On the admissible multiplication in $\alpha$-coefficient cohomology theories

Dedicated to Professor Ryoji Shizuma on his 60-th birthday

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Introduction

S. Araki and H. Toda [1] discussed the multiplicative structures in mod. $q$ generalized cohomology theories. In [2], the first named author discussed the multiplicative structures in $\alpha$-coefficient cohomology theories ($\alpha$ is stable map of spheres) and in the case $\alpha = \eta$ (a stable class of the Hopf map from $S^3$ to $S^2$) obtained a sufficient condition for existence of admissible multiplication in $\tilde{h}^\bullet(\ ; \alpha)$ for any reduced multiplicative generalized cohomology theory $\{\tilde{h}^\bullet, \sigma\}$ defined on the category of finite CW-complexes or of the same homotopy type with base points. In [3], a sufficient condition for existence of admissible multiplication in the case $\alpha = \eta^2$ is obtained.

In this paper we discuss the existence of the admissible multiplication in a more general case including the case $\alpha = \eta$ and $\eta^2$.

For the stable map $\alpha \in \{S^{r+k-1}, S^r\}$ satisfying the condition "$k$ is an odd integer or $2\alpha = 0$", we obtained a sufficient condition for existence of admissible multiplication in $\tilde{h}^\bullet(\ ; \alpha)$ for any multiplicative generalized cohomology theory $\tilde{h}^\bullet$.

In §1, we define the notion of admissible multiplication in $\alpha$-coefficient cohomology theories. In §2, we compute some stable homotopy groups and make preparations to the existence theorem of admissible multiplication from homotopical points of view. The existence of admissible multiplication is proved in §3 by constructing a multiplication.
§ 1. Preliminaries

First we shall fix some notations:

- $X \wedge Y$ : the reduced join of two spaces $X$ and $Y$ with base points,
- $S^nX = X \wedge S^n$ : the iterated reduced suspension of $X$,
- $S^n f = f \wedge S^n$ : the iterated reduced suspension of $f$,
- $T = T(A, B) : A \wedge B \to B \wedge A$ : the map switching factors,
- $\{X, Y\}$ : the stable homotopy group of CW-complexes $X$ and $Y$ with base point preserving,
- $G_k = \lim_{r \to \infty} \pi_{r+k}(S^r)$ : the $k$-th stable homotopy group of the sphere.

Let $\{\tilde{h}^*, \sigma\}$ be a deduced cohomology theory defined on the category of finite CW-complexes and be equipped with an associative multiplication $\mu$. Let $\alpha$ be a stable homotopy class of a map from $S^{r+k-1}$ to $S^r$. Since the stable homotopy type of the reduced mapping cone of this map depends only on homotopy class $\alpha$, we denote as

$$C_{\alpha} = S^r \cup C(S^{r+k-1}).$$

The $\alpha$-coefficient cohomology theory $\{\tilde{h}^*; \alpha\}$ is defined by

$$\tilde{h}^i(X; \alpha) = \tilde{h}^{i+r+k}(X \wedge C_{\alpha})$$

and the suspension isomorphism

$$\sigma_\alpha : \tilde{h}^i(X; \alpha) \to \tilde{h}^{i+k}(SX; \alpha)$$

is defined as the composition

$$\sigma_\alpha = (1_X \wedge T)^* \sigma : \tilde{h}^i(X; \alpha) \to \tilde{h}^{i+k}(SX; \alpha),$$

where $T = T(S^r, C_{\alpha})$.

Let us denote by $i : S^r \to C_\alpha$ the canonical inclusion and let $\pi : C_\alpha \to S^{r+k}$ be the map collapsing $S^r$ to a point. Then we put

\begin{equation}
(1.1)
\rho_\alpha = (-1)^j(i^{r+k}(1_X \wedge \pi)^* \sigma^{r+k} : \tilde{h}^i(X) \to \tilde{h}^{i+j}(X; \alpha),
\end{equation}

$$\delta_{\alpha, 0} = (-1)^j(i^{r+k} \sigma^{-r} (1_X \wedge i)^* : \tilde{h}^i(X; \alpha) \to \tilde{h}^{i+k}(X)$$

and

$$\delta_\alpha = \rho_\alpha \delta_{\alpha, 0} : \tilde{h}^i(X; \alpha) \to \tilde{h}^{i+k}(X; \alpha)$$

which are natural and called the reduction mod. $\alpha$, the Bockstein homomorphism and the mod. $\alpha$ Bockstein homomorphism respectively.

Moreover we put

\begin{equation}
(1.2)
\mu_L = \mu : \tilde{h}^i(X) \otimes \tilde{h}^j(Y; \alpha) \to \tilde{h}^{i+j}(X \wedge Y; \alpha),
\end{equation}

$$\mu_R = (-1)^j(i^{r+k}(1_X \wedge T)^* \mu : \tilde{h}^i(X; \alpha) \otimes \tilde{h}^j(Y) \to \tilde{h}^{i+j}(X \wedge Y; \alpha)$$

where $T = T(Y, C_{\alpha})$. 
A multiplication
\[ \mu_\alpha : \tilde{h}^i(X ; \alpha) \otimes \tilde{h}^j(Y ; \alpha) \to \tilde{h}^{i+j}(X \wedge Y ; \alpha) \]
is said to be admissible (cf. [2] 1.6) if it satisfy the following properties

(A1) compatible with \( \mu_L \) and \( \mu_R \) through the reduction mod. \( \alpha \) i.e.,

\[ \mu_L = \mu_\alpha(\rho_\alpha \otimes 1) \quad \text{and} \quad \mu_R = \mu_\alpha(1 \otimes \rho_\alpha) ; \]

(A2) there exists a cohomology operation \( \chi_\alpha : \tilde{h}^i(\quad) \to \tilde{h}^{-k}(\quad ; \alpha) \) of degree \(-k\) satisfying the relation

\[ \chi_\alpha(x \otimes y) = (-1)^{ik} \mu_L(x \otimes \chi_\alpha(y)) = \mu_R(\chi_\alpha(x) \otimes y) \]
for \( x \in \tilde{h}^i(X) \) and it is related to \( \mu_\alpha \) by the following relation

\[ \delta_\alpha \mu_\alpha(x \otimes y) = \mu_L(\delta_\alpha(x) \otimes y) + (-1)^{ik} \mu_L(x \otimes \delta_\alpha(y)) - (-1)^{ik} \mu_R(\delta_\alpha(x) \otimes \delta_\alpha(y)) \]
for \( x \in \tilde{h}^i(X ; \alpha) \) and \( y \in \tilde{h}^j(Y ; \alpha) \);

(A3) it is quasi-associative in the sense that

\[ \mu_\alpha(\mu_L \otimes 1) = \mu_L(1 \otimes \mu_\alpha), \]

\[ \mu_\alpha(\mu_R \otimes 1) = \mu_R(1 \otimes \mu_\alpha), \]

\[ \mu_\alpha(\mu_\alpha \otimes 1) = \mu_L(1 \otimes \mu_\alpha) \]

\[ \mu_\alpha(\mu_\alpha \otimes 1) = \mu_R(1 \otimes \mu_\alpha). \]

§ 2. Stable homotopy groups of some complexes

Let \( t \) be an integer. Assume that \( ta = 0 \) for an element \( \alpha \in \pi_{r \pm k - 1}(S^r) \). Let \( C_\alpha \) be the reduced mapping cone of \( \alpha \). For simplicity we denote \( C = C_\alpha \). From Puppe's exact sequence and its dual associated with a cofibration

\[ S^r \xrightarrow{i} C \xrightarrow{\pi} S^{r+k} \]
we obtain the following table

