-,k,\lambda,\varepsilon}$ is obtained from $E_{1,\varepsilon}$ by solving the equation

$$\frac{dH_{2,i,\varepsilon}}{du}-\frac{1-kt\{\lambda u^{k-1}+\delta_{i,2}(k-1)u^{k-2}(u^k+\varepsilon)\}}{t(2ku^{k-1}(u^k+\varepsilon))}H_{2,i,}$$

$$=-\frac{1}{t}\,\frac{\partial E_{1,\varepsilon}}{\partial u}-\frac{\lambda u+\delta_{i,2}(k-1)(u^k+\varepsilon)}{2u(u^k+\varepsilon)}E_{1,\varepsilon},\quad \delta_{1,2}=0,\quad \delta_{2,2}=1.$$

A solution of this equation is

(46) $\qquad H_{2,i,\varepsilon,c}(t,u,v)$

$$=-\int_c^u\left[\exp\int_w^u\frac{1-kt\{\lambda x^{k-1}+\delta_{i,2}(k-1)x^{k-2}(x^k+\varepsilon)\}}{t(2kx^{k-1}(x+{}^k\varepsilon))}dx\right]\cdot$$

$$\left[\frac{1}{t}\,\frac{\partial E_{1,\varepsilon}}{\partial w}(t,w,v)-\frac{\lambda w+\partial_{i,2}(k-1)(w^k+\varepsilon)}{\lambda w(w^k+\varepsilon)}E_{1,\varepsilon}(t,w,v)\right]d\mathrm{w}.$$

Hence in $(\mathscr{F}_k)^*\otimes(\mathscr{H}_k)^*$, $lim_{\varepsilon\to 0}H_{2,i,\varepsilon,\varepsilon}{}^*E_{1,2}=0$ by (45) and we have the lemma.

Note. The fundamental solution of $\partial/\partial t-u^{-k}\partial^2/\partial u^2$ is given in [8] (cf. [8]', [9]). It takes similar form as our solution.

§ 9. Construction of the kernels of $\mathrm{e}^{-t\Delta_{i,-,k}}$, $i=1,2$, on the cylinder, II.

By the corollary 2 of lemma 13, we obtain

Lemma 14. *There exist fundamental solutions* $F_{i,-,k,\varepsilon,C}$ *of the equation* $\partial/\partial t-\Delta_{i,-,k,\varepsilon}$ *which tend to a fundamental solution* $F_{i,-,k,C}$ *of* $\partial/\partial t-\Delta_{i,-,k}$ *in* $(\mathscr{F}_k)^*\otimes(\mathscr{H}_k)^*\otimes H^s{}_C(Y)$ *for any* $C>0$, *and these fundamental solutions satisfy*

$$F_{i,-,k,\varepsilon,C'}|(\mathscr{F}_k)^*\otimes(\mathscr{H}_k)^*\otimes H^s{}_C(Y)=F_{i,-,k,\varepsilon,C},$$

$$Fi_{,-,k,C'}|(\mathscr{F}_k)^*\otimes(\mathscr{H}_k)^*\otimes H^s{}_C(Y)=F_{i,-,k,C},$$

if $C'>C$.

Proof. By (46), the convergence of $E_{1,\varepsilon}=E_{1,\varepsilon,\lambda}$ to $E_\varepsilon=E_{\varepsilon,\lambda}$ is uniform in λ *if* $|\lambda|\leqq C$. Hence we have the lemma.

Corollary. *There exist fundamental solutions $F_{i,-,k,\varepsilon}$ of the equation $\partial/\partial t-\Delta_{i,-,k,\varepsilon}$ densely defined in $(\mathscr{F}_k)^*\otimes(\mathscr{H}_k)^*\otimes H^s(Y)$ which tends to a fundamental solution $F_{i,-,k}$ of $\partial/\partial t-\Delta_{i,-,k}$ densely defined in $(\mathscr{F}_k)^*\otimes(\mathscr{H}_k)^*\otimes L^2(Y)$.*

We denote the inclusion from $\mathscr{F}_k\otimes\mathscr{H}_k\otimes H^s(Y)$ *into* $H^s(\mathbf{R}^+\times\mathbf{R}^+\times Y)$ by $i_{k,s}$. The dual of $i_{k,s}$ is denoted by $i_{k,s}^*$, and set

$$\ker.\, i_{k,s}^*=\mathscr{N}_{k,s}. \tag{47}$$

If $s\geqq n+1$ (dim. $Y=n-1$), then $\mathscr{N}_{k,s}\neq 0$. But, since we know

$$C_{c,}((1-k)/2)^\infty\subset\mathscr{H}_k^*,\ \ k<1,\ \ C_{c,((k-1)/2}^\infty\subset\mathscr{H}_k^*,\ \ k<1,$$

we have

$$C_{C,[2k]'}^\infty(\mathbf{R}\times\mathbf{R}\times Y)\cap\mathscr{N}_{k,s}=0, \tag{48}$$

Here, $C_{c,[2k]'}^\infty(\mathbf{R}\times\mathbf{R}\times Y)$ is considered to be a subspace of $H^s(\mathbf{R}^+\times\mathbf{R}^+\times Y)$.

