Indexes of Some Degenerate Operators

By AKIRA ASADA
Department of Mathematics, Faculty of Science
Shinshu University
(Received July 13, 1977)

§ 5. Fundamental solutions on the cylinder.

Let $A : C^\infty(Y, E) \to C^\infty(Y, E)$ be a 1-st order selfadjoint elliptic operator, where $Y$ is a compact oriented Riemannian manifold and $E$ is a (complex) vector bundle over $Y$. On $Y \times \mathbb{R}^+$, we consider differential operators $D_+ = D_{+, k}$ and $D_- = D_{-, k}$ given by

\begin{align*}
D_{+, k} &= -\frac{\partial}{\partial u} + u^k A, \quad k > 0, \\
D_{-, k} &= u^k \frac{\partial}{\partial u} + A, \quad k > 0.
\end{align*}

By definition, $D = D_{+, k}$ or $D_{-, k}$ is a differential operator from $C^\infty(Y \times \mathbb{R}^+, E)$ into itself. Here $E = \pi^*(E)$ is the induced bundle of $E$ on $Y \times \mathbb{R}^+$ by the projection $\pi : Y \times \mathbb{R}^+ \to Y$.

Since $u^k \neq 0$ on $Y \times (\mathbb{R}^+ - \{0\})$, $D_+ (= D_{+, k})$ and $D_- (= D_{-, k})$ are both elliptic on $Y \times (\mathbb{R}^+ - \{0\})$ and their formal adjoints are given by

\begin{align*}
D_{+, k}^* &= -\frac{\partial}{\partial u} + u^k A, \\
D_{-, k}^* &= -u^k \frac{\partial}{\partial u} - ku^k + A.
\end{align*}

The eigenvalues and eigenfunctions of $A$ are denoted by $\lambda$ and $\phi_{\lambda}$. The projection from $C^\infty(Y, E)$ onto the space spanned by $\{\phi_{\lambda} | \lambda \geq 0\}$ is denoted by $P$. We set $C^\infty(Y \times \mathbb{R}^+, E ; P) = \{f(y, u) | (Pf)(y, 0) = 0\}$. $C^\infty(Y \times \mathbb{R}^+, E ; P)$, etc., are similarly defined. The adjoint condition of $(Pf)(y, 0) = 0$ is $(J-P)f)(y, 0) = 0$. The space $C^\infty(Y \times \mathbb{R}^+, E ; I-P)$, etc., are similarly defined.

As in § 2, $L^2 = L^2(Y \times \mathbb{R}^+)$, $H^t = H^t(Y \times \mathbb{R}^+)$, etc., mean the Hilbert space, $t$-th Sobolev space, etc., on $Y \times \mathbb{R}^+$, etc.. $C_{c, \infty}$ means the space of compact support $C^\infty$-functions (or cross-sections) and set $(n = \dim Y + 1)$.
\[ H^s = \{ f \mid f \in H^s, f(y,0) = \frac{\partial f}{\partial u}(y,0) = \cdots = \frac{\partial^s f}{\partial u^s}(y,0) = 0 \}, s \leq t - \left[ \frac{n}{2} \right], \]

\[ C_{(s)}^\infty = \{ f \mid f \in C^\infty, f(y,0) = \frac{\partial f}{\partial u}(y,0) = \cdots = \frac{\partial^s f}{\partial u^s}(y,0) = 0 \}, C_{(s)}^\infty \cap C_{(s)}^\infty. \]

**Definition.** Let \( g(y,u) \) be a cross-section of \( E \) such that \( g(y,bl) = gR(za) \). Then we define operators \( Q_{+,k} \) and \( Q_{-,k} \) by

\[ Q_{+,k}(g) = \sum_l Q_{l,k}(g_l)(u)\phi_l(y), \quad Q_{-,k}(g) = \sum_l Q_{l,-k}(g_l)(u)\phi_l(y), \quad 0 < k < 1, \]

\[ g_l \in C_{\infty}(y \times \mathbb{R}^+, E), \]

\[ Q_{-,k}(g) = \sum_l Q_{l,-k}(g_l)(u)\phi_l(y), \quad k \geq 1, \quad g_l \in C_{\infty}(Y \times \mathbb{R}^+, E). \]

**Proposition 1.** \( Q_{\pm, k} \) are the fundamental solutions of \( D_{\pm, k} \) with the following properties.

(a). The kernels \( Q_{\pm, k}(y, u ; z, v) \) of \( Q_{\pm, k} \) are \( C^\infty \) for \( u \neq v, u \neq 0, v \neq 0 \).

(b). (i). \( Q_{+, k} \) and \( Q_{-, k}, 0 < k < 1 \), are defined on \( C_{\infty}(Y \times \mathbb{R}^+, E) \) and map it into \( C^\infty(Y \times \mathbb{R}^+ \setminus \{0\}, E) \).

(ii). \( Q_{-, k}, k \geq 1 \), is defined on \( C_{\infty}(Y \times \mathbb{R}^+ \setminus \{0\}, E) \) and maps it into \( C^\infty(Y \times \mathbb{R}^+, E) \cap C^\infty(Y \times \mathbb{R}^+, E ; P) \).

(c). For any \( 0 < m < M \), \( Q_{\pm, k} \) is extended to a continuous map \( L^2(Y \times [m, M] \to L^2_{\text{loc}}) \). More precisely, we have

(i). \( Q_{+, k} \) is extended to a continuous map \( Q_{+, k} : L^2 \to L^2_{\text{loc}} \).

(ii). \( Q_{-, k}, k \geq 1 \), is extended to a continuous map \( Q_{-, k} : H_{\{k' - 1 \}}[0, L] \to L^2_{\text{loc}}, \) for any \( L > 0 \).

**Proof.** Except (a), the proposition follows from lemma 2 and lemma 6. To show (a), as \([4]\), we set

\[ K_A(u) = Y(u)e^{u|A|P} - Y(-u)e^{u|A|(I - P)}, \]

\( Y(u) \) is the characteristic function of \( \mathbb{R}^+ \), \( |A| = AP - A(I - P) \).

Then, it is known that the kernel \( E_A(y, z, u) \) of \( K_A \) is a \( C^\infty \) function on \( Y \times Y \times \mathbb{R}^+ \) ([4], I). Then, since

\[ Q_{+, k}(y, u ; z, v) = E_{(k+1)A}(y, z, u, h^{k+1} - v^{k+1}), \]

\[ Q_{-, k}(y, u ; z, v) = v^{-k}E_{(k-1)A}(y, z, u^{1-k} - v^{1-k}), \quad k \neq 1, \]

\[ Q_{-, 1}(y, u ; z, v) = v^{-1}E_A(y, z, \log u - \log v), \]

we have the proposition.
Corollary. \( D_{\pm,k} \) and \( D_{\pm,k}^* \) have closed extensions \( \mathcal{D}_{\pm,k} \) and \( \mathcal{D}_{\pm,k}^* \). \( \mathcal{D}_{+,k} \) and \( \mathcal{D}_{+,k}^* \), \( \mathcal{D}_{-,k} \) and \( \mathcal{D}_{-,k}^* \) are adjoints in \( L^2 \) each others.

\[ \text{§ 6. A lemma on Volterra's integral equation.} \]

It is known (cf. [5]) that if the fundamental solution of the heat equation on \( \mathbb{R}^+ \times D \) with time variable \( t \) and space variables \( x \) given by

\[ \frac{\partial f}{\partial t} + Lf = 0, \ L \text{ is an operator on } D, \text{ the condition is given at } t = 0, \text{ is given by} \]

\[ G(t, x, \xi) = \int_D g(t, x, \eta) \psi(\xi) d\eta, \]

then the fundamental solution \( E \) of the equation

\[ \frac{\partial f}{\partial t} + (L + K)f = 0, \ K \text{ is an operator on } D, \]

is obtained in the form

\[ E = G + G^*H, \ G^*H = \int_0^t \int_D G(t - s, x, \eta)H(s, \eta, \xi) d\eta ds. \]

Here, \( H \) is the solution of the following Volterra type integral equation

\[ H + K_x G + K_x (G^*H) = 0. \]

Lemma 8. In (20), if \( G \) satisfies

\[ \lim_{t \to 0} \frac{\partial^n}{\partial t^n} G(t, x, \xi) = 0, \ x \neq \xi, \ n \leq N, \]

and assume \( H \) satisfies the condition

\[ \lim_{t \to 0} (1 + K_x) \frac{\partial^n}{\partial t^n} H(t, x, \xi) = 0 \text{ implies } \lim_{t \to 0} \frac{\partial^n}{\partial t^n} H(t, x, \xi) = 0, \ n \leq N, \ x \neq \xi, \]

then

\[ \lim_{t \to 0} \frac{\partial^n}{\partial t^n} H(t, x, \xi) = 0, \ n \leq N, \ x \neq \xi. \]

Proof. Since \( \lim_{t \to 0} (H + KG + K(G^*H)) = \lim_{t \to 0} (H + KH) = 0 \), we have \( \lim_{t \to 0} H(t, x, \xi) = 0, \ x \neq \xi \) by (c)0. Then we get

\[ G_t^*H = -G_t^*H = \left[-G(t-s)H(s)\right]_{s=0}^t + G^*H_s = G^*H_s. \]

Hence we obtain

\[ H_t = -(KG_t + KH + K(G^*H_s)). \]
(23) shows \( \lim_{h \to 0} H(t, x, \xi) = 0 \), \( x \neq \xi \), by (c). In general, we assume that we have

\[
(22)_n \quad \lim_{t \to 0} \frac{\partial^m}{\partial t^m} H(t, x, \xi) = 0, \quad x \neq \xi, \quad m \leq n.
\]

\[
(23)_n \quad H(n) = -(KG(n) + nKH(n-1) + K(G*H(n))), \quad F(n) \text{ means } \frac{\partial^n F}{\partial t^n},
\]

then, since \( G^*H(n) = -G^*H(n) = \left[-G^*H(n)\right]_0 + G^*H(n+1) \), we get

\[
H(n+1) = -(KG(n+1) + nKH(n) + KH(n) + K(G^*H(n+1)))
\]

\[
= -(KG(n+1) + (n+1)KH(n) + K(G^*H(n+1))).
\]

Hence we obtain (23)\(_{n+1}\) and therefore we have (22)\(_{n+1}\) if \( n + 1 \leq N \) by assumption.

**Note.** If \( (1 + Kx)f = 0 \) implies \( f = 0 \), for example, \( Kx \) is an operator given by multiplying a function, then (22) holds under the assumption (21).

**Lemma 9.** If \( G \) satisfies (21)\(_N\) and \( F = F(t, x, \xi) \) is a solution of the equation

\[
(24) \quad (1 + tKx)F(t, x, \xi) = -KxG(t, x, \xi),
\]

then \( F \) also satisfies (21)\(_N\) if \( F \) is continuous on \( t \geq 0 \).

