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Extended Borel Transformations and Elementary Solutions on Polydisks of Linear Partial Differential Operators with Holomorphic Coefficients

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Introduction.

Borel transformation is originally defined by

$$\mathscr{B}[\varphi(\zeta)](z)$$

$$=\frac{1}{(2\pi\sqrt{-1})^n}\int_{|\zeta_1|=\varepsilon_1,\cdots,|\zeta_n|=\varepsilon_n}\varphi(\zeta)\frac{1}{\zeta_1\cdots\zeta_n}\exp.\left(\frac{z_1}{\zeta_1}+\cdots+\frac{z_n}{\zeta_n}\right)d\zeta_1\cdots d\zeta_n,$$

where φ is holomorphic on $\{\zeta\,|\,|\zeta_i|\leq\varepsilon_i,\ i=1,\cdots,n\}$, and it has following properties.

$$\mathscr{B}[a\varphi+b\psi]=a\mathscr{B}[\varphi]+b\mathscr{B}[\psi],\quad \mathscr{B}[\varphi\cdot\psi]=\mathscr{B}[\varphi]\sharp\mathscr{B}[\psi],$$

$$\mathscr{B}[\varphi\otimes\psi]=\mathscr{B}[\varphi]\otimes\mathscr{B}[\psi],$$

$$\frac{\partial}{\partial z_i}\mathscr{B}[\varphi(\zeta)](z)=\mathscr{B}[\zeta_i^{-1}\varphi(\zeta)](z),$$ *if* $\zeta_i^{-1}\varphi(\zeta)$ *is holomorphic at* $\{0\}$, *the origin of* C^n,

where $(f\sharp g)(z)$ is given by $\partial^n/\partial z_1\cdots\partial z_n\int_0^{z_1}\cdots\int_0^{z_n}f(z-t)g(t)dt_1\cdots dt_n$, and $(\varphi\otimes\psi)(\zeta_1,\cdots,\zeta_{k+s})$ is given by $\varphi(\zeta_1,\cdots,\zeta_k)\psi(\zeta_{k+1},\cdots,\zeta_{k+s})$.

But, although $\varphi(\zeta)$ is meromorphic at $\{0\}$, $\mathscr{B}[\varphi]$ is defined by using a path γ, given by $\{\zeta\,|\,|\zeta_i|=\varepsilon_i,\ i=1,\cdots,n\}$ such that φ is holomorphic on γ (the existence of such path is shown in [1], cf. Lemma 1 of this paper). Then this definition does not depend on the choice of γ, if γ defines non vanishing element of $H_{\{0\},c,n}(U-Y\cup\{z|\ z_1\cdots z_n=0\},\mathbf{R})$, the n-dimensional compact carrier local homology group of $U-Y\cup\{z|\ z_1\cdots z_n=0\}$ at $\{0\}$ with the coefficients in $\mathbf{R}$. Here, U is a neighborhood of $\{0\}$ and φ has poles on Y ([1]). In [1], we show this extended Borel transformation also has same properties as above. Moreover, since we get

$$\sum_{n=o}^{\infty}\frac{t^n}{n!}\log(x)^{\#n}=\frac{e^{-\gamma t}}{(1+t)}x^t,\ \gamma \textit{ is Euler's constant,}$$

to define

$$\mathscr{B}[\log\zeta](z)=\log z+\gamma,$$

we can extend Borel transformation for the functions which involve $\log z_i$, to have the above properties ([1]). Then, since we obtain

$$\mathscr{B}[\zeta^t](z)=\frac{1}{\Gamma(1+t)}z^t,\ t\neq \textit{negative integer},$$

$$\mathscr{B}[\zeta^{-n}\log\zeta](z)=(-1)^{n-1}(n-1)!z^{-n},\ n\geqq 1,$$

we may conside any meromorphic function or algebroid function can be expressed as a Borel transformation image ([2]). Hence, using this extended Borel trans formation, we can solve Cauchy problem with the meromorphic data or the problem of division for the category of meromorphic functions for the constant coefficients linear partial differential operators ([2]).

In this paper, we use extended Borel transformation for the construction of elementary solutions on polydisks of linear partial differential operators with holomorphic coefficients and show the following theorem (cf. [9], [11], [13]).

Theorem. *For* $\delta=(\delta_1,\cdots,\delta_n)$, $\delta_1>0,\cdots,\delta_n>0$, $n\geqq 2$, *we set*

$$\Gamma(\delta)=\{z\,|\,|z_1+kz_2|<\delta_1,\ |z_1-kz_2|<\delta_2,\ |z_3|<\delta_3,\cdots,\ |z_n|<\delta_n\},$$

$$\gamma(\delta)=\{z\,|\,|z_1+kz_2|=\delta_1,\ |z_1-kz_2|=\delta_2,\ |z_3|=\delta_3,\cdots,\ |z_n|=\delta_n\},$$

and assume $P(z,\partial/\partial z)$ *is given by*

$$P\left(z,\frac{\partial}{\partial z}\right)=\frac{\partial^m}{\partial z_1{}^m}+P_1\left(z,\frac{\partial}{\partial z_2},\cdots,\frac{\partial}{\partial z_n}\right)\frac{\partial^{m-1}}{\partial z_1{}^{m-1}}+\cdots+P_m\left(z,\frac{\partial}{\partial z_2},\cdots,\frac{\partial}{\partial z_n}\right),$$

$$P_i\left(z,\frac{\partial}{\partial z_2},\cdots,\frac{\partial}{\partial z_n}\right)=\sum_{j_2+\cdots+j_n\leqq i}a_{i,j_2}\cdots_{j_n}(z)\frac{\partial^{j_2+\cdots+j_n}}{\partial z_2{}^{j_2}\cdots\partial z_n{}^{j_2}},$$

$$a_{i,j_2,}\cdots_{j_n}(z)\ \textit{is holomorphic on}\ \overline{\Gamma(\delta)}.$$

Then there exists an analytic function $E(z,\zeta)$ *on* $\overline{\Gamma(\delta)}\times\overline{\Gamma(\varrho)}$ *such that to set*

$$Ef(z)=\int_{\gamma(\delta)}E(z,\zeta)f(\zeta)d\zeta,$$

Ef has the following properties.

(i) $Ef(z)$ *is holomorphic on* $\Gamma(\delta)$ *if* $f(z)$ *is holomorphic on* $\Gamma(\delta)$ *and continuous on* $\overline{\Gamma(\delta)}$,

(ii) $P\left(z,\dfrac{\partial}{\partial z}\right)Ef(z)=f(z)$, $z\varepsilon\Gamma(\delta)$, *if* $f(z)$ *is holomorphic on* $\overline{\Gamma(\delta)}$,

(iii) $$Ef(z)|_{z_1=0}=\frac{\partial}{\partial z_1}Ef(z)|_{z_1=0}=\cdots=\frac{\partial^{m-1}}{\partial z_1^{m-1}}Ef(z)|_{z_1=0}=0.$$

Similar result for the domain $\Gamma(\delta, \delta')$ given by

$$\Gamma(\delta, \delta')=\{z|\delta_1'<|z_1+kz_2|<\delta_1, \ \delta_2'<|z_1-kz_2|<\delta_2,$$
$$\delta_3'<|z_3|<\delta_3, \cdots, \delta_n'<|z_n|<\delta_n\},$$

where $\delta_1'+\delta_2'$ is sufficiently small, is also shown.

The outline of this paper is as follows: In §1, we give integral formulas for inverse Borel transformations of meromorphic functions and Borel transformations of the functions which involve $\log z_i$. The starting point of these formulas are the fact that

$$\mathscr{B}[\log(\zeta+\lambda)](z)=\gamma+\log(z)-\mathrm{Ei}\left(-\frac{z}{\lambda}\right), \ \mathrm{Ei}(-z)=\int_z^\infty e^{-t}\, t^{-1}dt.$$

Hence, if $f(z)$ has poles on Y and γ is a (representation of) an element of n-dimensional compact carrier local homology group $\mathrm{H}_{\{0\},c,n}(U-Y\cup\{z|z_1\cdots z_n=0\}, \mathbf{R})$ of $U-Y\cup\{z|z_1\cdots z_n=0\}$ at $\{0\}$ with coefficients in $\mathbf{R}$, of the form

$$\gamma=\gamma(\varepsilon)=\{z|\,|z_1|=\varepsilon_1, \cdots, \ |z_n|=\varepsilon_n\},$$

then to set $\mathrm{F}_m(X)=\sum_{k=0}^m k!X^k$, we may define inverse Borel transformation of f (with respect to γ) by

$$\mathscr{B}_\gamma^{-1}[f(z_1, \cdots, z_n)](\zeta_1, \cdots, \zeta_n)$$
$$=\lim_{m_1\to\infty, \cdots, m_n\to\infty}\frac{1}{(2\pi\sqrt{-1})^n}\int_\gamma f(z_1, \cdots, z_n)\prod_{i=1}^n[\zeta_i^{-1}\exp\left(-\frac{z_i}{\zeta_i}\right)\log\zeta_i$$
$$+z_i^{-1}\,\mathrm{F}_{m_i}\left(\frac{\zeta_i}{z_i}\right)]dz_1\cdots dz_n.$$

On the other hand, if $g(\zeta)$ is given by

$$g(\zeta_1, \cdots, \zeta_n)=g_0(\zeta_1, \cdots, \zeta_n)$$
$$+\sum_{1\leq i_1<\cdots<i_k\leq n} g_{i_1, \cdots, i_k}(\zeta_1, \cdots, \zeta_n)\log\zeta_{i_1}\cdots\log\zeta_{i_k},$$

where each $g_0(\zeta)$ or $g_{i_1,\cdots,i_k}(\zeta)$ has singularities only on Y, then we may define its Borel transformation (with respect to $\gamma=\gamma(\varepsilon)$) by

$$\mathscr{B}_\gamma[g(\zeta_1, \cdots, \zeta_n)](z_1, \cdots, z_n)$$
$$=\lim_{\varepsilon_1\to 0, \cdots, \varepsilon_n\to 0, \varepsilon_1<|\lambda_1|\,\varepsilon_n<|\lambda_n|}\frac{1}{(2\pi\sqrt{-1})^n}\int_{\gamma(\varepsilon)}\{g_0(\zeta)$$
$$+\sum_{1\leq i_1<\cdots<i_k\leq n} g_{i_1, \cdots, i_k}(\zeta)\log(\zeta_{i_1}+\lambda_{i_1})\cdots\log(\zeta_{i_k}+\lambda_{i_k})\}$$

$$\frac{1}{\zeta_1\cdots\zeta_n}\exp\left(\frac{z_1}{\zeta_1}+\cdots+\frac{z_n}{\zeta_n}\right)d\zeta_1\cdots d\zeta_n,\quad Re.\,\lambda_1 z_1>0,\ \cdots,\ Re.\,\lambda_n z_n>0.$$

Then, for same $\gamma\in H_{\{0\},c,n}(U-Y\cup\{z\,|\,z_1\cdots z_n=0\},\ \mathbf{R})$, we have

$$\mathscr{B}_\gamma[\mathscr{B}_\gamma^{-1}[f(z)](\zeta)](z)=f(z).$$

In § 1, we also calculate

$$\mathscr{B}_{\gamma_1}^{-1}\left[\frac{1}{z_1^2-k^2z_2^2}\right](\zeta_1,\ \zeta_2)=\frac{\zeta_1\log\zeta_1}{k^2\zeta_2^2-\zeta_1^2},\quad \gamma_1\subset\{\zeta\,|\,|\zeta_1|>|k|\,|\zeta_2|\},$$

$$\mathscr{B}_{\gamma_2}^{-1}\left[\frac{1}{z_1^2-k^2z_2^2}\right](\zeta_1,\ \zeta_2)=\frac{k\zeta_2\log\zeta_2}{k^2\zeta_2^2-\zeta_1^2},\quad \gamma_2\subset\{\zeta\,|\,|\zeta_1|<|k|\,|\zeta_2|\}$$

In § 2, we construct the elementary solution of a constant coefficients operator on $\Gamma(\delta)$ as follows: Since we know

$$f(z)=\frac{-1}{2k(2\pi\sqrt{-1})^n}\int_{\gamma(\delta)}\frac{f(\zeta)}{\{(z_1-\zeta_1)^2-k^2(z_3-\zeta_3)^2\}(z_2-\zeta_2)\cdots(z_n-\zeta_n)}d\zeta_1\cdots d\zeta_n,$$

$z\in\Gamma(\delta)$, *$f(\zeta)$ is holomorphic on* $\overline{\Gamma(\delta)}$,

we set

$$B_\gamma(z)=\mathscr{B}_\gamma\left[\frac{1}{P(\zeta^{-1})}\cdot\frac{k\zeta_2\log\zeta_2\log\zeta_3\cdots\log\zeta_n}{(k^2\zeta_2^2-\zeta_1^2)\zeta_3\cdots\zeta_n}\right](z),$$

$$P(\zeta)=\zeta_1^m+P_1(\zeta_2,\cdots,\zeta_n)\,\zeta_1^{m-1}+\cdots+P_m(\zeta_2,\cdots,\zeta_n),\quad \text{deg. } P_k\leqq k.$$

Here the representation of γ is contained in

$$D_\gamma=\{\zeta\,|\,|\zeta_1P_1(\zeta_2^{-1},\cdots,\zeta_n^{-1})|+\cdots+|\zeta_1^mP_m(\zeta_2^{-1},\cdots,\zeta_n^{-1})|<1\}.$$

Then $B_\gamma(z)$ should be an elementary solution of $P(\partial/\partial z)$ on $\mathscr{A}(\overline{\Gamma(\delta)})$, the space of all holomorphic functions on $\overline{\Gamma(\delta)}$ with the normally convergence topology. We treat detailed properties of $B_\gamma(z)$ in §2 and show

$$B_\gamma(z)\,|_{z_1=0}=\frac{\partial}{\partial z_1}B_\gamma(z)\,|_{z_1=0}=\cdots=\frac{\partial^{m-1}}{\partial z_1^{m-1}}B_\gamma(z)\,|_{z_1=0}=0.$$

It is also shown that $B_\gamma(z)$ is defined on (some covering of) $\mathbf{C}^n-\{z\,|\,z_1=\pm kz_2\}\cup\{z\,|\,z_3\cdots z_n=0\}$ and holomorphic on D_γ. Therefore, to set

$$P^{-1}f(z)=\frac{-1}{2k(2\pi\sqrt{-1})^n}\int_{\gamma(\delta)}B_\gamma(z-\zeta)f(\zeta)d\zeta,$$

where the branch of $B_\gamma(z)$ is taken to coincide original $B_\gamma(z)$ on $D_\gamma\cap\Gamma(\delta)$, $P^{-1}f$ is the elementary solution of P on $\mathscr{A}(\overline{\Gamma(\delta)})$ with the 0 -Cauchy data.

