Some Extensions of Borel Transformation

By AKIRA ASADA

Department of Mathematics, Faculty of Science,
Shinshu University
(Received Oct. 5, 1974)

Introduction.

In this note, we give some extensions of Borel transformation.

Borel transformation is defined by

\[ \mathcal{B} \left[ \varphi(z) \right] \zeta = \sum_{i_1, \ldots, i_n} \frac{a_{i_1, \ldots, i_n}}{i_1! \cdots i_n!} \zeta_1^{i_1} \cdots \zeta_n^{i_n}, \]

where \( \varphi(z) = \sum a_{i_1, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n} \) is a germ of holomorphic functions at the origin.

To denote the ring of germs of holomorphic functions at the origin by \( \mathcal{O}_n \), \( \mathcal{B} \) gives a ring isomorphism of \( \mathcal{O}_n \) and \( \text{Exp}(\mathbb{C}^n) \), where \( \text{Exp}(\mathbb{C}^n) \) is the ring of finite exponential type functions on \( \mathbb{C}^n \) with the multiplication \( f \# g \), where

\[ (f \# g)(\zeta) = \frac{d}{d\zeta} \left[ \int_0^\zeta f(\zeta - r)g(r)dr \right]. \]

Since the algebraic closure \( \overline{\mathbb{A}}_n \) of the quotient field of \( \mathcal{O}_n \) is the field of convergence Puiseux series, \( \mathcal{B} \) is extended to a map of \( \overline{\mathbb{A}}_n \) if we define \( \mathcal{B} \left[ z_1^{1/p} \right] \). This is done to define \( \mathcal{B} \left[ z_1^{1/p} \right] = (1/p)(1+1/p)\zeta_1^{1/b} \), because we get

\[ \zeta_a \# \zeta_b = \frac{\Gamma(a + 1)\Gamma(b + 1)}{\Gamma(a + b + 1)} \zeta_a^{a+b}. \]

But, since some elements of the quotient field of \( \text{Exp}(\mathbb{C}^n) \) is not a function, we define \( \mathcal{B} \) on \( \overline{\mathbb{A}}_n \) to satisfy \( \mathcal{B} \left[ \varphi \right] \) to be a function. Then, the solution of Cauchy problem \( P(\partial/\partial \zeta) f = 0, \quad \partial^k f / \partial \zeta^k |_{\zeta=0} = g_{k+1} \in \text{Exp}(\mathbb{C}^{n-1}), \quad k = 0, 1, \ldots, m-1 \), \( P(z) = z_1^m + P_1(z_1, \ldots, z_n) z_1^{m-1} + \cdots + P_m(z_1, \ldots, z_n) \) is given by

\[ f(\zeta) = \mathcal{B} \left[ \sum_{l=1}^{m} \sum_{r \neq r_l} (1 - z_l \partial_l (z_2^{-1}, \ldots, z_n^{-1}))^{-1} \varphi_{i_l, r_l (z_2, \ldots, z_n)} \right] (\zeta), \]
\[ P(z) = \prod_i \left( z_i - \sigma_i(z_k, \cdots, z_n) \right)^{r_i}, \sum_i \sum_{\nu_i} c_{h, \nu_i}(\sigma_i^{k} \varphi_i, \nu_i) = \mathcal{B}^{-1}[g_k], \]

where \( c_{h, \nu_i} \) is given by \((1 - x)^{-\nu_i} = \sum_k c_{h, \nu_i} x^k(\S 1)\).

Moreover, since we get
\[
\sum_{n} \frac{t^n}{n!} (\log x)^n n! = \frac{e^{-\gamma}}{\Gamma(1 + t)} x^t, \quad \gamma \text{ is Euler's constant,}
\]
to define
\[
\mathcal{B} \left[ \log x \right](\zeta) = \log \zeta + \gamma,
\]
we can extend Borel transformation for the functions which involve \( \log z \) (Appendix).

In \$2\$, we consider topological extension of Borel transformation. In fact, if \( F(D) \) is a function space on \( D \subset \mathbb{R} \) such that \( F(D) \) contains \( \text{Exp}(\mathbb{C}^n) \) (by the restriction map), \( \text{Exp}(\mathbb{C}^n) \) is dense in \( F(D) \) and if \( \{ f_m \}, \ f_m \in \text{Exp}(\mathbb{C}^n) \) converges uniformly to \( f \) on \( \mathbb{C}^n \) (in wider sense), then \( \{ f_m \} \) converges to \( f \) by the topology of \( F(D) \), then we can construct the largest subspace \( F(D)_{\mathcal{B}} \) of \( F(D) \) such that Cauchy problem is solved and well posed for the data in \( F(D)_{\mathcal{B}} \) and the smallest space \( F(D)_{\mathcal{B}} \) such that there is a homomorphism from \( F(D)_{\mathcal{B}} \) onto \( F(D) \) and for given operator, Cauchy problem is solvable and well posed for the data in \( F(D)_{\mathcal{B}} \), and Borel transformation is extended to have \( F(D)_{\mathcal{B}} \) (or \( F(D)_{\mathcal{B}} \)) to be its image and the solution of the Cauchy problem is written explicitly by this extended Borel transformation.

\section*{\S 0 \ Review of the properties of Borel transformation}

1. In this \$\S\$, we review the definition and properties of Borel transformation.

\textbf{Definition.} Let \( \varphi(z) \) be a germ of holomorphic function at the origin of \( \mathbb{C}^n \), the \( n \)-dimensional complex euclidean space, given by \( \varphi(z) = \sum_{i_1, \cdots, i_n} a_{i_1 \cdots i_n} z_1^{i_1} \cdots z_n^{i_n} \), then its Borel transformation \( \mathcal{B}[\varphi](\zeta) \) is a power series in \( \zeta = (\zeta_1, \cdots, \zeta_n) \) given by

\[
\mathcal{B}[\varphi](\zeta) = \sum_{i_1, \cdots, i_n} \frac{a_{i_1 \cdots i_n}}{i_1! \cdots i_n!} \zeta_1^{i_1} \cdots \zeta_n^{i_n}.
\]

By definition, Borel transformation has the following properties.

(i). If \( \varphi(z) \) converges on \( \{ z | |z_i| \leq \varepsilon_i \} \), then
\[\mathcal{B}[\varphi](\zeta) = \frac{1}{(2\pi i)^n} \int \cdots \int_{|z_1|<1} \cdots \int_{|z_n|<1} \frac{\zeta^m}{2 \pi i} e^{\varphi(z)} dz_1 \cdots dz_n,\]

\[\zeta = \frac{\zeta_1}{z_1} + \cdots + \frac{\zeta_n}{z_n}.\]

(ii). \(\mathcal{B}[\varphi](\zeta)\) is a finite exponential type function on \(C^n\) and if \(f(\zeta)\) is a finite exponential type function on \(C^n\), then there is unique germ of holomorphic function \(\phi(z)\) at the origin of \(C^n\) such that \(f(\zeta) = \mathcal{B}[\phi](\zeta)\).