**Proof.** Since

\[
\frac{\partial^n}{\partial t^n}(1 + tKx)F = (1 + tKx)\frac{\partial^n F}{\partial t^n} + nKx\frac{\partial^{n-1} F}{\partial t^{n-1}},
\]

we have the lemma by induction.

**Lemma 10.** If \( G \) is real analytic in \( t, t > 0 \), and satisfies (21)\(_\infty\), \( F \), a continuous solution of (24) on \( t \geq 0 \), is also real analytic in \( t, t > 0 \), and \( G^*F \) exists, then \( F \) is a solution of (20) if \( F \) satisfies (c)\(_\infty\). Conversely, under the same assumptions on \( G \), if a solution \( H \) of (20) is real analytic in \( t, t > 0 \), and satisfies (c)\(_\infty\), then \( H \) is a solution of (24).

**Proof.** To show the first assertion, it is sufficient to show

\[
(26) \quad tKxF(t, x, \xi) = Kx(G^*F(t, x, \xi)).
\]

But since \( F(t, x, \xi) = \lim_{h \to 0} \int_D G(h - s, x, \eta)F(s + t, \eta, \xi)d\eta \), to set

\[
tKxF(r + t, x, \xi) = \int_r^{r+t} \int_D G(r + t - s, x, \eta)F(s + r + t, \eta, \xi)d\eta ds + f(r, t),
\]

we have
\[ f(r, t) = O(r), \quad r \rightarrow 0, \quad \text{and} \quad O(t), \quad t \rightarrow 0, \quad \text{and for} \quad r > 0, \quad f(r, t) \text{ is real analytic in } t \quad \text{although at } t = 0. \]

On the other hand, since
\[
\int_0^s \int_D \frac{\partial^n G}{\partial t^n} (r + t - s, x, \eta) F(s + r + t, \eta, \xi) d\eta ds = \int_0^s \int_D G(r + t - s, x, \eta) \frac{\partial^n F}{\partial s^n} (s + r + t, \eta, \xi) d\eta ds + O(t),
\]
for any \( n \) by lemma 9, we get by the same reason as in the proof of lemma 8
\[
\frac{\partial^n F}{\partial t^n} (r + t, x, \xi) + K_x \frac{\partial^n G}{\partial t^n} (r + t, x, \xi) + nK_x \frac{\partial^n G}{\partial t^n} (r + t, x, \xi) + K_x \int_0^s \int_D G(r + t - s, x, \eta) \frac{\partial^n F}{\partial s^n} (s + r + t, \eta, \xi) d\eta ds + \frac{\partial^n f(r, t)}{\partial t^n} + o(t) = 0,
\]
because \( F \) satisfies \((c)\). But by \((25)\), we also obtain
\[
\frac{\partial^n F}{\partial t^n} (r + t, x, \xi) + K_x \frac{\partial^n G}{\partial t^n} (r + t, x, \xi) + nK_x \frac{\partial^n G}{\partial t^n} (r + t, x, \xi) + K_x \int_0^s \int_D G(r + t - s, x, \eta) \frac{\partial^n F}{\partial s^n} (s + r + t, \eta, \xi) d\eta ds + \frac{\partial^n f(r, t)}{\partial t^n} + o(t) = 0.
\]
Hence for any \( n \), \( \frac{\partial^n}{\partial t^n} f(r, t) = o(t) \). This shows \( f(r, t) = 0, \quad r > 0 \), because \( f(r, t) \) is real analytic in \( t \) although at \( t = 0 \), for \( r > 0 \). Therefore we get
\[
(25)' \quad TK_x F(r + t, x, \xi) = \int_0^s \int_D G(r + t - s, x, \eta) F(s + r + t, \eta, \xi) d\eta ds.
\]
Tends \( r \) to 0 in \((25)\)', we obtain \((26)\) which shows the first assertion.

To show the second assertion, we note that we obtain
\[
\frac{H(n)}{n!} r^n = -(Kx(G(n)_{n!}) + Kx(H(n-1)_{(n-1)!}) + r^n + Kx(G^*H(n)_{n!}))
\]
by lemma 8 (and \(23)\)). Hence we get
\[
\sum_{n=0}^{\infty} \frac{H(n)}{n!} r^n = -(Kx(S_{n=0}^{\infty} G(n)_{n!} r^n + Kx(S_{n=0}^{\infty} H(n)_{n!} r^n)) + Kx(S_{n=0}^{\infty} G^*(n)_{n!} r^n))
\]
because \( G \) and \( H \) are both real analytic in \( t, \quad t > 0 \), by assumption. Therefore we have
In (27), $t$ tends to 0 and change $r$ to $t$, $H$ satisfies (24). Hence we obtain the lemma.

Corollary. If $G$ is real analytic in $t$, $t > 0$, and satisfies (21)$^\infty$, $K_x$ is a real analytic coefficients differential operator (may be degree is 0), then (20) has a solution $H$ which satisfies (22)$^\infty$ if (24) has a solution which satisfies (c)$^\infty$. Especially, if deg. $K = 0$, then (20) has a solution $H$ which satisfies (22)$^\infty$.

§ 7. Construction of the kernels of $e^{t\Delta_{l_1,+,k}}$, $i = 1, 2$, on the cylinder.

As in § 3, we set $A_{l_1, \pm, k} = \mathscr{D} \pm, k \mathscr{D} \pm, k$ and $A_{l_2, \pm, k} = \mathscr{D} \pm, k \mathscr{D} \pm, k^\ast$.

Definition. For any $\varepsilon > 0$, we set on $Y \times R^+$

\begin{align*}
(28)_+ & \quad D_{+, k, \varepsilon} = \frac{\partial}{\partial u} + (u^k + \varepsilon)A, \\
(28)_- & \quad D_{-, k, \varepsilon} = (u^k + \varepsilon)\frac{\partial}{\partial u} + A.
\end{align*}

By definitions, $D_{\pm, k, \varepsilon}$ are elliptic on $Y \times R^+$. Their closures (in $L^2$) $\mathscr{D} \pm, k, \varepsilon$ and their adjoints $\mathscr{D} \pm, k, \varepsilon^\ast$ are defined and we set

$$
A_{l_1, \pm, k, \varepsilon} = \mathscr{D} \pm, k, \varepsilon \mathscr{D} \pm, k, \varepsilon, \quad A_{l_2, \pm, k, \varepsilon} = \mathscr{D} \pm, k, \varepsilon \mathscr{D} \pm, k, \varepsilon^\ast.
$$

Similarly, $D_{l_1, \pm, \varepsilon}$, $D_{l_2, \pm, \varepsilon}$, $A_{l_1, \pm, \varepsilon}$, $A_{l_2, \pm, \varepsilon}$, $i = 1, 2$, etc., are defined. Explicitly, they take the forms

\begin{align*}
(29)_+ & \quad A_{l_1, \pm, k, \varepsilon} = -\frac{\partial^2}{\partial u^2} + (-1)^i k u^{k-1} + \lambda^2(u^k + \varepsilon)^2, \quad i = 1, 2, \\
(29)_{+, 1} & \quad A_{l_2, -, k, \varepsilon} = -(u^k + \varepsilon)^2 \frac{\partial^2}{\partial u^2} - 2k u^{k-1}(u^k + \varepsilon)\frac{\partial}{\partial u} - \lambda k u^{k-1} + \lambda^2, \\
(29)_{-, 2} & \quad A_{l_2, -, k, \varepsilon} = -(u^k + \varepsilon)^2 \frac{\partial^2}{\partial u^2} - 2k u^{k-1}(u^k + \varepsilon)\frac{\partial}{\partial u} - k(k - 1)u^{k-2}(u^k + \varepsilon) \\
& \quad - \lambda k u^{k-1} + \lambda^2.
\end{align*}

The boundary conditions for these operators are

\begin{align*}
(30)_{l_1, +, \varepsilon} & \quad f_l(0) = 0, \lambda \geq 0, \frac{df}{du} + \varepsilon \lambda f|_{u=0} = 0, \lambda < 0, \text{ for } A_{l_1, +, \varepsilon}, \\
(30)_{l_2, +, \varepsilon} & \quad f_l(0) = 0, \lambda \leq 0, \frac{df}{du} + \varepsilon \lambda f|_{u=0} = 0, \lambda > 0, \text{ for } A_{l_2, +, \varepsilon}.
\end{align*}
Indexes of Some Degenerate Operators

\[ f(0) = 0, \quad \lambda \geq 0, \quad \left( \frac{d}{du} f + \lambda f \right) \bigg|_{u=0} = 0, \quad \lambda < 0, \quad \text{for } A_{i,-,k,j,\epsilon}, \quad k < 1, \]

\[ \text{for } A_{i,-,k,j,\epsilon}, \quad k \geq 1, \]

\[ f(0) = 0, \quad \lambda \leq 0, \quad \left( \frac{d}{du} f + \lambda f \right) \bigg|_{u=0} = 0, \quad \lambda > 0, \quad \text{for } A_{i,-,k,j,\epsilon}, \quad k < 1, \]

\[ \text{for } A_{i,-,k,j,\epsilon}, \quad k \geq 1. \]

To construct the elementary solutions of the heat equations associated to
\[ A_{i,\pm,k,j,\epsilon}, \]
we set

\[ A_{i,+,k,j,\epsilon} = -\frac{\partial^2}{\partial t^2} + \epsilon^2 \lambda^2 + K, \]

\[ K = K_{i,+,k,j,\epsilon} = (-1)^i k \lambda k^{k-1} + \lambda^2 (u^{2k} + 2\epsilon u^k), \quad i = 1, 2, \]

\[ A_{i,-,k,j,\epsilon} = -\epsilon^2 \frac{\partial^2}{\partial u^2} + \lambda^2 + K, \quad K = K_{i,-,k,j,\epsilon}, \quad i = 1, 2, \]

\[ K_{i,-,k,j,\epsilon} = -(u^{2k} + 2\epsilon u^k) \frac{\partial^2}{\partial u^2} - 2ku^{k-1}(u^k + \epsilon) \frac{\partial}{\partial u} - \lambda ku^{k-1}, \]

\[ K_{i,-,k,j,\epsilon} = -(u^{2k} + 2\epsilon u^k) \frac{\partial^2}{\partial t^2} - 2ku^{k-1}(u^k + \epsilon) \frac{\partial}{\partial t} \]

\[-h(k-1)u^{k-2}(u^k + \epsilon) - \lambda ku^{k-1}. \]