We note that for the usual Cauchy kernel $1/z_1\cdots z_n$, we get $\mathscr{B}^{-1}[1/z_1\cdots z_n](\zeta)=\log\zeta_1\cdots\log\zeta_n/\zeta_1\cdots\zeta_n$ (since $H_{\{0\},c,n}(U-\{z\,|\,z_1\cdots z_n=0\},\ \mathbf{R})\cong\mathbf{R}$, we need not to denote γ) and to set $B'(z)=[1/P(\zeta^{-1}).\ \log\zeta_1\cdots\log\zeta_n/\zeta_1\cdots\zeta_n]$, we have

$$B'(z)|_{z_1=0}=\frac{\partial}{\partial z_1}B'(z)|_{z_1=0}=\cdots=\frac{\partial^{m-2}}{\partial z_1{}^{m-2}}B'(z)|_{z_1=0}=0,$$

but $\partial^{m-1}/\partial z_1{}^{m-1}B'(z)|_{z_1=0}$ can not be equal to 0.

In § 3, we treat the holomorphic coefficients operators. For a holomorphic coefficients operator $P(z, \partial/\partial z)$, we set $P(\partial/\partial z)=P(0, \partial/\partial z)$, and set

$$P\left(z, \frac{\partial}{\partial z}\right)=P\left(\frac{\partial}{\partial z}\right)+Q\left(z, \frac{\partial}{\partial z}\right).$$

Then, on $\mathscr{A}(\overline{\Gamma(\delta)})$, the operator QP^{-1} is a compact operator and it is shown

$$||QP^{-1}||_{\overline{\Gamma\delta}}=O((\delta_1+\delta_2)||\delta||),\quad ||\delta||=\sqrt{\sum_{i=1}^{n}\delta_i{}^2}.$$

Therefore, if $||\delta'||$ is small, we may set

$$Ef(z)=(P^{-1}(I+QP^{-1})^{-1}f)(z),\quad on\ \Gamma(\delta'),$$

where $(I+QP^{-1})^{-1}$ is defined by the Neuman series. Then, by virtue that we consider only in the category of holomorphic functions and Fredholm theory, we can show that $(I+QP^{-1})^{-1}$ can be continued on $\Gamma(\delta)$. Therefore we have the existence of E on $\Gamma(\delta)$. The existence of E on $\Gamma(\delta, \delta')$ is also shown by the same method.

We note that the existence of E on $\Gamma(\delta)$ also gives following integral expression of the holomorphic solution of $P(z, \partial/\partial z)u=0$ on $\Gamma(\delta)$ (cf. [6], [3']).

Theorem. *If $u(z)\in\mathscr{A}(\overline{\Gamma(\delta)})$ and $P(z, \partial/\partial z)u=0$, then $u(z)$ is written*

$$u(z_1, \cdots, z_n)$$
$$=\sum_{i=0}^{m-1} z_1{}^i u_i(z_2, \cdots, z_n)-\int_{\gamma(\delta)} E(z, \zeta)\left[P(\zeta, \frac{\partial}{\partial\zeta})\{\sum_{i=0}^{m-1}\zeta_1{}^i u_i(\zeta_2, \cdots, \zeta_n)\}\right]d\zeta,$$

$$u_1, \cdots, u_m\in\mathscr{A}(\overline{\Gamma(\delta)}\cap\{z|z_1=0\}).$$

These may by related to some analytic problems of analytic equations (cf. [5], [10]). We also note that using other integral kernels for the expression of holomorphic functions, we may calculate elementary solutions of holomorphic coefficients linear partial differential equations in the category of holomorphic functions on other types of domains (cf. [3], [4], [8], [12], [14], [15].

§ 1 Borel transformations and inverse Borel transformations of germs of meromorphic functions

1. ***Lemma 1.*** *Let $f(z)$ be a meromorphic functions on U, a neighborhood of the origin of $\mathbf{C}^n$, then there exists a neighborhood V of the origin of $\mathbf{C}^n$ which is contained in U and a real $(2n-1)$-dimensional (real analytic) subvariety Γ of V*

such that $V-\Gamma$ *has finite number of connected components and if* D *is a connected component of* $V-\Gamma$, *then* $\overline{D}\ni\{0\}$ *and* $f(z)$ *is expressed as a Laurent series on* D *for each* D.

Proof. By Weierstrass' preparation theorem, we need only to show the lemma for the functions of the form

$$f(z)=\frac{1}{(z_n{}^m+g_1(z_1,\cdots,z_{n-1})z_n{}^{m-1}+\cdots+g_m(z_1,\cdots,z_{n-1}))},$$

where each g_i is a meromorphic function and to show the lemma, we use induction about n and m.

Since the lemma is true if $n=1$, we assume the lemma is true for $(n-1)$-variables meromorphic functions. Then, if $m=1$, we get

$$f(z)=z_n{}^{-1}(1+\frac{1}{z_n}g_1(z_1,\cdots,z_{n-1}))^{-1},\quad |z_n|>|g_1(z_1\ z_n,\cdots,{}_1)|,$$

$$f(z)=\frac{1}{g_1(z_1,\cdots,z_{n-1})}\Big(1+\frac{z_n}{g_1(z_1,\cdots,z_{n-1})}\Big),$$

$$|z_n<|g_1(z_1,\cdots,z_{n-1})|.$$

Hence to denote V' and Γ' the neighborhood and subvariety determined by this lemma for g_1 (in $\mathbf{C}^{n-1}$) and take ε to satisfy $V'\times B_\varepsilon\subset U$, where $B_\varepsilon=\{z_n|\ |z_n|<\varepsilon\}$, we have the lemma for $f(z)$ by setting

$$V=V'\times B_\varepsilon,\quad \Gamma=\Gamma'\times B_\varepsilon\cup\{z|\ |z_n|=|g_1(z_1,\cdots,z_{n-1})|\}.$$

If the lemma is true for $m\leqq k-1$, then to set

$$z_n{}^k+g_s(z_1,\cdots,z_{n-1})\ z_n{}^{k-s}+\cdots+g_k(z_1,\cdots,z_{n-1})$$

$$=z_n{}^k+g_s(z_1,\cdots,z_{n-1})\,h(z_1,\cdots,z_n),\quad g_s(z_1,\cdots,z_{n-1})\neq 0,$$

the lemma is true for $(g_sh)^{-1}$ by inductive assumption (we may assume the existence of g_s because if $g_1=\cdots=g_k=0$, then the lemma is true). Then, since

$$f(z)=z_n{}^{-k}(1+z_n{}^{-k}g)_sh)^{-1},\quad |z_n|^k>|(g_sh)(z_1,\cdots,z_n)|,$$

$$f(z)=(g_sh)^{-1}\ (1+(g_sh^{-1}z_n{}^k))^{-1},\quad |z_n|^k<|(g_sh)\,z_1,\cdots,z_n)|,$$

if V' and Γ' are the neighborhood and subvariety determined by the lemma for g_sh and $V'\subset U$, then we have the lemma for $f(z)$ by setting

$$V=V',\quad \Gamma=\Gamma'\cup\{z|\ |z_n|^k=|(g_sh)(z_1,\cdots,z_n)|\}.$$

Corollary 1. *Let* Y *be an analytic subvariety of* U, *a neighborhood of* $\{0\}$, *the origin, of* $\mathbf{C}^n$, *and* $\{0\}\in Y$, *then there exists a neighborhood* V *of* $\{0\}$ *in* U *and a real* $(2n-1)$ *-dimensional real analytic subvariety* $\Gamma=\Gamma_Y$ *which contains* $V\cap Y$ *and*

$$(1)\qquad V-\Gamma=\bigcup_{i=1}^{m}D_i,\quad \{0\}\in\bigcap_{i=1}^{m}\overline{D}_i,\quad D_i\cap D_j=\phi,\quad i\neq j,$$

where each D_i *is a relative complete Reinhaldt domain, such that there exists a meromorphic function* f *on* U *with poles on* Y *whose Laurent expansion is distinct on each* D_i. *Moreover, to denote* $\mathrm{H}_{\{0\},c,n}(U-Y\cup z|\ z_1\cdots z_n=0\},\ \mathbf{R})_*$, *the subgroup of* $\mathrm{H}_{\{0\},c,n}(U-Y\cup\{z|\ z_1\cdots z_n=0\},\ \mathbf{R})$, *the compact carrier local n-dimensional homology group of* $U-Y\cup\{z|\ z_1\cdots z_n=0\}$ *at* $\{0\}$ *with coefficients in* $\mathbf{R}$, *generated by the class of those cycles* γ *such that*

$$(2)\qquad \gamma=\{z|\ |z_1|=\varepsilon_1,\cdots,|z_n|=\varepsilon_n\},$$

we have

$$(3)\qquad m=dim.\ \mathrm{H}_{\{0\},c,n}(U-Y\cup\{z|\ z_1\cdots z_n=0\},\ \mathbf{R})_*$$
$$\leqq dim.\ \mathrm{H}_{\{0\},c,n}(U-Y\cup\{z|\ z_1\cdots z_n=0\},\ \mathbf{R}).$$

Proof. By the uniqueness of Laurent expansion on a relative complete Reinhaldt domain and lemma 1, we have the first assertion. Hence we get $m\leqq dim.$ $\mathrm{H}_{\{0\},c,n}(U-Y\cup\{z|\ z_1\cdots z_n0\},\ \mathbf{R})_*$. On the other hand, if the class of a cycle γ with the form of (2) does not vanish in $\mathrm{H}_{\{0\},c,n}(U-Y\cup\{z|z_1\cdots z_n=0\},\ \mathbf{R})$, then by using Cauchy kernel and the chain $\gamma(\varepsilon,\ \varepsilon')$ given by

$$\gamma(\varepsilon,\ \varepsilon')=\sum_{\{i_1,\cdots,i_k\}\cup\{j_1,\cdots,j_{n-k}\}=\{1,\cdots,n\}}(-1)^{n-k}\{z|\ |z_{i_1}|=\varepsilon_{i_1},$$
$$\cdots|z_{i_k}|=\varepsilon_{i_k},\ |z_{j_1}|=\varepsilon_{j_1}',\cdots,|z_{j_{n-k}}|=\varepsilon_{j_{n-k}}'\},$$

where $\varepsilon_i>\varepsilon_i'$, $i=1,\cdots,n$ *and* $\gamma(\varepsilon)=\{z|\ |z_i|=\varepsilon_i,\ i=1,\cdots,n\}$ and $\gamma(\varepsilon')=\{z|\ |z_i|=\varepsilon_i'\}$ are both belonges in the same class as γ, γ defines a Laurent expansion of any meromorphic function f with poles on Y. Hence γ is contained in $\overline{D}_i$ for some i and if γ and γ' both contained in same $\overline{D}_i$, then γ and γ' must contained in the same class in $\mathrm{H}_{\{0\},c,n}(U-Y\cup\{z|\ z_1\cdots z_n=0\},\ \mathbf{R})$. Therefore we have $m\geqq dim.$ $\mathrm{H}_{\{0\},c,n}(U-Y\cup\{z|\ z_1\cdots z_n=0\},\ \mathbf{R})_*$.