(iii). If \(\varphi, \psi\) are germs and \(a, b\) are constants, then

\[\mathcal{B}[a \varphi + b \psi] = a \mathcal{B}[\varphi] + b \mathcal{B}[\psi],\]

where \(f g\) is given by

\[\mathcal{B}[f g](\zeta) = \frac{\partial^n}{\partial \zeta_1 \cdots \partial \zeta_n} \int_0^\zeta \cdots \int_0^\zeta f(\zeta_1 - \tau, \ldots, \zeta_n - \tau_n) g(\tau_1, \ldots, \tau_n) d\tau_1 \cdots d\tau_n.\]

(iv). To define \(\mathcal{B} \otimes \mathcal{D}(z_1, \ldots, z_{n+m}) = \mathcal{D}(z_1, \ldots, z_n) \mathcal{D}(z_{n+1}, \ldots, z_{n+m}), \) etc., we have

\[\mathcal{B}[\varphi \otimes \psi] = (\mathcal{B}[\varphi]) \times (\mathcal{B}[\psi]).\]

(v). For any \(i\), we get

\[\frac{\partial}{\partial \zeta_i} \mathcal{B}[\varphi](\zeta) = \mathcal{B}[(z_i^{-1} \varphi)_i]\](\zeta),\]

\[\int_0^{\zeta_i} \mathcal{B}[\varphi](\zeta) d\zeta_i = \mathcal{B}[z_i \varphi](\zeta).\]

Here for \(\phi(z) = \sum_{i_1=0, \ldots, i_n=0}^{i_1, \ldots, i_n=\infty} a_{i_1, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n} \phi_+,\) means

\[\phi_+(z) = \sum_{i_1 \geq 0, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n}.\]

(vi). For any \(i\), we get

\[\zeta_i \mathcal{B}[\varphi(z)](\zeta) = \mathcal{B}[z_i \varphi(z)] + z_i \frac{2 \partial \varphi(z)}{\partial z_i}(\zeta).\]

By (ii) and (iii), to denote \(\mathcal{O}^n\) the ring over \(C\) of germs of holomorphic functions of \(C^n\) at the origin with the usual addition and multiplication and by \(\text{Exp}(C^n)\) the ring over \(C\) of finite exponential type functions on \(C^n\) with the usual addition and the \(\otimes\) product, we get a ring isomorphism \(\mathcal{B}\) over \(C\) by
\( \mathcal{B} : \mathbb{C}^n \to \text{Exp}(\mathbb{C}^n) \), \( \mathcal{B}[\phi] \) is the Borel transformation of \( \phi \).

2. As usual, we denote by \( \mathcal{E} \mathbb{R}^n \), the space of compact carrier distributions on \( \mathbb{R}^n \). For \( T \in \mathcal{E} \mathbb{R}^n \), we define a map \( \iota_\alpha \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \), is fixed, by

\[
\iota_\alpha(T)(x) = \frac{1}{(2\pi \sqrt{-1})^n} T \left[ \frac{1}{1 - \alpha_1 z_1} \ldots \frac{1}{1 - \alpha_n z_n} \right].
\]

We note that to define \( \iota(T)(w) \) by

\[
\iota(T)(w) = \frac{1}{w_1 \ldots w_n} \iota_\alpha(T) \left( \frac{1}{\alpha_1 z_1}, \ldots, \frac{1}{\alpha_n z_n} \right), \quad w_i = \frac{1}{\alpha_i z_i},
\]

that is, \( \iota(T)(w) = 1/(2\pi \sqrt{-1})^n \cdot T \left[ 1/(w_1 - \zeta_1) \ldots (w_n - \zeta_n) \right] \), we get

\[
T[f] = \lim_{\epsilon_1, \ldots, \epsilon_n \to 0} \left( \sum_{\epsilon_1 = 0}^1 \ldots \sum_{\epsilon_n = 0}^1 (-1)^{\epsilon_1 + \cdots + \epsilon_n} \iota(T)(x_1 + (-1) \epsilon_1 z_1, \ldots, x_n + (-1)^n \epsilon_n z_n) f(x_1, \ldots, x_n) dx_1 \ldots dx_n \right),
\]

if \( f \in \mathcal{E} \mathbb{R}^n \) \([4], \ [9], \ [10]\).

By the definitions of \( \mathcal{B} \) and \( \iota_\alpha \), if we take \( \alpha = -2\pi \sqrt{-1} = (-2\pi \sqrt{-1}, \ldots, -2\pi \sqrt{-1}) \), we have

\[
\mathcal{F}[T] = \mathcal{B}[\iota_{-2\pi \sqrt{-1}}(T)].
\]

Where \( \mathcal{F} \) is the Fourier transformation of \( T \). In other word, we have the following commutative diagram.

\[
\begin{array}{ccc}
\mathcal{E} \mathbb{R}^n & \xrightarrow{\iota_{-2\pi \sqrt{-1}}} & \mathbb{C}^n \\
\alpha & \downarrow \mathcal{B} & \uparrow \mathcal{F} \\
& \mathcal{F}[\cdot] & \text{Exp}(\mathbb{C}^n). \\
\end{array}
\]

Note. We denote by \( A^n \) and \( \mathfrak{A}^n \) the spaces of real analytic functions on \( \mathbb{R}^n \) and entire functions on \( \mathbb{C}^n \) with the normally convergence topology. Then, since \( \mathcal{E} \mathbb{R}^n \supseteq A^n \supseteq \mathfrak{A}^n \), we have \( \mathcal{E} \mathbb{R}^n \subset A^n \subset \mathfrak{A}^n \), where \( A^n \) and \( \mathfrak{A}^n \) are the dual spaces of \( A^n \) and \( \mathfrak{A}^n \), and \( \iota_\alpha \) is defined on \( A^n \) and \( \mathfrak{A}^n \). Moreover, we know \([5], \ [7]\),

\[
iota_\alpha : \mathfrak{A}^n \subseteq \mathbb{C}^n,
\]

and the duality between \( \mathfrak{A}^n \) and \( \mathfrak{A}^n \) is given by

\[
\iota_\alpha : \mathfrak{A}^n \subseteq \mathbb{C}^n,
\]
Some Extensions of Borel Transformation

\[
\langle f, \varphi \rangle = \frac{(-1)^n}{(2\pi i)^n} \int \cdots \int_{|z_1| = \epsilon_1}^{\cdots} \cdots \int_{|z_n| = \epsilon_n}^{\cdots} \frac{1}{z_1 \cdots z_n} f(z_1, \ldots, z_n) \\
\varphi \left( \frac{1}{z_1}, \ldots, \frac{1}{z_n} \right) dz_1 \cdots dz_n \quad , \quad f \in \mathbb{C}^n, \varphi \in \mathcal{O}_n ,
\]

if \( \varphi \) is holomorphic on \( \{ z | |z| \leq \varepsilon_i \} \).

\section{Algebraic extension of Borel transformation}

3. In this \( \S \), we extend Borel transformation to be a map from the algebraic closure (of the quotient field) of \( \mathcal{O}_n \) to the algebraic closure (of the quotient field) of \( \text{Exp}(\mathbb{C}^n) \).

First we note that the algebraic closure \( \mathcal{M}_n \) of \( \mathcal{M}_n \), the quotient field of \( \mathcal{O}_n \), is the (convergence) Puiseux series field of \( n \)-variables over \( \mathbb{C} \), that is

(14) \( \text{Gal}(\mathcal{M}_n, \mathcal{M}_n) = \mathbb{Q}/\mathbb{Z} \oplus \cdots \oplus \mathbb{Q}/\mathbb{Z} \).

This can be shown by algebraic method (cf. [6]). But here we give an analytic proof. For this purpose, we use

\textbf{Lemma 1.} If \( f(z) \) is holomorphic on \( \{ z | |z| < \varepsilon_i \} \), then there exist \( 0 \leq \varepsilon_i < \varepsilon'_i = a_i, \ i = 1, \ldots, n \) such that \( f(z) \neq 0 \) if \( \varepsilon_i < |z_i| < \varepsilon_i' \), unless \( f(z) \) is identically equal to 0.

\textbf{Proof.} Since the lemma is true for \( n = 1 \), we use induction and assume the lemma is true for \((n-1)\)-variables functions. Then to set \( f(z) = z_1 h(z), \ h(0, z_2, \ldots, z_n) \) is not identically equal to 0, there exist \( a_i > 0, \ i = 2, \ldots, n \), such that \( h(0, z_2, \ldots, z_n) \) does not vanish on \( T = \{ z \ | \ |z_1| = 0, |z_i| = a_i, \ i \geq 2 \} \). Then, since \( \min_{z \in T} |h(z)| \geq 0 \), there exists \( \varepsilon'_i > 0 \) such that \( |z_i| < \varepsilon'_i \), \( |z_i| = a_i, \ i \geq 2 \). This shows the lemma.