The fundamental solutions of the equation \[ \frac{\partial}{\partial t} - \frac{\partial^2}{\partial u^2} + \epsilon^2 \lambda^2 \] (and \[ \frac{\partial}{\partial t} - \frac{\partial^2}{\partial u^2} + \lambda^2 \]) with the boundary conditions (30)\[ i,\pm,\epsilon, \quad i = 1, 2, \] (and (30)\[ i,\pm,\epsilon, \quad i = 1, 2, \]) are given in [4] and they satisfy the assumptions of lemma 10. Hence we may construct the fundamental solutions of the equation \[ \frac{\partial}{\partial t} - A_{i,\pm,k,j,\epsilon}, \quad i = 1, 2, \] (and \[ \frac{\partial}{\partial t} - A_{i,-,k,j,\epsilon}, \quad i = 1, 2, \]) with the boundary conditions (30)\[ i,\pm,\epsilon, \quad i = 1, 2, \] (and (30)\[ i,-,\epsilon, \quad i = 1, 2, \]) by lemma 10. In this.§, we treat \[ \frac{\partial}{\partial t} - A_{i,\pm,k,j,\epsilon}, \quad i = 1, 2. \]

Since \[ K_{i,\pm,k,j,\epsilon}, \quad i = 1, 2, \] are the operators of order 0, the solutions of the equation (24) is given by

\[ F(t, u, v) = F_{i,\pm,k,j,\epsilon}(t, u, v) \]

\[ = -\frac{(-1)^i k \lambda k^{k-1} + \lambda^2 (u^{2k} + 2\epsilon u^k))}{1 + ((-1)^i k \lambda k^{k-1} + \lambda^2 (u^{2k} + 2\epsilon u^k))} G(t, u, v), \quad i = 1, 2, \]

Here, \[ G = G_{i,\pm,k,j,\epsilon}(t, u, v), \quad i = 1, 2, \] are the kernels of the fundamental solutions of the equation \[ \frac{\partial}{\partial t} - \frac{\partial^2}{\partial u^2} + \epsilon^2 \lambda^2 \] with the boundary conditions (30)\[ i,\pm,\epsilon, i = 1, 2, \] given by [4]

\[ G_{i,\pm,k,j,\epsilon}(t, u, v) = \frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} \left\{ \exp\left(\frac{-(u-v)^2}{4t}\right) - \exp\left(\frac{-(u+v)^2}{4t}\right) \right\}, \]

\[ \lambda \geq 0, \quad \text{for } i = 1, \quad \lambda \leq 0, \quad \text{for } i = 2. \]
\[ G_{i,+,s}(t,u,v) = \frac{e^{-\epsilon^2 t t}}{\sqrt{4\pi t}} \left\{ \exp\left( -\frac{(u-v)^2}{4t} \right) + \exp\left( -\frac{(u+v)^2}{4t} \right) \right\} - \epsilon |\lambda| e^{t|\lambda|^2} \text{erfc}\left( \frac{u+v}{2\sqrt{t}} + \epsilon |\lambda| \sqrt{t} \right) \]

erfc(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt, \quad \lambda < 0, \quad \text{for } i = 1, \quad \lambda > 0, \quad \text{for } i = 2.

Hence \( \lim_{t \to 0} G_{i,+,s} = G_{i,+,0} \) in \( H^s \) for any \( s \geq 0 \), where \( G_{i,+,0} \) means the fundamental solution of \( \partial/\partial t - \partial^2/\partial u^2 \) with the boundary condition (14). Therefore we have

\[
\lim_{t \to 0} F_{i,+,s,k,\lambda}(t,u,v) = \frac{((1)k^{2h^{-1}} + \lambda^2 u^2 k)}{1 + ((1)k^{2h^{-1}} + \lambda^2 u^2 k)} t, \quad i = 1, 2,
\]

where the right hand side is the solution of (24) with \( G = G_{i,+,0} \) and \( K = K_{i,+,h,\lambda} = ((1)k^{2h^{-1}} + \lambda^2 u^2 k), \quad i = 1, 2 \). Then, since

(32)_i \quad 1 + ((1)k^{2h^{-1}} + \lambda^2 u^2 k) t \leq 1 + ((1)k^{2h^{-1}} + \lambda^2 (u^2 k + 2\epsilon u^k)) t,

(32)_ii \quad 1 + ((1)k^{2h^{-1}} + \lambda^2 u^2 k) t \geq 1, \quad k \leq 1, \quad \text{or } k > 1 \text{ and } (-1)^{k} \geq 0,

(32)_iii \quad 1 + ((1)k^{2h^{-1}} + \lambda^2 u^2 k) t \geq 1 - \frac{t(k + 1)}{2} \left( \frac{k - 1}{2} \right)^{(k^{-1})(k+1)} |\lambda|^{2(k+1)},

we obtain

Lemma 11. (i). If \( k \leq 1 \), or \( k > 1 \) and \( (-1)^{k} \geq 0 \), the fundamental solutions of the equations \( \partial/\partial t - \Delta_{i,+,s,k,\lambda} \), \( i = 1, 2 \), with the boundary conditions (30)\(_i,+,s, \), \( i = 1, 2 \), tend to the fundamental solutions of the equations \( \partial/\partial t - \Delta_{i,+,k,\lambda} \), \( i = 1, 2 \), with the boundary conditions (14)\(_i \) in \( H^s \) for any \( s \geq 0 \) and this convergence is uniform in \( \lambda \).

(ii). If \( k > 1 \) and \( (-1)^{k} \lambda < 0 \), we set \( L^2_C \) (or \( H^s_C \)) the subspace of \( L^2 \) (or \( H^s \)) spanned by \( \{ \phi \lambda | |\lambda| < C \} \), then the fundamental solution of \( \partial/\partial t - \Delta_{i,k,\lambda} \) with the boundary condition (30)\(_i,+,s, \) tends to the fundamental solution of the equation \( \partial/\partial t - \Delta_{i,k,\lambda} \) with the boundary condition (14)\(_i, \) in \( L^2_C \) (or \( H^s_C \)) on the interval

(33) \[ 2\left( \frac{2}{k - 1} \right)^{(k^{-1})(k+1)} C^{-2(k+1)} t \geq 0, \]

and this convergence is uniform in \( \lambda(|\lambda| < C) \).

Proof. By (32)_i and (32)_ii, to show (i), we only need to show the uniformity of the convergence in \( \lambda \). But, since \( G_{i,+,s,\lambda} \) tends to 0 at least in the order of \( \exp(-\epsilon^2 \lambda^2) \) because \( \text{erfc}(x) = O(\exp(-x^2)) \), we have the uniformity in \( \lambda \).
On the other hand, by (32)i and (32)iii, $F_{i,k,\lambda}$ is continuous on $0 \leq t < 2(2/(k-1))(k^{-1}+k+1)|2|^{-2/(k+2)}$, we get (ii).

**Corollary.** (i). If $k \leq 1$, denote the kernels of $\exp(-t\Delta_{i,k}) - \exp(-t\Delta_{2,k})$ and $\exp(-t\Delta_{i,k}) - \exp(-t\Delta_{2,k})$ in $L^2$ by $G_{i,k}(t,y,u)$ and $F_{k}(t,y,u)$, we have in $L^2$

$$\lim_{t \to 0} F_{i,k}(t,y,u) = F_{k}(t,y,u), \quad 0 \leq t < \infty.$$ 

(ii). If $k > 1$, denote the kernels of $\exp(-t\Delta_{i,k}) - \exp(-t\Delta_{2,k})$ and $\exp(-t\Delta_{i,k}) - \exp(-t\Delta_{2,k})$ in $L^2_c$ by $F_{i,k,c}(t,y,u)$ and $F_{k,c}(t,y,u)$, we have in $L^2_c$

$$\lim_{t \to 0} F_{i,k,c}(t,y,u) = F_{k,c}(t,y,u), \quad 0 \leq t < 2\left(\frac{2}{k-1}\right)^{(k-1)/(k+1)} C^{-2}/(k+2).$$

On the other hand, since $\lim_{t \to 0} F_{i,k,\lambda}(t,u,v) = 0$, for $\epsilon \geq 0$, to set

$$\int_0^\infty \int_Y F_{i,k,\lambda}(t,y,u)dydu = F_{i,k,\lambda}(t), \quad k \leq 1,$$

$$\int_0^\infty \int_Y F_{i,k,c}(t,y,u)dydu = F_{i,k,c}(t), \quad k > 1,$$

we obtain

$$\lim_{t \to 0} F_{i,k,\lambda}(t) = \lim_{t \to 0} \int_0^\infty \int_Y \{G_{i,k,\lambda}(t,y,u) - G_{2,k,\lambda}(t,y,u)\}dydu, \quad k \leq 1,$$

$$\lim_{t \to 0} F_{i,k,c}(t) = \lim_{t \to 0} \int_0^\infty \int_Y \{G_{i,k,c}(t,y,u) - G_{2,k,c}(t,y,u)\}dydu, \quad k > 1.$$

Here, $G_{i,k,\lambda}$ and $G_{i,k,c}$ are the kernels of $\exp(-t\Delta_{i,k})$ on $L^2$ or on $L^2_c$, where $\Delta_{i,k}$ mean $\partial^2/\partial u^2 + \epsilon^2 A^2$ with the boundary conditions

$$\tag{36}_1 \quad (Pf)(y,0) = 0, \quad (I - P)f \left(\left\{\frac{\partial}{\partial u} + A\right\}f\right)(y,0) = 0, \quad \epsilon = 1,$$

$$\tag{36}_2 \quad ((I - P)f)(y,0) = 0, \quad P\left(\left\{\frac{\partial}{\partial u} + A\right\}f\right)(y,0) = 0, \quad \epsilon = 2.$$

Then by [4], to set $G_{i,k}(t) = \int_0^\infty \int_Y G_{i,k,\lambda}(t,y,u) - G_{2,k,\lambda}(t,y,u)dydu$ and $G_{i,k,c}(t) = \int_0^\infty \int_Y G_{i,k,c}(t,y,u) - G_{2,k,c}(t,y,u)dydu$, they are both defined on $0 \leq t < \infty$ and we get
\[ \int_{0}^{\infty} \left( G_{s, \epsilon}(t) + \frac{h}{2} \right) t^{s-1} dt = -\frac{1}{2s\sqrt{\pi}} \Gamma(s + \frac{1}{2}) e^{-2s\eta(2s)}, \]

where \( h = \dim \ker A \), \( \eta(s) \) is the \( \eta \)-function of \( A \) given by

\[ \sum_{i \neq 0, \lambda \in \text{spec } A} \sigma \nu \lambda |\lambda|^{-s} \] and \( \eta_c(s) = \sum_{i \neq 0, \lambda \in \text{spec } A} \lambda |\lambda|^{-s} \). Hence, if \( G_{s, \epsilon}, G_{s, \epsilon, c}, F_{s, k, \epsilon} \) and \( F_{s, k, \epsilon, c} \) have asymptotic expansions at \( t \to 0 \) of the forms

\[ G_{s, \epsilon}(t) = \sum_{m \in -n} a_{s, m, \epsilon} t^{m/2}, \quad F_{s, k, \epsilon}(t) = \sum_{m \in -n} b_{s, m, k, \epsilon} t^{m/2}, \quad k \leq 1, \]

\[ G_{s, \epsilon, c}(t) = \sum_{m \in -n} a_{s, m, \epsilon, c} t^{m/2}, \quad F_{s, k, \epsilon, c}(t) = \sum_{m \in -n} b_{s, m, k, \epsilon, c} t^{m/2}, \quad k > 1, \]

we have by (35) and (37)

\[ \gamma(0) = -2(2a_{s, \epsilon} + h) = -(2b_{s, \epsilon} + h), \quad k \leq 1, \]

\[ \gamma_c(0) = -(2a_{s, \epsilon, c} + h) = -(2b_{s, \epsilon} + h), \quad k > 1. \]

Therefore we obtain

**Proposition 2.** (i). If \( k \leq 1 \) and \( G_{s, \epsilon}, F_{s, k, \epsilon} \) have asymptotic expansions of the form (38)i, then their coefficients of the terms of order 0 do not depend on \( \epsilon \) and \( k \).

