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Some Extensions of Borel Transformation

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Introduction.

In this note, we give some extensions of Borel transformation.

Borel transformation is defined by

$$\mathscr{B}[\varphi(z)](\zeta) = \sum_{i_1, \cdots, i_n} \frac{a_{i_1, \cdots, i_n}}{i_1! \cdots i_n!} \zeta_1^{i_1} \cdots \zeta_n^{i_n},$$

where $\varphi(z) = \sum a_{i_1, \cdots, i_n} z_1^{i_1} \cdots z_n^{i_n}$ is a germ of holomorphic functions at the origin. To denote the ring of germs of holomorphic functions at the origin by $\mathscr{O}_n$, $\mathscr{B}$ gives a ring isomorphism of $\mathscr{O}^n$ and $\mathrm{Exp}(\mathbf{C}^n)$, where $\mathrm{Exp}(\mathbf{C}^n)$ is the ring of finite exponential type functions on $\mathbf{C}^n$ with the multiplication $f \sharp g$, where

$$(f \sharp g)(\zeta) = \frac{d}{d\zeta} \int_0^\zeta f(\zeta - \tau) g(\tau) d\tau.$$

Since the algebraic closure $\tilde{\mathscr{M}}_n$ of the qtotient field of $\mathscr{O}_n$ is the field of (convergence) Puiseux series, $\mathscr{B}$ is extended to a map of $\tilde{\mathscr{M}}_n$ if we define $\mathscr{B}[z_1^{1/p}]$. This is done to define $\mathscr{B}[z_1^{1/p}] = (1/\Gamma(1+1/p))\zeta_1^{1/p}$, because we get

$$\zeta^a \sharp \zeta^b = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)} \zeta^{a+b}.$$

But, since some elements of the quotient field of $\mathrm{Exp}(\mathbf{C}^n)$ is not a function, we define $\mathscr{B}$ on $\tilde{\mathscr{M}}_n$ to satisfy $\mathscr{B}[\varphi]$ to be a function. Then, the solution of Cauchy problem $P(\partial/\partial\zeta) f = 0$, $\partial^k f / \partial^k \zeta_1 |_{s_1=0} = g_{k+1} \in \mathrm{Exp}(\mathbf{C}^{n-1})$, $k = 0, 1, \cdots, m-1$, $P(z) = z_1^m + P_1(z_2, \cdots, z_n) z_1^{m-1} + \cdots + P_m(z_2, \cdots, z_n)$ is given by

$$f(\zeta) = \mathscr{B}[\sum_i \sum_{1 \leq \rho_i = r_i} (1 - z_1 \sigma_i (z_2^{-1}, \cdots, z_n^{-1}))^{-\rho_i} \varphi_{i, \rho_i}(z_2, \cdots, z_n)](\zeta),$$

$$P(z)=\prod_i (z_1-\sigma_i(z_2,\cdots,z_n))^{r_1},\ \sum_i\sum_{\rho i} c_{k,\rho_1}(\sigma_i{}^k\varphi_{i,\rho_i})=\mathscr{B}^{-1}[g_k],$$

where c_{k,ρ_i} is given by $(1-x)^{-\rho_i}=\sum_k c_{k,\rho_i}x^k$(§ 1).

Moreover, since we get

$$\sum_n \frac{t^n}{n!}(\log x)^{\#n}=\frac{e^{-\gamma t}}{\Gamma(1+t)}x^t\quad,\quad \gamma \textit{ is Euler's constant},$$

to define

$$\mathscr{B}[\log z](\zeta)=\log\zeta+\gamma,$$

we can extend Borel transformation for the functions which involve $\log z$ (Appendix).

In § 2, we consider topological extension of Borel transformation. In fact, if $F(D)$ is a function space on D $(\subset \mathbf{R})$ such that $F(D)$ contains $\mathrm{Exp}(\mathbf{C}^n)$ (by the restriction map), $\mathrm{Exp}(\mathbf{C}^n)$ is dense in $F(D)$ and if $\{f_m\}$, $f_m\in \mathrm{Exp}(\mathbf{C}^n)$ converges uniformly to f on $\mathbf{C}^n$ (in wider sense), then $\{f_m\}$ converges to f by the topology of $F(D)$, then we can construct the largest subspace $F(D)_S$ of $F(D)$ such that Cauchy problem is solved and well posed for the data in $F(D)_S$ and the smallest space $F(D)^S$ such that there is a homomorphism from $F(D)^S$ onto $F(D)$ and for given operator, Cauchy problem is solvable and well posed for the data in $F(D)^S$, and Borel transformation is extended to have $F(D)_S$ (or $F(D)^S$) to be its image and the solution of the Cauchy problem is written explicitly by this extended Borel transformation.

§ 0 Review of the properties of Borel transformation

1. In this §, we review the definition and properties of Borel transformation.

Definition. *Let $\varphi(z)$ be a germ of holomorphic function at the origin of $\mathbf{C}^n$, the n-dimensional complex euclidean space, given by* $\varphi(z)=\sum_{i_1,\cdots,i_n} a_{i_1,\cdots,i_n} z_1{}^{i_1}\cdots z_n{}^{i_n}$, *then its Borel transformation $\mathscr{B}[\varphi](\zeta)$ is a power series in $\zeta=(\zeta_1,\cdots,\zeta_n)$ given by*

$$(1)\qquad \mathscr{B}[\varphi](\zeta)=\sum_{i_1,\cdots,i_n}\frac{a_{i_1,\cdots,i_n}}{i_1!\cdots i_n!}\zeta_1{}^{i_1}\cdots\zeta_n{}^{i_n}.$$

By definition, Borel transformation has the following properties.

(i). *If $\varphi(z)$ converges on $\{z\mid |z_i|\leq\varepsilon_i\}$, then*

(2) $$\mathscr{B}[\varphi](\zeta)=\frac{1}{(2\pi\sqrt{-1})^n}\int_{|z_1|=\varepsilon_1}\cdots\int_{|z_n|=\varepsilon_n}\frac{1}{z_1\cdots z_n}e^{\frac{\zeta}{z}}\varphi(z)dz_1\cdots dz_n\ ,$$

$$\frac{\zeta}{z}=\frac{\zeta_1}{z_1}+\cdots+\frac{\zeta_n}{z_n}\ .$$

(ii). *$\mathscr{B}[\varphi](\zeta)$ is a finite exponential type function on $\mathbf{C}^n$ and if $f(\zeta)$ is a finite exponential type function on $\mathbf{C}^n$, then there is unique germ of holomorphic function $\psi(z)$ at the origin of $\mathbf{C}^n$ such that $f(\zeta)=\mathscr{B}[\psi](\zeta)$.*

(iii). If φ, Ψ *are gems and* a, b *are constants, then*

(3)$_i$ $$\mathscr{B}[a\varphi+b\psi]=a\mathscr{B}[\varphi]+b\mathscr{B}[\psi]\ ,$$

(3)$_{ii}$ $$\mathscr{B}[\varphi\cdot\psi]=\mathscr{B}[\varphi]\sharp\mathscr{B}[\psi]\ ,$$

where $f\sharp g$ is given by

(4) $$(f\sharp g)(\zeta)=\frac{\partial^n}{\partial\zeta_1\cdots\partial\zeta_n}\int_0^{\zeta_1}\cdots\int_0^{\zeta_n}f(\zeta_1-\tau_1,\cdots,\zeta_n-\tau_n)g(\tau_1,\cdots,\tau_n)\,d\tau_1\cdots d\tau_n.$$

(iv). *To define $\varphi\otimes\Psi(z_1,\cdots,z_{n+m})=\varphi(z_1,\cdots,z_n)\Psi(z_{n+1},\cdots,z_{n+m})$, etc., we have*

(5) $$\mathscr{B}[\varphi\otimes\Psi]=(\mathscr{B}[\varphi])\times(\mathscr{B}[\psi])\ .$$

(v). *For any i, we get*

(6)$'$ $$\frac{\partial}{\partial\zeta_i}\mathscr{B}[\varphi](\zeta)=\mathscr{B}[(z_i{}^{-1}\varphi)_+](\zeta)\ ,$$

(7) $$\int_0^{\zeta_i}\mathscr{B}[\varphi](\zeta)\,d\zeta_i=\mathscr{B}[z_i\varphi](\zeta)\ .$$

Here for $\psi(z)=\sum_{i_1=-\infty,\cdots,i_n=-\infty}^{i_1=\infty,\cdots,i_n=\infty}a_{i_1,\cdots,i_n}z_1{}^{i_1}\cdots z_n{}^{i_n}$, ψ_+ means

(8) $$\psi_+(z)=\sum_{i_1\geqq 0,\cdots,i_n\geqq 0}a_{i_1,\cdots,i_n}z_1{}^{i_1}\cdots z_n{}^{i_n}\ .$$