**Lemma 2.1.** The groups \( \{S^r+i, C\} \) and \( \{C, S^{r+i}\} \) are isomorphic to the corresponding groups in the following table:

<table>
<thead>
<tr>
<th>{S^r, C}</th>
<th>{S^{r+k}, C}</th>
<th>{S^{r+i}, C}</th>
<th>{C, S^{r+k}}</th>
<th>{C, S^r}</th>
<th>{C, S^{r+i}}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z )</td>
<td>( Z + (G_k/\eta_\alpha) )</td>
<td>( \text{finite group for } j \geq 0, k )</td>
<td>( Z )</td>
<td>( Z + (G_k/\eta_\alpha)\pi )</td>
<td>( \tilde{i} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where \( \tilde{i} \) and \( \tilde{i} \) are defined by \( \tilde{t} = t \gamma^{r+k} \) and \( \tilde{i} = t \gamma^r \).
Moreover we may chose \( \tilde{i} \) and \( \tilde{i} \) such that the relation
\[
(2.2) \quad \tilde{i} \tilde{t} + \tilde{i} \pi = t 1_c
\]
holds in \( \{C, C\} \), where \( 1_c \) is the homotopy class of the identity of \( C \).

From Lemma 2.1 and dual Puppe's exact sequence associated with (2.1) it follows that

**Lemma 2.2.** The groups \( \{S^iC, C\} \) is isomorphic to the corresponding groups in the following table:

| \( \{C, S^4C\} \) | \( Z \) | \( (S^4)i\pi \) |
| \( \{C, C\} \) | \( Z + Z + i(G/iG) \) | \( \tilde{i} \pi \) (or \( i\tilde{i} \)), \( 1_c \) |
| \( \{S^4C, C\} \) | \( Z + \text{finite group} \) |
| \( \{S^jC, C\} \) | \( \text{finite group for } j \neq k, 0 \) and \( -k \) |

From dual Puppe's exact sequence associated with (2.1), we obtain the following exact sequence
\[
\cdots \rightarrow \{S^{r+2k}, C\} \xrightarrow{(S^{k+2})^*} \{S^{k}, C\} \xrightarrow{(S^{k+1})^*} \{S^{r+k}, C\} \xrightarrow{(S^{k})^*} \{S^{r+2k-1}, C\} \rightarrow \cdots
\]
where groups \( \{S^{r+2k-1}, C\} \) and \( \{S^{r+2k}, C\} \) are finite by Lemma 2.1.

If \( (S^k\alpha)\tilde{t} = \tilde{t}(S^k\alpha) = 0 \), then there exists an element \( \delta \) of \( \{S^kC, C\} \) which satisfy the following relations
\[
(2.3) \quad \delta(S^k) = \tilde{t} \quad \text{and} \quad \pi \delta = S^k \tilde{t}
\]
and we can take \( \delta \) as generator of free part in \( \{S^kC, C\} \).

In the following, we consider \( \alpha \in \pi_{r+k-1}(S^r) \) only as the element satisfying
\[
(2.4) \quad k \text{ is an odd integer or } 2k = 0.
\]

**Lemma 2.3.** (Lemma 3.5 of [4]) Let \( \alpha \) be an element of \( \pi_{r+k-1}(S^r) \) satisfying (2.4). Assume that \( k \leq 2r - 1 \), then there exists an element \( \alpha' \) of \( \pi_{2r+k-1}(S^{2r}) \) such that equality
\[
(2.5) \quad 1_c \wedge \alpha = (S^{r+1}\alpha)(S^{r+k-1})
\]
holds in the homotopy set \( [S^{r+k-1}C, S^rC] \).

Under the condition (2.4), from Lemma 2.3, we shall see that \( C \wedge C \) is homotopy equivalent in stable range to the following mapping cone
\[
(2.6) \quad \overline{N}_\alpha = S^rC_a \cup \overline{C}(S^{r+k-1}C_a)
\]
where \( \overline{g} = (S^{r+1}\alpha')(S^{r+k-1}) \).

We denote also by \( N_\alpha \) a subcomplex of \( \overline{N}_\alpha \) obtained by removing the \((2r+k)\)
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- cell $S^rC - S^3r$, i.e.,

$$(2.7) \quad \bar{N}_\alpha = S^r \cup C(S^{r+k-1}C_\alpha)$$

where $\bar{g} = \alpha'(S^{r+k-1}C_\alpha)$.

The cell structures of $\bar{N}_\alpha$ and $N_\alpha$ can be interpreted as follows:

$$(2.8) \quad \bar{N}_\alpha = (S^rC \cup S^{2r+k}) \cup e^{2r+2k}, \quad N_\alpha = (S^{2r} \cup S^{2r+k}) \cup e^{2r+2k},$$

where $e^{2r+2k}$ is attached to $S^{2r} \cup S^{2r+k}$ by a map represented the sum of $\alpha' \in \{S^{2r+k-1}, S^{2r}\}$ and $\alpha \in \{S^{r+k-1}, S^r\}$.

We use the following notations:

$$(2.9) \quad \bar{i}_0 : \bar{N}_\alpha \longrightarrow \bar{N}_\alpha, \quad \bar{i}_0 : S^rC \longrightarrow \bar{N}_\alpha, \quad \bar{i}_0 : S^r \longrightarrow N_\alpha,$$

$$\pi_0 : \bar{N}_\alpha \longrightarrow S^{r+k}C, \quad \pi_0 : N_\alpha \longrightarrow S^{r+k}C, \quad \text{the map collapsing } S^rC \text{ or } S^r,$$

$$(2.10) \quad \bar{i}_1 : S^{r+k} \longrightarrow \bar{N}_\alpha, \quad \bar{i}_1 : S^{r+k} \longrightarrow N_\alpha, \quad \text{the inclusions},$$

$$\pi_1 : N_\alpha \longrightarrow Q = S^{2r} \cup e^{2r+2k}, \quad \text{the map collapsing } S^{2r+k}.$$

Hereafter, these mapping will be fixed as to satisfy the following relations:

$$(2.11) \quad \bar{i}_0 j = \pi_0, \quad j i_0 = \bar{i}_0(S^r i), \quad \pi_0 i_1 = S^{r+k} i = \pi_0 i_1, \quad \bar{i}_1 = j i_0, \quad \bar{i}_0 = \pi_i \quad \text{and} \quad i_0 \alpha' = - i_1(S^{r+k} \alpha).$$

**Lemma 2.4.** There exists an element $\zeta$ of $\{N_\alpha, C \wedge C\}$ satisfying the following three conditions:

(i) $\zeta$ is a homotopy equivalence, i.e., there is an inverse $\xi \in \{C \wedge C, \bar{N}_\alpha\}$ of $\zeta$ such that $\xi \zeta = 1$ and $\zeta \xi = 1$,

(ii) $\bar{\zeta}(1_c \wedge i) = \bar{\bar{\zeta}(1_c \wedge i)}$; thus $\bar{\xi}(1_c \wedge i) = \bar{\bar{\zeta}(1_c \wedge i)}$

and

(iii) $\bar{\xi}(1_c \wedge i) = \bar{\bar{\zeta}(1_c \wedge i)}$, thus $\bar{\zeta}(1_c \wedge i) = \bar{\bar{\zeta}(1_c \wedge i)}$.