(ii). If \( k > 1 \) and \( G_{s, \epsilon, c}, F_{s, k, \epsilon, c} \) have asymptotic expansions of the form (38)ii, then their coefficients of the terms of degree 0 do not depend on \( \epsilon \) and \( k \) and their limits at \( C \to \infty \) exist.

In the rest, we set these constants by \( a_{s, \epsilon} \) and \( a_{s, \epsilon, c} \). Hence we get

\[ \gamma(0) = -(2a_{s, \epsilon} + h), \]

\[ \gamma_c(0) = -(2a_{s, \epsilon, c} + h), \quad \lim_{\epsilon \to \infty} a_{s, \epsilon, c} = a_{s, \epsilon}. \]

---

**8. Construction of the kernels of** \( e^{-\xi \partial - \lambda^2} \), \( i = 1, 2 \), **on the cylinder, I.**

**Lemma 12.** The fundamental solutions of \( \partial / \partial t - \epsilon^2 \partial^2 / \partial u^2 + \lambda^2 \) with the boundary conditions (30)i, \( i = 1, 2 \), tends to the fundamental solution of \( \partial / \partial t + \gamma^2 \) on \( (t, u) \) space if \( u > 0 \).

**Proof.** Since the fundamental solutions are given

\[ G_{i, -\lambda, \epsilon}(t, u, v) = \frac{1}{\epsilon} \frac{e^{-\lambda \sqrt{t}}}{\sqrt{4\pi t}} \left[ \exp \left( \frac{-(u - v)^2}{4\epsilon^2 t} \right) - \exp \left( \frac{-(u + v)^2}{4\epsilon^2 t} \right) \right], \]

\( \lambda \geq 0 \) for \( i = 1 \), \( \lambda \leq 0 \) for \( i = 2 \).
Indexes of Some Degenerate Operators

\[ G_{i,-\lambda,\epsilon}(t,u,v) = \frac{1}{\epsilon} \left[ \frac{e^{-\epsilon^2 t}}{4\epsilon^2 t} \left( \exp\left\{ -\frac{(u-v)^2}{4\epsilon^2 t} \right\} + \exp\left\{ -\frac{(u+v)^2}{4\epsilon^2 t} \right\} \right) \right. \\
- |\lambda| e^{-\lambda \epsilon (u+v)} \text{erfc}\left( \frac{u+v}{2\epsilon \sqrt{t}} + |\lambda| \sqrt{t} \right), \]

\[ \lambda < 0 \text{ for } i = 1, \lambda > 0 \text{ for } i = 2, \]

and

\[ \frac{|\lambda|}{\epsilon} \int_{0}^{\infty} e^{-\lambda \epsilon (u+v)} \text{erfc}\left( \frac{u+v}{2\epsilon \sqrt{t}} + |\lambda| \sqrt{t} \right) f(v) dv \\
= 2|\lambda| \sqrt{t} \int_{u/2\epsilon \sqrt{t}}^{\infty} e^{2|\lambda| \sqrt{t} w} \text{erfc}\left( w + |\lambda| \sqrt{t} \right) f(2\epsilon \sqrt{t} w - u) dw, \]

\[ \int_{0}^{\infty} e^{2|\lambda| \sqrt{t} w} \text{erfc}\left( w + |\lambda| \sqrt{t} \right) dw = \frac{1}{2|\lambda| \sqrt{t}} (e^{-|\lambda| \sqrt{t}} - \text{erfc}\left( |\lambda| \sqrt{t} \right)), \]

we have

1

\[ \lim_{\epsilon \to 0} G_{i,-\lambda,\epsilon}(t,u,v) = e^{-u} \delta_u, \quad u \neq 0, \]

\[ \lim_{\epsilon \to 0} G_{i,-\lambda,\epsilon}(t,0,v) = 0, \quad \lambda \geq 0 \text{ for } i = 1, \lambda \leq 0 \text{ for } i = 2, \] (40i)

\[ \lim_{\epsilon \to 0} G_{i,-\lambda,\epsilon}(t,u,v) = e^{-\lambda t} \delta_u, \quad u \neq 0, \]

\[ \lim_{\epsilon \to 0} G_{i,-\lambda,\epsilon}(t,0,v) = (e^{-\lambda t} + \text{erfc}\left( |\lambda| \sqrt{t} \right)) \delta_u, \]

\[ \lambda < 0 \text{ for } i = 1, \lambda > 0 \text{ for } i = 2, \]

in \( (C^1_c)^* \), the dual space of compact support \( C^1 \)-class functions. Here, \( \delta_u \) means the Dirac measure concentrated at \( u \). Because we get

\[ \lim_{\epsilon \to 0} \int_{u/2\epsilon \sqrt{t}}^{\infty} e^{-\epsilon^2 t} f(2\epsilon \sqrt{t} w - u) dw \]

\[ = \lim_{\epsilon \to 0} \int_{u/2\epsilon \sqrt{t}}^{\infty} e^{2|\lambda| \sqrt{t} w} \text{erfc}\left( w + |\lambda| \sqrt{t} \right) f(2\epsilon \sqrt{t} w - u) dw = 0, \]

\[ u \neq 0, \quad f \in C^1_c. \]

Hence we obtain the lemma.

**Corollary.** Let \( H_{t,\lambda,\epsilon} = H_{i,\lambda,\epsilon} \) be a solution of the equation

\[ \frac{\partial^2}{\partial u^2} H_{i,\lambda,\epsilon}(t,u,v) - \frac{1}{tu^{1/2}} \frac{\partial}{\partial u} H_{i,\lambda,\epsilon}(t,u,v) = -\frac{1}{t} \frac{\partial^2}{\partial u^2} G_{i,-\lambda,\epsilon}(t,u,v), \quad i = 1, 2, \]

and assume \( G_{\epsilon} + G_{\epsilon}^* H_{t} \) and \( \lim_{\epsilon \to 0} G_{\epsilon} + G_{\epsilon}^* H_{t} \) both exist and \( H_{t} \) tends to a solution
of $\partial^2 H_\lambda / \partial u^2 - H_\lambda / u^{2k} = -(e^{-2t} / t) \delta_u^{(2)}$. Then the fundamental solution of $\partial / \partial t - (u^{2k} + e^2) e^2 / \partial u^2 + \lambda^2$ given by $G_1 + G_2 + H_2$ tends to a fundamental solution of $\partial / \partial t - u^{2k} \partial^2 / \partial u^2 + \lambda^2$ on $u > 0$ if $\lim_{t \to 0} H_2 (t, u, v) = u^{2k} \delta_u^{(2)}$. Here, $G_i = G_1, \ldots, G_{2k}$, means $G_i^{(-2k)}$.

Proof. First we note that $H_2$ is given by $e^{-2t} H_2$ if $H_2$ exists. Then, since $t u^{2k} H_{2u} = H_2$, we have $u^{2k} H_{2u} = H_{2u}^{(2)}$, and therefore

$$u^{2k} H_{2u} - H_{2u}^{(2)} = \lambda^2 H_2.$$  

On the other hand, since we get by (40)

$$\lim_{t \to 0} G_{2u}^{(-2k)} H_2 = e^{-2t} G_2^{(-2k)} + G_2^{(-2k)} H_2,$$

we have

$$\left( \frac{\partial}{\partial t} - u^{2k} \frac{\partial^2}{\partial u^2} + \lambda^2 \right) \left( \lim_{t \to 0} G_{2u}^{(-2k)} H_2 \right) = H_2 (t, u, v) - u^{2k} e^{-2t} \lambda^2 H_2 (s, u, v) ds.$$  

Hence $\partial / \partial t \left( e^{2t} \left( \frac{\partial}{\partial t} - u^{2k} \frac{\partial^2}{\partial u^2} + \lambda^2 \right) \left( \lim_{t \to 0} G_{2u}^{(-2k)} H_2 \right) \right) = 0$ by (42). Then, since to set

$$\left( \frac{\partial}{\partial t} - u^{2k} \frac{\partial^2}{\partial u^2} + \lambda^2 \right) \left( \lim_{t \to 0} G_{2u}^{(-2k)} H_2 \right) = e^{-2t} C,$$

$C$ is given by $\lim_{t \to 0} \left( H_2 (t, u, v) - u^{2k} e^{-2t} \lambda^2 H_2 (s, u, v) ds \right) = \lim_{t \to 0} \left( H_2 (t, u, v) - u^{2k} \delta_u^{(2)} \right)$, we obtain the corollary.

To solve the equation (41), we set

$$y_1 (t, u) = \sqrt{u} J_{\alpha - 2k} \left( \sqrt{-\frac{1}{t}} \frac{u^{1-k}}{1-k} \right), \quad y_2 (t, u) = \sqrt{u} Y_{\alpha - 2k} \left( \sqrt{-\frac{1}{t}} \frac{u^{1-k}}{1-k} \right),$$

where $J_\alpha$ and $Y_\beta$ are $\alpha$-th Bessel function and $\beta$-th Bessel function of the second kind. Then $y_1$ and $y_2$ are the solutions of the equation $d^2 y / du^2 - y / u^{2k} = 0$ ([10], [18]) and their Wronskians $W(y_1, y_2)$ are given by

$$W(y_1, y_2) = \frac{2(1-k)\sqrt{-1}}{\pi \sqrt{t}}, \quad W(y_1, y_2) = \frac{1+4}{t}, \quad k = 1.$$  

Hence a solution of (41) is given by
(43) \[ H_{\varepsilon}(t,u,v) = -\frac{1}{t} G_\varepsilon(t,u,v) - \frac{1}{t^2} \int_0^u \frac{1}{w^{\frac{1}{2}}} g(t,u,w) G_\varepsilon(t,w,v) \, dw, \]

\[ g(t,u,v) = \frac{1}{W(y_1,y_2)} \left\{ y_1(t,v) y_2(t,u) - y_1(t,u) y_2(t,v) \right\}. \]

By (43) and (40), we have

(44) \[ \lim_{\varepsilon \to 0} H_{\varepsilon}(t,u,v) = -e^{-\frac{k}{2}} \left\{ \frac{\partial u}{t} + \frac{Y(u-v)}{t^2 \varepsilon^k} g(t,u,v) \right\}, \quad u > 0, \quad v > 0, \quad \text{in } (C_\varepsilon)^*. \]