(vi). *For any i, we get*

(9) $$\zeta_i\mathscr{B}[\varphi(z)](\zeta)=\mathscr{B}[z_i\varphi(z)'+z_1\frac{2\partial\varphi(z)}{\partial z_i}](\zeta)\ .$$

By (ii) and (iii), to denote $\mathscr{O}^n$ the ring over $\mathbf{C}$ of germs of holomorphic functions of $\mathbf{C}^n$ at the origin with the usual addition and multiplication and by Exp$(\mathbf{C}^n)$ the ring over $\mathbf{C}$ of finite exponential type functions on $\mathbf{C}^n$ with the usual addition and the $\sharp$ -product, we get a ring isomorphism $\mathscr{B}$ over $\mathbf{C}$ by

$\mathscr{B}: \mathscr{O}^n \to \mathrm{Exp}(\mathbf{C}^n)$, *$\mathscr{B}[\varphi]$ is the Borel transformation of φ.*

2. As usual, we denote by $\mathscr{E}_{\mathbf{R}^n}'$, the space of compact carrier distributions on $\mathbf{R}^n$. For $T \in \mathscr{E}_{\mathbf{R}^n}'$, we define a map ι_α, $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbf{C}^n$, is fixed, by

$$(10) \qquad \iota_\alpha(\mathrm{T})(z) = \frac{1}{(2\pi\sqrt{-1})^n} \mathrm{T}_\varsigma \left[\frac{1}{(1-\alpha_1\zeta_1 z_1)\cdots(1-\alpha_n\zeta_n z_n)}\right] .$$

We note that to define $\iota(T)(w)$ by

$$\iota(T)(w) = \frac{1}{w_1\cdots w_n}\iota_\alpha(T)\left(\frac{1}{\alpha_1 z_1}, \cdots, \frac{1}{\alpha_n z_n}\right), \quad w_i = \frac{1}{\alpha_i z_i},$$

that is, $\iota(T)(w) = 1/(2\pi\sqrt{-1})^n \cdot T[1/(w_1-\zeta_1)\cdots(w_n-\zeta_n)]$, we get

$$(11) \qquad T[f] = \lim_{\varepsilon_1, \cdots, \varepsilon_n \to 0} \frac{1}{(2\pi\sqrt{-1})^n} \int_{\mathbf{R}^n} \Big(\sum_{\sigma_1=0, \cdots, \sigma_n=0} (-1)^{\sigma_1+\cdots+\sigma_n} \iota(T)(x_1 + (-1)^{\sigma_1}\sqrt{-1}\varepsilon_1, \cdots, x_n + (-1)^{\sigma_n}\sqrt{-1}\varepsilon_n)\Big) f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n ,$$

if $f \in \mathscr{E}_{\mathbf{R}^n}$([4], [9], [10]).

By the definitions of $\mathscr{B}$ and ι_α, if we take $\alpha = -2\pi\sqrt{-1}$ $(=(-2\pi\sqrt{-1}, \cdots, -2\pi\sqrt{-1}))$, we have

$$(12) \qquad \mathscr{F}[T] = \mathscr{B}[\iota_{-2\pi\sqrt{-1}}(T)],$$

Where $\mathscr{F}$ is the Fourier transformation of T. In other word, we have the following commutative diagram.

$$\begin{array}{ccc} \mathscr{E}_{\mathbf{R}^n}' & \xrightarrow[\cong]{\iota_{-2\pi\sqrt{-1}}} & \mathscr{O}_n \\ & \searrow^{\mathscr{F}} & \downarrow \mathscr{B} \\ & & \mathrm{Exp}(\mathbf{C}^n) \end{array} .$$

Note. We denote by A^n and $\mathfrak{A}^n$ the spaces of real analytic functions on $\mathbf{R}^n$ and entire functions on $\mathbf{C}^n$ with the normally convergence topology. Then, since $\mathscr{E}_{\mathbf{R}^n} \supset A^n \supset \mathfrak{A}^n$, we have $\mathscr{E}_{\mathbf{R}^n}' \subset \mathbf{A}^{n\prime} \subset \mathfrak{A}^{n\prime}$, where $A^{n\prime}$ and $\mathfrak{A}^{n\prime}$ are the dual spaces of A^n and $\mathfrak{A}^n$, and ι_α is defined on $A^{n\prime}$ and $\mathfrak{A}^{n\prime}$. Moreover, we know ([5], [7]),

$$(13) \qquad \iota_\alpha : \mathfrak{A}^{n\prime} \cong \mathscr{O}_n ,$$

and the duality between $\mathscr{O}_n$ and $\mathfrak{A}^n$ is given by

$$<f, \varphi> = \frac{(-1)^n}{(2\pi\sqrt{-1})^n}\int_{|z_1|=\varepsilon_1^{-1}}\cdots\int_{|z_n|=\varepsilon_n^{-1}} \frac{1}{z_1\cdots z_n} f(z_1, \cdots, z_n)$$

$$\varphi\left(\frac{1}{z_1}, \cdots, \frac{1}{z_n}\right) dz_1, \cdots dz_n \quad , \qquad f \in \mathfrak{A}^n,\ \varphi \in \mathcal{O}_n \ ,$$

if φ is holomorphic on $\{z|\ |z_i| \leqq \varepsilon_i\}$.

§1 Algebraic extension of Borel transformation

3. In this §, we extend Borel transformation to be a map from the algebraic closure (of the quotient field) of $\mathcal{O}_n$ to the algebraic closure (of the quotient field) of $\mathrm{Exp}(\mathbf{C}^n)$.

First we note that the algebraic closure $\widetilde{\mathcal{M}}_n$ of $\mathcal{M}_n$, the quotient field of $\mathcal{O}_n$, is the (convergence) Puiseux series field of n-variables over $\mathbf{C}$, that is

(14) $$Gal\ (\widetilde{\mathcal{M}}_n/\mathcal{M}_n) = \overbrace{\mathrm{Q/Z} \oplus \cdots \oplus \mathrm{Q/Z}}^{n},$$

(14)′ $$\widetilde{\mathcal{M}}_n = \mathcal{M}_n(z_1^{1/2},\ z_1^{1/3}, \cdots,\ z_1^{1/p}, \cdots,\ z_2^{1/2}, \cdots,\ z_n^{1/2}, \cdots, z_n^{1/p}, \cdots).$$

This can be shown by algebraic method (cf. [6]). But here we give an analytic proof. For this purpose, we use

Lemma 1. *If $f(z)$ is holomorphic on $\{z|\ |z_i| < a_i\}$, then there exist $0 \leqq \varepsilon_i < \varepsilon_i' = a_i$, $i = 1, \cdots, n$ such that $f(z) \neq 0$ if $\varepsilon_i < |z_i| < \varepsilon_i'$, unless $f(z)$ is identically equal to* 0.

Proof. Since the lemma is true for $n = 1$, we use induction and assume the lemma is true for $(n-1)$-variables functions. Then to set $f(z) = z_1{}^k h(z)$, $h(0, z_2, \cdots, z_n)$ is not identically equal to 0, there exist $\alpha_1 > 0$, $i = 2, \cdots, n$, such that $h(0, z_2, \cdots, z_n)$ does not vanish on $T = \{z|\ z_1 = 0,\ |z_i| = \alpha_1,\ i \geqq 2\}$. Then, since $min._{z \in T} |h(z)| \gneqq 0$, there exists $\varepsilon_1' > 0$ such that $h(z) \neq 0$ if $|z_1| < \varepsilon_1'$, $|z_i| = \alpha_i$, $i \geqq 2$. This shows the lemma.