\textbf{Corollary.} If \( g(z) \in \mathcal{M}_n \), then \( g(z) \) is expressed as

(15)' \( g(z) = \sum_{i_1 = -\infty, \ldots, i_n = -\infty}^{i_1 = \infty, \ldots, i_n = \infty} a_{i_1, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n}, \ \varepsilon_i < |z_i| < \varepsilon'_i . \)

\textbf{Note.} Since \( g(z) \) is meromorphic, although there may by \( \sup \lim_{i \to \infty} |a_{i_1, \ldots, i_n} | \neq 0 \), there exists an integer \( M \) such that

(16) \( a_{i_1, \ldots, i_n} = 0, \ if \ i_1 + \cdots + i_n < M \).

\textbf{Proof of (14).} If \( w \) is algebraic over \( \mathcal{M}_n \), then by lemma 1, \( w \) has no
singularity or branching point on $\Gamma = \{z| e_i < |z| < e_i' \text{ for some } 0 < e_i < e_i'\}$.

Then, since $\pi_1(\Gamma) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ and the Riemann surface $\hat{\Gamma}$ of $w$ over $\Gamma$ covers $\Gamma$ only finite times, there exist integers $r_1 \geq 1, \cdots, r_n \geq 1$ such that to set $G(\Gamma)$ the subgroup of $\pi_1(\Gamma)$ generated by $r_1 e_1, \cdots, r_n e_n, e_1, \cdots, e_n$ are the generator of $\pi_1(\Gamma)$, $\hat{\Gamma}/G(\Gamma)$ covers $\hat{\Gamma}$, where $\hat{\Gamma}$ is the universal covering space of $\Gamma$. Then, since $\hat{\Gamma}/G(\Gamma)$ and its projection $p: \hat{\Gamma}/G(\Gamma) \to \Gamma$ are given by

$$\hat{\Gamma}/G(\Gamma) = \{y | r_i \sqrt{e_i} < |y| < r_i \sqrt{z_i e_i'}\},$$

$$p((y_1, \cdots, y_n)) = (y_1 r_1, \cdots, y_n r_n), \text{ or } y_i = z_i^{1/r_i}, \ i = 1, \cdots, n,$$

$w$ can be expressed as a Puiseux series by (15)$'$, that is

$$w = \sum_{i_1=1, \cdots, i_n=1}^{a_{i_1}, \cdots, a_{i_n} \in \mathbb{Z}} a_{i_1} \cdots, a_{i_n} z_1^{i_1/r_1} \cdots z_n^{i_n/r_n},$$

$$a_{i_1}, \cdots, a_{i_n} = 0, \text{ if } i_1 + \cdots + i_n < M \text{ for some } M.$$

By (14)$'$ and (3)$'$ and (5)$'$, to extend Borel transformation on $\mathbb{R}^n$, it is sufficient to define Borel transformation of $z_i^{1/p}$ for any $i$ and $p$.

4. **Lemma 2.** If $\Re a > -1, \Re b > -1$, then

$$\zeta^a \# \zeta^b = \frac{\Gamma(a + 1) \Gamma(b + 1)}{\Gamma(a + b + 1)} \zeta^{a+b}.$$

Here, in the definition of $\#$-product, integral is taken along the path $\{t \zeta, 0 < t \leq 1\}$.

**Proof.** By definition, we get

$$\zeta^a \# \zeta^b = \frac{d}{d \zeta} \int_0^1 (\zeta - \tau)^{a+b} d\tau = \frac{d}{d \zeta} \int_0^1 \zeta^{a+b+1} (1 - \sigma)^a d\sigma \quad \text{(a = }) \frac{\tau}{\zeta} \text{ )}$$

$$= (a + b + 1) B(a + 1, b + 1) \zeta^{a+b} = \frac{\Gamma(a + 1) \Gamma(b + 1)}{\Gamma(a + b + 1)} \zeta^{a+b}.$$

**Corollary.** For any natural number $p$, we have

$$\zeta^{1/p} \# \zeta^{1/\overline{p}} = \{\Gamma(\frac{1}{p} + 1)\} \zeta.$$

**Proof.** By (17)$'$, we get

$$(\zeta^{1/p}) \# \zeta = \zeta^{1/p} \# \zeta^{1/\overline{p}} = \{\Gamma(\frac{1}{p} + 1)\} \zeta.$$
Some Extensions of Borel Transformation

\[ \zeta = \left( \Gamma \left( \frac{1}{p} + 1 \right) \right)^b \zeta. \]

Since \( \zeta^{(r-1)} \gamma f(\zeta) = df(\zeta)/d\zeta \) by (6), we have by (17)

\[ \zeta a \gamma z b = \frac{\Gamma(a + 1) \Gamma(b + 1)}{\Gamma(a + b + 1)} \zeta a+b, \]

where \( a \) and \( b \) are not negative integers.

By (14), (18) and (17), the algebraic closure \( \mathcal{E} \mathcal{V}(C^n) \) of the quotient field \( \mathcal{E} \mathcal{V}(C^n) \) of \( \operatorname{Exp}(C^n) \) is generated by \{ \( \zeta_1 \cdots \zeta_n = \zeta_1, \cdots, \zeta_n \) are rational numbers and none of \( \rho_i \) is a negative integer\} and \{ \( \zeta_1, \cdots, \zeta_m = \partial \zeta_1, \cdots, \partial \zeta_m \), \( k \geq 1, k + m = n, \{ i_1, \cdots, i_k \} \cup \{ j_1, \cdots, j_m \} = \{ 1, \cdots, n \} \) and none of \( \rho_i, i \in \{ i_1, \cdots, i_m \} \) is a negative integer\} as a \( \mathbb{C} \)-module.

We denote by \( \mathcal{E} \mathcal{V}(C^n)_+ \) the submodule of \( \mathcal{E} \mathcal{V}(C^n) \) consisted by those elements that are realized by (some multi-valued) function. That is, the element of \( \mathcal{E} \mathcal{V}(C^n) \) whose (any) Puiseux expansion does not involve the term which involves \( \zeta_i \) for some \( i \) and \( k \). By definition there is a projection \( \pi_\ast \) (as a \( \mathbb{C} \)-module) from \( \mathcal{E} \mathcal{V}(C^n) \) onto \( \mathcal{E} \mathcal{V}(C^n)_+ \).

Note. By definition, we have

\[ \mathcal{E} \mathcal{V}(C^n)_+ \cap \mathcal{E} \mathcal{V}(C^n) = \operatorname{Exp}(C^n), \]

and the integral closure \( \hat{\operatorname{Exp}}(C^n) \) (in \( \mathcal{E} \mathcal{V}(C^n) \)) is contained in \( \mathcal{E} \mathcal{V}(C^n)_+ \).

Since Borel transformation \( \mathcal{B} \) gives an isomorphism from \( C^n \) onto \( \operatorname{Exp}(C^n) \), it is extended to an isomorphism \( \mathcal{B} : \hat{\mathcal{M}}_n \xrightarrow{\cong} \mathcal{E} \mathcal{V}(C^n)_+ \). By (18) (and (17)), explicitly, \( \mathcal{B} \) is given by

\[ \mathcal{B} [z_i^{1/p}] = \frac{1}{\Gamma \left( \frac{1}{p} + 1 \right)} \zeta_i^{1/p}, \quad \mathcal{B} [z_i^{-n}] = \partial^n \zeta_i^{-n}. \]

To fix the above \( \mathcal{B} \), we define

**Definition.** The Borel transformation \( \mathcal{B} \) of \( \hat{\mathcal{M}}_n \) is the map from \( \hat{\mathcal{M}}_n \) onto \( \mathcal{E} \mathcal{V}(C^n)_+ \) given by

\[ \mathcal{B} [w] = \pi_\ast \mathcal{B} [w]. \]
By definition, if \( w \) is given by Puiseux series (15), then
\[
\mathcal{B} [w] (\zeta) = \sum_{i_1 = -\infty, \ldots, i_n = -\infty}^{i_1 = \infty, \ldots, i_n = \infty} \frac{a_{i_1, \ldots, i_n}}{\Gamma(i_1/r_1 + 1) \cdots \Gamma(i_n/r_n + 1)} \zeta_1^{i_1/r_1} \cdots \zeta_n^{i_n/r_n},
\]
where \( 1/\Gamma(i_1/r_1 + 1) \cdots \Gamma(i_n/r_n + 1) = 0 \) if some of \( i_k/r_k \) is a negative integer.