In other word, \( \lim_{\varepsilon \to 0} H_{\varepsilon}(t,u,v) \) tends to a solution of the equation

\[ \partial \mathcal{H}/\partial \varepsilon - H/\varepsilon = -(e^{-\frac{k}{2}}/t) \partial u^{(2)} \quad \text{in } (C_\varepsilon)^* \text{ if } u > 0. \]

**Definition.** For \( k > 0 \), we set

\[ (T_kf)(w) = f((1-k)v)^{1/(1-k)}, \quad v = \frac{w^{1-k}}{1-k}, \quad k \neq 1, \]

\[ (T_1f)(w) = f(e^w), \quad v = \log w, \quad k = 1, \]

and define the subspaces \( \mathcal{E}_k \) of the space of continous functions in \( v \)-space by

\[ \mathcal{E}_k = \{ f | \text{ } T_k \left( \frac{f}{\sqrt{v}} \right)(w) \text{ is continuous on } 0 \leq \arg w \leq \frac{\pi}{2}, \text{ holomorphic on } 0 < \arg w < \frac{\pi}{2}, \text{ } w \neq 0, \text{ } |T_k \left( \frac{f}{\sqrt{v}} \right)(re^{-i\theta})| = O(e^{-r^{1+k}}), \text{ } r \to \infty, \text{ for some } \epsilon > 0, \text{ } w = re^{-i\theta}, \text{ } 0 \leq \theta \leq \frac{\pi}{2}, \text{ } k \neq 1, \]

\[ \mathcal{E}_1 = \{ f | \text{ } T_1 \left( \frac{f}{\sqrt{v}} \right)(w) \text{ is continuous on } 0 \leq \arg w \leq \pi, \text{ holomorphic on } 0 < \arg w < \pi, \text{ } |T_1 \left( \frac{f}{\sqrt{v}} \right)(re^{-i\theta})| \in L'(0, \infty) \}. \]

We denote by \( \mathcal{L} \) the Laplace transform and by \( H_\nu \), the \( \nu \)-th Hankel transform given by \( H_\nu(f)(x) = \int_0^\infty x^{-\nu} y^{\nu+1} J_\nu(xy)f(y) \, dy \), \( \text{Re. } \nu \geq -1/2 \). We know that \( H_\nu(H_\nu(f)) = f \) if \( f \) is \( C^\infty \) and rapidly decreasing at \( \infty \) ([2], [6], [18]).

**Definition.** We define the subspaces \( \mathcal{D}_k \) of the space of continuous functions in \( t \)-space by

\[ \mathcal{D}_k = \{ \varphi | \varphi = H_{1/2-k}(f) \left( \sqrt{-\frac{1}{t}} \right), \text{ } f \in \mathcal{E}_k \}, \quad k \neq 1, \]
\[ \mathcal{H}_k \text{ and } \mathcal{F}_k \text{ are both considered to be topological vector spaces (not complete) with the uniform convergence topology. Then we obtain by (44) and the corollary of lemma 12.} \]

**Lemma 13.** There exist fundamental solutions \( E_\varepsilon \) of the equation \( \frac{\partial}{\partial t} - (u^k + \varepsilon^2) \frac{\partial^2}{\partial u^2} + \lambda^2 \) (with the boundary conditions (30), \( \gamma \)) which tend to a fundamental solution \( E \) of \( \frac{\partial}{\partial t} - u^k \frac{\partial^2}{\partial u^2} + \lambda^2 \) in \( (\mathcal{F}_k)^* \otimes (\mathcal{H}_k)^* \), and this \( E \) satisfies

\[ \text{Supp. } E \subset \mathbb{R}^1 \times \{ (u, v) | u \geq v \} \]

**Proof.** By Hankel’s inversion theorem ([2], [6], [18]), \( \delta_a \) is approximated by the solutions of \( \frac{\partial^2 H}{\partial u^2} - H/u^k = 0 \) in \( (\mathcal{F}_k)^* \otimes (\mathcal{H}_k)^* \). On the other hand, \( (e^{-it}/t^k) g(t, u, v) \) is a solution of \( \partial^2 H/\partial u^2 - H/u^k = 0 \). Then, since

\[ g(t, u, v) = \frac{2(k - 1)}{\pi \sqrt{-1}} \sqrt{tu^k} v^k \left[ \sin \left( \sqrt{-1} t \right) \frac{1}{1 - k} (u^{1-k} - v^{1-k}) \right] \]

\[ + \frac{1 - k}{2\sqrt{-1}} \sqrt{t} (u^{k-1} - v^{k-1}) \cos \left( \sqrt{-1} t \right) \frac{1}{1 - k} (u^{1-k} - v^{1-k}) \cdots, \]

\[ \lim_{t \to 0} \left[ G_{\mathcal{F}_k} \right] \text{is defined as an element of} \ (\mathcal{F}_k)^* \otimes (\mathcal{H}_k)^* \text{if } u > 0. \]

**Corollary 1.** There exist fundamental solutions \( E_{1, \varepsilon} \) of the equations \( \partial/\partial t - (u^k + \varepsilon) \partial^2/\partial u^2 + \lambda^2 \) which tend to a fundamental solution \( E \) of \( \partial/\partial t - u^k \partial^2/\partial u^2 + \lambda^2 \) in \( (\mathcal{F}_k)^* \otimes (\mathcal{H}_k)^* \), which satisfies (45).

**Proof.** By lemma 10, a fundamental solution of \( \partial/\partial t - (u^k + \varepsilon) \partial^2/\partial u^2 + \lambda^2 \) is obtained from \( E_\varepsilon \) by solving the equation

\[ \frac{\partial^2 H_{1, \varepsilon}}{\partial u^2} - \frac{H_{1, \varepsilon}}{2\lambda^k} = -\frac{1}{t} \partial^2 E_\varepsilon. \]

The solution \( H_{1, \varepsilon} \) of this equation is given by

\[ H_{1, \varepsilon}(t, u, v) = -\frac{1}{t} \int_{-\infty}^{\infty} g_{\varepsilon}(t, u, w) E_{1, \varepsilon}(w, w) dw, \]

\[ g_{\varepsilon}(t, u, v) = \frac{\pi \sqrt{\xi t}}{(2 - k) \sqrt{-1}} I_{1/2-k} \left( \sqrt{-1} \frac{u^{1-k^2}}{2\xi t} \right) Y_{1/2-k} \left( \sqrt{-1} \frac{u^{1-k^2}}{2\xi t} \right) \]
Indexes of Some Degenerate Operators

\[ -I_{V|2-k}\left(\sqrt{\frac{1}{2\lambda t}} \frac{u^{1-k/2}}{2-k}, k \neq 2, \right) Y_{V|2-k}\left(\sqrt{\frac{1}{2\lambda t}} \frac{v^{1-k/2}}{2-k}, \right) \]

\[ g_e(t,u,v) = \sqrt{\frac{2\lambda t uv}{1 + 2\lambda t}} \left\{ \left(\frac{u}{v}\right)^{1/4 + 1/8kt} - \left(\frac{v}{u}\right)^{1/4 + 1/8kt}, \right\}, k = 2. \]

Hence \( \lim_{t \to 0} H_1,\epsilon, \epsilon^*E_1 = 0 \) by (45) because \( \lim_{t \to 0} g_e(t,u,u) = 0. \)

**Corollary 2.** There exist fundamental solutions \( E_{2,\epsilon} \) of the equation \( \partial / \partial t - A_{i,2,k,1,\lambda} \) on \( (t,u) \)-space which tend to a fundamental solution \( E \) of the equation \( \partial / \partial t - u^2 \partial^2 / \partial u^2 + \lambda^2 \in (\mathcal{S}_k)^* \otimes (\mathcal{R}_k)^* \) and this \( E \) satisfies (45).

**Proof.** By lemma 10, a fundamental solution of \( \partial / \partial t - A_{i,2,k,1,\lambda} \) is obtained from \( E_{1,\epsilon} \) by solving the equation

\[ \frac{dH_{2,i,\epsilon}}{du} = \frac{1 - \lambda t \left( \lambda u + \partial_{i,\epsilon}(k-1)(u^{k+\epsilon}) \right)}{u(2\lambda t u^{k+\epsilon})} H_{2,i}, \]

\[ = - \frac{\lambda t \partial E_{1,\epsilon}}{du} - \lambda u + \partial_{i,\epsilon}(k-1)(u^{k+\epsilon}) \frac{2\lambda u^{k+\epsilon}}{2\lambda u^{k+\epsilon}} E_{1,\epsilon}, \quad \partial_{1,\epsilon} = 0, \quad \partial_{2,\epsilon} = 1. \]

A solution of this equation is

\[ H_{2,i,\epsilon}(t,u,v) \]

\[ = - \int_c^u \left[ \exp \left( \frac{1 - \lambda t \left( \lambda x + \partial_{i,\epsilon}(k-1)x^{k+\epsilon} \right)}{u(2\lambda t u^{k+\epsilon})} \right) dx \right]. \]

\[ \left[ \frac{1}{\lambda t} \frac{\partial E_{1,\epsilon}}{\partial w}(t,w,v) - \frac{\lambda w + \partial_{i,\epsilon}(k-1)(w^{k+\epsilon})}{\lambda w(w^{k+\epsilon})} E_{1,\epsilon}(t,w,v) \right] dw. \]

Hence in \( (\mathcal{S}_k)^* \otimes (\mathcal{R}_k)^* \), \( \lim_{t \to 0} H_{2,i,\epsilon,\epsilon^*} = 0 \) by (45) and we have the lemma.

**Note.** The fundamental solution of \( \partial / \partial t - u^2 \partial^2 / \partial u^2 \) is given in [8] (cf. [8]', [9]). It takes similar form as our solution.

§ 9. Construction of the kernels of \( e^{-tA_{i,\lambda}} \), \( i = 1,2 \), on the cylinder, II.

By the corollary 2 of lemma 13, we obtain

**Lemma 14.** There exist fundamental solutions \( F_{i,-,\epsilon,\epsilon^*} \) of the equation \( \partial / \partial t - A_{i,-,\epsilon} \) which tend to a fundamental solution \( F_{i,-,\epsilon} \) of \( \partial / \partial t - A_{i,-,\epsilon} \) in \( (\mathcal{S}_k)^* \otimes (\mathcal{R}_k)^* \otimes \mathcal{H}_C(Y) \) for any \( C > 0 \), and these fundamental solutions satisfy

\[ F_{i,-,\epsilon,\epsilon^*}(\mathcal{S}_k)^* \otimes (\mathcal{R}_k)^* \otimes \mathcal{H}_C(Y) = F_{i,-,\epsilon,\epsilon^*}, \]

\[ F_{i,-,\epsilon,\epsilon^*}(\mathcal{S}_k)^* \otimes (\mathcal{R}_k)^* \otimes \mathcal{H}_C(Y) = F_{i,-,\epsilon,\epsilon^*}, \]

if \( C > 0 \).
Proof. By (46), the convergence of \( E_{i, e} = E_{i, e, \lambda} \) to \( E_{i, e, 2} \) is uniform in \( \lambda \) if \( |\lambda| \leq C \). Hence we have the lemma.