We put

$$(2.12) \quad \zeta_0 = \zeta \bar{i}_1 \in \{S^{2r+k}, C \wedge C\} \quad \text{and} \quad \xi_0 = \bar{\bar{\zeta}} \in \{C \wedge C, S^{2r+k}\}.$$

Then it follows from (ii) and (iii) of (2.11) that

$$(2.13) \quad (1_c \wedge i) \zeta_0 = S^{r+k} i, \quad \bar{\xi}_0(1_c \wedge i) = \bar{\bar{\xi}}(1_c \wedge i) = S^r \bar{\bar{\pi}}.$$

We consider the Puppe's exact sequence associated with cofibration

$$(2.14) \quad C \wedge S^r \xrightarrow{1_c \wedge i} C \wedge C \xrightarrow{1_c \wedge \bar{\bar{\pi}}} C \wedge S^{r+k}.$$
Lemma 2.5. The groups \( \{S_i, C \cap C\} \) and \( \{C \cap C, S_i\} \) are isomorphic to the corresponding groups in the following table:

| \( \{S^{sr}, C \cap C\} \) | \( Z \) | \( i \wedge i \) |
| \( \{S^{sr+k}, C \cap C\} \) | \( Z + Z + (i \wedge i)(G_k/\eta_0) \) | \( i \wedge i, \zeta_0 \) |
| \( \{S^{sr+2k}, C \cap C\} \) | \( Z + \) finite group |
| \( \{S_i, C \cap C\} \) | finite group \( j \approx 2r, 2r + h \) and \( 2r + 2k \) |
| \( \{C \cap C, S^{sr+2k}\} \) | \( Z \) | \( \pi \wedge \pi \) |
| \( \{C \cap C, S^{sr+k}\} \) | \( Z + Z + (\eta_0/\eta_0)(\pi \wedge \pi) \) | \( i \wedge \pi, \xi_0 \) |
| \( \{C \cap C, S^{sr}\} \) | \( Z + \) finite group |
| \( \{C \cap C, S^r\} \) | finite group for \( j \approx 2r, 2r + h \) and \( 2r + 2k \) |

where \( \zeta_0 \) and \( \xi_0 \) are elements satisfying \( (1_C \wedge \pi)\zeta_0 = S^{r+k} \) and \( \xi_0(1_C \wedge i) = S^{r+2k} \).

From the Puppe's exact sequence associated with (2.14) and Lemma 2.5 we can see easily the following lemma:

Lemma 2.6.

(i) \( \{C \cap C, C \cap S^{sr+k}\} = \{1_C \wedge \pi\} + \{i \wedge i, \pi\} + \{(S^{sr+k})\xi_0\} + i(G_k/\eta_0)(\pi \wedge \pi) \)
\[ \approx Z + Z + \) finite group, \]

(ii) \( \{C \cap S^r, C \cap C\} = \{1_C \wedge i\} + \{i \wedge i\} + \{\xi_0(S^r)\} + i(G_k/\eta_0)\pi \)
\[ \approx Z + Z + \) finite group. \]

Lemma 2.7. Let \( \xi \in \{C \cap C, \overline{N}_\alpha\} \) be an element satisfying (2.11)

(i) Any element \( \xi' \in \{C \cap C, \overline{N}_\alpha\} \) satisfies (2.11) if and only if
\[ \xi' = \xi + i_0 \omega(1_C \wedge \pi) \]
for some \( \omega \in \{S^{r+k}C, S^rC\} \).

(ii) For any element \( \xi \in G_k/\eta_0 \), put \( \xi'_0 = \xi_0 + \xi(\pi \wedge \pi) \) where \( \xi_0 = p \xi \). Then there exists \( \xi' \in \{C \cap C, \overline{N}_\alpha\} \) such that satisfying (2.11) and (2.12).

Proof. (i) Assume that \( \xi \) and \( \xi' \) satisfy (2.11). Since \( (1_C \wedge i)^*(\xi' - \xi) = 0 \), there exists \( r \in \{S^{r+k}C, \overline{N}_\alpha\} \) such that \( (1_C \wedge \pi)^*r = \xi' - \xi \). From (ii) of (2.11), \( (1_C \wedge \pi)^*(\pi_0 r) = 0 \). On the other hand the homomorphism \( (1_C \wedge \pi)^*: \{S^{r+k}C, S^{r+k}C\} \rightarrow \{C \cap C, S^{r+k}C\} \) is a monomorphism. Thus \( \pi_0 r = 0 \). Therefore \( r \) is contained in the image of \( \pi_0: \{S^{r+k}C, S^rC\} \rightarrow \{S^{r+k}C, \overline{N}_\alpha\} \).

Conversely, if \( \xi \) satisfies (ii) and (iii) of (2.11), then so dose \( \xi' \). Put \( \zeta' = \zeta - (1_C \wedge \pi) \omega \pi_0 \), then \( \xi' \) is a homotopy inverse of \( \xi' \).
(ii) The element \( \xi' = \xi + \overline{t}_0(1_C \wedge \xi')(1_C \wedge \pi) \) is the required element.

We consider the ordinary homology group. Let \( \zeta \) be an element of \( \{ \overline{N}_a, C \wedge C \} \) satisfying (2.11) and \( \xi \) be a homotopy inverse of \( \zeta \). Let

\[
\begin{pmatrix}
\hat{e}_r \wedge \hat{e}_r & \hat{e}_r \wedge \hat{s}_r \\
\hat{e}_r \wedge \hat{e}_r + k & \hat{e}_r + k \wedge \hat{e}_r \\
\hat{e}_r + k \wedge \hat{e}_r + k
\end{pmatrix}
= \begin{pmatrix}
\hat{e}_r \wedge \hat{s}_r & \hat{e}_r \wedge \hat{e}_r + k \\
\hat{e}_r \wedge \hat{e}_r + k & \hat{e}_r + k \wedge \hat{e}_r + k \\
\hat{e}_r + k \wedge \hat{e}_r + k
\end{pmatrix},
\]

be generators of the groups \( H_*(C \wedge C) \) and \( H_*(\overline{N}_a) \) respectively, where \( \hat{e}_i \wedge \hat{e}_j \) and \( \hat{e}_i \wedge \hat{s}_j \) is a generator of \( (i + j') \)-dim. group resp.

Using (2.11), for the ordinary homology map \( \xi_\# \) and \( \zeta_\# \) induced by \( \xi \) and \( \zeta \) resp., we obtain that

\[
\xi_\# \begin{pmatrix}
\hat{e}_r \wedge \hat{e}_r & \hat{e}_r \wedge \hat{s}_r \\
\hat{e}_r \wedge \hat{e}_r + k & \hat{e}_r + k \wedge \hat{e}_r \\
\hat{e}_r + k \wedge \hat{e}_r + k
\end{pmatrix} = \begin{pmatrix}
\hat{e}_r \wedge \hat{s}_r & \hat{e}_r \wedge \hat{e}_r + k \\
\hat{e}_r \wedge \hat{e}_r + k & \hat{e}_r + k \wedge \hat{e}_r + k \\
\hat{e}_r + k \wedge \hat{e}_r + k
\end{pmatrix},
\]

and

\[
\zeta_\# \begin{pmatrix}
\hat{e}_r \wedge \hat{s}_r & \hat{e}_r \wedge \hat{e}_r + k \\
\hat{e}_r \wedge \hat{e}_r + k & \hat{e}_r + k \wedge \hat{e}_r + k \\
\hat{e}_r + k \wedge \hat{e}_r + k
\end{pmatrix} = \begin{pmatrix}
\hat{e}_r \wedge \hat{s}_r & \hat{e}_r \wedge \hat{e}_r + k \\
\hat{e}_r \wedge \hat{e}_r + k & \hat{e}_r + k \wedge \hat{e}_r + k \\
\hat{e}_r + k \wedge \hat{e}_r + k
\end{pmatrix},
\]

for some integer \( n \).