Corollary. *If $g(z) \in \mathcal{M}_n$, then $g(z)$ is expressed as*

(15)′ $$g(z) = \sum_{i_1=-\infty, \cdots, i_n=-\infty}^{i_1=\infty, \cdots, i_n=\infty} a_{i_1, \cdots, i_n} z_1{}^{i_1} \cdots z_n{}^{i_n},\quad \varepsilon_i < |z_i| < \varepsilon_i' \ .$$

Note. Since $g(z)$ is meromorphic, although there may by $sup - lim_{i_k \to -\infty} |a_{i_1}, \cdots, {}_{i_n}| \neq 0$, there exists an integer M such that

(16) $$a_{i_1}, \cdots, {}_{i_n} = 0,\ \ if\ i_1 + \cdots + i_n < M\ .$$

***Proof of* (14).** If w is algebraic over $\mathcal{M}_n$, then by lemma 1, w has no singularity or branching point on $\Gamma=\{z|\ \varepsilon_i<|z_i|<\varepsilon_i'\ \textit{for some}\ 0<\varepsilon_i<\varepsilon_i'\}$. Then, since $\pi_1(\Gamma)=\overbrace{Z\oplus\cdots\oplus Z}^{n}$ and the Riemann surface $\hat{\Gamma}$ of w over Γ covers Γ only finite times, there exist integers $r_1\geqq 1,\cdots, r_n\geqq 1$ such that to set $G(r)$ the subgroup of $\pi_1(\Gamma)$ generated by $r_1e_1,\cdots, r_ne_n$, $e_1,\cdots, e_n$ are the generator of $\pi_1(\Gamma)$, $\tilde{\Gamma}/G(r)$ covers $\hat{\Gamma}$, where $\tilde{\Gamma}$ is the universal covering space of Γ. Then, since $\tilde{\Gamma}/G(r)$ and its projection $p:\tilde{\Gamma}/G(r)\rightarrow\Gamma$ are given by

$$\tilde{\Gamma}/G(r)=\{y|\ r_i\sqrt{\varepsilon_i}<|y_i|<r_i\sqrt{\varepsilon_{-i}'}\}\ ,$$

$$p((y_1,\cdots,y_n))=(y_1{}^{r_1},\cdots,y_n{}^{r_n}),\quad or\ y_i=z_i{}^{1/r_i},\ i=1,\cdots,n\ ,$$

w can be expressed as a Puiseux series by $(15)'$, that is

$$w=\sum_{i_1=-\infty,\cdots,i_n=-\infty}^{i_1=\infty,\cdots,i_n=\infty}a_{i_1,\cdots,i_n}z_1{}^{i_1/r_1}\cdots z_n{}^{i_n/r_n}\ , \tag{15}$$

$$a_{i_1,\cdots,i_n}=0,\quad if\ i_1+\cdots+i_n<M\ for\ some\ M\ .$$

By $(14)'$ (and $(3)_i$ and (5)), to extend Borel transformation on $\tilde{\mathscr{M}}_n$, it is sufficient to define Borel transformation of $z_i{}^{1/p}$ for any i and p.

4. ***Lemma 2.*** *If* $Re.a>-1$, $Re.b>-1$, *then*

$$\zeta^a\ \sharp\ \zeta^b=\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}\zeta^{a+b}\ . \tag{17'}$$

Here, in the definition of $\sharp$ *-product, integral is taken along the path* $\{t\zeta,\ 0\leqq t\leqq 1\}$.

Proof. By definition, we get

$$\zeta^a\ \sharp\ \zeta^b=\frac{d}{d\zeta}\int_0^\zeta(\zeta-\tau)^a\tau^bd\tau=\frac{d}{d\zeta}\int_0^1\zeta^{a+b+1}(1-\sigma)^ad\sigma\quad(\sigma=\frac{\tau}{\zeta})$$

$$=(a+b+1)B(a+1,b+1)\zeta^{a+b}=\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}\zeta^{a+b}\ .$$

Corollary. *For any natural number* p, *we have*

$$(\zeta^{1/p})^{\sharp p}=\overbrace{\zeta^{1/p}\sharp\cdots\sharp\zeta^{1/p}}^{p}=\{\Gamma(\frac{1}{p}+1)\}^p\zeta. \tag{18}$$

Proof. *By* $(17)'$, *we get*

$$(\zeta^{1/p})^{\sharp p}=\left\{\Gamma\left(\frac{1}{p}+1\right)\right\}^p\frac{\left(\frac{2}{p}+1\right)\cdots\left(\frac{p}{p}+1\right)\Gamma\left(\frac{2}{p}+1\right)\cdots\Gamma\left(\frac{p-1}{p}+1\right)}{\left(\frac{2}{p}+1\right)\cdots\left(\frac{p}{p}+1\right)\Gamma\left(\frac{2}{p}+1\right)\cdots\Gamma\left(\frac{p-1}{p}+1\right)\Gamma(2)}\zeta$$

$$= \left\{\Gamma\left(\frac{1}{p}+1\right)\right\}^{p}\zeta.$$

Since $\zeta^{\#(-1)}\# f(\zeta) = df(\zeta)/d\zeta$ by (6)′, we have by (17)′

$$\zeta^a \# \zeta^b = \frac{\Gamma(a+1)\,\Gamma(b+1)}{\Gamma(a+b+1)}\zeta^{a+b}\ , \tag{17}$$

$$\zeta^{-n}\#\zeta^a = \frac{d^n\zeta^a}{d\zeta^n}, \qquad \zeta^{-n}\#\zeta^{-m} = \frac{d^{n+m}}{d\zeta^{n+m}},$$

$$\zeta^a\#\zeta^{-a-n} = \Gamma(a+1)\Gamma(-a-n+1)\frac{d^n}{d\zeta^n},$$

where a and b are not negative integers.

By (14), (18) and (17), the algebraic closure $\widetilde{\mathscr{E}\mathscr{A}}(\mathbf{C}^n)$ of the quotient field $\mathscr{E}\mathscr{A}(\mathbf{C}^n)$ of $\mathrm{Exp}(\mathbf{C}^n)$ is generated by $\{\zeta_1^{\rho_1}\cdots\zeta_n^{\rho_n}$, *$\rho_1, \cdots, \rho_n$ are rational numbers and none of ρ_i is a negative integer*$\}$ and $\{\zeta_{j_1}^{\rho_1}\cdots\zeta_{j_m}^{\rho_m}\partial^{n_1+\cdots+n_k}/\partial\zeta_{i_1}^{n_1}\cdots\partial\zeta_{i_k}^{n_k}$, $k\geq 1$, $k+m=n$, $\{i_1, \cdots, i_k\}\cup\{j_1, \cdots, j_m\}=\{1, \cdots, n\}$ *and none of* ρ_i, $i\in\{j_1, \cdots, j_m\}$ *is a negative integer*$\}$ *as a* $\mathbf{C}$-module.

We denote by $\widetilde{\mathscr{E}\mathscr{A}}(\mathbf{C}^n)_+$ the submodule of $\widetilde{\mathscr{E}\mathscr{A}}(\mathbf{C}^n)$ consisted by those elements that are realized by (some multi-valued) function. That is, the element of $\widetilde{\mathscr{E}\mathscr{A}}(\mathbf{C}^n)$ whose (any) Puiseux expansion does not involve the term which involves $\partial^k/\partial\zeta_i^k$ for some i and k. By definition there is a projection π_+ (as a $\mathbf{C}$-module) from $\widetilde{\mathscr{E}\mathscr{A}}(\mathbf{C}^n)$ onto $\widetilde{\mathscr{E}\mathscr{A}}(\mathbf{C}^n)_+$.

Note. By definition, wehave

$$\widetilde{\mathscr{E}\mathscr{A}}(\mathbf{C}^n)_+ \cap \mathscr{E}\mathscr{A}(\mathbf{C}^n) = \mathrm{Exp}(\mathbf{C}^n)\ ,$$

and the integral closure $\widehat{\mathrm{Exp}}(\mathbf{C}^n)$ (in $\mathscr{E}\mathscr{A}(\mathbf{C}^n)$) is contained in $\widetilde{\mathscr{E}\mathscr{A}}(\mathbf{C}^n)_+$.

Since Borel transformation $\mathscr{B}$ gives an isomorphism from $\mathscr{O}_n$ onto $\mathrm{Exp}(\mathbf{C}^n)$, it is extended to an isomorphism $\tilde{\mathscr{B}} : \tilde{\mathscr{M}}_n \overset{\longrightarrow}{\cong} \widetilde{\mathscr{E}\mathscr{A}}(\mathbf{C}^n)$. By (18) (and (17)), explicitly, $\tilde{\mathscr{B}}$ is given by

$$\tilde{\mathscr{B}}[z_i^{1/p}] = \frac{1}{\Gamma\left(\frac{1}{p}+1\right)}\zeta_i^{1/p}\ , \quad \tilde{\mathscr{B}}[z_i^{-n}] = \frac{\partial^n}{\partial\zeta_1^{\,n}}\ . \tag{19}$$

To fix the above $\tilde{\mathscr{B}}$, we define

Definition. *The Borel transformation* $\mathscr{B}$ *of* $\tilde{\mathscr{M}}_n$ *is the map from* $\tilde{\mathscr{M}}_n$ *onto* $\widetilde{\mathscr{E}\mathscr{A}}(\mathbf{C}^n)_+$ *given by*

$$\mathscr{B}[w] = \pi_+\tilde{\mathscr{B}}[w]\ . \tag{20}$$

By definition, if w is given by Puiseux series (15), then

$$(20)' \qquad \mathscr{B}[w](\zeta) = \sum_{i_1=-\infty,\cdots,i_n=-\infty}^{i_1=\infty,\cdots,i_n=\infty} \frac{a_{i_1},\cdots,i_n}{\Gamma(i_1/r_1+1)\cdots\Gamma(i_n/r_n+1)} \zeta_1{}^{i_1/r_n}\cdots\zeta_n{}^{i_n/r_n} ,$$

where $1/\Gamma(i_1/r_1+1)\cdots T(i_n/r_n+1)=0$ if some of i_k/r_k is a negative integer.