Corollary 2. *Using same notations as corollary 1, if* f *is holomorphic on* $U-Y$ *(may not be meromorphic on* U*), then* f *is expanssed as a Laurent series on each* D_i.

2. *Lemma 2.* *If* $Re.\,\lambda\zeta$ *is positive, then we have*

$$(4)\qquad \lim_{\varepsilon\to 0}\mathscr{B}[\log(z+\lambda)](\zeta)=\log\zeta+\gamma,\quad \gamma \textit{ is Euler's constant},$$

and this convergence is uniform on $\{\zeta|\ |\zeta|>\varepsilon,\ \theta\leqq arg.\ \zeta\lambda\leqq\theta'\}$ *for any* $\varepsilon>0$,

$\theta > -\pi/2$, $\theta' < \pi/2$ *if arg.* $\lambda =$ *constant.*

Proof. Since $\log(z+\lambda) = \log \lambda + \log(1+z/\lambda)$, we get

$$\mathscr{B}[\log(z+\lambda)](\zeta) = \log \lambda - \sum_{n=1}^{\infty} \frac{1}{n!n}\left(-\frac{\zeta}{\lambda}\right)^n.$$

On the other hand, since we know

$$\mathrm{Ei}\left(-\frac{\zeta}{\lambda}\right) = \gamma + \log\left(\frac{\zeta}{\lambda}\right) + \sum_{n=1}^{\infty} \frac{1}{n!n}\left(-\frac{\zeta}{\lambda}\right)^n,$$

where $\mathrm{Ei}(-\zeta) = \int_{\zeta}^{\infty} e^{-t}\, t^{-1}\, dt$ ([7]), we have

$$\mathscr{B}[\log(z+\lambda)](\zeta) = \gamma + \log \zeta - \mathrm{Ei}\left(-\frac{\zeta}{\lambda}\right).$$

Since $\mathrm{Ei}(-\zeta/\lambda)$ tends to 0 uniformly on $\{\zeta \mid |\zeta| > \varepsilon,\ \theta \leqq \arg.\ \zeta\lambda \leqq \theta'\}$ for any $\varepsilon > 0$, $\theta > -\pi/2$, $\theta' < \pi/2$ if *arg.* $\lambda =$constant and λ tends to 0, we get the lemma.

Corollary, *If* $Re.\, \lambda\zeta$ *is positive, then we have*

$$\lim_{\lambda \to 0} [z^{-n} \log(z+\lambda)](\zeta) = \frac{(-1)^{n-1}}{(n-1)!}\zeta^{-n}, \tag{5}$$

and this convergence is uniform on $\{\zeta \mid |\zeta| > \varepsilon,\ \theta \leqq \arg.\ \zeta\lambda \leqq \theta'\}$ *for any* $\varepsilon > 0$, $\theta > -\pi/2$, $\theta' < \pi/2$ *if arg.* $\lambda =$ *constant.*

Note. In [1] and [2], we show that to difine

$$\mathscr{B}[\log z](\zeta) = \log \zeta + \gamma, \quad \mathscr{B}[z^{-n} \log z](\zeta) = \frac{(-1)^{n-1}}{(n-1)!}\zeta^{-n},$$

Borel transformation can be extended to have same properties as usual Borel transformation for the functions which involve $\log z$. Hence (4) and (5) may be written

$$\lim_{\lambda \to 0} \mathscr{B}[\log(z+\lambda)](\zeta) = \mathscr{B}[\log z](\zeta), \quad if\ Re.\, \lambda\zeta > 0, \tag{4$'$}$$

$$\lim_{\lambda \to 0} \mathscr{B}[z^{-n} \log(z+\lambda)](\zeta) = \mathscr{B}[z^{-n} \log z](\zeta), \quad if\ Re.\, \lambda\zeta > 0. \tag{5$'$}$$

For simple, in the rest, we assume Y always containes the subvariety of $\mathbf{C}^n$ defined by $z_1 \cdots z_n = 0$. Hence $\mathrm{H}_{\{0\},c,n}(U-Y, \mathbf{R})_* \neq \{0\}$.

Definition. *Let* $\gamma_1, \cdots \gamma_m$ *be a basis of* $\mathrm{H}_{\{0\},c,n}(U-Y,\ \mathbf{R}_*$ *such that each* γ_i *is (represented by the chains of) the form of (2),* $\mathbf{F}_m(X)$ *is a polynomial of the form*

$$\mathbf{F}_m(X) = \sum_{k=0}^{m} k!X^k,$$

then, for a germ of analytic function with singularities on Y *at* $\{0\}$ *(that is, a germ at* $\{0\}$ *of a holomorphic function on* $U-Y$, *cf.* [3]), *denoted by* $f(z)$, *we define its inverse Borel transformation* $\mathscr{B}_{\gamma_i}{}^{-1}[f(z)](\zeta)$ *with respect to* γ_i, *by*

(6) $\mathscr{B}_{\gamma_i}{}^{-1}[f z_1, \cdots, z_n)](\zeta_1, \cdots, \zeta_n)$

$$= \lim_{m_1, \cdots, m_n \to \infty} \frac{1}{(2\pi\sqrt{-1})^n} \int_{\gamma_i} f(z_1, \cdots, z_n) \prod_{i=1}^{n} \left[\zeta_i^{-1} \exp\left(-\frac{z_i}{\zeta_i}\right) \log \zeta_i + z_i^{-1} \mathbf{F}_{m_i}\left(\frac{\zeta_i}{z_i}\right) \right] dz_1 \cdots dz_n.$$

By definition, we may set

(7) $$\mathscr{B}_{\gamma_i}{}^{-1}[f(z_1, \cdots, z_n)](\zeta_1, \cdots, \zeta_n)$$

$$= \varphi_0(\zeta_1, \cdots, \zeta_n) + \sum_{1 \leq i_1 < \cdots < i_k \leq n} \varphi_{i_1, \cdots, i_k}(\zeta_1, \cdots, \zeta_n) \log \zeta_{i_1} \cdots \log \zeta_{i_k},$$

where $\varphi_0(\zeta)$ involves only positive (may be equal to 0) powers of $\zeta_1, \cdots, \zeta_n$ and to divide $\{1, \cdots, n\} = \{i_1, \cdots, i_k\} \cup \{j_1, \cdots, j_{n-k}\}$, each $\varphi_{i_1, \cdots, i_k}(\zeta)$ involves only negative powers of $\zeta_{i_1}, \cdots, \zeta_{i_k}$ and only positive (may be equal to 0) powers of $\zeta_{j_1}, \cdots, \zeta_{j_{n-k}}$ as formal Laurent series.

We note that in the usual definition of Borel transformation, $\mathscr{B}^{-1}[f]$ should be φ_0, and it does not depend on the choice of γ.

We also set

(7)' $$\frac{1}{(2\pi\sqrt{-1})^n} \int_{\gamma_i} f(z_1, \cdots, z_n) \prod_{i=1}^{n} \left[\zeta_i^{-1} \exp\left(-\frac{z_i}{\zeta_i}\right) \log \zeta_i + z_i^{-1} \mathbf{F}_{m_i}\left(\frac{\zeta_i}{z_i}\right) \right] dz_1 \cdots dz_n$$

$$= \varphi_0{}^{m_1, \cdots, m_n}(\zeta) + \sum_{1 \leq i_1 < \cdots < i_k \leq n} \varphi_{i_1, \cdots, i_k}{}^{m_1, \cdots, m_n}(\zeta) \log \zeta_{i_1} \cdots \log \zeta_{i_k}.$$

Then, each $\varphi_0{}^{m_1, \cdots, m_n}(\zeta)$ or $\varphi_{i_1, \cdots, i_k}{}^{m_1, \cdots, m_n}(\zeta)$ converges on $\mathbf{C}^n$ or on $\mathbf{C}^n - \{\zeta \mid \zeta_{i_1} \cdots \zeta_{i_k} = 0\}$ and we have

(8) $$\varphi_0(\zeta) = \lim_{m_1, \cdots, m_n \to \infty} \varphi_0{}^{m_1, \cdots, m_n}(\zeta),$$

$$\varphi_{i_1, \cdots, i_k}(\zeta) = \lim_{m_1, \cdots, m_n \to \infty} \varphi_{i_1, \cdots, i_k}{}^{m_1, \cdots, m_n}(\zeta),$$

as formal Laurent serieses. For simple, we set

$$\mathscr{B}_{\gamma_i}{}^{-1}{}_{m_1, \cdots, m_n}[f(z)](\zeta)$$

$$= \varphi_0{}^{m_1, \cdots, m_n}(\zeta) + \sum_{1 \leq i_1 < \cdots < i_k \leq n} \varphi_{i_1, \cdots, i_k}{}^{m_1, \cdots, m_n}(\zeta) \log \zeta_{i_1} \cdots \log \zeta_{i_k}.$$

Definition. *We denote the representative of γ_i contained in the bowl*

$\{z| \, ||z|| <\delta\}$ *by* $\gamma_i<\delta>$. Then, *if* $g(\zeta)$ *is a (germ of) function at* $\{0\}$ *of the form*

$$g(\zeta)=g_0(\zeta)+\sum_{1\leq i_1<\cdots<i_k\leq n} g_{i_1,\cdots,i_k}(\zeta)\log\zeta_{i_1}\cdots\log\zeta_{i_k},$$

where each $g_0(\zeta)$ or $g_{i_1,\cdots,i_k}(\zeta)$ is analytic at $\{0\}$ with singularities on Y, we define its Borel transformation $\mathscr{B}_{\gamma_i}[g(\zeta)](z)$ with respect to γ_i by

(9) $$\mathscr{B}_{\gamma_i}[g(\zeta)](z)$$

$$=\lim_{\lambda_1 0,\cdots,\lambda_n\to 0,\ \delta<min.\ (|\lambda_1|,\cdots,|\lambda_n|)} \frac{1}{(2\pi\sqrt{-1})^n}\int_{\gamma_i<\delta>} g_0(\zeta)$$

$$+\sum_{1\leq i_1<\cdots<i_k\leq n} g_{i_1,\cdots,i_k}(\zeta)\log(\zeta_{i_1}+\lambda_{i_1})\cdots\log(\zeta_{i_n}+\lambda_{i_n})\}\frac{1}{\zeta_1\cdots\zeta_n}$$

$$\exp\left(\frac{z_1}{\zeta_1}+\cdots+\frac{z_n}{\zeta_n}\right)d\zeta_1\cdots d\zeta_n,\quad Re.\,\lambda_1 z_1>0,\cdots,Re.\,\lambda_n z_n>0.$$

By definition and lemma 2 (cf. [1], [2]), we have

Theorem 1. *We denote by* D_i *the (germ of) Reinhaldt domain which containes* γ_i, f *a (germ of) analytic function on* U *with singularities on* Y, *then*

(10) $$\lim_{m_1\to\infty,\cdots,m_n\to\infty}\mathscr{B}_{\gamma_i}[\mathscr{B}_{\gamma_i}{}^{-1}{}_{m_1,\cdots,m_n}[f(z)](\zeta)](z)=f(z).$$

Especially, if each $\varphi_0(\zeta)$ *or* $\varphi_{i_1,\cdots,i_k}(\zeta)$ *determines a (germ of) analytic function at* $\{0\}$ *(with singularities on* Y*), then*

(10)′ $$\mathscr{B}_{\gamma_i}[\mathscr{B}_{\gamma_i}{}^{-1}[f(z)](\zeta)](z)=f(z).$$

Note. Originally, (10) *or* (10)′ is shown (by using Laurent expansion) only on D_i. But, since so set

$$\mathscr{B}_{\gamma_i,\lambda_1,\cdots\lambda_n}[g(\zeta)](z)$$

$$=\frac{1}{(2\pi\sqrt{-1})^n}\int_{\gamma_i<\delta>}\{g_0(\zeta)+\sum g_{i_1,\cdots,i_k}(\zeta)\log(\zeta_{i_1}+\lambda_{i_1})\log(\zeta_{i_n}+\lambda_{i_n})\}$$

$$\frac{1}{\zeta_1\cdots\zeta_n}\exp\left(\frac{z_1}{\zeta_1}+\cdots+\frac{z_n}{\zeta_n}\right)d\zeta_1\cdots d\zeta_n,\quad \delta<min.\ (|\lambda_1|,\cdots,|\lambda_n|),$$

$\mathscr{B}_{\gamma_i,\lambda_1,\cdots,\lambda_n}[g]$ is holomorphic on $\mathbf{C}^n$ (cf. [1]), and (by the calculus of residue) we have

$$\mathscr{B}_{\gamma_i,\lambda_1,\cdots,\lambda_n}[g](z_0+h)$$

$$=\sum_{j_1\geq 0,\cdots,j_n\geq 0}\frac{1}{j_1!\cdots j_n!}\left(\frac{\partial^{j_1+\cdots+j_n}}{\partial z_1{}^{j_1}\cdots\partial z_n{}^{j_n}}\mathscr{B}_{\gamma_1,}{}^{\lambda_1,\cdots,\lambda_n}[g]|_{z=z_0}\right)h_1{}^{j_1}\cdots h_n{}^{j_n},$$

(10) or (10)′ is hold on any domain on which f is defined.