Using an element \( \xi_\# = \rho \xi \) satisfying (2.13), we can put

\[
(*) \left( \frac{1}{t} \cdot \frac{1}{n} \right) T = \left( \frac{1}{t} \cdot \frac{1}{n} \right) \xi = \left( \frac{1}{t} \cdot \frac{1}{n} \right) \xi_\# \mod \left( \frac{1}{t} \cdot \frac{1}{n} \right) \xi_\#(C \wedge C),
\]

for some integers \( a, b \) and \( c \) by Lemma 2.6, where \( T = T(C, C) \). The homology maps induced by \( 1_C \wedge \pi T \), \( 1_C \wedge \pi, \overline{t} \wedge \pi \) and \( (S^{r+k})_\xi \) can be expressed as follows:

\[
(1_C \wedge \pi)_T T_\# \begin{pmatrix}
\hat{e}_r \wedge \hat{e}_r & \hat{e}_r \wedge \hat{s}_r \\
\hat{e}_r \wedge \hat{e}_r + k & \hat{e}_r + k \wedge \hat{e}_r \\
\hat{e}_r + k \wedge \hat{e}_r + k
\end{pmatrix} = \begin{pmatrix}
\hat{e}_r \wedge \hat{s}_r & \hat{e}_r \wedge \hat{e}_r + k \\
\hat{e}_r \wedge \hat{e}_r + k & \hat{e}_r + k \wedge \hat{e}_r + k \\
\hat{e}_r + k \wedge \hat{e}_r + k
\end{pmatrix},
\]

\[
(1_C \wedge \pi)_\# \begin{pmatrix}
\hat{e}_r \wedge \hat{e}_r & \hat{e}_r \wedge \hat{s}_r \\
\hat{e}_r \wedge \hat{e}_r + k & \hat{e}_r + k \wedge \hat{e}_r \\
\hat{e}_r + k \wedge \hat{e}_r + k
\end{pmatrix} = \begin{pmatrix}
\hat{e}_r \wedge \hat{s}_r & \hat{e}_r \wedge \hat{e}_r + k \\
\hat{e}_r \wedge \hat{e}_r + k & \hat{e}_r + k \wedge \hat{e}_r + k \\
\hat{e}_r + k \wedge \hat{e}_r + k
\end{pmatrix},
\]

\[
(i \overline{t} \wedge \pi)_\# \begin{pmatrix}
\hat{e}_r \wedge \hat{e}_r & \hat{e}_r \wedge \hat{s}_r \\
\hat{e}_r \wedge \hat{e}_r + k & \hat{e}_r + k \wedge \hat{e}_r \\
\hat{e}_r + k \wedge \hat{e}_r + k
\end{pmatrix} = \begin{pmatrix}
\hat{e}_r \wedge \hat{s}_r & \hat{e}_r \wedge \hat{e}_r + k \\
\hat{e}_r \wedge \hat{e}_r + k & \hat{e}_r + k \wedge \hat{e}_r + k \\
\hat{e}_r + k \wedge \hat{e}_r + k
\end{pmatrix},
\]

\[
(S^{r+k})_\xi T_\# = \begin{pmatrix}
\hat{e}_r \wedge \hat{e}_r & \hat{e}_r \wedge \hat{s}_r \\
\hat{e}_r \wedge \hat{e}_r + k & \hat{e}_r + k \wedge \hat{e}_r \\
\hat{e}_r + k \wedge \hat{e}_r + k
\end{pmatrix},
\]

\[
(i \overline{t} \wedge \pi)_\# T_\# = \begin{pmatrix}
\hat{e}_r \wedge \hat{s}_r & \hat{e}_r \wedge \hat{e}_r + k \\
\hat{e}_r \wedge \hat{e}_r + k & \hat{e}_r + k \wedge \hat{e}_r + k \\
\hat{e}_r + k \wedge \hat{e}_r + k
\end{pmatrix},
\]

\[
(i \overline{t} \wedge \pi)_{\#} = 0 \text{ for any } g \in G_k \wedge \pi.
\]

From (*) we obtain the identity
of homology maps. Applying this to \(e_{r+k}/e_{r+h}, e_r/e_{r+k} \) and \(e_{r+k}/e_r\), we have

\[
\begin{align*}
(-1)^{r+k}e_{r+k}/e_{r+h} &= ae_{r+k}/e_{r+h}, \\
0 &= ae_r/e_{r+k} + be_r/e_{r+h} - ce_r/e_{r+k}, \\
(-1)^{r(r+k)}e_r/e_{r+k} &= ce_r/e_{r+k}.
\end{align*}
\]

This is, \(a = (-1)^{r+k}, \ c = (-1)^{r(r+k)} \) and \(b = -((-1)^{r+k} - (-1)^{r(r+k)})/t\) and

\[
(1_c/\pi)T = (-1)^{k+r}(1_c/\pi) - n'(i\bar{i}/\pi) + (-1)^{r(r+k)}(S^{r+k})\xi_0
\]

mod. \(i(G_{\text{h/k}})(\pi/\pi)\), where \(n' = (-1)^{r+k} - (-1)^{r(r+k)}/t\).

Here we can put

\[
(1_c/\pi)T = (-1)^{k+r}(1_c/\pi) - n'(i\bar{i}/\pi) + (-1)^{r(r+k)}(S^{r+k})\xi_0 + iG(\pi/\pi)
\]

for some \(g \in G_{\text{h/k}}\). If \(g \neq 0\), put \(\xi'_0 = \xi_0 + (-1)^{r(r+k)}g(\pi/\pi)\), then \(\xi'_0\) satisfies (2.13) and the equality

\[
(1_c/\pi)T = (-1)^{k+r}(1_c/\pi) - n'(i\bar{i}/\pi) + (-1)^{r(r+k)}(S^{r+k})\xi'_0
\]

hold.

From (ii) of Lemma 2.7, there exists \(\xi' \in \{C \cap C, \ N_\sigma\}\) such that satisfy (2.11), (2.12) and induce the same homology maps as \(\xi\).

Let \(\xi' \in \{N_\sigma, C \cap C\}\) be the homotopy inverse of \(\xi'\) and \(\xi'_0 = \xi'_1\). Then \(\xi'\) and \(\xi'_0\) induce the same homology map as \(\xi\) \ and \(\xi_0\). Making use of \(\xi'\), by a similar calculation we see that

\[
T(1_c/\pi) = (-1)^{r}(1_c/\pi) - n''(\bar{i}/i/\pi) + (-1)^{r(r+k)}\xi'_0(S^{r+k}) + (i/\pi)\xi
\]

for some \(g \in G_{k/\pi}\), where \(n'' = (-1)^{r} - (-1)^{r(r+k)}/t\).

Hence we get the Lemma:

**Lemma 2.8.** There exists \(\xi \in \{C \cap C, \ N_\sigma\}\) and its inverse \(\xi \in \{N_\sigma, C \cap C\}\) which satisfy (2.11) and the following relations :

(i) \(\xi = \xi_0\) and \(\xi'_0 = \xi'_1\).

Hence we get the Lemma:

**Lemma 2.8.** There exists \(\xi \in \{C \cap C, \ N_\sigma\}\) and its inverse \(\xi \in \{N_\sigma, C \cap C\}\) which satisfy (2.11) and the following relations :

\[
\begin{align*}
(1_c/\pi)T &= (-1)^{r+k}(1_c/\pi) - n'(i\bar{i}/\pi) + (-1)^{r(r+k)}(S^{r+k})\xi_0, \\
T(1_c/\pi) &= (-1)^{r+k}(1_c/\pi) - n''(\bar{i}/i/\pi) + (-1)^{r(r+k)}\xi'_0(S^{r+k}) + (i/\pi)\xi
\end{align*}
\]

for some \(g \in G_{k/\pi}\), where \(n' = (-1)^{r+k} - (-1)^{r(r+k)}/t\), \(n'' = (-1)^{r} - (-1)^{r(r+k)}/t\), \(\xi_0 = \beta\xi\) and \(\xi'_0 = \xi'_1\).