Corollary. There exist fundamental solutions \( F_{i, -k, e} \) of the equation \( \partial_t - \Delta_{i, -k, e} \) densely defined in \( (\mathcal{F}_{k})^* \otimes (\mathcal{H}_{k})^* \otimes H^s(Y) \) which tends to a fundamental solution \( F_{i, -k} \) of \( \partial_t - \Delta_{i, -k} \) densely defined in \( (\mathcal{F}_{k})^* \otimes (\mathcal{H}_{k})^* \otimes L^2(Y) \).

We denote the inclusion from \( \mathcal{F}_{k} \otimes \mathcal{H}_{k} \otimes H^s(Y) \) into \( H^s(\mathbb{R}^s \times \mathbb{R}^s \times Y) \) by \( i_{k, s} \). The dual of \( i_{k, s} \) is denoted by \( i_{k, s}^* \), and set

\[
(47) \quad \ker i_{k, s}^* = \mathcal{F}_{k, s}.
\]

If \( s \geq n + 1 \) (dim. \( Y = n - 1 \)), then \( \mathcal{F}_{k, s} \neq 0 \). But, since we know

\[
C_{\epsilon, ((1 - k)/2)^\infty} \subset \mathcal{H}_{k}^s, \quad k < 1, \quad C_{\epsilon, ((1 - k)/2)^\infty} \subset \mathcal{H}_{k}^s, \quad k < 1,
\]

we have

\[
(48) \quad C_{\epsilon, ((1 - k)/2)^\infty}(\mathbb{R} \times \mathbb{R} \times Y) \cap \mathcal{F}_{k, s} = 0.
\]

Here, \( C_{\epsilon, ((1 - k)/2)^\infty}(\mathbb{R} \times \mathbb{R} \times Y) \) is considered to be a subspace of \( H^s(\mathbb{R}^s \times \mathbb{R}^s \times Y) \).

In \( (\mathcal{F}_{k})^* \otimes (\mathcal{H}_{k})^* \otimes H^s(Y) \), \( \lim_{s \to 0} \text{trace} [\exp (-t \Delta_{i, -k, e}) - \exp (-t \Delta_{i, -k, e})] \) coincides to \( \lim_{s \to 0} \left[ \int_0^\infty \int_Y [G_{1, -i}(t, y, u) - G_{2, -i}(t, y, u)] dy du \right] \), because \( \lim_{s \to 0} (E_{i} - G_{s})(t, y, u) = \lim_{s \to 0} H_{i, e}(t, u, u) = \lim_{s \to 0} H_{i, s}(t, u, u) = 0 \). Hence we get

\[
(49) \quad \lim_{s \to 0} \text{trace}[e^{-t \Delta_{i, -k, e}} - e^{-t \Delta_{i, -k, e}}] = -\sum_{l \in \text{spec. } \Lambda, \lambda \neq 0} \frac{\text{sign} \lambda}{2} \text{erfc}(\vert \lambda \vert \sqrt{t}), \quad k < 1,
\]

\[
= \sum_{l \in \text{spec. } \Lambda, \lambda \neq 0} \frac{\text{sign} \lambda}{2} \text{erfc}(\vert \lambda \vert \sqrt{t}), \quad k \geq 1,
\]

because \( G_{i, -i}(t, y, u) = \frac{1}{2} G_{i}(t, y, u/\epsilon) \), where \( G_{i}(t, y, u) \), \( i = 1, 2 \), are the kernels of \( \partial_t - \Delta_{i} \), \( i = 1, 2 \), with the boundary conditions (36), \( i = 1, 2 \) (cf. [4]). Therefore, to set

\[
F_{-k, e}(t) = \int_0^\infty \int_Y \left[ F_{1, -k, e}(t, y, u) - F_{2, -k, e}(t, y, u) \right] dy du,
\]

if \( F_{-k, e}(t) \) has asymptotic expansion at \( t \to 0 \),

\[
F_{-k, e}(t) \sim \sum_{m \geq n} b_{-m, k, e} t^{m/2},
\]

we have by (49)
\[ v(0) = \lim_{\varepsilon \to 0} \frac{(2b_-, \phi, h, \varepsilon + h)}{k < 1}, k \geq 1. \]

Summarising these, we obtain

**Proposition 3.** If \( F_+, h, \varepsilon \) has asymptotic expansion (50) at \( t \to 0 \), then \( \lim_{t \to 0} b_+, h, \varepsilon \) exists and we have

\[ \lim_{t \to 0} b_+, h, \varepsilon = a_0, \quad k < 1, \]
\[ = -a_0, \quad k \geq 1, \]

where \( a_0 = a_+, 0 \) is determined by the asymptotic expansion of \( G(t) \) at \( t \to 0 \) given by

\[ G(t) \sim \sum_{m \geq -n} a_{n} t^{n/2}. \]

Here \( G(t) \) means \( \int_{Y} \left[ G_1(t, y, u) - G_2(t, y, u) \right] dydu. \)

§ 10. Indexes of degenerate operators, I.

Let \( X \) be a real analytic \( n \)-dimensional compact Riemannian manifold with boundary \( Y \) and \( D = D_+, h, k \) or \( D_-, h, k \) be first order differential operators defined on \( C^\infty(X, E) \) and map it into \( C^\infty(X, F) \) such that on a neighborhood \( Y \times I \) (\( I = [0, 1] \)) of the boundary of \( X \)

\[ D_+, h, k = \sigma \left( \frac{\partial}{\partial u} + u^k A \right), \quad k > 0, \]
\[ D_-, h, k = \sigma \left( u^k \frac{\partial}{\partial u} + A \right), \quad k > 0. \]

Here, \( u \in I \) is the (real analytic) normal coordinate, \( \sigma = \sigma_D \left( du \right) \) is the bundle isomorphism \( E \to F \), \( A = A_u : C^\infty(Y, E_u) \to C^\infty(Y, E_u) \to C^\infty(Y, F_u) \) is a first order selfadjoint elliptic operator on \( Y \) which is independent of \( u \).

We denote by \( \hat{X} \) the double of \( X \). Then \( D_+, h, k \) and \( D_-, h, k \) define differential operators \( \hat{D}_+, h, k \) and \( \hat{D}_-, h, k \) both defined on \( C^\infty(\hat{X}, \hat{E}) \) and map it into \( C^0(\hat{X}, \hat{F}) \). They are elliptic on \( \hat{X} - Y \) but degenerate on \( Y \).

**Definition.** We define differential operators \( D_+, h, \varepsilon \) and \( D_-, h, \varepsilon \) on \( X \) by

\[ D_+, h, \varepsilon = \left( \frac{\partial}{\partial u} + (u^k + \varepsilon) A \right), \quad \varepsilon > 0, \text{ on } Y \times I, \]
\[ D_+, h, \varepsilon = \sigma, \quad \text{on } X - Y \times I, \]
\[ D_{-,k,e} = (u^k + \varepsilon \varepsilon \frac{\partial}{\partial u} + A), \quad \varepsilon > 0, \text{ on } Y \times I, \quad D_{-,k,e} = D_{-,k}, \text{ on } X - Y \times I. \]

Here \( e = e(y,u) \) is a \( C^\infty \)-function given by \( e(y,u) = e_i(u) \) where \( e_i(u) \) is a \( C^\infty \)-function on \( I \) such that

\[
0 \leq e_i(u) \leq 1, \quad e_i(u) = 1, \quad 0 \leq u \leq \frac{1}{3}, \quad e_i(u) = 0, \quad \frac{2}{3} \leq u \leq 1.
\]

**Definition.** For \( 0 < \varepsilon' < 1 \), we set

\[
u_k: = (1 - \varepsilon_i(u)), \quad \nu_k: = (1 - \varepsilon_i(u))\left(1 - c_i\left(\frac{u}{\varepsilon_i}\right)\right), \quad 0 \leq u \leq \varepsilon', \quad \text{if } k \text{ is not an even integer},
\]

and define differential operators \( D_+, k, e, e, \) and \( D-, k, e, e, \) by

\[
D_+, k, e, e, = a \left(\frac{\partial}{\partial u} + (u^k + \varepsilon e)A\right), \quad \text{on } Y \times I, \quad D_+, k, e, e, = D_+, k, \text{ on } X - Y \times I,
\]

\[
D-, k, e, e, = a \left(\frac{\partial}{\partial u} + (u^k + \varepsilon e)A\right), \quad \text{on } Y \times I, \quad D-, k, e, e, = D-, k, \text{ on } X - Y \times I.
\]

By definitions, the operators \( \tilde{D}_+, k, s, e, \) and \( \tilde{D}_-, k, s, e, \) defined from \( D_+, k, s, e, \) and \( D-, k, s, e, \) on \( \tilde{X} \) are \( C^\infty \)-coefficients elliptic operators on \( \tilde{X} \). Hence under the boundary conditions \( Pf(0,y) = 0, \) for \( D_+, k, s, e, \) and \( D-, k, s, e, \), \( k < 1, \) and \( (I-P)f(0,y) = 0, \) for \( D-, k, s, \), \( k \geq 1, \) they have finite indexes and there exist differential forms \( \alpha_{+,k,s,e}(x)dx \) and \( \alpha_{-,k,s,e}(x)dx \) on \( X \) such that (cf. [3], [4], [13]),

\[
(53)_+ \quad \text{index } D_+, k, s, e, = \int_X \alpha_{+,k,s,e}(x)dx - \frac{h + \gamma(0)}{2},
\]

\[
(53)_-1 \quad \text{index } D-, k, s, e, = \int_X \alpha_{-,k,s,e}(x)dx - \frac{h + \gamma(0)}{2}, \quad k < 1,
\]

\[
(53)_-2 \quad \text{index } D-, k, s, e, = \int_X \alpha_{-,k,s,e}(x)dx + \frac{\gamma(0) - h}{2}, \quad k \geq 1.
\]

Here, \( \text{index } D \) means the index of \( D \) with the boundary condition \( (I-P)f(0,y) = 0. \)

**Lemma 15.** Under the boundary conditions \( Pf(0,y) = 0, \) for \( D_+, k, s, e, \) and \( D-, k, s, \), \( k < 1, \) and \( (I-P)f(0,y) = 0, \) for \( D-, k, s, \), \( k \geq 1, \) \( D_+, k, s \) and \( D-, k, s \) have finite indexes and we have

\[
(54)'_+ \quad \text{index } D_+, k, s, = \text{index } D_+, h, s, s_e,
\]