Lemma 3. (1). *$\mathscr{B}[w]$ converges on $\Gamma=\{z|\ \varepsilon_i<|z_i|<\varepsilon_i'\}$ if w is given by* (15) *and it converges on Γ.*

(ii). *If u is integral over $\mathscr{O}_n$, then the Riemann surface of $\mathscr{B}[u]$ covers $\mathbf{C}^n$.*

(iii). *If Ψ belongs in $\mathscr{E}xp(\mathbf{C}^n)$, then*

$$(2)' \qquad \mathscr{B}[\Psi](\zeta) = \frac{1}{(2\pi\sqrt{-1})^n}\int_{|z_1|=\varepsilon_1}\cdots\int_{|z_n|=\varepsilon_n}\frac{1}{z_1\cdots z_n}e^{\zeta/z}\Psi(z)dz_1\cdots dz_n ,$$

if $\Psi(z)$ is holomorphic on $\{z|\ |z_i|=\varepsilon_i\}$.

Proof. (i) follows from (16). Since $\mathscr{B}[u]$ satisfies the equation

$$\mathscr{B}[u]^{\#m}+\mathscr{B}[\varphi_1]\#\mathscr{B}[u]^{\#(m-1)}+\cdots+\mathscr{B}[\varphi_m]=0 ,$$

if u satisfies the equation $u^m+\varphi_1u^{m-1}+\cdots+\varphi_m=0$, we have (ii) by (18) and the fact that each $\mathscr{B}[\varphi_i]$ converges on $\mathbf{C}^n$). (2)$'$ follows from the definition.

Note. On $\widetilde{\Gamma}/G(r)$, to set $y_i=z_i{}^{1/r_i}, i=1,\cdots,n$, we set

$$e^{r_1,\cdots,r_n}_{\cdots 1,\cdots,m_n} = \sum_{i_1\geqq m_1,\cdots,i_n\geqq m_n}\frac{y_1{}^{i_1}\cdots y_n{}^{i_n}}{\Gamma(i_1/r_1+1)\cdots\Gamma(i_nr_n+1)} ,$$

where $1/\Gamma(i_1/r_1+1)\cdots\Gamma(i_n/r_n+1)=0$ if some of i_k/r_k is a negative integer, then to set $\eta_i=\zeta_i{}^{1/r_i}$, we have

$$(2)'' \qquad \mathscr{B}[w] = \lim_{m_1\to-\infty,\cdots,m_n\to-\infty}\frac{1}{(2\pi\sqrt{-1})^n}\int_{|y_1|=\varepsilon_1}\cdots\int_{|y_n|=\varepsilon_n}\frac{1}{y_1\cdots y_n}\cdot$$

$$e^{r_1,\cdots,r_n}_{m_1,\cdots,m_n}\left(\frac{\eta_1}{y_1},\cdots,\frac{\eta_n}{y_n}\right)w(y_1,\cdots,y_n)dy_1\cdots dy_n .$$

By (2)$''$, we can show analytically if $\sum a_{i_1},\cdots,{}_{i_n}{}'z_1{}^{i_1/r_1}\cdots z_n{}^{i_n/r_n}$ is an analytic continuation of w, then $\sum a_{i_1},\cdots,{}_{i_n}{}'/\Gamma(i_1/r_1+1)\cdots\Gamma(i_n/r_n+1)\zeta_1{}^{i_1/r_1}\cdot\cdots\zeta_n{}^{i_n/r_n}$ is an analytic continuation of $\mathscr{B}[w]$. In fact, since the branching points and poles of w are given by $\varphi(z)=0$, $z\in\mathscr{O}_n$, to set

$$\varphi(z)=\sum_I z^I\varphi_I(z),\ \ I=(i_1,\cdots,i_n),\ z^I=z_1{}^{i_1}\cdots z_n{}^{i_n},\ \ \varphi_I(0)\neq 0 ,$$

any Puiseux expansion of w covers a connected component Γ_i of $U(0)-\{z|\ |z^I|=$

$= |z^J|$ *for some* I, $I\}$. But if $z_0 \in \partial\Gamma_i$ and $\varphi(z_0) \neq 0$, the Riemann surface of w which covers such Γ_j that $z_0 \in \partial\Gamma_j$ can be extended to cover z_0 and since on which (2)″ is hold, we have the assertion.

5. Definition. We set $\tilde{\mathscr{B}}^{-1}\pi_+\tilde{\mathscr{B}} = p_+$ and set

(21) $$p_+ w = w_+ \ .$$

Theorem 1. *Borel transformation (of* $\tilde{\mathscr{M}}_n$*) has the following properties.*

(i). $\mathscr{B}[w]=0$ *if and only if* w *belongs in* $ker.\, p_+$, *that is, each term of Puiseux expansion of* w *involve negative power of some* z_i.

(ii). *If* v, $w \in \tilde{\mathscr{M}}_n$ *and* a, b *are constants, then*

(3)i′ $$\mathscr{B}[av+bw] = a\mathscr{B}[v] + b\mathscr{B}[w] \ ,$$

(3)ii′ $$\mathscr{B}[vw] = \pi_+(\mathscr{B}[v]\sharp\mathscr{B}[w]) \ .$$

In (3)ii′, *if* v, w *both contained in* $\hat{\mathscr{O}}_n$, *the integral closure of* $\mathscr{O}_n$, *then*

(3)*ii* $$\mathscr{B}[vw] = \mathscr{B}[v]\sharp\mathscr{B}[w] \ .$$

(iii). *To define* $v\otimes w$, *etc., similarly as* $\varphi\otimes\psi$, *we have*

(5)′ $$\mathscr{B}[v\otimes w] = \mathscr{B}[v]\otimes\mathscr{B}[w] \ .$$

(iv). *For any* i, *we get*

(6) $$\frac{\partial}{\partial\zeta_i}\mathscr{B}[w] = \mathscr{B}[z_i^{-1}w] \ .$$

(9)′ $$\zeta_i\mathscr{B}[w] = \mathscr{B}[z_i w + z_i\frac{2\partial w}{\partial z_i}] \ .$$

Theorem 2. *If* $P(\varrho/\partial\zeta)$ *is a constant coefficients partial differential operator given by*

$$P\left(\frac{\partial}{\partial\zeta}\right)=\frac{\partial^m}{\partial\zeta_1{}^m}+P_1\left(\frac{\partial}{\partial\zeta_2},\cdots,\frac{\partial}{\partial\zeta_n}\right)\frac{\partial^{m-1}}{\partial\zeta_1{}^{m-1}}+\cdots+P_m\left(\frac{\varrho}{\partial\zeta_2},\cdots,\frac{\partial}{\partial\zeta_n}\right) \ ,$$

then its solution with the data

$$\frac{\partial f}{\varrho\zeta_1{}^k}(0,\zeta_2,\cdots,\zeta_n) = g_{k+1}(\zeta_2,\cdots,\zeta_n),\ 0\leq k\leq m-1,\ \ g_k\in \mathrm{Exp}(\mathbf{C}^{n-1}) \ ,$$

is given by

(22) $$f(\zeta) = \mathscr{B}[\sum_i \sum_{1\leq\rho_i<r_i}(1-z_1\sigma_i(z_2^{-1},\cdots,z_n^{-1}))^{-\rho_i}\varphi_{i,\rho_i}(z_2,\cdots,z_n)](\zeta) \ .$$

This $f(\zeta)$ *is holomorphic on* $\mathbf{C}^n$ *if deg.* $P_i\leq m-i$ *for each* i. *Here*

$$P(z)=\prod_i (z_1-\sigma_i(z_2,\cdots,z_n))^{r_i},$$

$$\sum_i \sum_{1\leq\rho_i\leq r_i} c_{k,\rho_i}(\sigma_i{}^k\varphi_{i,\rho_i})(z_2,\cdots,z_n)=\mathscr{B}^{-1}[g_k](z_2,\cdots,z_n),\quad 0\leq k\leq m-1,$$

where c_{k,ρ_i} *is given by* $(1-x)^{-\rho_i}=\sum_k c_{k,\rho_i}x^k$.