**Lemma 3.** (1). If \( \mathcal{B} [w] \) converges on \( \Gamma = \{ z | |z_i| < \varepsilon_i \} \) if \( w \) is given by (15) and it converges on \( \Gamma' \).

(ii). If \( u \) is integral over \( \partial \), then the Riemann surface of \( \mathcal{B} [u] \) covers \( \mathbb{C}^n \).

(iii). If \( \Psi \) belongs in \( \mathcal{B} \mathcal{P}(\mathbb{C}^n) \), then
\[
\mathcal{B} [\Psi] (\zeta) = \frac{1}{(2\pi \sqrt{-1})^n} \int_{|z_1| = \varepsilon_1} \cdots \int_{|z_n| = \varepsilon_n} \frac{1}{z_1 \cdots z_n} e^{\zeta/z} \Psi(z) dz_1 \cdots dz_n,
\]
if \( \Psi(z) \) is holomorphic on \( \{ z | |z_i| = \varepsilon_i \} \).

**Proof.** (i) follows from (16). Since \( \mathcal{B} [u] \) satisfies the equation
\[
\mathcal{B} [u] + \mathcal{B} [\varphi_1] + \mathcal{B} [u] + \cdots + \mathcal{B} [\varphi_m] = 0,
\]
if \( u \) satisfies the equation \( u^m + \varphi_1 u^{m-1} + \cdots + \varphi_m = 0 \), we have (ii) by (18) and the fact that each \( \mathcal{B} [\varphi_i] \) converges on \( \mathbb{C}^n \). (2)' follows from the definition.

Note. On \( \mathcal{F} (\mathbb{C}^n) \), to set \( y_i = z_i^{1/r_i} \), we set
\[
y_1 \cdots y_n = \sum_{i_1 \geq m_1, \ldots, i_n \geq m_n} \frac{y_1^{i_1} \cdots y_n^{i_n}}{\Gamma(i_1/r_1 + 1) \cdots \Gamma(i_n/r_n + 1)},
\]
where \( 1/\Gamma(i_1/r_1 + 1) \cdots \Gamma(i_n/r_n + 1) = 0 \) if some of \( i_k/r_k \) is a negative integer, then to set \( \eta_i = \xi_i^{1/r_i} \), we have
\[
\mathcal{B} [w] = \lim_{m_1, \ldots, m_n \to \infty} \left( \frac{1}{2\pi \sqrt{-1}} \right)^n \int_{|y_1| = \varepsilon_1} \cdots \int_{|y_n| = \varepsilon_n} \frac{1}{y_1 \cdots y_n} \sum_{r_1, \ldots, r_n} \frac{\eta_1^{r_1} \cdots \eta_n^{r_n}}{y_1^{r_1} \cdots y_n^{r_n}} w(y_1, \ldots, y_n) dy_1 \cdots dy_n.
\]

By (2)', we can show analytically if \( \sum a_{i_1, \ldots, i_n} z_1^{i_1/r_1} \cdots z_n^{i_n/r_n} \) is an analytic continuation of \( w \), then \( \sum a_{i_1, \ldots, i_n} / \Gamma(i_1/r_1 + 1) \cdots \Gamma(i_n/r_n + 1) \zeta_1^{i_1/r_1} \cdots \zeta_n^{i_n/r_n} \) is an analytic continuation of \( \mathcal{B} [w] \). In fact, since the branching points and poles of \( w \) are given by \( \varphi(z) = 0 \), \( z \in \partial \), to set
\[
\varphi(z) = \sum_I z^I \varphi(z), \quad I = (i_1, \ldots, i_n), \quad z^I = z_1^{i_1} \cdots z_n^{i_n}, \quad \varphi(z) \neq 0,
\]
any Puiseux expansion of \( w \) covers a connected component \( \Gamma_i \) of \( U(0) - \{ z | |z| = \})
Some Extensions of Borel Transformation

$= |z^j| \text{ for some } I, I'$. But if $z_0 \in \partial I_i$ and $\phi(z_0) \neq 0$, the Riemann surface of $w$ which covers such $I'_j$ that $z_0 \in \partial I'_j$ can be extended to cover $z_0$ and since on which $(2)^\prime$ is hold, we have the assertion.

5. Definition. We set $\widehat{B}^{-1} \pi_* \widehat{B} = \rho$, and set

\[(21) \rho, w = w_* \text{.}\]

**Theorem 1.** Borel transformation (of $\widehat{M}$) has the following properties.

(i). $B[w] = 0$ if and only if $w$ belongs in ker. $p_\pi$, that is, each term of Puiseux expansion of $w$ involve negative power of some $z_i$.

(ii). If $v, w \in \widehat{M}$ and $a, b$ are constants, then

\[(3) ii. f(a v + b w) = a f[v] + b f[w], \]

\[(3) ii. f(v w) = \pi_\pi (B[v] \# B[w]). \]

In (3)ii', if $v, w$ both contained in $\widehat{\mathcal{M}}$, the integral closure of $\mathcal{O}_n$, then

\[(3) ii. f(v w) = f[v] \# f[w]. \]

(iii). To define $v \otimes w$, etc., similarly as $\phi \otimes \phi$, we have

\[(5) f(v \otimes w) = f[v] \otimes f[w]. \]

(iv). For any $i$, we get

\[(6) \frac{\partial}{\partial z_i} B[w] = B[z_i^{-1} w]. \]

\[(9) \zeta_i B[w] = B[z_i w + z_i \frac{2 \partial w}{\partial z_i}]. \]

**Theorem 2.** If $P(\partial/\partial \zeta)$ is a constant coefficients partial differential operator given by

\[P(\frac{\partial}{\partial \zeta}) = \frac{\partial^m}{\partial \zeta^m} + P_1(\frac{\partial}{\partial \zeta_1}, ..., \frac{\partial}{\partial \zeta_n}) \frac{\partial^{m-1}}{\partial \zeta_1 \partial \zeta_m} + ... + P_m(\frac{\partial}{\partial \zeta_1}, ..., \frac{\partial}{\partial \zeta_n}) \text{,} \]

then its solution with the data

\[\frac{\partial f}{\partial \zeta^k}(0, \zeta_2, ..., \zeta_n) = g_{k+1}(\zeta_2, ..., \zeta_n), 0 \leq k \leq m - 1, \text{ } g_k \in \text{Exp}(C^{n-1}) \text{,} \]

is given by

\[(22) f(\zeta) = B \left[ \sum_i \sum_{1 \leq i \leq n} \left(1 - z_i \sigma_i(z_2, ..., z_n)\right)^{-i} + \phi_i(z_2, ..., z_n) \right] \zeta. \]

This $f(\zeta)$ is holomorphic on $C^n$ if deg. $P_i \leq m - i$ for each $i$. Here
\[ P(z) = \prod_{i} \left( z_1 - \sigma_i (z_2, \ldots, z_n) \right)^{r_i}, \]

\[
\sum_{i} \sum_{1 \leq r_i \leq r} c_{k, r} (\sigma_i \varphi_i, \zeta) (z_2, \ldots, z_n) = G^{-1} [g_i] (z_2, \ldots, z_n), \quad 0 \leq k \leq m - 1,
\]

where \( c_{k, r} \) is given by \((1 - x)^{-r} = \sum_{k} c_{k, r} x^k.\)

In the rest, we set

\[
T_{r_1, \ldots, r_s}^{(r_1, \ldots, r_s)} = \begin{pmatrix}
1, \\
\sigma_1, \\
\sigma_1^2, \\
\sigma_1^m, \\
\sigma_1^2, \\
\sigma_2, \\
\sigma_2^2, \\
\sigma_2^m, \\
\sigma_2^2, \\
\sigma_2, \\
\sigma_2^2, \\
\sigma_2, \\
\sigma_2^2, \\
\sigma_2, \\
\sigma_2^2
\end{pmatrix}.
\]