In $(\mathscr{F}_k)^*\otimes(\mathscr{H}_k)^*\otimes H^s(Y)$, $lim_{\varepsilon\to0}trace\,[\exp(-t\Delta_{1,-,k,\varepsilon})-\exp(-t\Delta_{2,-,k,\varepsilon})]$ coincides to $lim_{\varepsilon\to0}\int_0^\infty\int_Y G_{1,-,\varepsilon}(t,y,n)-G_{2,-,\varepsilon}(t,y,u)\}dydu$, because $lim_{\varepsilon\to0}(E_\varepsilon-G_\varepsilon)(t,y,u)=lim_{\varepsilon\to0}H_{1,\varepsilon}(t,u,u)=lim_{\varepsilon\to0}H_{2,\varepsilon}(t,u,u)=0$. Hence we get

$$\lim_{\varepsilon\to0}\mathrm{trace}[e^{-t\Delta_{1,-,k,\varepsilon}}-e^{-t\Delta_{2,-,k,\varepsilon}}]=-\sum_{\lambda\in\mathrm{spec.}A,\ \lambda\neq0}\frac{\mathrm{sign}\lambda}{2}\mathrm{erfc.}(|\lambda|\sqrt{t}),\ \ k<1, \tag{49}$$

$$=\sum_{\lambda\in\mathrm{spec.}A,\ \lambda\neq0}\frac{\mathrm{sign}\lambda}{2}\mathrm{erfc.}(|\lambda|\sqrt{t}),\ \ k\geqq1,$$

because $G_{i,-,\varepsilon}(t,y,u)=\frac{1}{\varepsilon}G_i(t,y,u/\varepsilon)$, where $G_i(t,y,u)$, $i=1,2$, are the kernels of $\partial/\partial t-\Delta_i$, $i=1,2$, with the boundary conditions $(36)_i$, $i=1,2$ (cf. [4]). Therefore, to set

$$F_{-,k,\varepsilon}(t)=\int_0^\infty\int_Y\Big\{F_{1,-,k,\varepsilon}(t,y,u)-F_{2,-,k,\varepsilon}(t,y,u)\Big\}dydu,$$

if $F_{-,k,\varepsilon}(t)$ has asymptotic expansion at $t\to0$,

$$F_{-,k,\varepsilon}(t)\sim\sum_{m\geqq n}b_{-,m,k,\varepsilon}t^{m/2}, \tag{50}$$

we have by (49)

(51)
$$\eta(0) = \lim_{\varepsilon\to 0} -(2b_{-,0,k,\varepsilon}+h),\quad k<1,$$
$$= \lim_{\varepsilon\to 0}(2b_{-,0,k,\varepsilon}+h),\quad k\geqq 1.$$

Summarising these, we obtain

Proposition 3. *If $F_{-,k,\varepsilon}$ has asymptotic expansion (50) at $t\to 0$, then $\lim_{\varepsilon\to 0}b_{-,0,k,\varepsilon}$ exists and we have*

(52)
$$\lim_{\varepsilon\to 0} b_{-,0,k,\varepsilon} = a_0,\quad k<1,$$
$$= -a_0,\quad k\geqq 1,$$

where $a_0 = a_{+,0}$ is determined by the asymptotic expansion of $G(t)$ at $t\to 0$ given by

$$G(t)\sim \sum_{m\geqq -n} a_m t^{m/2}.$$

Here $G(t)$ means $\int_0^\infty\int_Y \{G_1(t,y,u)-G_2(t,y,u)\}dydu.$

§ 10. Indexes of degenerate operators, I.

Let X be a real analytic n-dimensional compact Riemannian manifold with boundary Y and $D=D_{+,k}$ or $D_{-,k}$ be first order differential operators defined on $C^\infty(X,E)$ and map it into $C^\infty(X,F)$ such that on a neighborhood $Y\times\mathbf{I}$ ($\mathbf{I}=[0,1]$) of the boundary of X

$$D_{+,k}=\sigma\Big(\frac{\partial}{\partial u}+u^kA\Big),\quad k>0,$$

$$D_{-,k}=\sigma\Big(u^k\frac{\partial}{\partial u}+A\Big),\quad k>0.$$

Here, $u\in\mathbf{I}$ is the (real analytic) normal coordinate, $\sigma=\sigma_D\ (du)$ is the bundle isomorphism $E\to F$, $A=A_u: C^\infty(Y,E_u)\to C^\infty(Y,E_u)\to C^\infty(Y,F_u)$ is a first order selfadjoint elliptic operator on Y which is independet of u.