In the rest, we set

$$T\binom{r_1,\cdots,r_s}{\sigma_1,\cdots,\sigma_s}=\begin{pmatrix} 1,\cdots\cdots\cdots\cdots\cdots,1 \\ \sigma_1,\cdots\cdots,c_{1,r_1}\sigma_1,\cdots\cdots,c_{1,r_s}\sigma_s \\ \sigma_1{}^2,\cdots\cdots,c_{2,r_1}\sigma_1{}^2,\cdots\cdots,c_{2,r_s}\sigma_s{}^2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sigma_1{}^{m-1},\cdots,c_{m,r_1}\sigma_1{}^{m-1},\cdots,c_{m,r_s}\sigma_s{}^{m-1} \end{pmatrix}.$$

Note. If in $\mathrm{Exp}(\mathbf{C}^n)$, a system of constant cofficiente partial differential operators is given, then by normalization theorem ([11]), is equivalent to the system of operators

$$P_i\left(\frac{\partial}{\partial\zeta}\right)=\frac{\partial^{m_i}}{\partial\zeta_i{}^{m_i}}+P_{i,1}\left(\frac{\partial}{\partial\zeta_{h+1}},\cdots,\frac{\partial}{\partial\zeta_m}\right)\frac{\partial^{m_i-1}}{\partial\zeta_i{}^{m_i-1}}+\cdots+$$

$$+P_{im_i}\left(\frac{\partial}{\partial\zeta_{h+1}},\cdots,\frac{\partial}{\partial\zeta_m}\right),\quad 1\leq i\leq h,$$

by a change of variables. Then the solution of the overdetermined system $\mathfrak{P}$ with the data

$$\frac{\partial^{k_1+\cdots+k_h}f}{\partial\zeta_1{}^{k_1}\cdots\partial\zeta_h{}^{k_h}}(0,\cdots,0,\zeta_{j+1},\cdots,\zeta_n)=g_{k_1+1,\cdots k_h+1}(\zeta_{j+1},\cdots,\zeta_n),$$

$$0\leq k_i\leq m_i-1,\quad g_{k_1,\cdots,k_h}\in \mathrm{Exp}(\mathbf{C}^{n-h}),$$

is given by

$$(22)'\qquad f(\zeta)=\mathscr{B}[\sum_{(i,j)}\sum_{1\leq\rho_{i,j}\leq r_{i,j}}(1-z_1\sigma_{1,j_1}(z_{h+1}{}^{-1},\cdots,z_n{}^{-1}))^{-\rho_{1,j_1}}\cdots$$

$$(1-z_h\sigma_{h,j}(z_{h+1}{}^{-1},\cdots,z_n{}^{-1}))^{-\rho_{h,j_h}}\varphi_{\rho_{1,j_1},\cdots,\rho_{h,j_h}}(z)](\zeta),$$

$$P^i(z)=\prod_j (z_i-\sigma_{i,j}(z_{h+1},\cdots,z_n))^{r_{i,j}},$$

$$\sum_{(i,j)} \sum_{1\leq \rho_{i,j}\leq r_{i,j}} c_{k_{1},1,j_1}\cdots c_{k_h,1,j_h}((\sigma_1, i_1)^{k_1}\cdots(\sigma_h, j_h)^{k_h}$$
$$\varphi_{\rho_1, j_1,\cdots \rho_h, j_h})(z) = \mathscr{B}^{-1}[g_{k_1+1,\cdots,k_h+1}](z) \quad .$$

we note that this last coeficients matrix is given by $T\binom{r_{1,1},\cdots,r_{1,s_1}}{\sigma_{1,1},\cdots,\sigma_{1,s_1}}\otimes\cdots$

$\otimes T\binom{r_{h,1},\cdots,r_{h,s_h}}{\sigma_{h,1},\cdots,\sigma_{h,s_h}}$.

As in the single equation case, if *deg.* $P_{i,k}\leq m_i-k$ for each i and k, then this f is holomorphic on $\mathbf{C}^n$.

§2 Topological extension of Borel transformation

6. Let D be a subset of $\mathbf{R}^n$ such that *Int.* $D\neq\emptyset$ and $F(D)$ is a complete topological vector space (over $\mathbf{C}$) consisted by the functions on D and satisfy

(i). $r_D(f)=f|D$, *the restriction of* f *on* D *belongs in* $F(D)$ *if* $f\in \mathrm{Exp}(\mathbf{C}^n)$.

(ii). $\{r_D(f)\mid f\in\mathrm{Exp}(\mathbf{C}^n)\}$ *is dense in* $F(D)$.

We note that by assumption, $r_D:\mathrm{Exp}(\mathbf{C}^n)\to F(D)$ is an (into) isomorphism.

Definition. *To regard* $\mathrm{Exp}(\mathbf{C}^n)$ *to be a subspace of* $F(D)$ *by the map* r_D, *the induced topology of* $\mathrm{Exp}(\mathbf{C}^n)$ *from* $F(D)$ *is called* $F(D)-$*topology of* $\mathrm{Exp}(\mathbf{C}^n)$. *If a series* $\{f_m\}$ *of the elements of* $\mathrm{Exp}(\mathbf{C}^n)$ *converges to* f *by this topology, then we denote* $F(D)-lim_{m\to\infty}f_m=f$.

By definition, to denote the completion of $\mathrm{Exp}(\mathbf{C}^n)$ by $F(D)-$topology by $\mathrm{Exp}(\mathbf{C}^n)^*$ (or $(\mathrm{Exp}(\mathbf{C}^n))^{*F(D)}$), we have

(23) $\qquad r_D{}^*:\mathrm{Exp}(\mathbf{C}^n)^*\cong F(D)$.

Example. If D is a bounded domain, then for all p, $L^p(D)$ can be taken as $F(D)$. The $k-$th Sobolev space $L^{2k},(D)$ and $C^k(D)$ (with the C^k topology) can also be taken as $F(D)$. The $k-$th local Sobolev space $L^{2,k}{}_{loc.}(\mathbf{R}^n)$ or $C^k(\mathbf{R}^n)$ are also taken as $F(D)$. Here, k might be negative.

Since Borel transformation $\mathscr{B}$ is an isomorphism between $\mathscr{O}_n$ and $\mathrm{Exp}(\mathbf{C}^n)$, $\mathscr{B}^{-1}$ induces $F(D)-$ topology of $\mathrm{Exp}(\mathbf{C}^n)$ to $\mathscr{O}_n$. It is also called $F(D)-$topology of $\mathscr{O}_n$ and if $\{\varphi_m\}$, $\varphi_m\in\mathscr{O}_n$ converges to φ by $F(D)-$ topology, we also denote $F(D)-lim\varphi_m=\varphi$.

By n°2 and (23), to denote $\mathscr{O}_n{}^*$, *etc.*, the completions of $\mathscr{O}_n$, *etc.*, by $F(D)$ $-$ topology, we have the following commutative diagram.

$$\begin{array}{ccc}
\mathscr{E}_{\mathbf{R}^n}{}^* & \xrightarrow{\iota^*-2\pi\sqrt{-1}} & \mathscr{O}_n{}^* \\
\downarrow \mathscr{E}^* & & \cong\ \downarrow \mathscr{B}^* \\
F(D) & \underset{\cong}{\xleftarrow{r_D{}^*}} & \mathrm{Exp}(\mathbf{C}^n)^* \quad .
\end{array}$$

Note. If we consider $\mathrm{Exp}(\mathbf{C}^n)$ to be a topological vector space by the compact open topology (of $\mathbf{C}^n$), the completion of $\mathrm{Exp}(\mathbf{C}^n)$ is $\mathfrak{A}^n$, the space of entire functions on $\mathbf{C}^n$, and the completion of $\mathscr{O}_n$ by this topology (induced by $\mathscr{B}^{-1}$) is $\mathrm{Exp}(\mathbf{C}^n)'$, the dual space of $\mathrm{Exp}(\mathbf{C}^n)$, and the extended Borel transformation $\mathscr{B}^*$ is $\mathscr{B}'$, the dual map of $\mathscr{B}: \mathscr{O}_n \to \mathrm{Exp}(\mathbf{C}^n)$.

Lemma 4. *If $f \sharp g$ is defined in $F(D)$ for any f, g $F(D)$ and the $\sharp$-product is continuous in $F(D)$, then $\mathscr{O}_n{}^*$ is a ring (by the usual multiplication) and we have*

$$\mathscr{B}^*[\varphi\psi] = (r_D\mathscr{B}^*[\psi]) \sharp (r_D\mathscr{B}^*[\psi]) \quad .$$

7. *We set*

$$(24) \qquad \widetilde{\mathbf{R}}^n = (\mathbf{R}\times\mathbf{Z})^n = \overbrace{(\mathbf{R}\times\mathbf{Z})\times\cdots\times(\mathbf{R}\times\mathbf{Z})}^{n} ,$$

$$p((x_1, m_1), \cdots, (x_n, m_n)) = (x_1, \cdots, x_n) \in \mathbf{R}^n, \ ((x_1, m_1), \cdots, (x_n, m_n)) \in \widetilde{\mathbf{R}}^n .$$

By Definition, $\mathbf{Z}^n$ acts on $\mathbf{R}^n$ and we set

$$(24)' \qquad \mathbf{R}^n{}_{(r)} = \widetilde{\mathbf{R}}^n/G(r), \ \widetilde{D} = p^{-1}(D), \ D_{(r)} = \widetilde{D}/G(r), \ r = (r_1, \cdots, r_n).$$

The projection from $\mathbf{R}^n{}_{(r)}$ (or $\mathrm{D}_{(r)}$) onto $\mathbf{R}^n$(or D) is denoted by p_r.