Note. If in \( \text{Exp}(C^n) \), a system of constant coefficient partial differential operators is given, then by normalization theorem ([11]), is equivalent to the system of operators

\[
P_i \left( \frac{\partial}{\partial \zeta_j} \right) = \frac{\partial^{m_i}}{\partial \zeta_{i+1}^{m_i}} + P_{i, 1} \left( \frac{\partial}{\partial \zeta_{i+1}^{m_i}}, \ldots, \frac{\partial}{\partial \zeta_{m_i}} \right) \frac{\partial^{m_{i-1}}}{\partial \zeta_{i+1}^{m_{i-1}}} + \ldots +
\]

\[
+ P_{i, m_i} \left( \frac{\partial}{\partial \zeta_{i+1}^{m_i}}, \ldots, \frac{\partial}{\partial \zeta_{m_i}} \right), \quad 1 \leq i \leq h,
\]

by a change of variables. Then the solution of the overdetermined system \( \mathcal{B} \) with the data

\[
\frac{\partial^{k_i + \ldots k_h}}{\partial \zeta_{i+1}^{k_i} \ldots \partial \zeta_{i}^{k_i}} (0, \ldots, 0, \zeta_{j+1}, \ldots, \zeta_n) = g_{k_i+1}, \ldots, k_h+1 (\zeta_{j+1}, \ldots, \zeta_n),
\]

\[ 0 \leq k_i \leq m_i - 1, \quad g_{k_i}, \ldots, g_{k_h} \in \text{Exp}(C_n), \]

is given by

\[
f(\zeta) = \mathcal{B} \left[ \sum_{(i, j) \in \sigma_h} \sum_{j_h} \varphi_{j_h} (z_{h+1}^{-1}, \ldots, z_n^{-1})^{-\varphi_h, j_h} \ldots
\]

\[
(1 - z_{h+1}^{j_h}, \ldots, z_n^{j_h})^{-\varphi_h, j_h, j_h} (z) \zeta) \right],
\]

\[
P_i (z) = \prod_{j} (z_i - \sigma_i (z_{h+1}, \ldots, z_n))^{r_i, j},
\]
Some Extensions of Borel Transformation

\[
\sum_{(i, j) \leq (i', j')} \sum_{r, s \geq 1} c_{r, s, i, j} \cdots c_{r, s, i, j} \left( \sigma_{r, s, i, j} \right)^{k_{1}} \cdots \left( \sigma_{r, s, i, j} \right)^{k_{h}}
\]

we note that this last coefficients matrix is given by \( T \left( \sigma_{r, s, i, j} \right) \otimes \cdots \otimes T \left( \sigma_{r, s, i, j} \right) \).

As in the single equation case, if \( \text{deg. } P_{i, k} \leq m_{i} - k \) for each \( i \) and \( k \), then this \( f \) is holomorphic on \( \mathbb{C}^{n} \).

\section{2 Topological extension of Borel transformation}

6. Let \( D \) be a subset of \( \mathbb{R}^{n} \) such that \( \text{Int. } D \neq \emptyset \) and \( F(D) \) is a complete topological vector space (over \( \mathbb{C} \)) consisted by the functions on \( D \) and satisfy

(i). \( r_{D}(f) = f|_{D} \), the restriction of \( f \) on \( D \) belongs in \( F(D) \) if \( f \in \text{Exp}(\mathbb{C}^{n}) \).

(ii). \( \{ r_{D}(f) \mid f \in \text{Exp}(\mathbb{C}^{n}) \} \) is dense in \( F(D) \).

We note that by assumption, \( r_{D} : \text{Exp}(\mathbb{C}^{n}) \rightarrow F(D) \) is an (into) isomorphism.

**Definition.** To regard \( \text{Exp}(\mathbb{C}^{n}) \) to be a subspace of \( F(D) \) by the map \( r_{D} \), the induced topology of \( \text{Exp}(\mathbb{C}^{n}) \) from \( F(D) \) is called \( F(D) \) – topology of \( \text{Exp}(\mathbb{C}^{n}) \). If a series \( \{ f_{m} \} \) of the elements of \( \text{Exp}(\mathbb{C}^{n}) \) converges to \( f \) by this topology, then we denote \( F(D) - \lim m \rightarrow \infty f_{m} = f \).

By definition, to denote the completion of \( \text{Exp}(\mathbb{C}^{n}) \) by \( F(D) \) – topology by \( \text{Exp}(\mathbb{C}^{n})^{\#} \) (or \( \text{Exp}(\mathbb{C}^{n})^{\#} F(D) \)), we have

\[ r_{D}^{\#} : \text{Exp}(\mathbb{C}^{n})^{\#} \cong F(D). \]

**Example.** If \( D \) is a bounded domain, then for all \( p \), \( L^{p}(D) \) can be taken as \( F(D) \). The \( k \)-th Sobolev space \( L^{k}(D) \) and \( C^{k}(D) \) (with the \( C^{k} \) topology) can also be taken as \( F(D) \). The \( k \)-th local Sobolev space \( L^{k, k \text{loc.}}(\mathbb{R}^{n}) \) or \( C^{k}(\mathbb{R}^{n}) \) are also taken as \( F(D) \). Here, \( k \) might be negative.

Since Borel transformation \( \mathcal{B} \) is an isomorphism between \( \mathcal{O}_{n} \) and \( \text{Exp}(\mathbb{C}^{n}) \), \( \mathcal{B}^{-1} \) induces \( F(D) \) – topology of \( \text{Exp}(\mathbb{C}^{n}) \) to \( \mathcal{O}_{n} \). It is also called \( F(D) \) – topology of \( \mathcal{O}_{n} \) and if \( \{ \varphi_{m} \} \), \( \varphi_{m} \in \mathcal{O}_{n} \) converges to \( \varphi \) by \( F(D) \) – topology, we also denote \( F(D) - \lim m \rightarrow \infty \varphi_{m} = \varphi \).

By \( n^{\#} \) and (23), to denote \( \mathcal{O}_{n}^{\#} \), etc., the completions of \( \mathcal{O}_{n} \), etc., by \( F(D) \) – topology, we have the following commutative diagram.

\[
\begin{array}{ccc}
\mathcal{E} \mathbb{R}^{n} & \overset{\mathcal{E}^{\#}, -2\pi i}{\longrightarrow} & \mathcal{O}_{n}^{\#} \\
\downarrow \mathcal{E}^{\#} & & \mathcal{B}^{*} \downarrow \mathcal{B}^{*} \equiv \mathcal{O}^{*} \equiv \mathcal{O}^{*} \\
F(D) & \overset{r_{D}^{\#}}{\longrightarrow} & \text{Exp}(\mathbb{C}^{n})^{\#} \end{array}
\]
Note. If we consider $\text{Exp}(C^n)$ to be a topological vector space by the compact open topology of $C^n$, the completion of $\text{Exp}(C^n)$ is $\mathbb{B}^n$, the space of entire functions on $C^n$, and the completion of $\mathcal{O}_n$ by this topology (induced by $\mathbb{B}$) is $\text{Exp}(C^n)'$, the dual space of $\text{Exp}(C^n)$, and the extended Borel transformation $\mathbb{B}$ is $\mathbb{B}^1$, the dual map of $\mathbb{B} : \mathcal{O}_n \to \text{Exp}(C^n)$.

Lemma 4. If $f \# g$ is defined in $F(D)$ for any $f, g \in F(D)$ and the $\#$ product is continuous in $F(D)$, then $\mathcal{O}_n^*$ is a ring (by the usual multiplication) and we have

$$\mathbb{B}^*[\varphi \psi] = (r_D \mathbb{B}^*[\varphi]) \# (r_D \mathbb{B}^*[\psi])$$

7. We set

$$\mathbb{R}^n = (\mathbb{R} \times \mathbb{Z})^n = (\mathbb{R} \times \mathbb{Z})^n \times (\mathbb{R} \times \mathbb{Z}),$$

$p((x_1, m_1), \ldots, (x_n, m_n)) = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $(x_1, m_1, \ldots, x_n, m_n) \in \mathbb{R}^2$.