We denote by $\hat{X}$ the double of X. Then $D_{+,k}$ and $D_{-,k}$ define differential operators $\hat{D}_{+,k}$ and $\hat{D}_{-,k}$ both defined on $C^\infty(\hat{X},\hat{E})$ and map it into $C^0\ (\hat{X},\hat{F})$. They are elliptic on $\hat{X}-Y$ but degenerate on Y.

Definition. *We define differential operators $D_{+,k,\varepsilon}$ and $D_{-,k,\varepsilon}$ on X by*

$$D_{+,k,\varepsilon}=(\frac{\partial}{\partial u}+(u^k+\varepsilon e)A),\ \ \varepsilon>0,\ \ on\ \ Y\times\mathbf{I},\ \ D_{+,k,\varepsilon}=D_{+,k},\ \ on\ \ X-Y\times\mathbf{I},$$

$$D_{-,k,\varepsilon} = ((u^k + \varepsilon e)\frac{\partial}{\partial u} + A),\ \ \varepsilon > 0,\ \ on\ Y \times \mathbf{I},\ \ D_{-,k,\varepsilon} = D_{-,k},\ \ on\ X - Y \times \mathbf{I}.$$

Here $e = e(y, u)$ is a C^∞–function given by $e(y, u) = e_1(u)$ where $e_1(u)$ is a C^∞–function on $\mathbf{I}$ such that

$$0 \leqq e_1(u) \leqq 1,\ \ e_1(u) = 1,\ \ 0 \leqq u \leqq \frac{1}{3},\ \ e_1(u) = 0,\ \ \frac{2}{3} \leqq u \leqq 1.$$

Definition. *For* $0 < \varepsilon' < 1$, *we set*

$$u^k{}_{\varepsilon'} = u^k,\ \ if\ k\ is\ an\ even\ integer,$$

$$u^k{}_{\varepsilon'} = u^k,\ \ u \geqq \varepsilon',\ \ u^k{}_{\varepsilon'} = u^k\Big(1 - e_1\Big(\frac{u}{\varepsilon'}\Big)\Big),\ \ 0 \leqq u \leqq \varepsilon',\ \ if\ k\ is\ not\ an\ even\ integer,$$

and define differential operators $D_{+,k,\varepsilon,\varepsilon'}$ and $D_{-,k,\varepsilon,\varepsilon'}$ by

$$D_{+,k,\varepsilon,\varepsilon'} = \sigma\Big(\frac{\partial}{\partial u} + (u^k{}_{\varepsilon'} + \varepsilon e)A\Big),\ \ on\ Y \times \mathbf{I},\ \ D_{+,k,\varepsilon,\varepsilon'} = \mathrm{D}_{+,k},\ \ on\ X - Y \times \mathbf{I},$$

$$D_{-,k,\varepsilon,\varepsilon'} = \sigma\Big((u^k{}_{\varepsilon'} + \varepsilon e)\frac{\partial}{\partial u} + A\Big),\ \ on\ Y \times \mathbf{I},\ \ D_{-,k,\varepsilon,\varepsilon'} = D_{-,k},\ \ on\ X - Y \times \mathbf{I}.$$

By definitions, the operators $\hat{D}_{+,k,\varepsilon,\varepsilon'}$ and $\hat{D}_{-,k,\varepsilon,\varepsilon'}$ defined from $D_{+,k,\varepsilon,\varepsilon'}$ and $D_{-,k,\varepsilon,\varepsilon'}$ *on* $\hat{X}$ are C^∞–cofficients elliptic operators on $\hat{X}$. Hence under the boundary conditions $Pf(0, y) = 0$, *for* $D_{+,k,\varepsilon,\varepsilon'}$ and $D_{-,k,\varepsilon,\varepsilon'}$, $k < 1$, and $(I - P)f(0, y) = 0$, for $D_{-,k,\varepsilon'}$, $k \geqq 1$, they have finite indexes and there exist differential forms $\alpha_{+,k,\varepsilon,\varepsilon'}(x)dx$ and $\alpha_{-,k,\varepsilon,\varepsilon'}(x)dx$ on X such that (cf. [3], [4], [13]),

$$(53)_+ \qquad \text{index } D_{+,k,\varepsilon,\varepsilon'} = \int_X \alpha_{+,k,\varepsilon,\varepsilon'}(x)dx - \frac{h + \eta(0)}{2},$$

$$(53)_{-,1} \qquad \text{index } D_{-,k,\varepsilon,\varepsilon'} = \int_X \alpha_{-,k,\varepsilon,\varepsilon'}(x)dx - \frac{h + \eta(0)}{2},\ \ k < 1,$$

$$(53)_{-,2} \qquad \text{index }_{-}D_{-,k,\varepsilon,\varepsilon'} = \int_X \alpha_{-,k,\varepsilon,\varepsilon'}(x)dx + \frac{\eta(0) - h}{2},\ \ k \geqq 1.$$

Here, $\text{index}_{-}D$ means the index of D with the boundary condition $(I-P)f(0, y)=0$.

