Note on $A'_n$-maps

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1. Introduction.

In [1] we have defined $A'_n$-spaces and $A'_n$-maps, and considered some of their properties. The purpose of the present note is to give certain complementary facts.

In §2, we show that $A'_2$-notions are homotopy-categorical, and $A'_3$-notions are categorical.

In §3, we consider conditions for a map of $A'_2$-spaces having the vanishing generalized Hopf homomorphism to be an $A'_2$-map. Our main theorem is as follows.

Theorem 3.1. Let $f: S^2 \rightarrow Y$ be a map into an $A'_2$-space. If $H(f) = 0$, then $f$ is an $A'_2$-map.

We use the same notations and definitions as in [1], in particular, we work in the category of based spaces having the homotopy-types of CW-complexes and based maps.

2. Categories $A'_2$ and $A'_3$.

As easily seen, $A'_2$-spaces and $A'_2$-maps constitute a category $A'_2$.

Definition 2.1. Two $A'_2$-maps $f_0, f_1: X \rightarrow Y$ are said to be $A'_2$-homotopic, in notation $f_0 \simeq f_1 (A'_2)$, if there exist a homotopy $F = H(f_0, f_1)$ and a homotopy $H(F): X \times I \times I \rightarrow Y \vee Y$ satisfying the following conditions:

$$H(F)|X \times I \times \{0\} = H'_2(f_0), \quad H(F)|X \times \{1\} = H'_2(f_1),$$

$$H(F)|X \times \{0\} \times I = (F \vee F') \circ \mu_x \quad \text{and} \quad H(F)|X \times \{1\} \times I = \mu_y \circ F.$$

An $A'_2$-map $f: X \rightarrow Y$ is an $A'_2$-homotopy-equivalence if there exists a homotopy-inverse $g$ such that we have $g \circ f \simeq 1 (A'_2)$ and $f \circ g \simeq 1 (A'_2)$.

Proposition 2.2. Let $f_0: X \rightarrow Y$ be an $A'_2$-map, and $f_1: X \rightarrow Y$ is a map which is homotopic to $f_0$. Then, we may define $H'_2(f_1)$ so that it holds $f_0 \simeq f_1 (A'_2)$.

Proof. Put $F = H(f_0, f_1)$, then the homotopy $H'_2(f_1)$ is given by
Let $G_0$ be the homotopy for $f_0$ in $(3, 2, 2^{\prime})$ of $[1]$, then the corresponding homotopy $G_1$ for $f_1$ is defined by the followings:

$$\begin{align*}
G_1(x, s, t) &= \begin{cases} 
G_0(x, 2s-1, 3t-1) & \text{for } \frac{1}{2} \leq s \leq 1, \quad \frac{1}{3} \leq t \leq \frac{2}{3} \\
(F(\cdot ; 1-2s) \times F(\cdot ; 1-2s')) \circ D'(x, s') & \text{for } s = \frac{2s' + t' - t s'}{2}, \quad t = \frac{t'}{2} \\
F(x, 1-2s) \times F(x, 1-2s) & \text{for } s = \frac{s'}{2}, \quad t = \frac{s' + 3t' - 2s'}{3} \\
D'(x, t') & \text{for } s = \frac{1 + s' - t' + 2s't'}{3}, \quad t = \frac{2 + t'}{3}
\end{cases}
\end{align*}$$

where $s'$ and $t'$ run from 0 through 1.

Now, let $X$ and $Y$ be simply-connected CW-complexes and $f: X \rightarrow Y$ be a cellular homotopy-equivalence. Since the mapping cylinder $M_f$ is a simply-connected CW-complex and $\pi_n(M_f, X) = 0$ for all $n \geq 2$, $X$ is a strong deformation retract of $M_f$. Let $r_1: M_f \rightarrow X$ be the retraction, $i_1: X \rightarrow M_f$ be the inclusion and $D_1 = H(i_1 \circ r_1, 1)$. On the other hand, $Y$ is a strong deformation retract of $M_f$, let $r_2: M_f \rightarrow Y$ be the retraction, $i_2: Y \rightarrow M_f$ be the inclusion and $D_2 = H(i_2 \circ r_2, 1)$. Then, we have $f = r_2 \circ i_1$, and $f' = r_1 \circ i_2$ is a homotopy-inverse of $f$, moreover we have $F = H(f' \circ f, 1) = r_1 \circ D_2(\circ i_2 \times 1)$ and $F' = H(f \circ f', 1) = r_2 \circ D_1(\circ i_1 \times 1)$. Define $F(\circ)x \times I \times I \rightarrow Y$ by $F(\circ)(x, s, t) = r_2 \circ D_2(\circ i_1(x), s, t)$, then $F(\circ)$ satisfies the following conditions:

$$
\begin{align*}
F(\circ)x \times I \times \{0\} &= f \circ F, \quad F(\circ)x \times \{0\} \times I = F' \circ (f \times 1), \\
F(\circ)[x, 0, 0] &= (f \circ f' \circ f)(x) \quad \text{and} \\
F(\circ)x \times I \times \{1\} &= F(\circ)x \times \{1\} \times I = f.
\end{align*}
$$

By abuse of language, we say that two homotopies $f \circ F$ and $F' \circ (f \times 1)$ are homotopic. Similarly, we have that two homotopies $f' \circ F'$ and $F' \circ (f' \times 1)$ are homotopic. In these situations, we say that $\{f, f', F, F'\}$ is nice.