By Definition, $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ and we set

$$\mathbb{R}^n(r) = \mathbb{R}^n(G(r), \tilde{D} = p^{-1}(D), D(r) = \tilde{D}/G(r), r = (r_1, \ldots, r_n).$$

The projection from $\mathbb{R}^n(r)$ (or $D(r)$) onto $\mathbb{R}^n$ (or $D$) is denoted by $p_r$.

Since $D(r)$ is a $G(r)$ direct sum of $D$, we define $F(D(r))$ as the $G(r)$ direct sum of $F(D)$. Then if $r|r'$, that is $r_i|r_i'$ for all $i$, there is a map $p^{r|r} : F(D(r)) \to F(D(r'))$ and since

$$p^{r|r'} p^{r'|r'} = p^{r|*,} \text{ if } r|r' \text{ and } r'|r'',$

we define $F(\tilde{D})$ by

$$F(\tilde{D}) = \lim[F(D(r)), p^{r|r}].$$

By definition, we can define $\tilde{r}_D : \text{Exp}(C^n) \to F(\tilde{D})$. More general, if $\varphi \in \mathbb{C}(C^n)$, has no singularity on $D$ and $F(D)$ satisfies (i) of $n_0$, then $\tilde{r}_D(\varphi)$ is defined and belongs in $F(\tilde{D})$.

Definition. Let $S$ be a subset of $\mathcal{H}_n$ which contains 1, $F(D)$ a function space such that

$$\tilde{r}_D(\mathbb{B}^*[\varphi \sigma]) \in F(\tilde{D}), \text{ if } \varphi \in \mathcal{O}_n, \sigma \in S.$$  

Then we call $\{f_m, f_m \in \text{Exp}(C^n), \text{ converges to } f \text{ by the } F(D) - \text{topology with respect to } S \text{ if } [\tilde{r}_D(\mathbb{B}^*[\mathbb{B}^{-1}[f_m]]), \text{ converges to } \tilde{r}_D(\mathbb{B}^*[\mathbb{B}^{-1}[f]] \text{ in } F(\tilde{D}) \text{ for any } \sigma \in S \text{ and denote } F(D)_S \text{ -lim } f_m = f.$$

If $F(D)_S \text{ -lim } \mathbb{B}_{[\varphi_m]} = \mathbb{B}_{[\varphi]}$, then we denote $F(D)_S \text{ -lim } \varphi_m = \varphi.$
Example. We take $C(R^1)$ as $F(D)$ $(n = 1)$. If $S = \{(1 + a^2)^ {-1} | a \in R\}$, we have

$$C(R^1)_S \lim_{n \to \infty} f_m = f \text{ if and only if } C(R^1) \lim_{n \to \infty} f_m = f.$$ 

On the other hand, if $S = \{(1 + \sqrt{-1}a)^ {-1} | a \in R\}$, we have

$$C(R^1)_S \lim_{n \to \infty} f_m = f \text{ if and only if } \{f_m\} \text{ converges uniformly to } f \text{ on } C^i.$$ 

These may be two extremal cases and in the rest, we assume $F(D)$ satisfies (iii). If $\{f_m\}$ converges uniformly to $f$ on $C^n$, then $r_D(f_m)$ converges to $r_D(f)$ in $F(D)$.

8. By (iii), denoting $U(0)$ a neighborhood of $\{0\}$ by $F(D)$-topology, to set

$$U_S(0) = \{g|B \frac{1}{2} [B^{-1}g, \sigma] \in U(0), \sigma \in S\},$$

$U_S(0)$ contains $\{g | |g(z)| < \epsilon, z \in K, a \text{ compact set in } C^n\}$ for some $\epsilon > 0$ and $K \neq \emptyset$.

We denote the vector space of all Cauchy sequences of the elements of $\text{Exp} (C^n)$ by $F(D)$-topology by $F(D) - \text{Exp}(C^n)$. We consider $F(D) - \text{Exp}(C^n)$ to be a topological vector space to take

$$U(\{f_m\}) = \{g_m | g_m - f_m \in U_m(0), U_m(0) \text{ is a neighborhood of } 0 \text{ by } F(D) \text{-topology and } U_m(0) \supseteq U_{m+1}(0), \cap_{m} U_m(0) = \{0\} \}.$$ 

On the other hand, to take

$$U_S(\{f_m\}) = \{g_m | g_m - f_m \in U_{m+1}(0), U_{m+1}(0) \text{ is a neighborhood of } 0 \text{ by } F(D) \text{-topology and } U_{m+1}(0) \supseteq U_{m+2}(0), \cap_{m} U_m(0) = \{0\} \},$$

to be the neighborhood basis of $F(D) - \text{Exp}(C^n)$, $F(D) - \text{Exp}(C^n)$ also becomes a topological vector space. This space is denoted by $F(D) - \text{Exp}(C^n)_S$.

In $F(D) - \text{Exp}(C^n)$, we set

$$F(D)_S - \text{Exp}(C^n) = \{\{f_m\} | \{f_m\} \text{ is a Cauchy sequence with respect to } S\},$$

$$F(D)_S - \text{Exp}(C^n)0 = \{\{f_m\} | F(D)_S \lim_{n \to \infty} f_m = 0\},$$

$$F(D)_S - \text{Exp}(C^n)0 = \{\{f_m\} | F(D)_S \lim_{n \to \infty} f_m = 0\}.$$ 

The same spaces regarded as the subspaces of $F(D) - \text{Exp}(C^n)_S$ are denoted by $F(D)_S - \text{Exp}(C^n)_{S,S}, F(D) - \text{Exp}(C^n)_{S,S}$ and $F(D)_S - \text{Exp}(C^n)_{S,S}$.

**Lemma 5.** $F(D)_S - \text{Exp}(C^n)$ and $F(D)_S - \text{Exp}(C^n)_{S,S}$ are equal to $F(D)_{S,S} - \text{Exp}(C^n)_{S,S}$ and $F(D)_S - \text{Exp}(C^n)_{S,S}$ as topological vector spaces.

**Proof.** Since $F(D)_S - \lim(f_m - g_m) = 0$ if $\{g_m\} \in U(f_m)$ in $F(D)_S - \text{Exp}(C^n)$, $\{g_m\}$ should belong some $U_S(f_m)$ and we have the lemma.

**Lemma 6.** To set
We have the following commutative diagram with exact (as) topological vector spaces) columns and rows. Here the maps are induced by the natural inclusions and projections.

\[
\begin{array}{cccc}
0 & \rightarrow & F(D)_S & \rightarrow & F(D)_S & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & N_S & \rightarrow & F(D)^S & \rightarrow & F(D) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & N_S & \rightarrow & F(D)^S & \rightarrow & F(D)/F(D)_S & \rightarrow & 0 \\
\end{array}
\]

**Proof.** Since \( F(D) = F(D) - \text{Exp}(C^n) \) by the condition (ii) of \( n'\), we know

\[
(25) \quad F(D)_S - \text{Exp}(C^n) = F(D)_S - \text{Exp}(C^n) \cap F(D) - \text{Exp}(C^n) \,.
\]

we have the lemma by lemma 5.

In the rest, we denote by \( F(D)_S \) and \( F(D)^S \), the spaces constructed from \( F(D)_S \) and \( F(D)^S \) similarly as \( F(D) \).