(54)'}_-1 \quad \text{index } D-, k, s, = \text{index } D-, h, s, s_e,
Proof. Take $a$ to satisfy $\varepsilon > a > \varepsilon'$ and

$$2(u^k + \varepsilon) \geq u^k + \varepsilon \geq u_k + \varepsilon, \quad 0 \leq u \leq a,$$ for $D_{+, k, \varepsilon, \varepsilon'},$

$$\frac{2}{u^k + \varepsilon} \geq \frac{2}{u_k + \varepsilon} \geq \frac{1}{u^k + \varepsilon}, \quad 0 \leq u \leq a,$$ for $D_{-, k, \varepsilon, \varepsilon'}.$

Then, since on $Y \times [0, a]$, the equations $D_{+, k, \varepsilon, \varepsilon'}f = 0$ and $D_{-, k, \varepsilon, \varepsilon'}g = 0$ reduce (on each eigenspace of $A$)

$$\frac{d}{du} f_{k, h, e} + \lambda(u^k + \varepsilon)f_{k, h, e} = 0,$$

$$\frac{d}{du} g_{k, h, e} + \lambda u_k + \varepsilon g_{k, h, e} = 0,$$ for $D_{+, k, \varepsilon, \varepsilon'}$, and $D_{-, k, \varepsilon, \varepsilon'}$, 

$$(u^k + \varepsilon) \frac{d}{du} f_{k, h, e} + \lambda h_{k, h, e} = 0,$$

$$(u_k + \varepsilon) \frac{d}{du} g_{k, h, e} + \lambda h_{k, h, e} = 0,$$ for $D_{-, k, \varepsilon}$ and $D_{-, k, \varepsilon'}$, the equations $D_{+, k, \varepsilon, \varepsilon'}g = 0$ with the boundary condition $g(a, y) = f(a, y)$ have unique solution on $Y \times [0, a]$ if $D_{+, k, \varepsilon, \varepsilon'} = 0$ by the choice of $a$. Moreover, since $f(u, y) = \sum_{\lambda \geq 0} f_{k, k, e}(u)\phi_{k, \varepsilon}^\lambda(y)$ if $Pf(0, y) = 0$ and $f(u, y) = \sum_{\lambda \geq 0} f_{k, k, e}(u)\phi_{k, \varepsilon}^\lambda(y)$ if $(I-P)f(0, y) = 0$ on $Y \times [0, a]$, this $g$ satisfies $Pg(0, y) = 0$ if $Pf(0, y) = 0$ and $(I-P)g(0, y) = 0$ if $(I-P)f(0, y) = 0$. Therefore, since $D_{+, k, \varepsilon, \varepsilon'} = D_{-, k, \varepsilon}$ on some neighborhood of $Y \times \{a\}$ in $X$, to define a function $\widetilde{g}$ on $X$ by

$$\widetilde{g} = f,$$ on $X - Y \times [0, a]$, \quad $\widetilde{g} = g,$ on $Y \times [0, a],$$

$\widetilde{g}$ is a solution of $D_{+, k, \varepsilon, \varepsilon'}$ on $X$ with the boundary condition $P\widetilde{g}(0, y) = 0$ (or $(I-P)\widetilde{g}(0, y) = 0$). This shows dim. ker. $D_{+, k, \varepsilon, \varepsilon'} \geq$ dim. ker. $D_{-, k, \varepsilon}$. Similarly, we get dim. ker. $D_{+, k, \varepsilon, \varepsilon'} \leq$ dim. ker. $D_{+, k, \varepsilon}$ and we have dim. ker. $D_{+, k, \varepsilon, \varepsilon'} = \dim. \ker. D_{+, k, \varepsilon}$. By the same reason, we have dim. ker. $D_{-, k, \varepsilon, \varepsilon'} = \dim. \ker. D_{-, k, \varepsilon}$.

§ 11. Indexes of degenerate operators, II.

Lemma 16. (i). Under the boundary condition $Pf(0, y) = 0$, $D_{+, k}$ has finite index and for sufficiently small $\varepsilon$, we have

$$\text{index } D_{+, k} = \text{index } D_{+, k, \varepsilon}.$$
(ii). Under the boundary condition

\[ (B_k) \quad Pf(0, y) = 0, \quad \text{for } D_{+, k}f = 0, \quad \lim_{u \to 0} (I - P)(u^k g(u, y)) = 0, \quad \text{for } D_{-, k}^* g = 0, \]

\( D_{-, k} \) has finite index for \( k < 1 \), and for sufficiently small \( \varepsilon \), we have

\[ \text{index}_{k} D_{-, k} = \text{index} D_{-, k, \varepsilon}. \]

Here \( \text{index}_{k} D \) means the index of \( D \) with the boundary condition \( (B_k) \).

(iii). Under the boundary condition

\[ (B_{-k}) \quad (I - P)f(0, y) = 0, \quad \text{for } D_{-, k}f = 0, \quad \lim_{u \to 0} P(u^k g(u, y)) = 0, \quad \text{for } D_{+, k} g = 0, \]

\( D_{-, k} \) has finite index for \( k \geq 1 \), and for sufficiently small \( \varepsilon \), we have

\[ \text{index}_{-k} D_{-, k} = \text{index}_{-k} D_{-, k, \varepsilon}. \]

Here \( \text{index}_{-k} D \) means the index of \( D \) with the boundary condition \( (B_{-k}) \).

**Proof.** Let \( \varepsilon < 1/5 \) and take \( b \) to satisfy \( 1 - \varepsilon > b > 3 \varepsilon \). On \( Y \times [0, b] \), the equations \( D_{+, k} = f = 0 \) and \( D_{+, k}^* g = 0 \) reduce

\[ \frac{d}{du} f_{+, k} + \lambda u^k f_{+, k} = 0, \quad \text{for } D_{+, k}, \quad u^k \frac{d}{du} f_{-, k} + \lambda f_{-, k} = 0, \quad \text{for } D_{-, k}, \]

\[ \frac{d}{du} g_{+, k} - \lambda u^k g_{+, k} = 0, \quad \text{for } D_{+, k}^*, \]

\[ u^k \frac{d}{du} g_{-, k} + (k u^k - \lambda) g_{-, k} = 0, \quad \text{for } D_{-, k}^*, \]

The solutions of these equations are

\[ f_{+, k} = c_2 e^{-(\lambda/1+k)u^k}, \quad f_{-, k} = c_2 e^{-(\lambda/1-k)u^k}, \quad k \neq 1, \quad f_{+, -1} = c_3 u^{-3}, \]

\[ g_{+, k} = c_2 e^{(\lambda+k-1)u^k}, \quad g_{-, k} = c_2 u^{-k} e^{(\lambda+k-1)u^k}, \quad k \neq 1, \quad g_{+, 1} = c_2 u^1. \]

Then, since

\[ 0 < e^{(\lambda+k+1)u^k} \leq e^{(\lambda+k+1)u^{k+1}} \leq e^{(\lambda+k+1)u^{\kappa(k+1)}} \leq e^{(\lambda+k+1)u^{k+1} + (\lambda/1+k)^k + 1}, \lambda \geq 0, \]

we obtain (i).

To show (ii) and (iii), we use the inequalities

\[ 0 \leq \exp \left( \int_0^u \frac{dv}{v^k + \varepsilon} \right) \leq e^{(\lambda/1-k)u^k}, \quad 0 \leq \exp \left( \int_0^u \frac{dv}{v^k + \varepsilon} \right) \leq e^{\alpha(b, k, \varepsilon)} \frac{dv}{v^k + \varepsilon}, \quad k < 1, \lambda \geq 0, \]
Indexes of Some Degenerate Operators

\[ \frac{u^{1-k}}{1-k} < \int_0^u \frac{dv}{v^k + \varepsilon}, \log u < \int_0^u \frac{dv}{v + \varepsilon}, \quad u < b, \]

\[ \frac{b^{1-k}}{1-k} > \int_b^{\beta(b, k, \varepsilon)} \frac{dv}{v^k + \varepsilon}, \log b > \int_b^{\beta(b, 1, \varepsilon)} \frac{dv}{v + \varepsilon}, \quad 0 < \beta(b, k, \varepsilon) < b, \]

where \( \alpha(b, k, \varepsilon) \) is the twice of infimum value of these \( \alpha \) that satisfy \( \int_0^u (1/(v^k + \varepsilon)) dv \leq C(u/v^k(v^k + \varepsilon))dv \), \( 0 \leq u \leq b \) and therefore \( \lim_{\varepsilon \to 0} \alpha(b, k, \varepsilon) = 0 \). Hence we get (ii). On the other hand, since \( \lim_{\varepsilon \to 0} \beta(b, k, \varepsilon) = 2^{1/(1-k)} 3b, k > 1 \), \( \lim_{\varepsilon \to 0} \beta(b, 1, \varepsilon) = \sqrt{b} \), to take \( b < 1/3 \) and set

\[ e_k(u) = e(c(k)u), \quad c(k) < 2^{1-k} 3b, \quad k > 1, \quad c(1) < \frac{3}{2} \sqrt{b}, \]

\[ D_{-k, k} = 0 \]

we have for sufficiently small \( \varepsilon \),

\[ \text{index}_{D_{-k, k}} = \text{index}_{-kD_{-k, k}}, \quad k \geq 1. \]

But since \( \text{index}_{D_{-k, k}} = \text{index}_{-kD_{-k, k}} \) by the same reason as lemma 16, we get (iii).

**Lemma 17.** To set

\[ H_k = \{0\} \cup \{f | D_{-, k}f = 0, \ f(0, y) \neq 0, \ A f(0, y) = 0\}, \]

\[ H_k^* = \{0\} \cup \{f | D_{-, k}^* f = 0, \ f(0, y) \neq 0, \ A f(0, y) = 0\}, \]

\[ \dim. H_k = h_k, \quad \dim. H_k^* = h_k^*, \]

we have

(i). \( h_k \leq h \), and \( h_k^* \leq h \).

(ii). If \( D_{-k} \) is a real analytic coefficients operator, then \( h_k \) does not depend on \( D_{-k} \) and \( h_k^* \) depends only on \( k \).

(iii). To set index_{D} the index of \( D \) with the 0-boundary condition, that is \( \lim_{\varepsilon \to 0} f(u, y) = 0 \), we have

\[ \text{index}_{D_{-k, k}} = \text{index}_{-kD_{-k, k}} - (h_k - h_k^*), \quad k \geq 1. \]

**Proof.** If \( f \in H_k \) (or \( H_k^* \)), then \( f(u, y) = \sum_i f_i(u) \phi_i(y), \ A \phi_i(y) = 0 \), on \( Y \times [0,1] \). Then, since \( D_{-k} \) and \( D_{-k}^* \) are first order elliptic operators, unique continuity is hold for the solutions of \( D_{-k} \) and \( D_{-k}^* \), we get (i).

If \( D_{-k} \) is a real analytic coefficients operator, then above \( f_i(u) \) are constants for all \( i \) along any integral curve of real analytic normal vector field of \( Y \) starts
from a point of \( Y \). By the same reason, \( H_k \) is determined by \( k \) and we obtain (ii).