3. In this n°, we give some examples of inverse Borel transformations.

Later, example 2 and note will be used.

Example 1. Let Y be given by $\{z|\ z_1z_2=0\}\cup\{z|\ z_1=z_2\}$ *in* $\mathbf{C}^2$, then we have

$$dim.\ \mathrm{H}_{\{0\},c,2}(U-Y,\ \mathbf{R})_*=2,$$

and its generators γ_1 and γ_2 are given by the representatives

$$\gamma_1=\{z|\ |z_1|=\varepsilon_1,\ |z_2|=\varepsilon_2\},\ \gamma_2=\{z|\ |z_1|=\varepsilon_2,\ |z_2|=\varepsilon_1\},\ \varepsilon_1>\varepsilon_2.$$

The corresponding Reinhaldt domains D_1 and D_2 are

$$D_1=\{z|\ |z_1|>|z_2|\},\ D_2=\{z|\ |z_1|<|z_2|\}.$$

If $f(z)=1/(z_1-z_2)$, we have

$$f(z)=\sum_{n\geqq 0} z_1^{-(n+1)}z_2^{n},\ on\ D_1,\ f(z)=-\sum_{n\geqq 0} z_1^{n}z_2^{-(n+1)},\ on\ D_2.$$

Hence we get

$$\mathscr{B}_{\gamma_i}{}^{-1}\Big[\frac{1}{z_1-z_2}\Big](\zeta_1,\ \zeta_2)=\frac{\zeta_1\log\zeta_1}{\zeta_1+\zeta_2},\ \ \mathscr{B}_{\gamma_2}{}^{-1}\Big[\frac{1}{z_1-z_2}\Big](\zeta_1,\ \zeta_2)=-\frac{\zeta_2\log\zeta_2}{\zeta_1+\zeta_2}.$$

Example 2. If Y is given by $\{z|\ z_1z_2=0\}\cup\{z|\ z_1=\pm z_2\}$ in $\mathbf{C}_2$, then we also have

$$dim.\ \mathrm{H}_{\{0\},c,2}(U-Y,\ \mathbf{R})_*=2,$$

and its generators γ_1, γ_2 and their corresponding Reinhaldt domains D_1, D_2 are same as example 1.

If $f(z)=1/(z_1^2-z_2^2)$, then we have

$$f(z)=\sum_{n\geqq 0} z_1^{-2(n+1)}z_2^{2n},\ on\ D_1,$$

$$f(z)=-\sum_{n\geqq 0} z_1^{2n}z_2^{-2(n+1)},\ on\ D_2.$$

Hence we get

$$\mathscr{B}_{\gamma_1}{}^{-1}\Big[\frac{1}{z_1^2-z_2^2}\Big](\zeta_1,\ \zeta_2)=\frac{\zeta_1\log\zeta_1}{\zeta_2^2-\zeta_1^2},\tag{11}$$

$$\mathscr{B}_{\gamma_2}{}^{-1}\Big[\frac{1}{z_1^2-z_2^2}\Big](\zeta_1,\ \zeta_2)=\frac{\zeta_2\log\zeta_2}{\zeta_2^2-\zeta_1^2}.$$

Note. Similarly, if $Y=\{z|\ z_1z_2=0\}\cup\{z|\ z_1=\pm kz_2\}$ in $\mathbf{C}^2$, then $dim.\ \mathrm{H}_{\{0\},c,2}(U-Y,\ \mathbf{R})_*=2$ and its generators and their corresponding Reinhaldt domains D_1, D_2 are given by

$$\gamma_1=\{z||z_1|=\varepsilon_1,\ |z_2|=\varepsilon_2\},\ \gamma_2=\{z||z_1|=\varepsilon_2,|z_2|=\varepsilon_1\},\ \varepsilon_1>|k|\varepsilon_2,$$

$$D_1=\{z|\ |z_1>|k||z_2|\},\ D_2=\{z||z_1|<|k||z_2\}.$$

Then we get

$$(11)' \qquad \mathscr{B}_{\gamma_1}{}^{-1}\left[\frac{1}{z_1^2-k^2z_2^2}\right](\zeta_1,\ \zeta_2)=\frac{\zeta_1\log\zeta_1}{k^2\zeta_2^2-\zeta_1^2},$$

$$\mathscr{B}_{\gamma_2}{}^{-1}\left[\frac{1}{z_1^2-k^2z_2^2}\right](\zeta_1,\ \zeta_2)=\frac{k\zeta_2\log\zeta_1}{k^2\zeta_2^2-\zeta_1^2}.$$

In general, if $Y=\{z|\ z_1\cdots z_n=0\}\cup\{z|\ z_1=\pm kz_2\}$ in $\mathbf{C}^n$, then *dim.* $H_{\{0\},c,n}(U-Y,\ \mathbf{R})_*=2$ and its generators $\gamma_1,\ \gamma_2$ are given by

$$\gamma_1=\{z|\ |z_1|=\varepsilon_1,\ \ |z_2|=\varepsilon_2,\ \ |z_3|=\varepsilon_3,\cdots,\ |z_n|=\varepsilon_n\},$$

$$\gamma_2=\{z|\ |z_1|=\varepsilon_2,\ \ |z_2|=\varepsilon_1,\ \ |z_3|=\varepsilon_3,\cdots,\ |z_n|=\varepsilon_n\},$$

$\varepsilon_1>|k|\varepsilon_2,\ \varepsilon_3,\cdots,\varepsilon_n$ *are arbitrary.*

Then we get

$$(11)'' \qquad \mathscr{B}_{\gamma_1}{}^{-1}\left[\frac{1}{(z_1^2-k^2z_2^2)\,z_3\cdots z_n}\right](\zeta_1,\ \zeta_2,\ \zeta_3,\cdots,\zeta_n)$$

$$=\frac{\zeta_1\log\zeta_1\log\zeta_3\cdots\log\zeta_n}{(k^2\zeta_2^2-\zeta_1^2)\,\zeta_3\cdots\zeta_n},$$

$$\mathscr{B}_{\gamma_2}{}^{-1}\left[\frac{1}{(z_1^2-k^2z_2^2)\,z_3\cdots z_n}\right](\zeta_1,\ \zeta_2,\ \zeta_3,\cdots,\zeta_n)$$

$$=\frac{k\zeta_2\log\zeta_2\log\zeta_3\cdots\log\zeta_n}{(k^2\zeta_2^2-\zeta_1^2)\,\zeta_3\cdots\zeta_n}.$$

We note that, for example, if $f(z)$ is holomorphic on $\bar{D}$, where $D=D\ (\delta_1,\cdots,\delta_n)=\{z|\ |z_1+kz_2|<\delta_1,\ \ |z_1-kz_2|<\delta_2,\ \ |z_3|<\delta_3,\cdots,\ |z_n|<\delta_n\}$, then we have for $z\in D$,

$$f(z)=\frac{-1}{2k(2\pi\sqrt{-1})^n}\int_{|\zeta_1+k\zeta_2|=\delta_1,\ |\zeta_1-k\zeta_2|=\delta_2,\ |\zeta_3|=\delta_3,\cdots,\ |\zeta_n|=\delta_n}$$

$$\frac{f(\zeta)}{\{(z_1-\zeta_1)^2-k^2(z_2-\zeta_2)^2\}\ (z_3-\zeta_3)\cdots(z_n-\zeta_n)}d\zeta_1\cdots d\zeta_n$$

§2 Elementary solutions of constant coeficients operators

4. Since we know

$$P\left(\frac{\partial}{\partial z}\right)\mathscr{B}_{\gamma}[f(\zeta)](z)=\mathscr{B}_{\gamma}[P(\zeta^{-1})f(\zeta)](z),$$

if $P(\partial/\partial z)$ is a constant coefficients linear partial differential operator, we have

$$(12) \qquad P\left(\frac{\partial}{\partial z}\right)\mathscr{B}_{\gamma}[(P(\zeta^{-1}))^{-1}\ \mathscr{B}_{\gamma}{}^{-1}[f(z)](\zeta)](z)=f(z).$$

Here, to take Y to be the union of the polar varieties of $P(z^{-1})$ and $f(z)$ and $\{z|\ z_1\cdots z_n=0\}$, γ is a generator of $H_{\{0\},c,n}\ (U-Y,\ \mathbf{R})_*$ and we denote by D_γ, the Reinhaldt domain corresponds to γ.

***Lemma* 3.** *Let Y be a subvariety of $\mathbf{C}^n$ given by*

$$Y=\{z|\ P(z^{-1})=0\}\cup\{z|\ z_1\cdots z_n=0\}\cup\{z|\ z_1=\pm kz_2\},$$

and denote L the hypersurface of $\mathbf{C}^n$ given by $z_1=0$. Then, for any $z_0\in L\cap U$, there is a relative complete Reinhaldt domain D_γ in $U-\Gamma_Y$ such that $z_0\in\overline{D}_\gamma$ and $\overline{D}_\gamma\cap L$ contains some non-empty open set of L. Moreover, we can take the path γ corresponding to D_γ to satisfy if γ is given by $\{|z_1|=\varepsilon_1\}\times\{|z_2|=\varepsilon_2\}\times\cdots\times\{|z_n|=\varepsilon_n\}$, then

(13) $\qquad$ *γ homologous to γ' in $H_{\{0\},c,n}\ (U-Y,\ \mathbf{R})_*$, if*

$$\gamma'=\{|z_1|=\varepsilon_1'\}\times\{|z_2|=\varepsilon_2\}\times\cdots\times\{|z_n|=\varepsilon_n\},\ \text{where}\ \varepsilon_1'<\varepsilon_1,$$

and for this γ, we have

$$\mathscr{B}_\gamma^{-1}\left[\frac{1}{(z_1^2-k^2z_2^2)z_3\cdots z_n}\right](\zeta_1,\ \zeta_2,\ \zeta_3,\cdots,\zeta_n)\tag{14}$$

$$=\frac{k\zeta_2\log\zeta_2\log\zeta_3\cdots\log\zeta_n}{(k^2\zeta_2^2-\zeta_1^2)\,\zeta_3\cdots\zeta_n}.$$

***Proof*.** Since $\overline{U-\Gamma_Y}=\overline{U}$ and *dim.* $H_{\{0\},c,n}\ (U-Y)_*<\infty$, we have the first assertion. Then, since D_γ is open, we have the second assertion. If γ satisfies (13), then γ should be the form of γ_2 in the notation of the note and example 2 of n°3, we have (14).

Example. If $P(z)$ is given by $z_1^m+P_1(z_2,\cdots,z_n)z_1^{m-1}+\cdots+P_m(z_2,\cdots,z_n)$, then D_γ is the Reinhaldt domain containning of the domain D given by

$$D=\{z|\ |z_1P_1(z_2^{-1},\cdots,z_n^{-1})|+\cdots+|z_1^mP_m(z_2^{-1},\cdots,z_n^{-1})|<1\}.$$

Hence D_γ contains $L-\{(0,\ z_2,\cdots,z_n)|z_2\cdots z_n=0\}$, that is $D\cap L$ is open dense in L.

For simple, we denote by $G(\zeta)$, $G_\gamma(\zeta)$ or $G_{\gamma,k}(\zeta)$, the right hand side of (14).

***Lemma* 4.** *If $P(\partial/\partial z)$ is given by*

$$P\left(\frac{\partial}{\partial z}\right)=\frac{\partial^m}{\partial z_1^m}+P_1\left(\frac{\partial}{\partial z_2},\cdots,\frac{\partial}{\partial z_n}\right)\frac{\partial^{m-1}}{\partial z_1^{m-1}}+\cdots+P_m\left(\frac{\partial}{\partial z_2},\cdots,\frac{\partial}{\partial z_n}\right),$$

then we have

$$\frac{\partial^k}{\partial z_1^k}\mathscr{B}_\gamma\left[\frac{1}{(P(\zeta^{-1}))}G_\gamma(\zeta)\right](z)|_{z_1=0}=0,\quad 0\leq k\leq m-1,\tag{15}$$

where $(\partial^0/\partial z_1^0)f$ means f and γ is determined by lemma 3.