**Proposition 2.3.** Let $f: X \rightarrow Y$ be a cellular homotopy-equivalence of simply-connected CW-complexes with a homotopy-inverse $f'$ such that $\{f, f', F = H(f' \circ f, 1), F' = H(f \circ f', 1)\}$ is nice. If $X$ is an $A'$-space, then we may define an $A'$-structure of $Y$ such that $f$ is an $A'$-homotopy-equivalence.

**Proof.** Put $\mu'(x) = (f' \circ f) \circ f' \circ f$ and $D'_Y = (F' \times F') \circ D'_Y \circ (f \times f) \circ D'_X \circ (f' \times 1)$,
then \( \{ t' Y, D' Y \} \) defines an \( A_{t'} \)-structure of \( Y \), i.e., we have \( D' Y = H(\partial Y, j_{t'} \circ t' Y) \). Moreover, we have \( H' Y(f) = -(f \vee f) \circ \partial \), and \( H' Y(f') = (F \vee F) \circ \partial \times 1 \).

Define \( G(f): X \times I \rightarrow Y \times X \) by

\[
G(f)(x, s, t) = \begin{cases} (f \times f) \circ D' Y F(x, 1 - t), & \frac{2s - l}{2 - l} \leq s \leq t, s \leq 0 \\ A \circ F(\partial)(x, 1 - 2s, 2 - 2t) & \frac{1}{2} \leq s \leq \frac{t}{2}, 0 \leq s \leq \frac{t}{2} \\ A \circ F(\partial)(x, 1 - 2s, 2t - 4s) & t \leq s + \frac{1}{2}, 0 \leq s \leq \frac{t}{2} \\ \end{cases}
\]

where \( F(\partial) \) is the homotopy from \( f \circ F \) to \( F \circ (f \times 1) \). Then, \( G(f) \) satisfies the condition (3. 2. 2) in [1], and \( f \) is an \( A'_{t'} \)-map. Similarly, \( G(f') \) is defined and \( f' \) is an \( A'_{t'} \)-map. As easily seen, we have \( f' \circ f \cong 1(x \times Y) \) and \( f \circ f' \cong 1(x \times Y) \), thus \( f \) is an \( (Y') \)-homotopy-equivalence.

**Proposition 2.4.** \( A'_{t'} \)-spaces and \( A'_{t'} \)-maps constitute a category \( \mathcal{A}'_{t'} \).

**Proof.** It suffices to define a homotopy \( H' Y(f \circ g): X \times K_3 \times I \rightarrow W_3(\mathcal{A}) \) satisfying the conditions (3. 2. 1~3) in [1] for \( A'_{t'} \)-maps \( f: X \rightarrow Y \) and \( g: Y \rightarrow Z \).

Let \( H' Y(f) \) and \( H' Y(\xi) \) be the homotopies for \( f \) and \( \xi \). Subdivide \( I \times I \) into four domains as in the following figure.

![Diagram](image)

Define \( H' Y(f \circ \xi) \times (I) \) using \( W_3(\xi) \circ H' Y(f) \) and \( H' Y(f \circ \xi) \times (II) \) using \( H' Y(\xi) \circ (f \times 1) \). Next, define \( H' Y(f \circ \xi) \times (III) \) using \( (H' Y(\xi) \circ \xi) \circ H' Y(f) \) and \( H' Y(f \circ \xi) \times (IV) \) using \( H' Y(\xi) \circ H' Y(\xi) \). Then, these maps coincide on the intersections of domains, therefore we may define \( H' Y(\xi) \) all over \( X \times K_3 \times I \). By the construction, \( H' Y(f \circ \xi) \) satisfies the condition (3. 2. 1). Since \( f \) and \( \xi \) are \( A'_{t'} \)-maps (3. 2. 2) is obvious, and (3. 2. 3) will be seen easily.

### 3. Generalized Hopf Homomorphism

At first, we recall the definition of the generalized Hopf homomorphism \( H(f) \)
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Let $X$ be an $A'$-space, then any map $u: X \to Y \vee Y$ is represented as $u = (u_1 \times u_2) \circ \delta_X$, and since there exists a homotopy $D'_X = H(\delta_X, j_X \circ \eta'_X)$, we obtain $u = (u_1 \times u_2) \circ \delta_X = (j_Y \circ (u_1 \vee u_2) \circ \eta'_X$, i.e., $j_Y*: [X; Y \vee Y] \to [X; Y \times Y]$ is surjective. Then, we have the following exact sequence ([1], Lemma 4.3):

$0 \to [CX, X; Y \times Y, Y \vee Y] \xrightarrow{r_*} [X, Y \vee Y] \to [X, Y \times Y] \to 0.$

And we have the isomorphism $\varphi*: [CX, X; Y \times Y, Y \vee Y] \cong [X; \Omega Y \vee \Omega Y]$.