Lemma 15. *Under the boundary conditions* $Pf(0, y) = 0$, *for* $D_{+,k,\varepsilon}$ *and* $D_{-,k,\varepsilon}$, $k < 1$, *and* $(I - P)f(0, y) = 0$, *for* $D_{-,k,\varepsilon}$, $k \geqq 1$, $D_{+,k,\varepsilon}$ *and* $D_{-,k,\varepsilon}$ *have finite indexes and we have*

$$(54)'_+ \qquad \text{index } D_{+,k,\varepsilon} = \text{index } D_{+,k,\varepsilon,\varepsilon'},$$

(54)$'_{-,1}$ $\quad$ $\text{index } D_{-,k,\varepsilon} = \text{index } D_{-,k,\varepsilon,\varepsilon'},\ k<1,$

(54)$'_{-,2}$ $\quad$ $\text{index}_{-} D_{-,k,\varepsilon} = \text{index}_{-} D_{-,k,\varepsilon,\varepsilon'},\ k \geqq 1.$

Proof. Take a to satisfy $\varepsilon > a > \varepsilon'$ and

$$2(u^k{}_{\varepsilon'} + \varepsilon) \geqq u^k + \varepsilon \geqq u^k{}_{\varepsilon'} + \varepsilon,\ 0 \leqq u \leqq a,\ \textit{for } D_{+,k,\varepsilon,\varepsilon'},$$

$$\frac{2}{u^k+\varepsilon} \geqq \frac{2}{u^k{}_{\varepsilon'}+\varepsilon} \geqq \frac{1}{u^k+\varepsilon},\ 0 \leqq u \leqq a,\ \textit{for } D_{-,k,\varepsilon,\varepsilon'}.$$

Then, since on $Y\times[0,a]$, the equations $D_{\pm,k,\varepsilon}f=0$ and $D_{\pm,k,\varepsilon,\varepsilon'}g=0$ reduce (on each eigenspace of A)

$$\frac{d}{du}f_{\lambda,k,\varepsilon} + \lambda(u^k+\varepsilon)f_{\lambda,k,\varepsilon} = 0,$$

$$\frac{d}{du}g_{\lambda,k,\varepsilon,\varepsilon'} + \lambda(u^k{}_{\varepsilon'}+\varepsilon)g_{\lambda,k,\varepsilon,\varepsilon'} = 0,\ \textit{for } D_{+,k,\varepsilon},\ \textit{and } D_{+,k,\varepsilon,\varepsilon'},$$

$$(u^k+\varepsilon)\frac{d}{du}f_{\lambda,-k,\varepsilon} + \lambda f_{\lambda,-k,\varepsilon} = 0,$$

$$(u^k{}_{\varepsilon'}+\varepsilon)\frac{d}{du}g_{\lambda,-k,\varepsilon,\varepsilon'} + \lambda g_{\lambda,-k,\varepsilon,\varepsilon'} = 0,\ \textit{for } D_{-,k,\varepsilon}\ \textit{and } D_{-k,\varepsilon,\varepsilon'},$$

the equations $D_{\pm,k,\varepsilon,\varepsilon'}g=0$ with the boundary condition $g(a,y)=f(a,y)$ have unique solution on $Y\times[0,a]$ if $D_{\pm,k,\varepsilon}f=0$ by the choice of a. Moreover, since $f(u,y)=\sum_{\lambda\geqq 0}f_{\lambda,\pm k,\varepsilon}(u)\phi_\lambda(y)$ if $Pf(0,y)=0$ and $f(u,y)=\sum_{\lambda\leqq 0}f_{\lambda,-k,\varepsilon}(u)\phi_\lambda(y)$ if $(I-P)f(0,y)=0$ on $Y\times[0,a]$, this g satisfies $Pg(0,y)=0$ if $Pf(0,y)=0$ and $(I-P)g(0,y)=0$ if $(I-P)f(0,y)=0$. Therefore, since $D_{\pm,k,\varepsilon,\varepsilon'}=D_{\pm,k,\varepsilon}$ on some neighborhood of $Y\times\{a\}$ in X, to define a function $\bar{g}$ on X by

$$\bar{g}=f,\ \textit{on } X-Y\times[0,a],\quad \bar{g}=g,\ \textit{on } Y\times[0,a],$$

$\bar{g}$ is a solution of $D_{+,k,\varepsilon,\varepsilon'}$ on X with the boundary condition $P\bar{g}(0,y)=0$ (or $(I-P)\bar{g}(0,y)=0$). This shows dim. ker. $D_{\pm,k,\varepsilon,\varepsilon'} \geqq$ dim. ker. $D_{\pm,k,\varepsilon}$. Similarly, we get dim. ker. $D_{\pm,k,\varepsilon,\varepsilon'} \leqq$ dim. ker. $D_{\pm,k,\varepsilon}$ and we have dim. ker. $D_{\pm,k,\varepsilon,\varepsilon'} =$ dim. ker. $D_{\pm,k,\varepsilon}$. By the same reason, we have dim. ker. $D_{\pm,k,\varepsilon,\varepsilon'}{}^* =$ dim. ker. $D_{\pm,k,\varepsilon}$. This shows the lemma.