9. For a series \( \{\varphi_m\} \) of the elements of \( \mathcal{O}_n \), we define \( F(D)_S - \lim_{m \to \infty} \varphi_m \) similarly as \( F(D)_S - \lim_{m \to \infty} \varphi_m \). Then we can define \( F(D) - \mathcal{O}_n, F(D)_S - \mathcal{O}_n, \) etc., similarly as \( F(D) - \text{Exp}(C^n), F(D)_S - \text{Exp}(C^n), \) etc., Then to define \( \mathcal{B} : F(D) - \mathcal{O}_n, F(D) - \text{Exp}(C^n), \) etc.,

\[
\mathcal{B} \left[ \{\varphi_m\} \right] = \left[ \mathcal{B} [\varphi_m] \right],
\]

\( \mathcal{B} \) maps \( F(D)_S - \mathcal{O}_n, F(D) - \mathcal{O}_n, \) etc., isomorphically onto \( F(D)_S - \text{Exp}(C^n), F(D) - \text{Exp}(C^n) \), etc. Moreover, \( \mathcal{B} \) can be regarded as the map from \( F(D) - \mathcal{O}_n, s \) onto \( F(D) - \text{Exp}(C^n) \). Hence to set \( \mathcal{O}_n, s = F(D)_S = \mathcal{O}_n, F(D)_S = \mathcal{O}_n, s \), \( \mathcal{B} \) induces maps

\[
\mathcal{B}_{s'} : \mathcal{O}_n, s \rightarrow F(D)_S,
\]

\[
\mathcal{B}_{s'} : \mathcal{O}_n \rightarrow F(D)^S.
\]

Then, if \( \varphi_S \, (\text{resp. } \varphi^S) \) is an element of \( \mathcal{O}_n, s \, (\text{resp. } \mathcal{O}_n^S) \) given by \( F(D)_S - \lim \varphi_m = \varphi_S \, (\text{resp. } F(D)_S - \lim \varphi_m = \varphi^S) \), for any \( \sigma \in S \), \( F(D)_S - \lim \varphi_{m \sigma} \, (\text{resp. } F(D)_S - \lim \varphi_{m \sigma}) \) exists as an element of \( F(D)_S \, (\text{resp. } F(D)^S) \), and to set
(26) \[ F(D)_S - \lim_{m \to \infty} \varphi_m = \varphi_S, \quad F(D)_{S^\delta} - \lim_{m \to \infty} \varphi_m = \varphi_{S^\delta}, \]

\( \mathcal{B}_S^I \) and \( \mathcal{B}_{S^\delta}^I \) are extended to maps
\[ \mathcal{B}_S^I : \mathcal{O}_{n, S} \langle S \rangle \to F(\hat{D})_S, \quad \mathcal{B}_{S^\delta}^I : \mathcal{O}_{n, S_\delta} \langle S \rangle \to F(\hat{D})_{S^\delta}. \]

Here, \( \mathcal{O}_{n, S} \langle S \rangle \) and \( \mathcal{O}_{n, S_\delta} \langle S \rangle \) are the completions of modules generated by \( \mathcal{O}_{n, S} \) (or \( \mathcal{O}_{n, S_\delta} \)) and \( S \) under the operations given by (26), by the topologies of \( F(D)_S \) and \( F(D)_{S^\delta} \).

**Theorem 3.** We assume \( D \) is given by \( \Omega \times k, \) where \( \Omega \) is an open set in \( \mathbb{R}^{n-1} \) (may be equal to \( \mathbb{R}^{n-1} \), \( K \) is a simply connected subset of \( \mathbb{C}^1 \) such that \( K \) contains either of intervals \( (a, b), [0, b), \) or \( (a, 0] \) \((a < 0 < b)\) in \( \mathbb{R}^1 \), and \( F(D) \) is given by
\[ F(D) := L(\Omega) \otimes_\pi A(K), \]
where \( V \otimes_\pi W \) means the completion of \( V \otimes W \) by \( \pi \) - topology (cf. [11]), \( A(K) \) is a space of analytic functions on \( K \) such that by the map \( r_K \), \( \text{Exp}(\mathbb{C}) \) is contained in \( A(K) \) with the variable \( \zeta_1, \) and \( L(\Omega) \) is a function space such that to satisfy (i), (ii) of \( n^\Omega \) and (iii) of \( n^\Omega \) for \( \text{Exp}(\mathbb{C}^{n-1}) \) with the variables \( \zeta_1, \ldots, \zeta_n \).

Let \( P(z) = z_1^m + P_1(z_2, \ldots, z_n)z_1^{m-1} + \cdots + P_m(z_2, \ldots, z_n) \) be a polynomial such that
\[ P(z) = \prod_{i} (z_1 - \sigma_i(z_2, \ldots, z_n))^{r_i}, \quad 1 \leq i \leq k, \quad \sum_{i=1}^{k} r_i = m, \]
and set
\[ S = \{(1 - z_1\sigma_i_1(z_2^{-1}, \ldots, z_n^{-1}))^{-\rho_i}, \ldots, (1 - z_1\sigma_i_1(z_2^{-1}, \ldots, z_n^{-1}))^{-\rho_i} \}, \quad 1 \leq i \leq k, \quad 1 \leq \rho_i \leq r_i \}.
\]

Then to set
\[ L(\Omega)_S = p_\partial(F(D)_S), \quad L(\Omega)_{S^\delta} = p_\partial(F(D)_{S^\delta}), \]
\[ p_\partial\{f_m\} = \{p_\partial f_m\}, \quad (p_\partial f)(\xi_2, \ldots, \xi_n) = f(0, \xi_2, \ldots, \xi_n), \]
for any data in \( L(\Omega)_S \) (rep. in \( L(\Omega)_{S^\delta} \)), the equation \( P(\partial/\partial \xi_1) f = 0 \) has unique solution in \( F(D)_S \) (resp. in \( F(D)_{S^\delta} \)) and it is well posed by the topology of \( F(D)_S \) (resp. \( F(D)_{S^\delta} \)).

**Proof.** By assumption, for the given data \( \{g_k\} \) in \( L(\Omega)_S \) (resp. in \( L(\Omega)_{S^\delta} \)), we can solve the equation
\[ \sum_k \sum_{1 \leq i \leq r_i} c_k, r_i (p_i, k\varphi, r_i) = \mathcal{B}_S^{-1}[g_k] \quad (\text{or } (\mathcal{B}_{S^\delta})^{-1}[g_k]), \quad 0 \leq k \leq m - 1, \]
Then to set
\[ f = \mathcal{B}_S^{-1} \left[ \sum_k \sum_{1 \leq i \leq r_i} (1 - z_1\sigma_i(z_2^{-1}, \ldots, z_n^{-1}))^{-\rho_i} \varphi, r_i \right] \]
(resp. \( s \sum_{l} \sum_{1 \leq i \leq L} (1 - z_{l}^{i} (z_{l}^{-1}, \cdots, z_{n}^{-1}))^{r} f_{l}^{i} \)) ,

we get a solution in \( F(D)_{S} \) (resp. in \( F(D)^{S} \)). But, since the solution is invariant under the covering transformation, \( f \) should belong in \( F(D)_{S} \) (resp. in \( F(D)^{S} \)).

Moreover, since \( T_{(r_{1}, \cdots, r_{s})} \) is ergular and operates continuously on \( L(\Omega)_{S}^{m} \), the \( m \)-direct sum of \( L(\Omega)_{S} \) (resp. on \( (L(\Omega)^{S})^{m} \)), we have the theorem.

**Note.** Similarly, starting from \( D = \Omega \times K, \Omega \subset \mathbb{R}^{n-k} \), \( K \subset \mathbb{C}^{k} \) and \( F(D) = L^{2}(\Omega) \otimes A(K) \), we get corresponding theorem for systems.

**Appendix. Borel transformation of \( \log z \).**

Since the universal covering space \( \tilde{\Gamma} \) of \( \Gamma = \{ z \mid \varepsilon_{1} < |z| < \varepsilon' \} \) is given by \( \{ w \mid \log \varepsilon_{1} < \Re w_{i} < \log \varepsilon' \} \) with the covering map \( (z_{1}, \cdots, z_{n}) = \exp w_{1}, \cdots, \exp w_{n} \), to extend Borel transformation for the functions on \( \tilde{\Gamma} \), it is sufficient to define \( \mathcal{B}[\log z] \). For this purpose, first we note, if \( \mathcal{B}[\log z] \) is defined, then by (9), \( \zeta \mathcal{B}[\log z] = [z \log z + z] \) and by (6), it must be

\[
\frac{d}{dz} \mathcal{B}[\log z] = 1.
\]

Therefore \( \mathcal{B}[\log z] = \log z + c \), if \( \mathcal{B}[\log z] \) is defined. To determine this constant, we use

**Lemma.** For \( t < 0 \), we get

\[
\sum_{n=0}^{\infty} t^{n} n! \frac{(\log z)^{#n}}{(1+t)^{n}} = \frac{e^{-t}}{1+t} x_{t},
\]

where \( \gamma \) is Euler's constant.