Now we consider the element \(a \in \pi_{r+k-1}(S^r)\) satisfying

\[
(2.15) \quad 1_c/\alpha = 0 \quad \text{and} \quad \bar{j}(S^k\alpha) = 0
\]

where \(t\alpha = 0\) for an integer \(t\).

Then the cell structure of \(\overline{N}_\sigma\) can be interpreted as follows:

\[
\overline{N}_\sigma = S^rC_\sigma \cap S^{r+k}C_\sigma.
\]
Thus there exists a map \( \pi_0^{-1} : S^{r+k}C \to \tilde{N}_a \) such that

\[ \pi_0^{-1}(S^{r+k}) = \tilde{t}_1 \] and \( \tilde{t}_0 \pi_0^{-1} = 1_{S^{r+k}C} \)

i.e., \( \pi_0^{-1} \) is the inclusion.

Making use of (2.15) we have the following commutative diagram associated with (2.14) and (2.1) in which all rows and all columns are exact:

\[
\begin{array}{cccc}
\ldots & \ldots & 0 & \\
\downarrow & \downarrow & \downarrow & \\
0 & \{S^{2r+2k}, S^{r}C\} & \{S^{2r+2k}, C/\Delta C\} & \{S^{2r+2k}, S^{r+k}C\} & \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \{S^{r+k}C, S^{r}C\} & \{1_{(C/\Delta C)}\} & \{S^{r+k}C, C/\Delta C\} & \{1_{(C/\Delta C)}\} & \{S^{r+k}C, S^{r+k}C\} & \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \{S^{2r+k}, S^{r}C\} & \{S^{2r+k}, C/\Delta C\} & \{S^{2r+k}, S^{r+k}C\} & \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & \\
\end{array}
\]

From (2.11), (2.12), and (2.16) we have

\[ (S^{r+k})^*(\xi_{\pi_0^{-1}}) = \zeta_0 \] and \( (1_{(C/\Delta C)})^*(\xi_{\pi_0^{-1}}) = 1_{S^{r+k}C} \).

From (2.2) and the commutativity of above diagram, we have

\[ (S^{r+k})^*(\Delta/\Delta) = \tilde{t}_1/\tilde{t}_1. \]

**Proposition 2.9.** Let \( k \) be an odd integer. Assume that \( a \in \pi_{r+k-1}(S^r) \) satisfies (2.15). Then there exists an element \( \gamma \in \{S^{r+k}C, C/\Delta C\} \) such that

(i) \( (1_{(C/\Delta C)})T^r = (-1)^{r+k}1_{S^{r+k}C} \)

(ii) \( (1_{(C/\Delta C)})T^r = 1_{S^{r+k}C} \)

and

(iii) \( T(1_{(C/\Delta C)} + (-1)^{r+k}(1_{(C/\Delta C)}) = (-1)^{k(r+k)}\varphi(S^{r+k})[(S^{r+k}) + (i/\Delta)] \varphi \)

where \( T = T(C, C) \) and some \( \varphi \in G_{k/\Delta} \).

**Proof.** From Lemma 2.8, we have

\[ T(1_{(C/\Delta C)} + (-1)^{r+k}(1_{(C/\Delta C)}) = (-1)^{k(r+k)}\varphi(S^{r+k})[(S^{r+k}) + (i/\Delta)] \varphi \]

where \( \varphi_0 = (-1)^{r+k}\zeta_0 + (-1)^{k+1}(k+1)n_0(i/\Delta), n_0 = (1 + (-1)^{k+1})/t. \)

We consider an element \( \gamma = (-1)^{r+k}\zeta_0^{-1} + (-1)^{k+1}(k+1)n_0(i/\Delta) \)

of \( \{S^{r+k}C, C/\Delta C\} \). Then, from (2.17) and (2.18), we obtain that \( (S^{r+k})^*\gamma = \gamma_0. \)
Thus we have (iii).

Using (2.17) and \((1_c \wedge \pi)(1_c \wedge i) = 0\), it follows that

\[
(2.20) \quad (1_c \wedge \pi)^r = (-1)^{r+k}1_{s^{r+k}C}.
\]

From (i) of Lemma 2.8, (2.20) and (2.3), we obtain

\[
(1_c \wedge \pi)T\pi = 1_{s^{r+k}C}.
\]

**Proposition 2.10.** Let \(k\) be an even integer. Assume that \(\alpha \in \pi_{r+k-1}(S^r)\) satisfies (2.15) and \(t = 2\). Then there exists an element \(r \in \{s^{r+k}C, C \wedge C\}\) such that

\[
\begin{align*}
(1) & \quad (1_c \wedge \pi \wedge i)^r = (-1)^{r+k}1_{s^{r+k}C}, \\
(2) & \quad (1_c \wedge \pi)T\pi = 1_{s^{r+k}C}
\end{align*}
\]

and

\[
(3) \quad T(1_c \wedge \pi) + (-1)^r(1_c \wedge \pi) = \tau(S^{r+k})\pi + (-1)^r(1_c \wedge \pi) + (1_c \wedge \pi)g\pi
\]

where \(T = T(C, C)\) and some \(g \in G_k, \pi\).

**Proof.** From (ii) of Lemma 2.8 and (2.2), we have

\[
(2.19') \quad T(1_c \wedge \pi) = (-1)^{r+k}(1_c \wedge \pi) + \gamma_0(S^r\pi) - (-1)^r(1_c \wedge \pi) + (1_c \wedge \pi)g\pi
\]

where \(\gamma_0 = (-1)^r(1 - n_3)(1_c \wedge \pi) + (-1)^{r+k}\zeta_0, n_3 = (1 + (-1)^{r+1})/t\).

We consider an element

\[
T = (-1)^{r+k}\zeta_0^{-1} + (-1)^r(1 - n_3)(1_c \wedge \pi)
\]

of \(\{s^{r+k}C, C \wedge C\}\).

By a similar calculation as in Proposition 2.9 we have the results.

Next we consider the element \(\alpha \in \pi_{r+k-1}(S^r)\) satisfying

\[
(2.21) \quad 1_c \wedge \alpha = (S^r\alpha)\alpha'(S^{r+k-1}\pi) \quad \text{and} \quad \gamma_0 = 0
\]

for some non trivial element \(\alpha' \in \pi_{2r+2k-1}(S^{2r})\) and the integer \(\ell\) such that \(\ell = 0\) (c.f., Lemma 2.3).