§ 11. Indexes of degenerate operators, II.

Lemma 16. (i). *Under the boundary condition $Pf(0,y)=0$, $D_{+,k}$ has finite index and for sufficiently small ε, we have*

(54)$_+$ $\quad$ $\text{index } D_{+,k} = \text{index } D_{+,k,\varepsilon}.$

(ii). *Under the boundary condition*

$$(B_k) \qquad Pf(0,y)=0, \text{ for } D_{-,k}f=0, \ \lim_{u\to 0}(I-P)(u^k g(u,y))=0, \text{ for } D_{-,k}{}^*g=0,$$

$D_{-,k}$ has finite index for $k<1$, and for sufficiently small ε, we have

$$\text{index}_k D_{-,k} = \text{index}\, D_{-,k,\varepsilon}. \tag{54}_{-,1}$$

Here $\text{index}_k D$ *means the index of D with the boundary condition* (B_k).

(iii). *Under the boundary condition*

$$(B_{-k}) \qquad (I-P)f(0,y)=0, \text{ for } D_{-,k}f=0, \ \lim_{u\to 0}P(u^k g(u,y))=0, \text{ for } D_{-,k}{}^*g=0,$$

$D_{-,k}$ has finite index for $k\geqq 1$, and for sufficiently small ε, we have

$$\text{index}_{-k} D_{-,k} = \text{index}_{-} D_{-,k,\varepsilon}. \tag{54}_{-,2}$$

Here $\text{index}_{-k} D$ *means the index of D with the boundary condition (B_{-k}).*

Proof. Let $\varepsilon<1/5$ and take b to satisfy $1-\varepsilon>b>3\varepsilon$. On $Y\times[0,b]$, the equations $D_{\pm,k}=f=0$ and $D_{\pm,k}{}^*g=0$ reduce

$$\frac{d}{du}f_{\lambda,k}+\lambda u^k f_{\lambda,k}=0, \ \textit{for } D_{+,k}, \ u^k\frac{d}{du}f_{\lambda,-k}+\lambda f_{\lambda,-k}=0, \ \textit{for } D_{-,k},$$

$$\frac{d}{du}g_{\lambda,k}-\lambda u^k g_{\lambda,k}=0, \ \textit{for } D_{-,k}{}^*,$$

$$u^k\frac{d}{du}g_{\lambda,-k}+(ku^{k-1}-\lambda)g_{\lambda,-k}=0, \ \textit{for } D_{-,k}{}^*.$$

The solutions of these equations are

$$f_{\lambda,k}=c_\lambda \mathrm{e}^{-(\lambda/k+1)u^{k+1}}, \ f_{\lambda,-k}=c_\lambda \mathrm{e}^{-(\lambda/1-)ku^{1-k}}, \ k\neq 1, \ f_{\lambda,-1}=c_\lambda u^{-\lambda},$$

$$g_{\lambda,k}=c_\lambda \mathrm{e}^{(\lambda/k+1)u^{k+1}}, \ g_{\lambda,-k}=c_\lambda u^{-k}\mathrm{e}^{(\lambda/1-k)u^{1-k}}, \ k\neq 1, \ g_{\lambda,1}=c_\lambda u^{\lambda-1}.$$

Then, since

$$0<\mathrm{e}^{(\lambda/k+1)u^{k+1}}\leqq \mathrm{e}^{(\lambda/k+1)(u^{k+1}+\varepsilon)}\leqq \mathrm{e}^{(\lambda/k+1)(u+\varepsilon/k+1)^{(1/k+1))^{k+1}}}, \ \lambda\geqq 0,$$

we obtain (i).

To show (ii) and (iii), we use the inequalities

$$0\leqq \exp\Bigl(\int_0^u\frac{dv}{v^k+\varepsilon}\Bigr)\leqq \mathrm{e}^{(\lambda/1-k)u^{1-k}}\leqq \exp\Bigl(\int_0^{u+\alpha(b,k,\varepsilon)}\frac{dv}{v^k+\varepsilon}\Bigr), \ k<1, \ \lambda\geqq 0,$$

$$\frac{u^{1-k}}{1-k} < \int_b^u \frac{dv}{v^k+\varepsilon}, \quad \log u < \int_b^u \frac{dv}{v+\varepsilon}, \quad u < b,$$

$$\frac{b^{1-k}}{1-k} > \int_b^{\beta(b,k,\varepsilon)} \frac{dv}{v^k+\varepsilon}, \quad \log b > \int_b^{\beta(b,1,\varepsilon)} \frac{dv}{v+\varepsilon}, \quad 0 < \beta(b,k,\varepsilon) < b,$$