**Proof.** To set \( \log x^{#}(\log x)^{n-1} = \sum_{k=0}^{n} a_{n,k}(\log x)^{k} \), we get

\[
a_{n,n} = 1, \quad a_{n,n-1} = 0, \quad a_{n,k} = \frac{(n-1)!}{k!(n-k-1)!} a_{n-k,0}, \quad 2 \leq k \leq n-1,
\]

\[
a_{n,0} = (-1)^{n-1}(n-1)! \zeta(n), \quad n \geq 2, \quad \zeta(n) = \sum_{m=1}^{\infty} \frac{1}{m^{n}},
\]

because \( \int_{0}^{x} \log (x-t)(\log t)^{n-1} dt = \log x \int_{0}^{x} (\log t)^{n-1} dt - \sum_{m=1}^{\infty} \frac{1}{mx^{m}} \int_{0}^{x} t^{m}(\log t)^{n-1} dt \).

Then, to set \( \log x^{#} = \sum_{k=0}^{n} b_{n,k}(\log x)^{k} \), we get
Some Extensions of Borel Transformation

\[ b_{n,n} = 1, \quad b_{n,n-1} = 0, \quad b_{n,k} = \frac{n!}{k!(n-k)!} b_{n-k,n}, \quad 2 \leq k \leq n - 1, \]
\[ b_{n,0} = \sum_{s=1}^{[n/2]} \sum_{j_1, \ldots, j_s = n, j_i \geq 2} (-1)^{n-s} \frac{n! \zeta(j_1) \cdots \zeta(j_s)}{j_1(j_1 + j_2) \cdots (j_1 + \cdots + j_s)}, \quad n \geq 2. \]

Hence we get
\[ \sum_{n=0}^{\infty} \frac{t^n}{n!} (\log x)^n \]
\[ = (1 + \sum_{n=2}^{\infty} \sum_{s=1}^{[n/2]} \sum_{j_1, \ldots, j_s = n, j_i \geq 2} (-1)^{n-s} \frac{\zeta(j_1) \cdots \zeta(j_s)}{j_1(j_1 + j_2) \cdots (j_1 + \cdots + j_s)} t^n). \]
\[ = (\sum_{n=0}^{\infty} \frac{t^n}{n!} (\log x)^n). \]

But since we know \( \log(1 + t) = -\gamma t + \sum_{m=2}^{\infty} \frac{(-1)^m \zeta(m)}{m} t^m \), we obtain
\[ 1 + \sum_{n=2}^{\infty} \sum_{s=1}^{[n/2]} \sum_{j_1, \ldots, j_s = n, j_i \geq 2} (-1)^{n-s} \frac{\zeta(j_1) \cdots \zeta(j_s)}{j_1(j_1 + j_2) \cdots (j_1 + \cdots + j_s)} t^n \]
\[ = 1 + \sum_{n=2}^{\infty} \sum_{s=1}^{[n/2]} \sum_{j_1, \ldots, j_s = n, j_i \geq 2} (-1)^{n-s} \frac{\zeta(j_1) \cdots \zeta(j_s)}{s! j_1 \cdots j_s} t^n \]
\[ = \exp \left[ \log \frac{e^{-\gamma t}}{1 + t} \right] \]
\[ = \frac{e^{-\gamma t}}{1 + t}. \]

Hence we have the lemma.

**Definition.** We define the Borel transformation \( \mathcal{B} [\log z] (\zeta) \) of \( \log z \) by
\[ \mathcal{B} [\log z] (\zeta) = \log \zeta + \gamma. \]

By definition, if \( f(z) = \sum z_I a^I f_I(z), \quad I = (i_1, \ldots, i_k), a^I = (i_1/r_1, \ldots, i_k/r_k), \)
\( z_I^a = z_{i_1/r_1} \cdots z_{i_k/r_k} \) and \( f_I(0) \neq 0, \) then
\[ \mathcal{B} [\log f(z)] = \sum_{j=1}^{k} \frac{i_j}{r_j} (\log \zeta - \gamma) + \mathcal{B} [\varphi_I], \quad \varphi_I \in \mathcal{I}^+, \]
\[ \zeta \in \mathcal{I}^+ (\alpha_1, \ldots, \alpha_m) = \left\{ \zeta \mid |\zeta_{I_m}^a| > |\zeta_{J_m}^a|, \quad \alpha_1, \alpha_j \in (\alpha_1, \ldots, \alpha_m) \right\}. \]

In the rest, the corresponding set of \( \mathcal{I}^+ (\alpha_1, \ldots, \alpha_m) \) in the \( z \) space is also denoted by same notation and set
\[
\pi^{-1}(I^{aI}(a_1, \ldots, a_m)) = \tilde{I}^{aI}(a_1, \ldots, a_n).
\]

We consider following class \( \mathcal{H}^m \) of holomorphic functions on \((w_1, \ldots, w_n)\)-space such that

\((*)\). \( f \) is holomorphic on some open set \( D \) in \( \{ w_1 \ Re. w_i < \rho_i \} \) for some \( \rho_1, \ldots, \rho_n \) such that for any \( \delta_1, \ldots, \delta_n \) there exist \( r_1 < r_1' \leq \delta_1, \ldots, r_n < r_n' \leq \delta_n \) such that \( D \) contains \( \{ w_1 \ Re. w_i < r_i \} = T(r_1, r_1', \ldots, r_n, r_n') \).

If \( f_1 \) and \( f_2 \) both belongs in \( \mathcal{H}^n \), then we denote \( f_1 \sim f_2 \) if for any \( \delta_1, \ldots, \delta_n \), there exist \( r_1 < r_1' \leq \delta_1, \ldots, r_n < r_n' \leq \delta_n \) such that

\[
 f_1 \mid_{T(r_1, r_1', \ldots, r_n, r_n')} = f_2 \mid_{T(r_1, r_1', \ldots, r_n, r_n')}. 
\]

The set of this equivalence classes form an integral domain \( \mathcal{H} \) by natural way and to set the quotient field of \( \mathcal{H} \) by \( \mathcal{N} \), the elements of \( \mathcal{H} \) and \( \mathcal{N} \) both considered to be the germs of multi-valued analytic functions at the origin of \( z \)-space, where \( w_i = \exp z_i, i = 1, \ldots, n \). Similarly, we define the germ of those functions which are holomorphic on each \( T^{aI}(a_1, \ldots, a_m), k = 1, \ldots, m \), for some \( \langle a_1, \ldots, a_m \rangle \). The set of those germs form an integral domain and its quotient field is denoted by \( \mathcal{E} \). As the elements of \( \mathcal{H} \), we consider the elements of \( \mathcal{E} \) to be the germs of multi-valued functions of \( \xi \)-space. Then by the above, we can define Borel transformation \( B \) for the elements of \( \mathcal{H} \) to be the map from \( \mathcal{H} \) into \( \mathcal{E} \) and it also satisfies (3)i, (3)ii, (5), (6), (7) and (9).

Note. In this extended Borel transformation, although \( f(z) \) is analytic near the origin, \( B[f] \) may not be analytic on any neighborhood of the origin. if \( n \geq 2 \). For example, we have

\[
B[\log(z_2 + z_3)](\zeta_1, \zeta_3) = \log \zeta_1 + \gamma, \quad |\zeta_1| > |\zeta_3|,
\]

\[
= \log \zeta_3 + \gamma, \quad |\zeta_3| > |\zeta_1|.
\]

References


Some Extensions of Borel Transformation