Since $\mathrm{D}_{(r)}$ is a $G(r)$ − direct sum of D, we define $F(D_{(r)})$ as the $G(r)$ − direct sum of $F(D)$. Then if $r|r'$, that is $r_i|r_i'$ for all i, there is a map $p^r{}_{r'}{}^*: F(D_{(r)}) \to F(D_{(r')})$ and since

$$p^r{}_{r'}{}^* p^{r'}{}_{r''}{}^* = p^r{}_{r''}{}^*, \quad if\ r|r'\ and\ r'|r'',$$

we define $F(\widetilde{D})$ by

$$F(\widetilde{D}) = \varprojlim[F(D_{(r)}),\ p^r{}_{r'}{}^*] \ .$$

By definition, we can define $\tilde{r}_D : \widehat{\mathrm{Exp}}(\mathbf{C}^n) \to F(\widetilde{D})$. More general, if $\varphi \in \widetilde{\mathscr{E}\mathscr{U}}(\mathbf{C}^n)_+$ has no singularity on D and $F(D)$ satisfies (i) of n°6, then $\tilde{r}_D(\varphi)$ is defined and belongs in $F(\widetilde{D})$.

Definition. *Let S be a subset of $\widetilde{\mathscr{M}}_n$ which contains* 1, *$F(D)$ a function space such that*

$$\tilde{r}_D(\mathscr{B}[\varphi\sigma]) \in F(\widetilde{D}),\ if\ \varphi \in \mathscr{O}_n,\ \sigma \in S \ .$$

Then we call $\{f_m\}$, $f_m \in \mathrm{Exp}(\mathbf{C}^n)$, converges to f by the $F(D)$ − topology with respect to S if $\{\tilde{r}_D(\mathscr{B}[\mathscr{B}^{-1}[f_m]\sigma])\}$ converges to $\tilde{r}_D(\mathscr{B}[\mathscr{B}^{-1}[f]\sigma)$ in $F(\widetilde{D})$ for any $\sigma \in S$ and denote $F(D)_S - \lim f_m = f$.

If $F(D)_S - \lim \mathscr{B}[\varphi_m] = \mathscr{B}[\varphi]$, then we denote $F(D)_S - \lim \varphi_m = \varphi$.

Example. We take $C(\mathbf{R}^1)$ as $F(D)$ $(n=1)$. If $S=\{(1+az)^{-1} | a \in \mathbf{R}\}$, we have

$$C(\mathbf{R}^1)_S - \lim_{m\to\infty} f_m = f \textit{ if and only if } C(\mathbf{R}^1) - \lim_{m\to\infty} f_m = f \quad .$$

On the other hand, if $S=\{(1+\sqrt{-1}az)^{-1} | a \in \mathbf{R}\}$, we have

$$C(\mathbf{R}^1)_S - \lim_{m\to\infty} f_m = f \textit{ if and only if } \{f_m\} \textit{ converges uniformly to } f \textit{ on } \mathbf{C}^1.$$

These may be two extremal cases and in the rest, we assume $F(D)$ satisfies

(iii). *If* $\{f_m\}$ *converges uniformly to* f *on* $\mathbf{C}^n$, *then* $r_D(f_m)$ *converges to* $r_D(f)$ *in* $F(D)$.

8. By (iii), denoting $U(0)$ a neighborhood of $\{0\}$ by $F(D)$ − topology, to set

$$U_S(0) = \{g | \mathscr{B}[\mathscr{B}^{-1}[g]\sigma] \in U(0),\ \sigma \in S\} \quad ,$$

$U_S(0)$ contains $\{g \mid |g(z)| < \varepsilon,\ z \in K,$ *a compact set in* $\mathbf{C}^n\}$ for some $\varepsilon > 0$ and $K \neq \emptyset$.

We denote the vector space of all Cauchy sequences of the elements of Exp $(\mathbf{C}^n)$ by $F(D)$ − topology by $F(D)-\mathrm{Exp}(\mathbf{C}^n)$. We consider $F(D)-\mathrm{Exp}(\mathbf{C}^n)$ to be a topological vector space to take

$$U(\{f_m\}) = \{\{g_m\} | g_m - f_m \in U_m(0),\ U_m(0) \textit{ is a neighborhood of } 0 \textit{ by } F(D) - \textit{topology and } U_m(0) \supset U_{m+1}(0),\ \bigcap_m U_m(0) = \{0\}\}.$$

On the other hand, to take

$$U_S(\{f_m\}) = \{\{g_m\} | g_m - f_m \in U_{m,S}(0),\ U_m(0) \textit{ is a neighborhood of } 0 \textit{ by } F(D) - \textit{topology and } U_m(0) \supset U_{m+1}(0),\ \bigcap_m U_m(0) = \{0\}\},$$

to be the neighborhood basis of $F(D)-\mathrm{Exp}(\mathbf{C}^n)$, $F(D)-\mathrm{Exp}(\mathbf{C}^n)$ also becomes a topological vector space. This space is denoted by $F(D)-\mathrm{Exp}(\mathbf{C}^n)_S$.

In $F(D)-\mathrm{Exp}(\mathbf{C}^n)$, we set

$$F(D)_S - \mathrm{Exp}(\mathbf{C}^n) = \{\{f_m\} | \{f_m\} \textit{ is a Cauchy sequence with respect to } S\},$$
$$F(D) - \mathrm{Exp}(\mathbf{C}^n)0 = \{\{f_m\} | F(D) - \lim_{m\to\infty} f_m = 0\},$$
$$F(D)_S - \mathrm{Exp}(\mathbf{C}^n)0 = \{\{f_m\} | F(D)_S - \lim_{m\to\infty} f_m = 0\}.$$

The same spaces regarded as the subspaces of $F(D)-\mathrm{Exp}(\mathbf{C}^n)_S$ are denoted by $F(D)_S - \mathrm{Exp}(\mathbf{C}^n)_S$, $F(D)-\mathrm{Exp}(\mathbf{C}^n)_{0,S}$ and $F(D)_S-\mathrm{Exp}(\mathbf{C}^n)_{0,S}$.

Lemma 5. $F(D)_S - \mathrm{Exp}(\mathbf{C}^n)$ *and* $F(D)_S - \mathrm{Exp}(\mathbf{C}^n)_0$ *are equal to* $F(D)_S - \mathrm{Exp}(\mathbf{C}^n)_S$ *and* $F(D)_S - \mathrm{Exp}(\mathbf{C}^n)_{0,S}$ *as topological vector spaces.*

Proof. Since $F(D)_S - \lim(f_m - g_m) = 0$ if $\{g_m\} \in U(f_m)$ in $F(D)_S - \mathrm{Exp}(\mathbf{C}^n)$, $\{g_m\}$ should belong some $U_S(f_m)$ and we have the lemma.

Lemma 6. *To set*

$$F(D)_S = F(D)_S - \mathrm{Exp}(\mathbf{C}^n) / F(D)_S - \mathrm{Exp}(\mathbf{C}^n)_0 \quad ,$$
$$F(D)^S = F(D) - \mathrm{Exp}(\mathbf{C}^n)_S / F(D)_S - \mathrm{Exp}(\mathbf{C}^n)_{0,S} \quad ,$$
$$N_S(F(D)) = F(D) - \mathrm{Exp}(\mathbf{C}^n)_{0,S} / F(D)_S - \mathrm{Exp}(\mathbf{C}^n)_{0,S} \quad ,$$

We have the following commmutative diagram with exact (as) topological vector spaces) columns and raws. Here the maps are induced by the natural inclusions and projections.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(D)_S & \xrightarrow{=} & F(D)_S & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \to N_S & \longrightarrow & F(D)^S & \longrightarrow & F(D) & \longrightarrow & 0 \\
\downarrow = & & \downarrow & & \downarrow & & \\
0 \to N_S & \to & F(D)^S/F(D)_S & \to & F(D)/F(D)_S & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Proof. Since $F(D) = F(D) - \mathrm{Exp}(\mathbf{C}^n)/F(D) - \mathrm{Exp}(\mathbf{C}^n)_0$ by the condition (ii) of $n^0 6$ and we know

$$(25) \qquad F(D)_S - \mathrm{Exp}(\mathbf{C}^n)_0 = F(D)_S - \mathrm{Exp}(\mathbf{C}^n) \cap F(D) - \mathrm{Exp}(\mathbf{C}^n)_0 \ ,$$

we have the lemma by lemma 5.