where $\alpha(b,k,\varepsilon)$ is the twice of infimum value of these α thet satisfy $\int_0^{u+\alpha}(1/(v^k+\varepsilon))\,dv > \int_0^u (\varepsilon/v^k(v^k+\varepsilon))dv$, $0 \leqq n \leqq b$ and therefore $lim_{\varepsilon\to 0}\alpha(b,k,\varepsilon)=0$. Hence we get (ii). On the other hand, since $lim_{\varepsilon\to 0}\beta(b,k,\varepsilon)=2^{1/(1-k)}3b, k>1$, $lim_{\varepsilon\to 0}\beta(b,1,\varepsilon)=\sqrt{b}$, to take $b<1/3$ and set

$$e_k(u)=e(c(k)u), \quad c(k)<2^{\frac{1}{1-k}}3b, \quad k>1, \quad c(1)<\frac{3}{2}\sqrt{b},$$

$$D_{-,k,\varepsilon_k}=\sigma((u^k+\varepsilon e_k)\frac{\partial}{\partial u}+A),$$

we have for sufficiently small ε,

$$\text{index}_- D_{-,k,\varepsilon k} = \text{index}_{-k} D_{-,k}, \quad k \geqq 1.$$

But since $\text{index}_- D_{-,k,\varepsilon k} = \text{index}_- D_{-,k,\varepsilon}$ by the same reason as lemma 16, we get (iii).

Lemma 17. *To set*

$$H_k = \{0\} \cup \{f \mid D_{-,k}f=0, \ f(0,y)\neq 0, \ Af(0,y)=0\},$$

$$H_{k^*} = \{0\} \cup \{f \mid D_{-k}{}^*f=0, \ f(0,y)\neq 0, \ Af(0,\mathrm{y})=0\},$$

$$\text{dim. } H_k = h_k, \quad \text{dim. } H_{k^*}=h_{k^*},$$

we have

(i). $h_k \leqq h$, *and* $h_{k^*} \leqq h$.

(ii). *If* $D_{-,k}$ *is a real analytic coefficients operator, then* h_k *does not depend on* $D^-{}_{,k}$ *and* h_{k^*} *depends only on* k.

(iii). *To set* $\text{index}_0 D$ *the index of* D *with the* 0*–boundary condition, that is* $lim_{u\to 0} f(u,y)=0$, *we have*

$$\text{index}_0 D_{-,k} = \text{index}_{-k} D_{-,k} - (h_k - h_{k^*}), \quad k \geqq 1. \tag{55}$$

Proof. If $f \in H_k$ (or H_{k^*}), then $f(u,y)=\sum_i f_i(u)\phi_i(y)$, $A\phi_i(y)=0$, on $Y\times[0,1]$. Then, since $D_{-,k}$ and $D_{-\ k}$ are first order elliptic operators, unique continuity is hold for the solutions of $D_{-,k}$ and $D_{-,k}{}^*$, we get (i).

If $D_{-,k}$ is areal analytic coefficients operator, then above $f_i(u)$ are constants for all i along any integral curve of real analytic normal vector field of Y starts from a point of Y. By the same reason, H_{k^*} is determined by k and we obtain (ii).

Let $D_{-,k}=0$ and $D_{-,k}^* g=0$, then to set $f=\sum_{\lambda\leq 0} f_\lambda(u)\phi_\lambda(y)$, $g=\sum_{\lambda\leq 0} g_\lambda(u)\phi_\lambda(y)$ on $Y\times[0,1]$, we get

$$\lim_{u\to 0} f_\lambda(u)=0,\ \lambda>0,\ \lim_{u\to 0} g_\lambda(u)=0,\ \lambda<0.$$

This shows (55).

By(53), lemma 15, lemma 16 and lemma 17, we obtain

Proposition 4. (i). *Under the boundary condition $Pf(0,y)=0$, $D_{+,k}$ has finite index and we have*

$$(56)_+ \qquad \text{index } D_{+,k}=\int_X \alpha_{+,k,\varepsilon,\varepsilon'}(x)dx-\frac{h+\eta(0)}{2}.$$

(ii). *Under the boundary condition (B_k), $D_{-,k}$, $k<1$, has finite index and we have*

$$(56)_{-,1} \qquad \text{index}_k D_{-,k}=\int_X \alpha_{-,k,\varepsilon,\varepsilon'}(x)dx-\frac{h+\eta(0)}{2},\quad k>1.$$

(iii). *Under the 0-bondary condition, $D_{-,k}$, $k\geq 1$, has finite index and we have*

$$(56)_{-,2} \qquad \text{index}_0 D_{-,k}=\int_X \alpha_{-,k,\varepsilon,\varepsilon'}(x)dx+\frac{\eta(0)-h}{2}-(h_k-h_{k*}),\quad k\geq 1.$$

Here $h_k\leq h$, $h_{k^*}\leq h$ and if $D_{-,k}$ is a real analytic coefficients operator, h_k does not depend on $D_{-,k}$ and h_{k^*} depends only on k.