Proof. First we note that, since $G_\gamma(\zeta)$ does not involve $\log\zeta_1$, we have

$$\frac{\partial^k}{\partial z_1{}^k}\mathscr{B}_\gamma\Big[\frac{1}{(P(\zeta^{-1}))}G_\gamma(\zeta)\Big](z)|_{z_1=0}$$

$$=\lim_{\lambda_2\to 0,\cdots,\lambda_n\to 0\ \delta<min.(|\lambda_2|,\cdots,|\lambda_n|)}\frac{1}{(2\pi\sqrt{-1})^n}\int_{\gamma<\delta>}\frac{1}{(P(\zeta^{-1}))}\zeta_1{}^{-k}$$

$$\frac{k\zeta_2\log(\zeta_2+\lambda_2)\cdots\log(\zeta_n+\lambda_n)}{(k^2\zeta_2{}^2-\zeta_1{}^2)\zeta_3\cdots\zeta_n}\frac{1}{\zeta_1\cdots\zeta_n}\exp\Big(\frac{z_2}{\zeta_2}+\cdots+\frac{z_n}{\zeta_n}\Big)d\zeta_1\cdots d\zeta_n,$$

$$Re.\ \lambda_2\zeta_2>0,\ \cdots,\ Re.\ \lambda_n\zeta_n>0.$$

Hence, to set

$$\frac{1}{(P(\zeta^{-1}))}\zeta_1{}^{-(k+1)}=\zeta_1{}^{m-(k+1)}\frac{1}{1+\zeta_1P_1(\zeta_2{}^{-1},\cdots,\zeta_n{}^{-1})+\cdots+\zeta_1{}^mP_m(\zeta_2{}^{-1},\cdots,\zeta_n{}^{-1})}$$

$$=\zeta_1{}^{m-(k+1)}\ (1+R(\zeta_1)),$$

we get

$$\int_{\gamma<\delta>}\frac{1}{(P(\zeta^{-1}))}\zeta_1{}^{-k}\frac{k\zeta_2\log(\zeta_2+\lambda_2)\cdots\log(\zeta_n+\lambda_n)}{(k^2\zeta_2{}^2-\zeta_1{}^2)\,\zeta_3\cdots\zeta_n}\frac{1}{\zeta_1\cdots\zeta_n}$$

$$\exp\Big(\frac{z_2}{\zeta_2}+\cdots+\frac{z_n}{\zeta_n}\Big)d\zeta_1\cdots d\zeta_n$$

$$=\int_{\gamma_{n-1}<\delta>}\Big[\int_{|\zeta_1|=\varepsilon_1}\zeta_1{}^{m-(k+1)}\,(1+R(\zeta_1))\frac{k}{(k^2\zeta_2{}^2-\zeta_1{}^2)}d\zeta_1\Big]$$

$$\frac{\zeta_2\log(\zeta_2+\lambda_2)\cdots\log(\zeta_n+\lambda_n)}{(\zeta_3\cdots\zeta_n)^2}\exp\Big(\frac{z_2}{\zeta_2}+\cdots+\frac{z_n}{\zeta_n}\Big)d\zeta_2\cdots d\zeta_n,$$

where $\gamma_{n-1}<\delta>$ is the path given by $\{|\zeta_2|=\varepsilon_3\zeta\times\cdots\times\}|\zeta_n|=\varepsilon_n\}$ if $\gamma<\delta>$ is given by $\{|\zeta_1|=\varepsilon_1\{\times\}\ |\zeta_2|=\varepsilon_2\{\times\cdots\times\}\ |\zeta_n|=\varepsilon_n\}$. But by (13), we obtain

$$\int_{\gamma<\delta>}\frac{1}{(P(\zeta^{-1}))}\zeta_1{}^{-k}\frac{k\zeta_2\log(\zeta_2+\lambda_2)\cdots\log(\zeta_n+\lambda_n)}{(k^2\zeta_2{}^2-\zeta_1{}^2)\ \zeta_3\cdots\zeta_n}\frac{1}{\zeta_1\cdots\zeta_n}$$

$$\exp\Big(\frac{z_2}{\zeta_2}+\cdots+\frac{z_n}{\zeta_n}\Big)d\zeta_1\cdots d\zeta_n$$

$$=\lim_{\varepsilon_1\to 0}\int_{\gamma_{n-1}<\delta>}\Big[\int_{|\zeta_1|=\varepsilon_1}\zeta_1{}^{m-(k+1)}\,(+R(\zeta_1))\frac{k}{(k^2\zeta_2{}^2-\zeta_1{}^2)}d\zeta_1\Big]$$

$$\frac{\zeta_2\log(\zeta_2+\lambda_2)\cdots\log(\zeta_n+\lambda_n)}{(\zeta_3\cdots\zeta_n)^2}\exp\Big(\frac{z_2}{\zeta_2}+\cdots+\frac{z_n}{\zeta_n}\Big)d\zeta_2\cdots d\zeta_n.$$

But, if $k, \geqq m - k$ and $(\zeta_2, \cdots, \zeta_n) \in \gamma_{n-1}\langle\delta\rangle$, then we have

$$\lim_{\varepsilon_1 \to 0} \int_{|\zeta_1| = \varepsilon_1} \zeta_1^{m-(k+1)} (1 + R(\zeta_1)) \frac{k}{k^2\zeta_2^2 - \zeta_1^2} d\zeta_1 = 0,$$

because $|\zeta_2| = \varepsilon_2 > 0$ (and fixed) and on $\gamma_{n-1}\langle\delta\rangle$, we have $|R(\zeta_1)| = o\ (|\zeta_1|)$. Therefore we obtain

$$\int_{\gamma\langle\delta\rangle} \frac{1}{(P(\zeta^{-1}))} \zeta_1^{-k} \frac{k\zeta_2 \log(\zeta_2 + \lambda_2) \cdots \log(\zeta_n + \lambda_n)}{(k^2\zeta_2^2 - \zeta_1^2)\zeta_3 \cdots \zeta_n} \frac{1}{\zeta_1 \cdots \zeta_n}$$

$$\exp\left(\frac{z_2}{\zeta_2} + \cdots + \frac{z_n}{\zeta_n}\right) d\zeta_1 \cdots d\zeta_n = 0, \quad 0 \leqq k \leqq m-1.$$

This proves the lemma.

5. In the rest, we denote $\mathscr{B}_\gamma[(1/P(\zeta^{-1}))\ G_\gamma(\zeta)](z)$ by $P^{-1}G_\gamma(z)$, $P^{-1}G_{\gamma,k}(z)$, $B_\gamma(z)$ or $B_{\gamma,k}(z)$.

Lemma 5. *On D_γ, $B_\gamma(z)$ is meromorphic and does not involve* log z_i *for all i, $1 \leqq i \leqq n$.*

Proof. Since we may assume

$$|\zeta_1 P_1(\zeta_2^{-1}, \cdots, \zeta_n^{-1})| + \cdots + |\zeta_1^m P_m(\zeta_2^{-1}, \cdots, \zeta_n^{-1})| < 1,$$

to consider the Laurent expansion of $1/P(\zeta^{-1})$ on D_γ by the example of lemma 3, the Laurent expansion of $1/P(\zeta^{-1})$ on D_γ should be the form

$$\frac{1}{P(\zeta^{-1})} = \sum_{m_1 \geqq k} \sum_{m_2 \leqq 0} \cdots \sum_{m_n \leqq 0} a_{m_1, m_2, \cdots, m_n} \zeta_1^{m_1} \zeta_2^{m_2} \cdots \zeta_n^{m_n}.$$

On the other hand, the Laurent expansion of $\zeta_2/(k^2\zeta_2^2 - \zeta_1^2)$ on D_γ is given by

$$\frac{\zeta_2}{(k^2\zeta_2^2 - \zeta_1^2)} = \frac{1}{k^2} \sum_{m=0}^{\infty} \zeta_1^{2m} \zeta_2^{-2m+1}.$$

Therefore, the Laurent expansion of $(1/P(\zeta^{-1}))\ (\zeta_2/\{(k^2\zeta_2^2 - \zeta_1^2))\ (1/\zeta_3 \cdots \zeta_n).\ (1/\zeta_1 \cdots \zeta_n)\}$ should be the form

$$\frac{\zeta_2}{P(\zeta^{-1})\,(k^2\zeta_2^2 - \zeta_1^2)\zeta_3 \cdots \zeta_n \zeta_1 \cdots \zeta_n}$$

$$= \sum_{m_1 \geqq k} \sum_{m_2 \leqq -1} \cdots \sum_{m_n \leqq -1} b_{m_1, m_2, \cdots, m_n} \zeta_1^{m_1} \zeta_2^{m_2} \cdots \zeta_n^{m_n}.$$

Hence we have the lemma, because $G_\gamma(\zeta)$ does not involve log ζ_1 and $\mathscr{B}[\zeta^{-k}\log\zeta] = (-1)^{k-1}(1/(k-1)!)z^{-k}$ if $k \geqq 1$.

Lemma 6. *We assume deg. $P = m$, that is deg. $P_i \leqq i$ for all i. Then, for any k, $k \neq 0$, to set*

$$A = A_k = \{z \mid z_1 = \pm k z_2\} \cup \{z \mid z_3 \cdots z_n = 0\},$$

$B_{r,k}(z)$ is defined on (some covering of) $\mathbf{C}^n - A_k$.

Proof. By assumption, $B_{r,k}(z)$ is defined at least on D_r. We assume $B_{r,k}$ is continued analytically on $\hat{D}_r$ (may be equal to D_r) and take $z_0 \in \partial\hat{D}$ such that $z_0 \not\ni A$. Then, since $D_r \cap L$ is open dense in L, we can choose vectros $\theta^1, \cdots, \theta^n$ and $\xi^1, \cdots, \xi^n$ such that

(i). $||\theta^1|| = \cdots = ||\theta^n|| = 1$, $P_0\ (\theta^1) \neq 0, \cdots, P_0(\theta^n) \neq 0$ *and* $\theta^1, \cdots, \theta^n$ *are linear independent over* $\mathbf{C}$, *where* P_0 *is the (homogeneous) maximal degree part of* P.

(ii). $\xi^1 \in D_r \cap L, \cdots, \xi^n \in D_r \cap L$.

(iii). T_0 *set* $\ell_j = \{(z\theta^j{}_1,\ z\theta^j{}_2 + \xi^j{}_2, \cdots,\ z\theta^j{}_n + \xi^j{}_n) |\ z \in \mathbf{C}, \}$ $\theta^j = (\theta^j{}_1, \dots, \theta^j{}_n)$, $\xi^j = (0, \xi^j{}_2, \cdots, \xi^j{}_n)$, *we have* $z_0 \in \cap_{j=1} \ell_j$.

Then, on each $\ell_j \cap D$, (the analytic continuation of) $B_{r,k}(z)$ is the solution of the ordinary differential equation

$$P\left(\theta^j \frac{d}{1dz}, \cdots, \theta^j \frac{d}{ndz}\right) u\ (z) = \frac{1}{(z_1{}^2 - k^2 z_2{}^2)\ z_3 \cdots z_n} | \ell_j, \tag{16}$$

with the Cauchy data

$$u(0) = \frac{d}{dz} u(0) = \cdots = \frac{d^{m-1}}{dz^{m-1}} u(0) = 0. \tag{16$'$}$$

But, since we have

$$P\left(\theta^j{}_1 \frac{d}{dz}, \cdots, \theta^j \frac{d}{ndz}\right) = P_0(\theta^j{}_1, \cdots, \theta^j{}_n) \frac{d^m}{dz^m} + \textit{lower order term},$$

$B_{r,k}(z) | \ell_j$ should be coincide to the unique solution of the equation (16) with the data (16)′ by lemma 4 and (i). Then, since (16) is a constant coefficients linear ordinary differential equation, its solution (with the data (16)′) is defined on (some covering of) $\ell_j - \ell_j \cap A_k$, $B_{r,k}\ (z) | \ell_j$ continued analytically to z_0 for any j. Therefore, since $\ell_1, \cdots, \ell_n$ gives an analytic coordinates at z_0, $B_{r,k}(z)$ is holomorphic at z_0 by Hartogs' theorem. Hence $B_{r,k}\ (z)$ is continued analytically to z_0. Since z_0. is arbitrary if $z_0 \not\ni A_k$, we have the lemma.

In the rest, we set

$$Y_1 = \{z |\ z_1 + kz_2 = 0\},\ \ Y_2 = \{z |\ z_1 - kz_2 = 0\},\ \ Y_i = \{z |\ z_i = 0\},\ \ i \geqq 3.$$

By definition, $Y_1, \cdots, Y_n$ are crossing normally at $\{0\}$ and $A_k = \bigcup_{j=1}^n Y_j$.

6. We set $w_1 = z_1 + kz_2$, $w_2 = z_1 - kz_2$, $w_3 = z_3, \cdots, w_n = z_n$, and regard $\mathbf{C}^n$ to be $(w_1, \cdots, w_n)$ -space. Then we set

$$S = \bigcup_{i=1}^n \{w_i |\ Im.\ w_i = 0,\ Re.\ w_i \leqq 0\} \times \mathbf{C}^{n-1}{}_i,$$

$\mathbf{C}^{n-1}{}_i$ *is* $(w_1, \cdots, w_{i-1},\ w_{i+1}, \cdots, w_n)$ *-space.*

Then, by definition, we have

$$S\supset A_k,\ dim.\ S=2n-1,\ dim.\ (S\cap L)=2n-3,\ \pi_1(\mathbf{C}^n-S)=\{0\}.$$

We choose the branch of $B_{\gamma,k}(z)$ on $\mathbf{C}^n-S$ to satisfy

$$B_{\gamma,k}(z)|_{z_1=0}\frac{\partial}{\partial z_1}B_{\gamma,k}(z)|_{z_1=0}=\cdots=\frac{\partial^{m-1}}{\partial {z_1}^{m-1}}B_{\gamma,k}(z)|_{z_1=0}=0,$$

where $z\in L-K\cap L$. We note thisr $B_{\gamma,k}(z)$ coincide to $\mathscr{B}_\gamma[(1/P(\zeta^{-1}))\,G_{\gamma,k}(\zeta)(z)$ on D_γ by lemma 5.