In the rest, we denote by $F(\widetilde{D})_S$ and $F(\widetilde{D})^S$, the spaces constructed from $F(D)_S$ and $F(D)^S$ similarly as $F(\widetilde{D})$.

9. For a series $\{\varphi_m\}$ of the elements of $\mathcal{O}_n$, we define $F(D)_S - lim\varphi_m$ *similarly* as $F(D)_S - lim\, f_m$. Then we can define $F(D) - \mathcal{O}_n$, $F(D)_S - \mathcal{O}_n$, *etc.*, similarly as $F(D) - \mathrm{Exp}(\mathbf{C}^n)$, $F(D)_S - \mathrm{Exp}(\mathbf{C}^n)$, *etc.*, Then to define $\mathscr{B}: F(D) - \mathcal{O}_n \to F(D) - \mathrm{Exp}(\mathbf{C}^n)$ by

$$\mathscr{B}[\{\varphi_m\}] = \{\mathscr{B}[\varphi_m]\} \ ,$$

$\mathscr{B}$ maps $F(D)_S - \mathcal{O}_n$, $F(D) - \mathcal{O}_{n0}$, *etc.*, isomorphiscally onto $F(D)_S - \mathrm{Exp}(\mathbf{C}^n)$, $F(D) - \mathrm{Exp}(\mathbf{C}^n)_0$, *etc.*. Moreover, $\mathscr{B}$ can be regarded as the map from $F(D) - \mathcal{O}_{n,s}$ onto $F(D) - \mathrm{Exp}(\mathbf{C}^n)_S$. Hence to set $\mathcal{O}_{n,s} = F(D)_S - \mathcal{O}_n/F(D)_S - \mathcal{O}_{n,0}$, $\mathcal{O}_n{}^S = F(D) - \mathcal{O}_n/F(D)_S - \mathcal{O}_{n,0}$, $\mathscr{B}$ induces maps

$$\mathscr{B}_S{}' : \mathcal{O}_{n,s} \overset{\rightarrow}{\cong} F(D)_S \ ,$$

$$\mathscr{B}^{S\prime} : \mathcal{O}_n{}^S \overset{\rightarrow}{\cong} F(D)^S \ .$$

Then, if φ_S (*resp.* φ^S) *is an element of* $\mathcal{O}_{n,s}$ (*resp.* $\mathcal{O}_n{}^S$) given by $F(D)_S - lim\,\varphi_m = \varphi_S$ (*resp.* $F(D)_S - lim\varphi^m = \varphi^S$), for any $\sigma \in S$, $F(D)_S - lim\varphi_m\sigma$ (*resp.* $F(D)_S - lim\varphi^m\sigma$) exists as an element of $F(\widetilde{D})_S$ (*resp.* $F(\widetilde{D})^S$), and to set

(26) $$F(D)_S - \lim_{m\to\infty} \varphi_m\sigma = \varphi_S\sigma, \qquad F(D)_S - \lim_{m\to\infty} \phi^m\sigma = \varphi^S\sigma ,$$

$\mathscr{B}_{S'}$ and $\mathscr{B}^{S'}$ are extended to maps

$$\mathscr{B}_S : \mathscr{O}_{n,S}\langle S\rangle \to F(\widetilde{D})_S , \quad \mathscr{B}^S \mathscr{O}_n^S \langle S\rangle \to \mathrm{F}(\widetilde{\mathrm{D}})^S .$$

Here, $\mathscr{O}_{n,S}\langle S\rangle$ *and* $\mathscr{O}_n^S\langle S\rangle$ are the completions of modules generated by $\mathscr{O}_{n,S}$ (or $\mathscr{O}_n^S$) and S under the operations given by (26), by the topologies of $F(D)_S$ and $F(D)^S$.

Theorem 3. *We assume D is given by $\Omega\times k$, where Ω is an open set in $\mathbf{R}^{n-1}$ (may be equal to $\mathbf{R}^{n-1}$, K is a simply connected subset of $\mathbf{C}^1$ such that K contains either of intervals (a, b), $[0, b)$ or $(a, 0]$ $(a<0<b)$ in $\mathbf{R}^1$, and $F(D)$ is given by*

$$F(D) = L(\Omega)\hat{\otimes}^\pi A(K) ,$$

where $V\hat{\otimes}_\pi W$ means the completion of $V\otimes W$ by π – topology (cf. [11]), *$A(K)$ is a space of analytic functions on K such that by the map r_K,* $\mathrm{Exp}(\mathbf{C}^1)$ *is containd in $A(K)$ with the variable ζ_1, and $L(\Omega)$ is a function space such that to satisfy* (i), (ii) *of n^o6 and* (iii) *of n^o7 for* $\mathrm{Exp}(\mathbf{C}^{n-1})$ *with the variables $\zeta_2, \cdots, \zeta_n$.*

Let $P(z) = z_1^m + P_1(z_2, \cdots, z_n)z_1^{m-1} + \cdots + P_m(z_2, \cdots, z_n)$ be a polynomial such that

$$P(z) = \prod_i (z_1 - \sigma_1(z_2, \cdots, z_n))^{r_i}, \quad 1\leqq i\leqq k, \quad \sum_{i=1}^k r_i = m ,$$

and set

$$S = \{(1 - z_1\sigma_{i_1}(z_2^{-1}, \cdots, z_n^{-1}))^{-\rho_{i_1}} \cdots (1 - z_1\sigma_{i_j}(z_2^{-1}, \cdots, z_n^{-1}))^{-\rho_{i_j}} |$$
$$1\leqq i_1 < \cdots < i_j \leqq k, \ 1\leqq \rho_i \leqq r_i\} .$$

Then to set

$$L(\Omega)_S = p_\Omega(F(D)_S) , \quad L(\Omega)^S = p_\Omega(F(D)^S) ,$$

$$p_\Omega(\{f_m\}) = \{p_\Omega f_m\}, \quad (p_\Omega f)(\xi_2, \cdots, \xi_n) = f(0, \xi_2, \cdots, \xi_n) ,$$

for any data in $L(\Omega)_S$ (rep. in $L(\Omega)^S$), the equation $P(\partial/\partial\xi_i) f = 0$ has unique solution in $F(D)_S$ (resp. in $F(D)^S$) and it is well posed by the topology of $F(D)_S$ (resp. $F(D)^S$).

Proof. By assumption, for the given data $\{g_k\}$ in $L(\Omega)_S$ (*resp.* in $L(\Omega)^S$), we can solve the equation

$$\sum_i \sum_{1\leqq\rho_i\leqq r_i} c_{k,\rho_i}(\rho_i, {}^k\varphi_{i\rho_i}) = \mathscr{B}_S^{-1}[g_k] \ (or\ (\mathscr{B}^S)^{-1}[g_k)], \quad 0\leqq k\leqq m-1.$$

Then to set

$$f = \mathscr{B}_S[\sum_i \sum_{1\leqq\rho_i\leqq r_i} (1 - z_1\sigma_i(z_2^{-1}, \cdots, z_n^{-1}))^{-\rho_i}\varphi_{i,\rho_i}]$$

$$(resp.\ \mathscr{B}^S[\sum_i \sum_{1\leq\rho_i\leq r_i}(1-z_1\sigma_i(z_2^{-1},\cdots,z_n^{-1}))^{-\rho_i}\varphi_{i,\rho_i}]) ,$$

we get a solutiom in $F(\widetilde{D})_S$ (*resp.* in $F(\widetilde{D})^S$). But, since the solution is invariant under the covering transformation, f should belong in $F(D)_S$ (*resp.* in $F(D)^S$. Moreover, since $T\binom{r_1,\cdots,r_s}{\sigma_1,\cdots,\sigma_s}$ is ergular and operates continuously on $L(\Omega)_S{}^m$, the m – direct sum of $L(\Omega)_S$ (*sesp.* on $(L(\Omega)^S)^m$), we have the theorem.

Note. Similarly, starting from $D=\Omega\times K$, $\Omega\subset\mathbf{R}^{n-k}$, $K\subset\mathbf{C}^k$ and $F(D)=L(\Omega)$ $\hat{\otimes}_\pi$ $A(K)$, we get corresponding theorem for systems.

Appendix. Borel transformation of $\log z$.