Let $f: X \to Y$ be any map $A'$-cogroups. Put $f_0 = (f \vee f) \circ \eta'_X$ and $f_1 = f'_Y \circ f$. Then, there exists a $[g'] \in [CX, X; Y \times Y, Y \vee Y]$ such that $r_*[g'] = [f_1] - [f_0]$. Define $H(f) = \varphi*[g'] = [g']$. If $f$ is an $A'$-map, then $H(f) = 0$, and if $H(f) = 0$, we have a homotopy $H'*(f) = H(f_0, f_1)$.

We attempt to consider these situations more precisely. Put $F' = -(f \times f) \circ D'_X + D'_Y \circ (f \times 1)$, then we have $F' = H(j_Y f_0, j_Y f_1)$. Define homotopies $F'' = H(j_Y f_0, j_Y f_1 - j_Y f_0)$ and $F''' = H*(j_Y f_0, j_Y f_1 - j_Y f_0)$ by $F'' = F' - j_Y f_0$ and $F''' = -j_Y f_0 \circ \eta'_Y \circ F''$, respectively. Then, there exists a homotopy $F: X \times I \to Y \vee Y$ such that we have $j_Y F = F'''$ and $F(x, 1) = f_1(x) - f_0(x)$. The above map $g$ is just a map defined by $g(x) = F(x, 0)$. Therefore, $H(f) = 0$ implies the existence of a homotopy $N_g: X \times I \to \Omega Y \vee \Omega Y$ such that $N_g(x, 0) = *$ and $N_g(x, 1) = g(x)$. Then, we may define $H'*(f)$.
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\[ H'(f) = f_0 \circ E'_L + ((N \circ F + f_0) \circ \nabla \circ (f_0 \circ (-f_0) \circ f_0)^L) \circ M_X, s \]

Now, consider the above diagram (the thick line segments represent the homotopy $j_1 \circ H'(f)$ and the broken line segments represent the homotopy $F'$): Squares (2)~(8) are homotopy-commutative by the similar argument as in Proposition 2.8.

If $X$ is a suspended space, say $X = SZ$, then the tetragon (1) is homotopy-commutative. In fact, if we define $E' : SZ \times K_3 \times I \to SZ$ by

\[
E'(a, z, t, s) = \begin{cases} 
\langle \frac{2a}{1+s}, z \rangle & \text{for } 0 \leq a \leq \frac{(1+s)t}{2} \\
\langle t, z \rangle & \text{for } \frac{(1+s)t}{2} \leq a \leq \frac{1+t-s+st}{2} \\
\langle \frac{2a+s-1}{1+s}, z \rangle & \text{for } \frac{1+t-s+st}{2} \leq a \leq 1 
\end{cases}
\]

then, we have $E' \mid SZ \times \{0\} \times I = H(s+1, 1)$, $E' \mid SZ \times \{1\} \times I = H(s+1, 1)$ and $E' \mid SZ \times K_3 \times \{1\} = 1_{SZ}$. Moreover, $E' : SZ \times K_3 \times \{0\}$ defines a homotopy $E'' = H(s+1, 1 + s)$ such that

\[
E''(a, z, t) = \begin{cases} 
\langle 2a, z \rangle & \text{for } 0 \leq a \leq \frac{t}{2} \\
\langle t, z \rangle & \text{for } \frac{t}{2} \leq a \leq \frac{1+t}{2} \\
\langle 2a-1, z \rangle & \text{for } \frac{1+t}{2} \leq a \leq 1 
\end{cases}
\]

Finally, we obtain a map $\tilde{G} : SZ \times I \times I \to SZ$ satisfying the conditions $\tilde{G}(a, z, t, 0) = \nabla \circ (1 \circ (v_0 \circ 1) \circ M'_{0, 0}(a, z, t))$ and $\tilde{G}(a, z, t, 1) = E''(a, z, t)$. In fact, $\tilde{G}$ is defined by the followings:

\[
\tilde{G}(a, z, t, s) = \begin{cases} 
\langle \frac{4(1-s+st)a}{1+t+st-s}, z \rangle & \text{for } 0 \leq a \leq \frac{1+t+st-s}{4} \\
\langle 1-s+st, z \rangle & \text{for } \frac{1+t+st-s}{4} \leq a \leq \frac{1+t+s-st}{4} \\
\langle -4a+2+t, z \rangle & \text{for } \frac{1+t+s-st}{4} \leq a \leq \frac{2+t-st}{4} 
\end{cases}
\]
Thus, we have the following

**Theorem 3.1.** Let $f : S^Z \rightarrow Y$ be a map into an $A'_z$-space. If $H(f) = 0$, then $f$ is an $A'_z$-map.

**Reference**