We put

\[
Q = S^{2r} \cup_{\alpha'} \bigcup_{\alpha} S^{2r+2k}
\]

and denote by

\[
(2.23) \quad i' : S^{2r} \to Q \quad \text{and} \quad \pi' : Q \to S^{2r+2k}
\]

the canonical inclusion and the map collapsing \(S^{2r}\) to a point resp. Then from (2.9) we have following cofibrations

\[
(2.24) \quad S^{2r} \xrightarrow{i'} Q \xrightarrow{\pi'} S^{2r+2k},
\]
Making use of (2.21) we have the following commutative diagram associated with (2.14) and (2.25) in which all rows and all columns are exact:

\[
\begin{array}{ccccccc}
0 & \rightarrow & \{Q, C \wedge S^r\} & \rightarrow & \{Q, C \wedge C\} & \rightarrow & \{Q, C \wedge S^{r+k}\} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \{N_\alpha, C \wedge S^r\} & \rightarrow & \{N_\alpha, C \wedge C\} & \rightarrow & \{N_\alpha, C \wedge S^{r+k}\} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \{S^{2r+k}, C \wedge S^r\} & \rightarrow & \{S^{2r+k}, C \wedge C\} & \rightarrow & \{S^{2r+k}, C \wedge S^{r+k}\} & \rightarrow & 0
\end{array}
\]

From the Puppe's exact sequence associated with the cofibration (2.24) and Lemma 2.1, we obtain that

\[
\{Q, C \wedge S^{r+k}\} = Z + \delta(G_k/\eta \alpha) \pi'
\]

and \((S^{r+k} \tilde{t}) \pi'\) is a generator of free part.

On the other hand, the right column in the above diagram splits. Thus, from (2.10), Lemma 2.1 and the relation \(\pi' \pi_1 = (S^{r+k} \pi) \pi_0\), we obtain that

\[
\{N_\alpha, C \wedge S^{r+k}\} = Z + Z + \delta(G_k/\eta \alpha) \pi \pi_0
\]

and \((\tilde{t} \pi) \pi_0\) and \(\pi_0\) are generators of free parts.

From (2.3) and (2.10), we obtain

\[
i_\ast^*(S' \delta) \pi_0 = (S' \delta) \pi_0 \delta_1 = (S' \delta)(S^{r+k} \delta) = S' \tilde{t}.
\]

Thus it follows from commutativity of the above diagram that

\[
i_\ast^*(1_{C \wedge i}(S' \delta) \pi_0) = \tilde{t} \wedge i.
\]

**Proposition 2.11.** Let \( k \) be an odd integer. Assume that an element \( \alpha \in \pi_{r+k-1}(S^r) \) satisfies (2.21). Then there exists an element \( \beta \) of \( \{N_\alpha, C \wedge C\} \) such that

(i) \( (1_{C \wedge i}) \beta = (-1)^{r+k} \pi_0 \),

(ii) \( (1_{C \wedge i}) T \beta = \pi_0 \)

and

(iii) \( T(1_{C \wedge i}) + (-1)^{r+1}(1_{C \wedge i}) = (-1)^{k(r+k)} \beta_1(S' \pi) + (i \wedge i) \pi \)

where \( T = T(C, C) \) and some \( \delta \in G_k/\eta \alpha \) (c.f., Lemma 2.8).
Proof. Let $\zeta$ be a homotopy equivalence given in Lemma 2.8. Then we put
\[ \beta = (-1)^{(r+k)(r+k)n_0}[1 + i)(S^r \delta)\pi_0 + (r+k)\zeta j \in \{ N, C \wedge C \} \]
where $n_0 = (1 + (-1)^{r+k})/t$. Then using (2.19), (2.28), (2.12) and Lemma 2.8, the proof of this proposition is completely parallel to it of Proposition 2.9.

Proposition 2.12. Let $k$ be an even integer. Assume that an element $\alpha \in \pi_{r+k-1}(S^r)$ satisfies (2.21) and $t = 2$. Then there exists an element $\beta$ of $\{ N, C \wedge C \}$ such that

\begin{enumerate}
  \item (i) $(l \sigma \sigma)$ $\beta = (-1)^{r+k}\pi_0$
  \item (ii) $(l \sigma \sigma)T \beta = \pi_0$
  \item (iii) $T(1 \sigma \sigma)^{-1}(-1)^{r+k}\pi_0 + (-1)^{r+k}\zeta j$
\end{enumerate}

where $T = T(C, C)$ and some $g \in G_{n+k}$ (c.f., Lemma 2.8).

Proof. For $s^r$ in Lemma 2.8, we put
\[ \beta = (-1)^{r+k}(1 - n_0)[1 + i)(S^r \delta)\pi_0 + (r+k)\zeta j \]
where $n_0 = (1 + (-1)^{r+k})/t$. Then using (2.19), (2.28) and (2.12), similarly as in Proposition 2.10 we have the results.

§ 3. Existence of the admissible multiplication in $\tilde{h}^*(X; \alpha)$

Let $\mu$ be an associative multiplication in a reduced generalized cohomology theory $\{ \tilde{h}^*, \sigma \}$. In this paragraph we define a multiplication $\mu_\alpha$ in $\tilde{h}^*(X; \alpha)$ for some $\alpha \in \pi_{r+k-1}(S^r)$, and give a sufficient condition for $\mu_\alpha$ to be admissible.

Let $A$ and $B$ be finite $CW$-complexes with base points, for any element $\varphi$ of $\{ A, B \}$, we define a homomorphism
\[ \varphi^{*+} : \tilde{h}^*(X ; \alpha) \longrightarrow \tilde{h}^*(X \wedge A) \]
by the formula
\[ \varphi^{*+} = \sigma^{-m}(1 \wedge f)\sigma^{*+} \]
where $f : S^m A \longrightarrow S^m B$ is a map representing $\varphi$. The definition of $\varphi^{*+}$ does not depend on the choice of $f$.

Making use of an element $\varphi \in \{ S^r \wedge C, C \wedge C \}$, we define a map
\[ \mu_\alpha : \tilde{h}^{i+j}(X ; \alpha) \otimes \tilde{h}^i(Y ; \alpha) \longrightarrow \tilde{h}^{i+j}(X \wedge Y ; \alpha) \]
as the composition
\[ \mu_\alpha = (-1)^{(r+k)\varphi - (r+k)(1 \wedge T) \wedge 1) \mu : \tilde{h}^{i+j}(X ; \alpha) \otimes \tilde{h}^i(Y ; \alpha) = \tilde{h}^{i+j}(X \wedge C) \otimes \tilde{h}^{i+j}(Y \wedge C) \]
\[ \longrightarrow \tilde{h}^{i+j}(X \wedge Y \wedge C) \]
\[ \longrightarrow \tilde{h}^{i+j}(X \wedge Y \wedge C) \]
On the admissible multiplication in $\alpha$-coefficient cohomology theories

\[ \rightarrow \tilde{h}^{i+j+2r+2k}(X \wedge Y \wedge C \wedge S^{r+k}) \]
\[ \rightarrow \tilde{h}^{i+j+r+k}(X \wedge Y \wedge C) = \tilde{h}^{i+j}(X \wedge Y ; \alpha) \]

where $T' = T(Y, C)$.

Obviously $\mu_\alpha$ is linear and natural with respect to both variable.

**Proposition 3.1.** Assume that $r \in \{S^{r+k}C, C \wedge C\}$ satisfies

\[ (-1)^{r+k}(1_C \wedge \pi)T = 1_{S^{r+k}C} = (1_C \wedge \pi)T' \]

where $T = T(C, C)$. Then the map $\mu_\alpha$ of (3.1) is a multiplication satisfying (A1) and (A3).

**Proof.** To prove (A1), putting $T' = T(Y, C), T = T(C, C)$.

By definition of $\rho_\alpha$ and $\mu_\alpha$, we have

\[ \mu_\alpha(\rho_\alpha \otimes 1) = \sigma^{-(r+k)}(1_X \wedge Y \wedge Y) \sigma^2(1_X \wedge T' \wedge 1_C) \mu((1_X \wedge \pi) \sigma^{r+k} \otimes 1_Y \wedge C) \]
\[ = \sigma^{-(r+k)}(1_X \wedge Y \wedge Y) \sigma^2(1_X \wedge T' \wedge 1_C) \sigma^{r+k} \mu \]
\[ = \sigma^{-(r+k)}((1_X \wedge \pi)T) \sigma^2 \sigma^{r+k} \mu \]
\[ = \mu = \mu_\alpha \text{ by (3.2).} \]

Similarly, using the relation $(1_C \wedge \pi)T = (-1)^{r+k}1_{S^{r+k}C}$, we obtain that

\[ \rho_\alpha(1 \otimes \rho_\alpha) = \mu_\alpha \]

Then it follows that $\rho_\alpha(1)$ is the bilateral unit of $\mu_\alpha$ (see [2]).