For real positive vectors $\delta=(\delta_1,\cdots,\delta_n)$ and $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_n)$, we set

$$\gamma=\gamma(\delta)=\{w_1|\ |w_1|=\delta_1\}\times\cdots\times\{w_n|\ |w_n|=\delta_n\},$$

$$\Gamma=\Gamma(\delta)=\{w_1|\ |w_1|<\delta_1\}\times\cdots\times\{w_n|\ |w_n|<\delta_n\}.$$

$$\gamma_\varepsilon=\gamma(\delta)_\varepsilon=\{w_1|\ |w_1|=\delta_1,\ 0\leqq arg.\,w_1<\pi-\varepsilon_1,\ 0\geqq arg.\,w_1>-\pi+\varepsilon_1\}\times\cdots$$
$$\times\{w_n|\ |w_n|=\delta_n,\ 0\leqq arg.\,w_n<\pi-\varepsilon_n,\ 0\geqq arg.\,w_n>-\pi+\varepsilon_n\}.$$

The correspondence domains and paths in ζ-space are also denoted by the same notations.

If $f(\zeta)$ is holomorphic on some neighborhood of γ (in $\mathbf{C}^n$), then we define $\int_\gamma B_{\gamma,k}(z-\zeta)f(\zeta)d\zeta$ by

$$(17)\qquad \int_\gamma B_{\gamma,k}(z-\zeta)\,f(\zeta)\,d\zeta=\lim_{\varepsilon\to 0}.\int_{\gamma_\varepsilon}B_{\gamma,k}(z-\zeta)\,f(\zeta)d\zeta.$$

By lemma 6 and the definition of γ, there exists a constant M such that

$$\sup_{z\in\gamma}.\,|B_{\gamma,k}(z)|\leqq M,\ \ for\ any\ \gamma.$$

Hence $\int_\gamma B_{\gamma,k}(z-\zeta)f(\zeta)d\zeta$ always exists and the convergence of the right hand side of (17) is uniform on K, if K satisfies $B_{\gamma,k}$ is bounded on $\cup_{z\in k}\{z-\gamma\}$, which is satisfied on some (small) relative compact neighborhood of $\Gamma(\delta')$ if $\delta'<\delta$, that is $\delta'_1<\delta_1,\cdots,\delta'_n<\delta_n$, where $\delta'=(\delta'_1,\cdots,\delta'_n)$. Therefore we have

$$(18)\qquad \frac{\partial^{i_1+\cdots+i_n}}{\partial {z_1}^{i_1}\cdots\partial {z_n}^{i_n}}\int_\gamma B_{\gamma,k}(z-\zeta)\,f(\zeta)\,d\zeta=\int_\gamma\frac{\partial^{i_1+\cdots+i_n}}{\partial {z_1}^{i_1}\cdots\partial {z_n}^{i_n}}B_{\gamma,k}(z-\zeta)\,f(\zeta)\,d\zeta,$$

if $z\in\Gamma(\delta)$.

Theorem 2. *If $f(z)$ is holomorphic on $\overline{\Gamma(\delta)}$, then to set*

$$(P^{-1}f)(z)=\frac{-1}{2k(2\pi\sqrt{-1})^n}\int_{\gamma(\delta)}B_{\gamma,k}(z-\zeta)\,f(\zeta)d\zeta,$$

we have

(19) $$P\Big(\frac{\partial}{\partial z}\Big)(P^{-1}f)(z)=f(z),\quad z\in\Gamma(\delta),$$

$$(P^{-1}f)(z)|_{z_1=0}=\frac{\partial}{\partial z_1}(P^{-1}f)(z)|_{z_1=0}=\cdots=\frac{\partial^{m-1}}{\partial {z_1}^{m-1}}(P^{-1}f)(z)|_{z_1=0}=0,$$

and this $P^{-1}f$ is holomorphic on $\Gamma(\delta)$.

Proof. By (18), we have

$$P\Big(\frac{\partial}{\partial z}\Big)\int_{\gamma(\delta)}B_{\gamma,k}(z-\zeta)f(\zeta)d\zeta=\int_{\gamma(\delta)}P\Big(\frac{\partial}{\partial z}\Big)B_{\gamma,k}(z-\zeta)f(\zeta)d\zeta,\quad z\in\Gamma(\delta).$$

Hence we have the first equality of (19). On the other hand, since $B_{\gamma,k}$ is holomorphic along $L-L\cap A_k$ by lemma 5 and $A_k=Y_1\cup\cdots\cup Y_n$, where $Y_1,\cdots,Y_n$ are crossing normally at $\{0\}$, we have by lemma 4

$$res._{z-Y_1,\cdots,z-Y_n}\Big[\frac{\partial^i}{\partial {z_1}^i}B_{\gamma,k}(z-\zeta)f(\zeta)d\zeta_1\wedge\cdots\wedge d\zeta_n\Big]=0,\quad 0\leqq i\leqq m-1,$$

if $z\in L$ and $f(\zeta)$ is holomorphic at $\zeta=z$. Hence we have the second equality of (19) by lemma 6 and the residue formula of composed residue ([3], [3]', [8], [12], [14]).

To get the last assertion, we first note that $P^{-1}f$ is holomorphic on $\Gamma(\delta)-S$ by definition. But, since to set

$$S_\theta=\bigcup_{i=1}^{n}\{w_i|\ arg.\,w_i=\theta_i\}\times C^{n-1}{}_i,\quad \theta=(\theta_1,\cdots,\theta_n),\quad 0\leqq\theta_i<2\pi,$$

and denoting $P^{-1}\theta f$ the corresponding function of $P^{-1}f$ obtained by using S_θ instead of S, we get same results for $P^{-1}\theta f$ as $P^{-1}f$, and we have

$$P^{-1}\theta f(z)=P^{-1}f(z),\quad z\in\Gamma(\delta)-(S\cup S_\theta),$$

by the uniqueness of the solution of Cauchy problem. Since $P^{-1}{}_\theta f$ is holomorphic on $\Gamma(\delta)-S_\theta$, $P^{-1}f$ should be holomorphic on $\Gamma(\delta)-A_k$ because we have $S\cap S_\theta=A_k$ if $\theta_i\neq\pi$, $i=1,\cdots,n$. But by (18), if $z\in A_k\cap\Gamma(\delta)$, then $P^{-1}f$ is holomorphic at z. Hence we have the theorem.

Corollary. *Let $\delta=(\delta_1,\cdots,\delta_n)$ and $\delta'=(\delta'_1,\cdots,\delta'_n)$ be positive vectors such that $\delta_i>\delta'_i$, $i=1,\cdots,n$. Then to set*

$$\Gamma(\delta,\ \delta')=\{w_1|\ \delta'_1<|w_1|<\delta_1\}\times\cdots\times\{w_n|\delta'_n<|w_n|<\delta_n\},$$

$$\gamma(\delta,\ \delta')=\sum_{\{i_1,\cdots,i_k\}\cup\{j_1,\cdots,j_{n-k}\}=\{1,\cdots,n\}}(-1)^{n-k}\cdot$$

$$\cdot\{w_{i_1}|\ \ |w_{i_1}|=\delta_{i_1}\}\times\cdots\times\{w_{i_k}|\ \ |w_{i_k}|=\delta_{i_k}\}$$

$$\times\{w_{j_1}|\ \ |w_{j_1}|=\delta'_{j_1}\}\times\cdots\times\{w_{j_{n-k}}|\ \ |w_{j_{n-k}}|=\delta'_{j_{n-k}}\},$$

we define for a holomorphic function f on $\overline{\Gamma(\delta, \delta')}$ the function $P^{-1}f$ by

$$(P^{-1}f)(z)=\frac{-1}{2k(2\pi\sqrt{-1})^n}\int_{\gamma(\delta,\delta')}B_{\gamma,k}(z-\zeta)f(\zeta)d\zeta,$$

where $\int_{\gamma(\delta,\delta')}B_{\gamma,k}(z-\zeta)f(\zeta)d\zeta$ *is defined similarly as* $\int_{\gamma(\delta)}B_{\gamma,k}(z-\zeta)f(\zeta)d\zeta$. *Then we have*

$$(19)' \qquad P\left(\frac{\partial}{\partial z}\right)(P^{-1}f)(z)=f(z), \quad z\in\Gamma(\delta, \delta'),$$

$$(P^{-1}f)(z)|_{z_1=0}=\frac{\partial}{\partial z_1}(P^{-1}f)(z)|_{z_1=0}=\cdots=\frac{\partial^{m-1}}{\partial z_1^{m-1}}(P^{-1}f)(z)|_{z_1=0}=0.$$

Proof. Since we know

$$f(z_1, \cdots, z_n)$$
$$=\frac{-1}{2k(2\pi\sqrt{-1})^n}\int_{\gamma(\delta,\delta')}\frac{f(\zeta_1,\cdots,\zeta_n)}{\{(z_1-\zeta_1)^2-k^2(z_2-\zeta_2)^2\}(z_3-\zeta_3)\cdots(z_n-\zeta_n)}d\zeta_1\cdots d\zeta_n,$$

if $z\in\Gamma(\delta, \delta')$, we have the corollary by the same reason as theorem 2.

We note that in this corollary, $P^{-1}f$ is holomorphic and bounded on $\Gamma(\delta, \delta')-S$, but may be many-valued on $\Gamma(\delta, \delta')$.

§ 3 Elementary solutions of analytic coefficients operators

7. We denote by $P(z, \partial/\partial z)$ an analytic coefficients linear partial dofferential operator of the form

$$(20) \qquad P\left(z, \frac{\partial}{\partial z}\right)=\frac{\partial^m}{\partial z_1^m}+P_1\left(z, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_n}\right)\frac{\partial^{m-1}}{\partial z_1^{m-1}}+\cdots+P_m\left(z, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_n}\right),$$

$$P_i\left(z, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_n}\right)=\sum_{j_2+\cdots+j_n\leq i}a^i{}_{j_2,\cdots,j_n}(z)\frac{\partial^{j_2+\cdots+j_n}}{\partial z_2^{j_2}\cdots\partial z_n^{j_n}},$$

where each $a^i{}_{j_2,\cdots,j_n}(z)$ is holomorphic on some neighborhood of $\{0\}$.

For this $P(z, \partial/\partial z)$, we set $P(\partial/\partial z)=P(0, \partial/\partial z)$. Then we have

$$P\left(z, \frac{\partial}{\partial z}\right)=P\left(\frac{\partial}{\partial z}\right)+Q\left(z, \frac{\partial}{\partial z}\right),$$

$$Q\left(0, \frac{\partial}{\partial z}\right)=0, \ Q\left(z, \frac{\partial}{\partial z}\right) \textit{ involves at most } \frac{\partial}{\partial z_1}, \cdots, \frac{\partial^{m-1}}{\partial z_1^{m-1}}.$$

We denote by $\mathscr{A}(\overline{D})$ the normed vector space of holomorphic functions on $\overline{D}$, that is, the space of holomorphic functions on some neighborhood of $\overline{D}$, where D is assumed to be relative compact, with the norm

$$||f||=\sup_{z\in D}|f(z)| \ (=\max_{z\in\overline{D}}|f(z)|).$$

Then, by theorem 2, P has the (continuous) inverse operator P^{-1} on $\mathscr{A}(\overline{\Gamma(\delta)})$ as an integral operator. Similarly, denoting $\mathscr{A}(\overline{\Gamma(\delta,\ \delta')}|_S)$ the normed vector space of (bounded) holomorphic functions on $\Gamma(\delta,\ \delta')-S$ which can be continued analytically on some neighborhood of $\Gamma(\delta,\ \delta')-S$ (may be many-valued) with the maximal norm, P also has the (continuous) inverse operator P^{-1} from $\mathscr{A}(\overline{\Gamma(\delta,\ \delta')})$ to $\mathscr{A}(\overline{\Gamma(\delta,\ \delta')}|_S)$ as an integral operator.