§ 12. Fundamental solutions of $\frac{\partial}{\partial t}+\hat{\Delta}$.

We denote the closed extensions of $\hat{D}_{\pm,k}$ and $\hat{D}_{\pm,k}^*$ by $\hat{\mathscr{D}}_{\pm,k}$ and $\hat{\mathscr{D}}_{\pm,k}^*$. Then set

$$\hat{\Delta}_{1,\pm,k}=\hat{\mathscr{D}}_{\pm,k}^*\hat{\mathscr{D}}_{\pm,k},\quad \hat{\Delta}_{2,\pm,k}=\hat{\mathscr{D}}_{\pm,k}\hat{\mathscr{D}}_{\pm,k}^*.$$

For simple, we denote $\hat{\Delta}$ instead of $\hat{\Delta}_{1,+,k}$, *etc.*. By definition, $\hat{\Delta}$ is elliptic on $\hat{X}-(Y\times[-1,1])$. On the other hand, $\hat{\Delta}_{i,+,k}$ and $\hat{\Delta}_{i,-,k}$, $k<1$, $i=1,2$, have smoothing operators in $L^2(Y\times[-1,1])$ by lemma 7, (i), and $\hat{\Delta}_{i,-,k}$, $k\geq 1$, $i=1,2$, have smoothing operators in $H_{([2k]')}{}^{[2k]'+2n}[Y\times[-1,1]]$, by lemma 7, (ii). Hence we have (cf. [3])

Lemma 18. (i). *$\hat{\Delta}_{i,+,k}$ and $\hat{\Delta}_{i,-,k}$, $k<1$, $i=1,2$, have parametrixes in $L^2(\hat{X})$.*
(ii). *$\hat{\Delta}_{i,-,k}$, $k\geq 1$, $i=1,2$, have parametrixes in $H_{([2k]')}{}^{[2k]'+2n}(\hat{X})$. Here $H_{([2k]')}{}^{[2k]'+2n}(\hat{X})$ is the Sobolev space of these functions on $\hat{X}$ that vanishes on Y at least order $[2k]'$.*

Corollary. (i). *$\partial/\partial t+\hat{\Delta}_{i,+,k}$ and $\partial/\partial t+\hat{\Delta}_{i,-,k}$, $k<1$, $i=1,2$, have fundamental solutions with C^∞-kernels on $(\hat{X}-Y)\times(\hat{X}-Y)\times(\mathbf{R}^+-\{0\})$ in $L^2(\hat{X})$.*
(ii). *$\partial/\partial t+\hat{\Delta}_{i,-,k}$, $k\geqq 1$, $i=1,2$, have fundamental solutions with C^∞-kernels on $(\hat{X}-Y)\times(\hat{X}-Y)\times(\mathbf{R}^+-\{0\})$ in $H_{([2k]')^{2k'+[2n]}}(\hat{X})$.*

We denote the kernels of the fundamental solution of $\partial/\partial t+\hat{\Delta}_{i,\pm,k}$ *by* $F_{i,\pm,k}(t,x)$, $i=1,2$.

By the definitions of $\hat{\Delta}_{i,\pm,k}$, on $Y\times[-a,a]$, we have

$$(57)_+\qquad \hat{\Delta}_{2,+,k}=\sigma_+(\hat{\Delta}_{1,+,k}+2k|u|^{k-1}A),$$

$$(57)_-\qquad \hat{\Delta}_{2,-,k}=\sigma_-(\hat{\Delta}_{1,-,k}-k(k-1)|u|^{2k-2}),\ k<1,$$
$$=\sigma_-(\hat{\Delta}_{1,-,k}+k(k-1)|u|^{2k-2}),\ k\geqq 1.$$

Here $\sigma_\pm$ are bundle isomorphisms. Hence by lemma 10, to define a C^∞-function e_2 on $\hat{X}$ by

$$e_2(u,y)=e_2(u),\ 0\leqq e_2\leqq 1,\ e_2(u)=0,\ |u|\leqq\frac{1}{2},$$
$$e_2(u)=1,\ |u|\geqq\frac{3}{4};\ on\ Y\times[-1,1],$$
$$e_2=1,\ on\ \hat{X}-Y\times[-1,1],$$

we have

$$F_{\pm,k}(t,x)e_2(x)\sim F_{\pm,k}(t,x)+H_{\pm,k}(t,x),\ \lim_{t\to 0}H_{\pm,k}(t,x)=0,$$
$$F_{\pm,k}(t,x)=F_{1,\pm,k}(t,x)-F_{2,\pm,k}(t,x).$$