The compatibility with suspension isomorphism $\sigma_\alpha$ and (A3) are verified directly from the definition of $\mu_\alpha$, $\sigma_\alpha$ and the associativity of $\mu$.

**Proposition 3.2.** If there exists an element $r \in \{S^{r+k}C, C \wedge C\}$ satisfying the relation

\[ (-1)^{r+k}(1_C \wedge \iota) = (-1)^{h(r+k)}(S^{r+k}\iota)(S^r \pi) + (-1)^{r(r+k)}(1 \wedge \iota)(S^r \pi) \]

for some $\iota \in \{C, S^r\}$, then map $\mu_\alpha$ of (3.1) satisfies (A3) with associated cohomology operation

\[ x_\alpha = (-1)^{i(r+k)}\chi^{*r} \sigma^r : \tilde{h}^{i}(\iota) \rightarrow \tilde{h}^{i-k}(\iota ; \alpha) \]

where $T = T(C, C)$.

**Proof.** We put $T' = T(Y, C)$. On $\tilde{h}^{i}(X ; \alpha) \otimes \tilde{h}^{j}(Y ; \alpha)$, we have

\[ \mu_\alpha(\rho_\alpha \otimes 1) = (-1)^{i+j+k} \mu_\alpha(1 \otimes \rho_\alpha) \]
\[ = (-1)^{i(r+k)} \sigma^{-r}(1_X \wedge Y \wedge 1_C \wedge \iota)^* \sigma(1_X \wedge Y \wedge T)^* (1_X \wedge T' \wedge 1_C)^* \mu \]
\[ + (-1)^{i(r+k)+r+k} \sigma^{-r}(1_X \wedge Y \wedge 1_C \wedge \iota)^* \sigma(1_X \wedge T' \wedge 1_C)^* \mu \]
\[ = (-1)^{i(r+k)} \sigma^{-r}(1_X \wedge Y \wedge 1_C \wedge \iota)^* \sigma(1_X \wedge T' \wedge 1_C)^* \mu \]
\[ + (-1)^{i(r+k)} \sigma^{-r}((-1)^{i(r+k)}(S^{r+k}\iota)(S^r \pi))^{*r} (1_X \wedge T' \wedge 1_C)^* \mu \]
\[ = (-1)^{i(r+k)} \sigma^{-r}((-1)^{i(r+k)}(1 \wedge \iota)(S^r \pi))^{*r} (1_X \wedge T' \wedge 1_C)^* \mu \]
\[\begin{align*}
&= (-1)^{i(r+k)}(1 \wedge \alpha)^* \sigma^h(1 \wedge \beta)^* \sigma^{-(r+k)}(1 \wedge \beta)^* (1 \wedge T'/1C)^* \mu \\
&\quad + (-1)^{i(r+k)+ir(r+k)} \chi^{*r} \sigma^r(i \wedge i)^* (1 \wedge T'/1C)^* \mu \\
&= \partial_i \mu - (-1)^{i(r+k)+ir(r+k)} \chi^{*r} \sigma^r(i \wedge i)^* (1 \wedge T'/1C)^* \mu.
\end{align*}\]

On the other hand we have
\[\mu(\partial_0 \otimes \partial_0) = (-1)^{i+j}(r+k)+ir(r+k) x^{*r} \sigma^r(i \wedge i)^* (1 \wedge T'/1C)^* \mu.\]

Here we put
\[X_\alpha = (-1)^{i(r+k)} \chi^{*r} \sigma^r : h^k( ; \alpha),\]
then we have
\[\partial_i \mu = \mu(\partial_0 \otimes 1) + (-1)^{i+k} \mu(1 \otimes \partial_0) - (-1)^{i+k} X_\alpha \mu(\partial_0 \otimes \partial_0).\]

Clearly \(X_\alpha\) is a cohomology operation and the relation
\[X_\alpha \mu = \mu(\partial_0 \otimes 1) = (-1)^{i+k} \mu(1 \otimes X_\alpha)\]
holds.

As a consequence of Proposition 2.9, 2.10, 3.1 and Proposition 3.2 we obtain the following theorem:

**Theorem 3.3.** Assume that an element \(\alpha \in \pi_{r+k-1}(S^r)\) satisfies (2.15) and \(t = 2\) if \(k\) is even. Then there exists an admissible multiplication \(\mu_\alpha\) in \(h^k( ; \alpha)\).

Now we consider an element \(\alpha \in \pi_{r+k-1}(S^r)\) satisfying (2.21), i.e., \(1C\wedge \alpha = (S^r)\alpha(S^{r+k-1})\) and \(\tilde{1}\alpha = 0\) for some \(\alpha' \in \pi_{2r+k-1}(S^{2r})\) and the integer \(t\) such that \(\tilde{t}\alpha = 0\).

Using the notation (2.9), cofibration
\[S^{2r} \xrightarrow{i_0} N_{\alpha} \xrightarrow{\pi_0} S^{r+k}C\]
yields, for any finite CW-complex \(W\) with a base point, a cofibration
\[(3.4) \quad W \wedge S^{2r} \xrightarrow{1 \wedge i_0} W \wedge N_{\alpha} \xrightarrow{1 \wedge \pi_0} W \wedge S^{r+k}C.\]

If \((\alpha'\sigma)^* = 0\) in \(\tilde{h}^n\), then the \(\tilde{h}^n\)-cohomology exact sequence associated to the above cofibration (3.4) breaks into the following exact sequence
\[(3.5) \quad 0 \rightarrow \tilde{h}^n(W \wedge S^{r+k}C) \xrightarrow{(1 \wedge \pi_0)^*} \tilde{h}^n(W \wedge N_{\alpha}) \xrightarrow{(1 \wedge i_0)^*} \tilde{h}^n(W \wedge S^{2r}) \rightarrow 0.\]

By (3.5) for \(W = S^n\) and \(n = 2r\), it follows that

**Lemma 3.4.** (i) If \((\alpha'\sigma)^* = 0\) in \(\tilde{h}^n\), then there exists \(\varphi_0 \in \tilde{h}^{2r}(N_{\alpha})\) satisfying
\[(3.6) \quad i_0^* \varphi_0 = \sigma^{2r} 1,\]

(ii) If \(\alpha'^{2r} = 0\) in \(\tilde{h}^n\), then there exists \(\varphi_0 \in \tilde{h}^{2r}(N_{\alpha})\) satisfying
\[(3.6') \quad i_0^* \varphi_0 = \sigma^{2r} 1 \quad \text{and} \quad i_1^* \varphi_0 = 0.\]
Proof. See Lemma 4.2 of [2].

Making use of \( \sigma \) of Lemma 3.4, hence at least under the assumption of \( (\alpha', \pi)^{**} = 0 \), we define a homomorphism

\[
\varphi_w : \tilde{h}^n(W \wedge N_\alpha) \longrightarrow \tilde{h}^n(W \wedge S^{r+k}C)
\]

by the formula

\[
(3.7) \quad \varphi_w(x) = (1_{W \wedge \pi_\alpha})^{* - 1}(x - \mu(\sigma^{-2r}(1_{W \wedge \pi_\alpha})^* x \otimes \varphi_0))
\]

for \( x \in h^n(W \wedge N_\alpha) \).