***Lemma* 7.** *If each coefficients of $P(z,\ \partial/\partial z)$ belongs in $\mathscr{A}(\overline{\Gamma(\delta)})$, then to define QP^{-1} by*

$$QP^{-1}f(z)=Q\left(z,\ \frac{\partial}{\partial z}\right)(P^{-1}f(z))=\frac{-1}{2k(2\pi\sqrt{-1})^n}\int_{\gamma(\delta)}Q\left(z,\ \frac{\partial}{\partial z}\right)B_{\gamma,k}(z-\zeta)f(\zeta)d\zeta,$$

QP^{-1} is a bounded linear (integral) operator on $\mathscr{A}(\overline{\Gamma(\delta)})$ and we have

$$\|QP^{-1}\|=O((\delta_1+\delta_2)\|\delta\|),\quad \|\delta\|=\sqrt{\sum_{i=1}^n\delta_i^2}. \tag{21}$$

***Proof*.** For $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_n)$, $\varepsilon_1>0,\cdots,\varepsilon_n>0$, we set $\delta+\varepsilon=(\delta_1+\varepsilon_1,\cdots,\delta_n+\varepsilon_n)$. Then, by definition, if $f\in\mathscr{A}(\overline{\Gamma(\delta)})$, f also belongs in $\mathscr{A}(\overline{\Gamma(\delta+\varepsilon)})$ for some ε. Denoting P^{-1} defined on $\mathscr{A}(\overline{\Gamma(\delta)})$ by $P^{-1}{}_\delta$, *etc.*, that is, to set $P^{-1}{}_\delta f(z)=(-1/2k(2\pi\sqrt{-1})^n)\int_{\gamma(\delta)}B_{\gamma,k}(z-(\zeta)f\zeta)d\zeta$, we have by the uniqueness of the solution of Cauchy problem and theorem 2

$$P^{-1}{}_\delta f(z)=P^{-1}{}_{\delta+\varepsilon}f(z),\quad f\in\mathscr{A}(\overline{\Gamma(\delta+\varepsilon)}),\quad z\in\Gamma(\delta). \tag{22}$$

Hence $P^{-1}{}_\delta f$ is (continued analytically on $\Gamma(\delta+\varepsilon)$ and) holomorphic on $\Gamma(\delta+\varepsilon)$. This shows $P^{-1}f\in\mathscr{A}(\overline{\Gamma(\delta)})$. Hence $QP^{-1}f$ also belongs in $\mathscr{A}(\overline{\Gamma(\delta)})$ by assumption.

By definition, QP^{-1} is an integral operator and since we have

$$Q\left(z,\ \frac{\partial}{\partial z}\right)B_{\gamma,k}(z-\zeta)\in\mathscr{A}_\zeta(\gamma(\delta)|_S),\quad z\in\Gamma(\delta),$$

where $\mathscr{A}(\Gamma(\delta)|_S)$ is defined similarly as $\mathscr{A}(\overline{\Gamma(\delta,\ \delta')}|_S)$ and there exists a constant M such that

$$\left\|Q\left(z,\ \frac{\partial}{\partial z}\right)B_{\gamma,k}(z-\zeta)\right\|\ \Gamma(\delta)|_S\geqq M,\quad z\in\Gamma(\delta),$$

QP^{-1} is a bounded operator. Moreover, since $Q(0,\ \partial/\partial z)=0$ and $Q(z,\ \partial/\partial z)$ involves at most $\partial/\partial z_1,\cdots,\ \partial^{m-1}/\partial z_1{}^{m-1}$, to set

$$Q\left(z,\ \frac{\partial}{\partial z}\right)=\sum_{i=1}^n z_iQ_i\left(z,\ \frac{\partial}{\partial z}\right),$$

we have

$$Q_i\left(z, \frac{\partial}{\partial z}\right)B_{\gamma,k}(z)|_{z_1=0}=0, \quad i=1, \cdots, n,$$

by lemma 4. Therefore we get

$$(23) \qquad Q\left(z, \frac{\partial}{\partial z}\right)B_{\gamma,k}(z)=O(z_1||z||), \quad z_1\to 0, \quad z\to 0, \quad ||z||=\sqrt{\sum_{i=1}^{n}|z_i|^2}.$$

Hence, if $f\in\mathscr{A}(\overline{\Gamma(\delta)})$ and $\delta'<\delta$, then by (22), we get

$$||P^{-1}{}_{\delta'} f||_{\Gamma(\delta')}=||P^{-1}{}_{\delta}f||_{\Gamma(\delta)}=O((\delta_1'+\delta_2')\,||\delta'||),$$

because $w_1=(z_1+z_2)/2$. But, since to define $r:\mathscr{A}(\overline{\Gamma(\delta)})\to\mathscr{A}(\overline{\Gamma(\delta')})$, $\delta>\delta'$, by $r(f)=f|\Gamma(\delta')$, $r(\mathscr{A}(\overline{\Gamma(\delta)}))$ is dense in $\mathscr{A}(\overline{\Gamma)\delta')})$, we have the lemma.

Corollary. *If in* (20), *each* $P_i\ (z,\ \partial/\partial z)$ *satisfies*

$$(24) \qquad a^i{}_{j_2,\cdots,j_n}(z)=z_1{}^{1-i}{}_b{}^i{}_{j^2,\cdots,j_n}(z), \quad b^i{}_{j_2,\cdots,j_n}(z)\in\mathscr{A}(\overline{\Gamma(\delta,\ \delta')}),$$
$$b^i{}_{j_2,\cdots,j_n}(z) \text{ is holomorphic if } z_1, z_2\to 0,$$

then, to set

$$P\left(z, \frac{\partial}{\partial z}\right)=P\left(\frac{\partial}{\partial z}\right)+Q\left(z, \frac{\partial}{\partial z}\right),$$

where $P(\partial/\partial z)$ *is a constant coefficients operator and involves* $\partial^m/\partial z_1{}^m$ *and* $Q(z,\ \partial/\partial z)$ *is a variable coefficients operator and involves at most* $\partial/\partial z_1, \cdots, \partial^{m-1}/\partial z_1{}^{m-1}$, QP^{-1} *is a bounded linear operator defined on* $\mathscr{A}(\overline{\Gamma(\delta,\ \delta')})$ *with the value in* $\mathscr{A}(\overline{\Gamma(\delta,\ \delta')}|_S)$ *and we have*

$$(21)' \qquad ||QP^{-1}||=O(\delta_1+\delta_2).$$

Proof. First we note that, if $f\in\mathscr{A}(\overline{\Gamma(\delta,\ \delta')})$, then $QP^{-1}f$ should be (many valued) meromorphic on $\overline{\Gamma(\delta,\ \delta')}$ with poles on $L\cap\overline{\Gamma(\delta,\ \delta')}$. But, since we have

$$(25) \qquad Q\left(z, \frac{\partial}{\partial z}\right)B_{\gamma,k}(z)|_{z_1=0}=0,$$

by lemma 4 *and the form of* $Q(z,\ \partial/\partial z)$, $QP^{-1}f$ is (many valued) holomorphic on $\overline{\Gamma(\delta,\ \delta')}$ by the same reason as above. Hence $QP^{-1}f$ maps $\mathscr{A}(\overline{\Gamma(\delta,\ \delta')})$ into $\mathscr{A}(\overline{\Gamma(\delta,\ \delta')}|_S)$. Moreover, by (24), we have

$$\left|Q\left(z, \frac{\partial}{\partial z}\right)B_{\gamma,k}(z)\right|=0(|z_1|), \quad z\in\Gamma(\delta,\ \delta').$$

Hence we obtain (21)$'$ by the same reason as above.

8. ***Lemma** 8. if* $P(z,\ \partial/\partial z)$ *satisfies the assumptions of lemma 7 and either of* $\delta_1+\delta_2$ *or* $||\delta||$ *is sufficiently small, then there exists an integral operator* $E=E_{p,\,\gamma(\delta)}$ *from* $\mathscr{A}(\overline{\Gamma(\delta)})$ *into* $\hat{\mathscr{A}}(\overline{\Gamma(\delta)})$, *the completion of* $\mathscr{A}(\overline{\Gamma(\delta)})$, *such that*

(26) $$P\left(z, \frac{\partial}{\partial z}\right)(Ef)(z)=f(z), \quad f\in \mathscr{A}(\overline{\Gamma(\delta)}), \quad z\in\Gamma(\delta),$$

$$Ef(z)|_{z_1=0}=\frac{\partial}{\partial z_1}Ef(z)|_{z_1=0}=\cdots=\frac{\partial^{m-1}}{\partial {z_1}^{m-1}}Ef(z)|_{z_1=0}=0.$$

Proof. By (20), if either $\delta_1+\delta_2$ or $||\delta||$ is sufficiently small, then we have

$$||QP^{-1}||<1.$$

Therefore, the Neumann series $\sum_{m\geqq 0}(-1)^m (QP^{-1})^m$ converges as an operator from $\mathscr{A}(\overline{\Gamma(\delta)})$ into $\hat{\mathscr{A}}(\overline{\Gamma(\delta)})$ and to set

$$(I+QP^{-1})^{-1}=\sum_{m=0}^{\infty}(-1)^m (QP^{-1})^m,$$

we have $(I+QP^{-1})(I+QP^{-1})^{-1}=I$, the identity map of $\mathscr{A}(\overline{\Gamma(\delta)})$. Hence to set

$$E=P^{-1}(I+QP^{-1})^{-1},$$

we have

$$P\left(z, \frac{\partial}{\partial z}\right)(Ef)(z)=(P\left(\frac{\partial}{\partial z}\right)+Q\left(z, \frac{\partial}{\partial z}\right))(Ef)(z)$$

$$=(P+Q)(P^{-1}(I+QP^{-1})^{-1}f)(z)=f(z),$$

for $f\in\mathscr{A}(\overline{\Gamma(\delta)})$, $z\in\Gamma(\delta)$. Moreover, since $(I+QP^{-1})^{-1}f$ is holomorphic on $\Gamma(\delta)$ and continuous on $\overline{\Gamma(\delta)}$, we have

$$Ef(z)|_{z_1=0}=\frac{\partial}{\partial z_1}Ef(z)|_{z_1=0}=\cdots=\frac{\partial^{m-1}}{\partial {z_1}^{m-1}}Ef(z)|_{z_1=0}=0,$$

by theorem 2.

We note that since we may consider P^{-1} to be an operator on $\hat{\mathscr{A}}(\Gamma(\delta))$, this E can also be regardedto be an operator on $\hat{\mathscr{A}}(\Gamma(\delta))$.

Corollary. *If $P(z, \partial/\partial z)$ satisfies the assumptions of the corollary of lemma 7 and assume each $b^i{}_{j_2,\cdots,j_n} \in\mathscr{A}(\overline{\Gamma(\delta, \delta')})$ where $\delta_1'+\delta_2'$ can be taken arbitraly small, and if $\delta_1+\delta_2$ is sufficiently small, then there exists an integral operator $E=E_{P,\Gamma\delta,\delta'}|_S$ from $\mathscr{A}(\overline{\Gamma(\delta, \delta')})$ into $\hat{\mathscr{A}}(\overline{\Gamma\delta, \delta')|_S})$, the completion of $\mathscr{A}(\overline{\Gamma(\delta, \delta')})|_S$, such that*

(26)′ $$P\left(z, \frac{\partial}{\partial z}\right)(Ef)(z)=f(z), \quad f\in\mathscr{A}(\overline{\Gamma(\delta, \delta')}), \quad z\in\Gamma(\delta, \delta'),$$

$$Ef(z)|_{z_1=0}=\frac{\partial}{\partial z_1}Ef(z)|_{z_1=0}=\cdots=\frac{\partial^{m-1}}{\partial {z_1}^{m-1}}Ef(z)|_{z_1=0}=0.$$

Proof. First we note that by the definition of P^{-1}, we can define P^{-1} for the elements of $\mathscr{A}(\overline{\Gamma(\delta, \delta')}|_S)$, and this extended P^{-1} is also a bounded operator. Hence, we can extend P^{-1} to be a bounded linear (integral operator from $\hat{\mathscr{A}}(\overline{\Gamma(\delta, \delta')}|_S)$ into $\hat{\mathscr{A}}(\overline{\Gamma(\delta, \delta')}|_S)$ and for this extended P^{-1}, the corollary of lemma *7* is also hold. Therefore, by (21)$'$, if $\delta_1+\delta_2$ is sufficiently small, then we have

$$||QP^{-1}|| < 1,$$

where P^{-1} is considered to be an operator on $\hat{\mathscr{A}}(\overline{\Gamma(\delta, \delta')}|_S)$. Hence, by the same reason as above, we have the theorem.