Then, since $\mathscr{F}_k\otimes\mathscr{H}_k\otimes H^s(Y)\cup H_{([2k])'^s}(X)$ if $s\geqq[2k]'+2n$ and if f satisfies 0-boundary condary condition and $D_{-,k}f=0$ or $D_{-,k}{}^*f=0$, then $f\in\mathscr{F}_k\otimes\mathscr{H}_k\otimes H^s(Y)$, $k\geqq 1$, we have by proposition 2, proposition 3, proposition 4 and lemma 18

Theorem (i). *For $D_{+,k}$, there exists a differential form $\alpha_{+,k}(x)dx$ on X such that*

$$(58)_+\qquad \text{index } D_{+,k}=\int_X\alpha_{+,k}(x)dx-\frac{h+\eta(0)}{2}.$$

(ii). *For $D_{-,k}$, $k<1$, there exists a differential form $\alpha_{-,k}(x)dx$ on X such that*

$$(58)_{-,1}\qquad \text{index }{}_kD_{-,k}=\int_X\alpha_{-,k}(x)dx-\frac{h+\eta(0)}{2}.$$

(iii). *For $D_{-,k}$, $k\geqq 1$, there exists a differential form $\alpha_{-,k}(x)dx$ on X such that*

$$\text{index}_0 D_{-,k} = \int_X \alpha_{-,k}(x)dx + \frac{h+\eta(0)}{2}. \tag{58$_{-,2}$}$$

Proof. We only need to show (ii). But since index $_0D_{-,k}=0$ if $k<1$, we have $\text{index}_k D_{-,k} = \text{index}_k D_{-,k} + \text{index}_0 D_{-,k}$, and by lemma 2, we have $\text{index}_0 D_{-,k} = \int_X \beta_k(x)dx - (h+\eta(0))/2)$for some differential form $\beta_k\ (x)dx$ on X and $\text{index}_k \hat{D}_{-,k} = \int_X \gamma_k(x)dx$ for some $\gamma_k(x)dx$, we obtain (ii).

Corollary. *Let* $\varepsilon > \varepsilon' > 0$ *and* ε *is sufficiently small, then*

$$\int_X \alpha_{+,k}(x)dx = \int_X \alpha_{+,k,\varepsilon,\varepsilon'}(x)dx, \tag{59$_+$}$$

$$\int_X \alpha_{-,k}(x)dx = \int_X \alpha_{-,k,\varepsilon,\varepsilon'}(x)dx, \quad k<1, \tag{59$_{-,1}$}$$

$$\int_X \alpha_{-,k}(x)dx = \int_X \alpha_{-,k,\varepsilon,\varepsilon'}(x)dx - (h_k - h_{k^*}), \quad k \geqq 1. \tag{59$_{-,2}$}$$

Note. If we consider

$$D_{(-k)} = \sigma\Big(\frac{\partial}{\partial u} + u^{-k}A\Big), \quad on\ Y\times[0,1],$$

instead of $D_{-,k}$, index $D_{(-k)}$ exists if $k<1$ and we have with suitable differential form $\alpha_{(-k)}(x)dx$,

$$\text{index } D_{(-k)} = \int_X \alpha_{(-k)}(x)dx - \frac{h+\eta(0)}{2}, \quad k<1.$$

On the other hand, since we get $D_{(-k)}{}^* = \sigma^{-1}D_{(-k)}\sigma^*$ on $L^2[0,1]\otimes \ker. A$, to set

$$H_{(k)} = \{0\} \cup \{f \mid D_{(-k)}f=0,\ f(0,y)\neq 0,\ Af(0,y)=0\},$$

$$H_{(k^*)} = \{0\} \cup \{g \mid D_{(-k)}{}^*g=0,\ g(0,y)\neq 0,\ Ag(0,y)=0\},$$

$H_{(k)}$ is isomorphic to $H_{(k^*)}$. Therefore we have

$$\text{index}_- D_{(-k)} = \text{index}_0 D_{(-k)}, \quad k\geqq 1. \tag{60}$$

Hence we get with suitable differential form $\alpha_{(-k)}(x)dx$ on X

$$\text{index}_- D_{(-k)} = \int_X \alpha_{(-k)}(x)dx + \frac{h-\eta(0)}{2}, \quad k\geqq 1,$$

and for this $\alpha_{(-k)}(x)dx$, we have

$$\int_X \alpha_{(-k)}(\mathrm{x})dx = \int_X \alpha_{(-k),\varepsilon,\varepsilon'}(x)dx, \quad k \geqq 1,$$

if ε and ε' are sufficiently small. Here $\alpha_{(-k),\varepsilon,\varepsilon'}(x)dx$ is the differential form constructed for Differential operator $D_{(-k),\varepsilon,\varepsilon'}$ given by

$$D_{(-k),\varepsilon,\varepsilon'} = \sigma\left(\frac{\partial}{\partial u} + \frac{A}{u^k{}_{\varepsilon'} + \varepsilon e}\right), \quad on \ \ Y \times \mathbf{I},$$

$$= D_{(-k)}, \quad on \ \ X - Y \times \mathbf{I}.$$