Since the universal covering space $\widetilde{\Gamma}$ *of* $\Gamma=\{z|\varepsilon_i<|z_i|<\varepsilon_i'\}$ is given by $\{w\mid \log\varepsilon_i<Re.w_i<\log\varepsilon_i'\}$ with the covering map $(z_1,\cdots,z_n)=\exp\ w_1,\cdots,\exp\ w_n)$, to extend Borel transformation for the functions on $\widetilde{\Gamma}$, it is sufficient to define $\mathscr{B}[\log z]$. For this purpose, first we note, if $\mathscr{B}[\log z]$ is defined, then by (9), $\zeta\mathscr{B}[\log z]=[z\log z+z]$ and by (6), it must be

$$\frac{d}{d\zeta}\mathscr{B}[\log z]=1 .$$

Therefore $\mathscr{B}[\log z]=\log\zeta+c$, if $\mathscr{B}[\log z]$ is defined. To determin this constant, we use

Lemma. *For* $t<0$, *we get*

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}(\log z)\#^n=\frac{e^{-rt}}{\Gamma(1+t)}x^t ,$$

where γ *is Euler's constant.*

Proof. To set $\log x\#(\log x)^{n-1}=\sum_{k=0}^{n}a_{n,k}(\log x)^k$, we get

$$a_{n,n}=1,\ a_{n,n-1}=0,\ a_{n,k}=\frac{(n-1)!}{k!(n-k-1)!}a_{n-k,0} ,\ 2\leq k\leq n-1 ,$$

$$a_{n,0}=(-1)^{n-1}(n-1)!\zeta(n),\quad n\geq 2,\ \zeta(n)=\sum_{m=lm}^{\infty}\frac{1}{n} ,$$

because $\int_0^x\log(x-t)(\log t)^{n-1}dt=\log x\int_0^x(\log t)^{n-1}dt-\sum_{m=1}^{\infty}1/mx^m\int_0^x t^m(\log t)^{n-1}$ dt.

Then, to set $(\log x)\#^n=\sum_{k=0}^{n}b_{n,k}(\log x)^k$, we get

$$b_{n,n}=1,\quad b_{n,n-1}=0,\quad b_{n,k}=\frac{n!}{k!(n-k)!}b_{n-k,0}\ ,\quad 2\leqq k\leqq n-1\ ,$$

$$b_{n,0}=\sum_{s=1}^{[n/2]}\sum_{j_1-\cdots+j_s=n,\, j_i\geqq 2}(-1)^{n-s}\frac{n!\zeta(j_1)\cdots\zeta(j_s)}{j_1(j_1+j_2)\cdots(j_1+\cdots+j_s)}\ ,\quad n\geqq 2\ .$$

Hence we get

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}(\log x)^{\#n}$$

$$=(1+\sum_{n=2}^{\infty}\sum_{s=1}^{[n/2]}\sum_{j_1+\cdots+j_s=n,\, j_1\leqq 2}(-1)^{n-s}\frac{\zeta(j_1)\cdots\zeta(j_s)}{j_1(j_1+j_2)\cdots(j_1+\cdots+j_s)})t^n).$$

$$(\sum_{n=0}^{\infty}\frac{t^n}{n!}(\log x)^n)\ .$$

But since we know $\log(1+t)=-\gamma t+\sum_{m=2}^{\infty}(-1)^m\zeta(m)/m\ t^m$ ([1]), *we obtain*

$$1+\sum_{n=2}^{\infty}(\sum_{s=1}^{[n/2]}\sum_{j_1+\cdots+j_s=n,\, j_i\geqq 2}(-1)^{n-s}\frac{\zeta(j_1)\cdots\zeta(j_s)}{j_1(j_1+j_2)\cdots(j_1+\cdots+j_s)})t^n$$

$$=1+\sum_{n=2}^{\infty}(\sum_{s=1}^{[n/2]}\sum_{j_1+\cdots+j_s=n,\, j_i\geqq 2}(-1)^{n-s}\frac{\zeta(j_1)\cdots\zeta(j_s)}{s!j_1\cdots j_s})t^n$$

$$=\exp[\log\frac{e^{-\gamma t}}{\Gamma(1+t)}]$$

$$=\frac{e^{-\gamma t}}{\Gamma(1+t)}\ .$$

Hence we have the lemma.

Definition. *We define the Borel transformation* $\mathscr{B}[\log z](\zeta)$ *of* $\log z$ *by*

$$\mathscr{B}[\log z](\zeta)=\log\zeta+\gamma\ .$$

By definition, if $f(z)=\sum_I z_I{}^{\alpha_I}f_I(z)$, $I=(i_1,\cdots,i_k), \alpha_I=(i_1/r_1,\cdots,i_k/r_k)$, $z_I{}^{\alpha_I}=z_{i_1}{}^{i_1/r_1}\cdots z_{i_k}{}^{i_k/r_k}$ and $f_I(0)\neq 0$, then

$$\mathscr{B}[\log f(z)]=\sum_{j=1}^{k}\frac{i_j}{r_j}(\log\zeta_{i_j}-\gamma)+\mathscr{B}[\varphi_I],\quad \varphi_I\in\widetilde{\mathscr{H}}\,,$$

$$\zeta\in\Gamma^{\alpha_I}(\alpha_{I_1},\cdots,\alpha_{I_m})=\{\zeta|\ |\zeta_{I_m}{}^{\alpha_I}|=>|\zeta_J{}^{\alpha_J}|,\ \alpha_I,\alpha_J\in(\alpha_{I_1},\cdots,\alpha_{I_m})\}\ .$$

In the rest, the corresponding set of $\Gamma^{\alpha_I}(\alpha_{I_1},\cdots,\alpha_{I_m})$ in the $z-$space is also denoted by same notation and set

$$\pi^{-1}(\Gamma^{\alpha_I}(\alpha_{I_1},\cdots,\alpha_{I_m}))=\tilde{T}^{\alpha_I}(\alpha_{I_1},\cdots,\alpha_{I_n}) \quad .$$

We consider following class $\mathscr{H}'$ of holomorphic functions on $(w_1,\cdots,w_n)$ —space such that

(*). *f is holomorphic on some open set D in* $\{w|\ Re.\,w_i<\rho_i\}$ *for some* $\rho_1,\cdots,\rho_n$ *such that for any* $\delta_1,\cdots,\delta_n$ *there exist* $r_1<r_1'\leqq\delta_1,\cdots,r_n<r_n'\leqq\delta_n$ *such that D contains* $\{w|\ r_i<Re.\ w_i<r_i'\}=T_{(r_1,r_1',\cdots,r_n,r_n')}$.

If f_1 and f_2 both belongs in $\mathscr{H}'$, then we denote $f_1\sim f_2$ if for any $\delta_1,\cdots,\delta_n$, there exist $r_1<r_1'\leqq\delta_1,\cdots,r_n<r_n'\leqq\delta_n$ such that

$$f_1\,|\,\Gamma_{(r_1,r_1',\cdots,r_n,r_n')}=f_2\,|\,T_{(r_1,r_1',\cdots r_n,r_n')} \quad .$$

The set of this equivalence classes form an integral domain $\mathscr{H}$ by natural way and to set the quotient field of $\mathscr{H}$ by $\hat{\mathscr{M}}$, the elements of $\mathscr{H}$ and $\hat{\mathscr{M}}$ both considered to be the germs of multi – valued analytic functions at the origin of z —space, where $w_i=\exp z_1, i=1,\cdots,n$. Similarly, we define the germ of those functions which are holomorphic on each $T^{\alpha_{I_k}}(\alpha_{I_1},\cdots,\alpha_{I_m})$, $k=1,\cdots,m$, for some $(\alpha_{I_1},\cdots,\alpha_{I_m})$. The set of those germs form an integral domain and its quotient field is denoted by $\hat{\mathscr{E}}$. As the elements of $\hat{\mathscr{M}}$, we consider the elements of $\hat{\mathscr{E}}$ to be the germs of multi – valued functions of ζ – space. Then by the above, we can define Borel transformation $\mathscr{B}$ for the elements of $\hat{\mathscr{M}}$ to be the map from $\hat{\mathscr{M}}$ into $\hat{\mathscr{E}}$ and it also satisfies (3)i, (3)ii, (5), (6), (7) and (9).

Note. In this extended Borel transformation, although $f(z)$ is analytic near the origin, $\mathscr{B}\,[f]$ may not be analytic on any neighborhood of the origin. if $n\geqq 2$. For example, we have

$$\begin{aligned}\mathscr{B}\,[\log(z_2+z_2)](\zeta_1,\zeta_2)&=\log\zeta_1+\gamma\ , \qquad |\zeta_1|>|\zeta_2| \ ,\\ &=\log\zeta_2+\gamma\ , \qquad |\zeta_2|>|\zeta_1| \ .\end{aligned}$$

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