Note. $\hat{\mathscr{A}}(\overline{\Gamma(\delta)})$ is the space of holomorphic functions on $\Gamma(\delta)$ which are continuous on $\overline{\Gamma(\delta)}$. On the other hand, $\hat{\mathscr{A}}(\overline{\Gamma(\delta, \delta')}|_S)$ is the space of those holomorphic functions f on $\Gamma(\delta, \delta')-S$ such that to define $v(f)$ by $v(f)(w_1, \cdots, w_n)=f(w_1^2, \dots, w_n^2)$, $\pi/2 < arg.\ w_i < \pi/2$, $i=1, \cdots, n$, $v(f)$ is extended to be a continuous function on

$$\{(w_1, \cdots, w_n)|\ \delta'_i < |w_i| < \delta_i,\ -\frac{\pi}{2} \leqq arg.\ w_i \leqq \frac{\pi}{2},\ i=1, \dots, n\}.$$

9. ***Theorem 3.*** *If each coefficients of $P(z, \partial/\partial z)$ belongs in $\mathscr{A}(\overline{\Gamma(\delta)})$, then there exists an integral operator E from $\mathscr{A}(\overline{\Gamma(\delta)})$ into $\hat{\mathscr{A}}(\overline{\Gamma(\delta)})$* such that (26) *is hold.*

Proof. We set $||QP^{-1}{}_{\delta'}||=A_{\delta'}$. Then, if $|\lambda|<(A_{\delta'})^{-1}$, the Neumann series $\sum_{n\geqq 0}(-1)^n\lambda^n(QP^{-1}{}_{\delta'})^n$ convrges on $\hat{\mathscr{A}}(\overline{\Gamma(\delta')})$ and equal to $(I+\lambda QP^{-1}{}_{\delta'})^{-1}$. Hence to set

$$E_{\delta', \lambda}=P^{-1}(I+\lambda QP^{-1}{}_{\delta'})^{-1},\ \delta' \leqq \delta,$$

$E_{\delta', \lambda}$ satisfies (26) as an operator from $\mathscr{A}(\overline{\Gamma(\delta')})$ into $\hat{\mathscr{A}}(\overline{\Gamma(\delta')})$ for the differential operator $P_\lambda(z, \partial/\partial z)=P(\partial/\partial z)+\lambda Q(z, \partial/\partial z)$. Moreover, by the definition of $E_{\delta', \lambda}$, there exists an analytic function $E_{\delta'}(z, \zeta, \lambda)$ on $\Gamma(\delta')\times\Gamma(\delta')\times\{\lambda|\ |\lambda|<(A_{\delta'})^{-1}\}$ such that $E_{\delta'}(z, \zeta, \lambda)$ is bounded on $\gamma(\delta')\times\gamma(\delta')$ for any (fixed) λ and

$$E_{\delta', \lambda}f(z)=\int_{\gamma(\delta')} E_{\delta'}(z, \zeta, \lambda)f(\zeta)d\zeta. \tag{27}$$

But, since $QP^{-1}{}_{\delta'}$ is a compact operator on $\hat{\mathscr{A}}(\Gamma(\delta'))$, by the theory of Fredholm, $E_{\delta'}(z, \zeta, \lambda)$ can be continued on whole λ-space as an meromorphic function ([16]). On the other hand, by the uniqueness of the solution of Cauchy proplem (of the operator $P(\partial/\partial z)$) and by theorem 2, we have

$$E_{\delta'}(z, \zeta, \lambda)|\Gamma(\delta'')\times\Gamma(\delta'')\times\{\lambda|\ |\lambda|<\frac{1}{A_{\delta'}}\} \tag{28}$$

$$=E_{\delta''}(z, \zeta, \lambda)|\Gamma(\delta'')\times\Gamma(\delta'')\times\{\lambda|\ |\lambda|<\frac{1}{A_{\delta'}}\},\ \delta\geqq\delta'\geqq\delta''.$$

Therefore, if $|\lambda|$ is small, then $E_{\delta'}(z, \zeta, \lambda)$ is continued analytically on $\Gamma(\delta)\times\Gamma(\delta)$ and bounded on $\gamma(\delta)\times\gamma(\delta)$. On the other hand, since $E_{\delta'}(z, \zeta, \lambda)$ is holomorphic at $\lambda=1$ if $\|\delta'\|$ or $\delta'_1+\delta'_2$ is sufficiently small, $E_\delta(z, \zeta, \lambda)$ is holomorphic at $\lambda=1$. Hence, by the theory of Fredholm, to set

$$Ef(z)=\int_{\gamma(\delta)}E_\delta(z, \zeta, 1)f(\zeta)d\zeta, \quad z\in\Gamma(\delta), \quad f\in\mathscr{A}(\overline{\Gamma(\delta)}), \tag{29}$$

we have

$$P\left(z, \frac{\partial}{\partial z}\right)Ef(z)=f(z), \quad z\in\Gamma(\delta).$$

On the other hand, since we have by lemma 8,

$$E_{\delta,\lambda}f(z)|_{z_1=0}=\frac{\partial}{\partial z_1}E_{\delta,\lambda}f(z)|_{z_1=0}=\cdots=\frac{\partial}{\partial z_1^{m-1}}E_{\delta,\lambda}f(z)|_{z_1=0}=0,$$

if $|\lambda|<(A_\delta)^{-1}$, we obtain

$$Ef(z)|_{z_1=0}=\frac{\partial}{\partial z_1}Ef(z)|_{z_1=0}=\cdots=\frac{\partial}{\partial z_1^{m-1}}Ef(z)|_{z_1=0}=0,$$

by the theorem of identity. Therefore, we obtain the theorem.

Corollary. *If $P(z, \partial/\partial z)$ is the form of (20) where each $a^i{}_{j_2,\dots,j_n}(z)$ satisfies (24) and δ'* satisfies the condition that *$\delta_1'+\delta_2'$ can be taken arbitrary small. then there exists an integral operator E from $\hat{\mathscr{A}}(\overline{\Gamma(\delta, \delta')}|_S)$ into $\hat{\mathscr{A}}(\overline{\Gamma(\delta, \delta')}|_S)$ such that if $f\in\mathscr{A}\overline{\Gamma(\delta, \delta')})$, then (26)' is hold.*

Proof. Since to set

$$P^{-1}{}_{\delta-\varepsilon,\delta'+\varepsilon'}f(z)=\frac{-1}{2k(2\pi\sqrt{-1})^n}\int_{\gamma(\delta-\varepsilon,\delta'+\varepsilon')}B_{\gamma,k}(z-\zeta)\,f(\zeta)d\zeta, \quad \varepsilon, \varepsilon'\geqq 0,$$

we may define $(I+\lambda QP^{-1}{}_{\delta-\varepsilon,\delta'+\varepsilon'})^{-1}=\sum_{n\geqq 0}(-1^n\lambda^n(QP^{-1}{}_{\delta-\varepsilon,\delta'+\varepsilon'})^n$ *if* $|\lambda|<\|QP^{-1}{}_{\delta-\varepsilon,\delta'+\varepsilon'}\|$, and we have

$$E_{\delta-\varepsilon,\delta'+\varepsilon',\lambda}f(z)=\int_{\gamma(\delta-\varepsilon,\delta'+\varepsilon')}E_{\delta-\varepsilon,\delta'+\varepsilon'}(z, \zeta, \lambda)\,f(\zeta)d\zeta,$$

where $E_{\delta-\varepsilon,\delta'+\varepsilon',\lambda}=P^{-1}(I+\lambda QP^{-1}{}_{\delta-\varepsilon,\delta'+\varepsilon'})^{-1}$ and $E_{\delta-\varepsilon,\delta'+\varepsilon'}(z, \zeta, \lambda)$ is analytic on $\Gamma(\delta-\varepsilon, \delta'+\varepsilon')\times\Gamma(\delta-\varepsilon, \delta'+\varepsilon')\times\{\lambda|\ |\lambda|<\|QP^{-1}{}_{\delta-\varepsilon,\delta'+\varepsilon'}\|\}$ and bounded on $\gamma(\delta-\varepsilon, \delta'+\varepsilon')$. Therefore, by the same reason as above, we have the corollary.

Note. Since we know that if $u(z)$ is holomorphic and $P(z, \partial/\partial z)\ u(z)=0$, then to set $v(z)=u(z)-\sum_{i=0}^{m-1}z_1^iu_i(z_2, \dots, z_n)$, where $u(z)=\sum_{i=0}^{m-1}z_1^iu_i(z_2, \dots, z_n)+O(|z_1|^m)$, $v(z)$ is the solution of the equation

$$P\left(z, \frac{\partial}{\partial z}\right)v(z)=-P\left(z, \frac{\partial}{\partial z}\right)\sum_{i=0}^{m-1}z_1^iu_i(z_2, \dots, z_n),$$

with the data $v(z)|_{z_1=0}=(\partial/\partial z_1)v(z)|_{z_1=0}=\cdots=(\partial^{m-1}/\partial z_1^{m-1})|_{z_1=0}=0$. Hence if $P(z, \partial/\partial z)$

satisfies the assumptions of theorem 3 or the corollary of theorem 3, then we have on $\Gamma(\delta)$, or on $\Gamma(\delta, \delta')$

(30) $$u(z)=\sum_{i=0}^{m-1} z_1{}^i u_i(z_2, \cdots, z_n)-E\left[P\left(\zeta, \frac{\partial}{\partial\zeta}\right)\left\{\sum_{i=0}^{m-1}\zeta_1{}^i u_i(\zeta_2, \cdots, \zeta_n)\right\}\right](z),$$

if $P(z, \partial/\partial z)\, u(z)=0$, $u\in\mathscr{A}(\overline{\Gamma(\delta)})$ and $u_1, \cdots, u_n\in\mathscr{A}(\overline{\Gamma(\delta)}\cap L)$ or $u\in\mathscr{A}(\overline{\Gamma(\delta, \delta')})$, $u_1, \cdots, u_n\in\mathscr{A}(\overline{\Gamma(\delta, \delta')}\cap L)$.

We denote the formal adjoint of $P(z, \partial/\partial z)$ by $P'(z, \partial/\partial z)$. Then, since

$$\int_{\gamma(\delta)}\left[\frac{\partial^{m_{i_1}+\cdots+m_{i_k}}}{\partial\zeta_{i_1}{}^{m_{i_1}}\cdots\partial\zeta_{i_k}{}^{m_{i_k}}}\{F(z, \zeta)\, f(\zeta)\}\right]d\zeta_1\wedge\cdots\wedge d\zeta_n$$

$$=\int_{\partial Y_{i_1},\cdots,Y_{i_k}\gamma(\delta)} res._{Y_{i_1},\dots,Y_{ik}}\left(\frac{\partial^{m_{i_1}+\cdots+m_{i_k}-k}}{\partial\zeta_{i_1}{}^{m_{i_1}-1}\cdots\partial\zeta_{ik}{}^{m_{i_k}-1}}F(z, \zeta)\right.$$

$$\left. f(\zeta)d\zeta_1\wedge\cdots\wedge d\zeta_n\right),$$

$$\partial_{Y_{i_1},\cdots,Y_{i_k}}\gamma(\delta)=\{w|\ w_{i_1}=\cdots=w_{i_k}=0,\ |w_{j_1}|=\delta_{j_1},\cdots,|w_{j_{n-k}}|$$

$$=\delta_{j_{n-k}}, \{i_1, \cdots, i_k\}\cup\{j_1, \cdots, j_{n-k}\}=\{1, \cdots, n\}\},$$

we can rewrite (30) as follows:

(31) $$u(z_1, \cdots, z_n)$$

$$=\sum_{i=0}^{m-1} z_1{}^i u_i(z_2, \cdots, z_n)-\int_{\gamma(\delta)}\left[P'\left(\zeta, \frac{\partial}{\partial\zeta}\right)E(z, \zeta)\right]\left(\sum_{i=0}^{m-1}\zeta_1{}^i u_i(\zeta_2, \cdots, \zeta_n)\right)d\zeta$$

$$+\sum_{1\leqq i_1<\cdots<i_k\leqq n}\int_{\partial Y_{i_1},\cdots,Y_{ik}\gamma(\delta)} res._{Y_{i_1},\cdots,Y_{ik}}\left[P_{i_1,\cdots,i_k}\left(\zeta, \frac{\partial}{\partial\zeta}\right)\right.$$

$$\left.(E(z, \zeta)\left(\sum_{i=0}^{m-1}\zeta_1{}^i u_i(\zeta_2, \cdots, \zeta_n)\right)d\zeta_1\wedge\cdots\wedge d\zeta_n\right],\quad E(z, \zeta)=E_\delta(z, \zeta, 1),$$

where $P_{i_1,\cdots,i_k}(\zeta, \partial/\partial\zeta)$ is a differential operator determined by $P(z, \partial/\partial z)$ and it has following properties.

(32) $$P_{i_1,\cdots,i_k}\left(\zeta, \frac{\partial}{\partial\zeta}\right)=0,\ \textit{if}\ P\left(z, \frac{\partial}{\partial z}\right)\textit{does not involve}$$

$$\frac{\partial^{m_{i_1}+\cdots+m_{i_k}}}{\partial z_{i_1}{}^{m_{i_1}}\cdots\partial z_{ik}{}^{m_{ik}}}\ \textit{if}\ m_{i_1}\geqq 1, \cdots, m_{ik}\geqq 1,$$

(32)′ $$deg._X\ P_{i_1,\cdots,i_k}(\zeta, X)\leqq M-k,$$

$$\textit{if}\ P\left(z, \frac{\partial}{\partial z}\right)\textit{does not involve}\ \frac{\partial^{m_{i_1}+\cdots+m_{i_k}}}{\partial z_{i_1}{}^{m_{i_1}}\cdots\partial z_{i_k}{}^{m_{i_k}}}\ \textit{if}\ m_{i_1}+\cdots+m_{ik}>M.$$

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