Let \( D_{-, k} = 0 \) and \( D_{-, k}^* \ g = 0 \), then to set \( f = \sum_{\lambda \leq 0} f_\lambda(y) \phi_\lambda(y) \), \( g = \sum_{\lambda \geq 0} g_\lambda(y) \phi_\lambda(y) \) on \( Y \times [0, 1] \), we get

\[
\lim_{n \to 0} f_\lambda(u) = 0, \quad \lambda > 0, \quad \lim_{n \to 0} g_\lambda(u) = 0, \quad \lambda < 0.
\]

This shows (55).

By (53), lemma 15, lemma 16 and lemma 17, we obtain

**Proposition 4.** (i) Under the boundary condition \( Pf(0, y) = 0 \), \( D_{+, k} \) has finite index and we have

\[
\text{index } D_{+, k} = \int_X \alpha_{+, k, e^v(x)}dx - \frac{h + \eta(0)}{2}.
\]

(ii) Under the boundary condition \((B_k)\), \( D_{-, k}, k < 1 \), has finite index and we have

\[
\text{index } D_{-, k} = \int_X \alpha_{-, k, e^v(x)}dx - \frac{h + \eta(0)}{2}, \quad k > 1.
\]

(iii) Under the 0-boundary condition, \( D_{-, k}, k \geq 1 \), has finite index and we have

\[
\text{index } D_{-, k} = \int_X \alpha_{-, k, e^v(x)}dx + \frac{\eta(0) - h}{2} - (h_k - h_{k_e}), \quad k \geq 1.
\]

Here \( h_k \leq h \), \( h_k^* \leq h \) and if \( D_{-, k} \) is a real analytic coefficients operator, \( h_k \) does not depend on \( D_{-, k} \) and \( h_k^* \) depends only on \( k \).

§ 12. Fundamental solutions of \( \frac{\partial}{\partial t} + \hat{A} \).

We denote the closed extensions of \( \hat{D}_{\pm, k} \) and \( \hat{D}_{\pm, k}^* \) by \( \hat{\Theta}_{\pm, k} \) and \( \hat{\Theta}_{\pm, k}^* \).

Then set

\[
\hat{A}_{\pm, k} = \hat{\Theta}_{\pm, k} \hat{\Theta}_{\pm, k}^*, \quad \hat{A}_{\pm, k} = \hat{\Theta}_{\pm, k}^* \hat{\Theta}_{\pm, k}.
\]

For simple, we denote \( \hat{A} \) instead of \( \hat{A}_{i,+, k}, \text{ etc.} \). By definition, \( \hat{A} \) is elliptic on \( \hat{X} = (Y \times [-1, 1]) \). On the other hand, \( \hat{A}_{i,+, k} \) and \( \hat{A}_{i,+, k}, \ k < 1, \ i = 1, 2 \), have smoothing operators in \( L^2(Y \times [-1, 1]) \) by lemma 7, (i), and \( \hat{A}_{i,+, k}, \ k \geq 1, \ i = 1, 2 \), have smoothing operators in \( H_{(12k)}(Y \times [-1, 1]) \), by lemma 7, (ii). Hence we have (cf. [3])

**Lemma 18.** (i) \( \hat{A}_{i,+, k} \) and \( \hat{A}_{i,+, k}, \ k < 1, \ i = 1, 2 \), have parametrixes in \( L^2(\hat{X}) \).

(ii) \( \hat{A}_{i,+, k}, \ k \geq 1, \ i = 1, 2 \), have parametrixes in \( H_{(12k)}(Y \times [-1, 1]) \). Here

\( H_{(12k)}(Y \times [-1, 1]) \) is the Sobolev space of these functions on \( \hat{X} \) that vanishes on \( Y \) at least order \( [2k] \).
**Corollary.** (i).\( \partial/\partial t + \hat{\Delta}_{i,k} \) and \( \partial/\partial t + \hat{\Delta}_{i,k} \), \( k < 1, \ i = 1, 2 \), have fundamental solutions with \( C^\infty \)-kernels on \((\hat{X} - Y) \times (\hat{X} - Y) \times (\mathbb{R}^+ - \{0\})\) in \( L^2(\hat{X})\).

(ii). \( \partial/\partial t + \hat{\Delta}_{i,k} \), \( k \geq 1, \ i = 1, 2 \), have fundamental solutions with \( C^\infty \)-kernels on \((\hat{X} - Y) \times (\hat{X} - Y) \times (\mathbb{R}^+ - \{0\})\) in \( H^{2k+2n,2k+2n}(\hat{X})\).

We denote the kernels of the fundamental solution of \( \partial/\partial t + \hat{\Delta}_{i,k} \) by \( F_{i,k}(t,x) \), \( i = 1, 2 \).

By the definitions of \( \hat{\Delta}_{i,k} \) on \( Y \times [-a,a] \), we have

\[
\hat{\Delta}_{i,k} = \sigma_i(\hat{\Delta}_{i,k} + 2k|u|^{-1}A),
\]

\[
\hat{\Delta}_{i,k} = \sigma_i(\hat{\Delta}_{i,k} + k(k-1)|u|^{2k-2}), \ k < 1,
\]

Here \( \sigma \) are bundle isomorphisms. Hence by lemma 10, to define a \( C^\infty \)-function \( e_2 \) on \( \hat{X} \) by

\[
es_2(u, y) = e_2(u), \ 0 \leq e_2 \leq 1, \ es(u) = 0, \ |u| \leq 1/2,
es_2(u) = 1, \ |u| \geq 3/4, \ \text{on} \ Y \times [-1,1],
es_2 = 1, \ \text{on} \ \hat{X} - Y \times [-1,1],
\]

we have

\[
F_{i,k}(t,x)e_2(x) \sim F_{i,k}(t,x) + H_{i,k}(t,x), \ \lim_{t \to 0} H_{i,k}(t,x) = 0,
\]

\[
F_{i,k}(t,x) = F_{i,k}(t,x) - F_{i,k}(t,x).
\]

Then, since \( \mathcal{F}_k \otimes \mathcal{H}_k \otimes H^s(Y) \cup H^{2k+2n}(X) \) if \( s \geq [2k] + 2n \) and if \( f \) satisfies 0-boundary condition and \( D_{-k}f = 0 \) or \( D_{-k}f = 0, \) then \( f \in \mathcal{F}_k \otimes \mathcal{H}_k \otimes H^p(Y), \ k \geq 1, \) we have by proposition 2, proposition 3, proposition 4 and lemma 18

**Theorem** (i). For \( D_{+k} \), there exists a differential form \( \alpha_{+,k}(x)dx \) on \( X \) such that

\[
\text{index } D_{+,k} = \int_X \alpha_{+,k}(x)dx - \frac{k + \eta(0)}{2}.
\]

(ii). For \( D_{-,k} \), \( k < 1 \), there exists a differential form \( \alpha_{-,k}(x)dx \) on \( X \) such that

\[
\text{index } D_{-,k} = \int_X \alpha_{-,k}(x)dx - \frac{k + \eta(0)}{2}.
\]

(iii). For \( D_{-,k} \), \( k \geq 1 \), there exists a differential form \( \alpha_{-,k}(x)dx \) on \( X \) such that
(58) \[ \text{index}_a D_{-h} = \int_X \alpha_{-h} k(x) dx + \frac{h + \gamma(0)}{2}. \]

**Proof.** We only need to show (ii). But since \( \text{index}_a D_{-h} = 0 \) if \( h < 1 \), we have \( \text{index}_b D_{-h} = \text{index}_b D_{-h} + \text{index}_a D_{-h} \), and by lemma 2, we have \( \text{index}_b D_{-h} = \int_X \beta_h(x) dx - (h + \gamma(0))/2 \) for some differential form \( \beta_h(x) dx \) on \( X \) and \( \text{index}_b D_{-h} = \int_X \gamma_h(x) dx \) for some \( \gamma_h(x) dx \), we obtain (ii).

**Corollary.** Let \( \varepsilon > \varepsilon' > 0 \) and \( \varepsilon \) is sufficiently small, then

\[
\begin{align*}
(59)_+ & \quad \int_X \alpha_{+h} k(x) dx = \int_X \alpha_{+h, \varepsilon, \varepsilon'}(x) dx, \\
(59)_- & \quad \int_X \alpha_{-h} k(x) dx = \int_X \alpha_{-h, \varepsilon, \varepsilon'}(x) dx, \quad k < 1, \\
(59)_- & \quad \int_X \alpha_{-h} k(x) dx = \int_X \alpha_{-h, \varepsilon, \varepsilon'}(x) dx - (h_k - h_k'), \quad k \geq 1.
\end{align*}
\]

**Note.** If we consider

\[ D_{-h} = D_{-h} A(x) = \sigma(\frac{D}{Dx} + u^{-h} A), \quad Y \times [0, 1], \]

instead of \( D_{-h} \), index \( D_{-h}^* \) exists if \( h < 1 \) and we have with suitable differential form \( \alpha_{-h} k(x) dx \),

\[ \text{index} D_{-h} = \int_X \alpha_{-h} k(x) dx - \frac{h + \gamma(0)}{2}, \quad k < 1. \]

On the other hand, since we get \( D_{-h}^* = -\sigma^{-1}D_{-h}^* \sigma^* \) on \( L^2[0, 1] \otimes \ker \ A \), to set

\[
\begin{align*}
H_{h} &= \{0\} \cup \{f | D_{-h} f = 0, \quad f(0, y) \neq 0, \quad A f(0, y) = 0\}, \\
H_{h}^* &= \{0\} \cup \{g | D_{-h}^* g = 0, \quad g(0, y) \neq 0, \quad A g(0, y) = 0\},
\end{align*}
\]

\( H_{h} \) is isomorphic to \( H_{h}^* \). Therefore we have

\[ \text{index} D_{-h} = \text{index} D_{-h}, \quad k \geq 1. \]

Hence we get with suitable differential form \( \alpha_{-h} k(x) dx \) on \( X \)

\[ \text{index} D_{-h} = \int_X \alpha_{-h} k(x) dx + \frac{h - \gamma(0)}{2}, \quad k \geq 1, \]

and for this \( \alpha_{-h} k(x) dx \), we have
\[
\int_X \alpha_{\lambda-\beta}(x) dx = \int_X \alpha_{\lambda-\beta}, \epsilon, \epsilon'(x) dx, \ k \geq 1,
\]

if \( \epsilon \) and \( \epsilon' \) are sufficiently small. Here \( \alpha_{\lambda-\beta}, \epsilon, \epsilon'(x) dx \) is the differential form constructed for Differential operator \( D_{\lambda-\beta}, \epsilon, \epsilon' \) given by

\[
D_{\lambda-\beta}, \epsilon, \epsilon' = \partial \left( \frac{\partial}{\partial u} + \frac{A}{u^{\lambda-\beta} + \epsilon \epsilon'} \right), \text{ on } Y \times I,
\]

\[
= D_{\lambda-\beta}, \text{ on } X - Y \times I.
\]