Since \( x - \mu(\sigma^{-2r}(1_{W \wedge \pi_\alpha})^* x \otimes \varphi_0) \) is in the kernel of \( (1_{W \wedge \pi_\alpha})^* \) and \( (1_{W \wedge \pi_\alpha})^* \) is monomorphomorphic, the map \( \varphi_w \) of (3.7) is well-defined homomorphism.

Lemma 3.5. (i) \( \varphi_w \) is a left inverse of \( (1_{W \wedge \pi_\alpha})^* \), i.e., \( \varphi_w(1_{W \wedge \pi_\alpha}) = \text{an identity map} \); hence the sequence of (3.5) splits:

\[
\tilde{h}^n(W \wedge N_\alpha) = \tilde{h}^n(W \wedge S^{r+k}C) \oplus \tilde{h}^n(W \wedge S^{2r}).
\]

(ii) \( \varphi_w \) is natural in the sense that

\[
(f \wedge S^{r+k}1_{C})^* \varphi_w = \varphi_w'(f \wedge 1_{N_\alpha})^*,
\]

where \( f : W' \longrightarrow W \).

(iii) \( \varphi_w \) is compatible with the suspension in the sense that

\[
(1_{W \wedge T})^* \sigma \varphi_w = \varphi_{SW}(1_{W \wedge T''})^* \sigma,
\]

where \( T' = T(S', S^{r+k}C) \) and \( T'' = T(S', N_\alpha) \).

(iv) The relation

\[
\mu(Y \otimes \varphi_w(x)) = \varphi_{Y \wedge W}(Y \otimes x)
\]

holds, where \( x \in \tilde{h}^n(W \wedge N_\alpha) \) and \( y \in \tilde{h}^m(Y) \).

(v) If \( \varphi_0 \) satisfies (3.6)' then the relation

\[
(1_{W \wedge S^{r+k}})^* \varphi_w = (1_{W \wedge \pi})^*
\]

holds for the inclusions \( i : S^r \longrightarrow C \) and \( i_1 : S^{2r} \longrightarrow N_\alpha \).

(vi) Assume that \( \mu \) is commutative. Then the relations hold:

(a) \( \mu(z \otimes \varphi_0) = T'^* \mu(\varphi_0 \otimes z) \),

where \( z \in \tilde{h}^i(Z) \) and \( T' = T(Z, N_\alpha) \).

(b) \( (1_{W \wedge T''})^* \mu(\varphi_w(x) \otimes z) = \varphi_{W \wedge Z}(1_{W \wedge T'})^* \mu(x \otimes z) \),

where \( x \in h^n(W \wedge N_\alpha) \), \( z \in h^i(Z) \), \( T' = T(Z, N_\alpha) \) and \( T'' = T(Z, S^{r+k}C) \).

Proof. By a similar calculation to Lemma 4.3, 4.4, 4.5 and Lemma 4.6 of [2], we have the results.
Making use of the homomorphism $\varphi_w$ defined by (3.7) and the element $\beta$ of \{\(N_\alpha, C\wedge C\)}, we define a map

\[(3.8) \quad \mu_\alpha : \tilde{h}^i(X; \alpha) \otimes \tilde{h}^j(Y; \alpha) \to \tilde{h}^{i+j}(X \wedge Y; \alpha)\]

as the composition

\[
\begin{align*}
\mu_\alpha &= (-1)^{i(r+k)}(-1)^{r+h} \varphi_{X \wedge Y}(1_X \wedge 1_Y)^* (1_X \wedge T' \wedge 1_C)^* \mu : \\
\tilde{h}^i(X; \alpha) \otimes \tilde{h}^j(Y; \alpha) &= \tilde{h}^{i+r+k}(X \wedge C) \otimes \tilde{h}^{j+r+k}(Y \wedge C) \\
&\to \tilde{h}^{i+j+r+2k}(X \wedge C \wedge Y \wedge C) \\
&\to \tilde{h}^{i+j+r+2k}(X \wedge Y \wedge N_\alpha) \\
&\to \tilde{h}^{i+j+r+2k}(X \wedge Y \wedge S^{r+k}C) \\
&\to \tilde{h}^{i+j+r+k}(X \wedge Y \wedge C) = \tilde{h}^{i+j}(X \wedge Y; \alpha),
\end{align*}
\]

where $T' = T(Y, C)$.

$\mu_\alpha$ is defined only if $(\alpha', \pi')^* = 0$.

The definition of $\mu_\alpha$ depends on the choices of $\varphi_w$ and $\beta$. However we fix during the subsequent proofs of properties of an admissible multiplication.

Clearly $\mu_\alpha$ is linear and natural with respect to both variables.

**Proposition 3.6.** If an element $\beta \in \{N_\alpha, C \wedge C\}$ satisfies

\[(3.9) \quad (1_C \wedge \pi)T \beta = \pi_0 = (-1)^{r+k}(1_C \wedge \pi)\beta,
\]

where $T = T(C, C)$, $\pi_0 : N_\alpha \to S^{r+k}C$ is the map collapsing $S^{2r}$, then the map $\mu_\alpha$ of (3.8) is a multiplication satisfying $(A_i)$.

**Proof.** (See Theorem 4.7 of [2]) From (i) of Lemma 3.5 and (3.9) we can see $(A_i)$ directly. From (ii) and (iii) of Lemma 3.5 it follows that the map $\mu_\alpha$ of (3.8) is compatible with the suspension isomorphism $\sigma_\alpha$.

**Proposition 3.8.** If $\mu$ is a commutative multiplication, then for any $\beta \in \{N_\alpha, C \wedge C\}$ the map $\mu_\alpha$ of (3.8) satisfies $(A_i)$.

**Proof.** (See Theorem 4.10 of [2]) It follows from (iv) and (vi) of Lemma 3.5 that $\mu_\alpha$ satisfy $(A_i)$.

**Proposition 3.8.** Let $\alpha'' = 0$ in $\tilde{h}^*$. Assume that $\beta \in \{N_\alpha, C \wedge C\}$ satisfies

\[(3.10) \quad T(1_C \wedge \iota) + (-1)^{r+k}(1_C \wedge \iota) = (-1)^{k(r+k)} \beta_1(S^{r+k}) + (-1)^{r+k}(\iota \wedge \iota)(S^r \chi)
\]

for some $\chi \in \{C, S^r\}$, where $T = T(C, C)$. Then the map $\mu_\alpha$ of (3.8) satisfies $(A_i)$ with associated cohomology operation

\[
\chi_\alpha = (-1)^{i(r+k)}\chi^* \sigma_\alpha : \tilde{h}^i(\ ) \to \tilde{h}^{i-k}(\ ); \alpha).
\]

**Proof.** (See Theorem 4.9 of [2]) It follows from (v) of Lemma 3.5 that satisfy $(A_i)$. 
As a consequence of Proposition 2.11, 2.12, 3.6, 3.7 and Proposition 3.8 we have

**Theorem 3.9.** Let \( \mu \) be a commutative, associative multiplication in a reduced generalized cohomology theory \( \tilde{h}^* \). Assume that \( \alpha \in \pi_{r+h-1}(S^r) \) satisfies (2.21) and the order of \( \alpha \) is two if \( k \) is an even integer. If \( \alpha^{**} = 0 \) in \( \tilde{h}^* \), then the admissible multiplication \( \mu_\alpha \) exist in \( \tilde{h}^* (\; ; \alpha) \).

**References**