JOUR. FAC. SCI., SHINSHU UNIV., Vol. 7, pp. 1—3, June, 1972

Note on the Special Sine Series

By TAKESHI KANO

Department of Mathematics, Faculty of Science,
Shinshu University
(Received April 30)

This is a simple and additional note to my reserch paper [4] with a view of explaining how we can attack actual problems of this field. All results stated here seem to have been escaped before as far as I am concerned.

§ 1 Let us consider for example the behaviour of the series

$$s(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{\pi(n)} \tag{1}$$

where $\pi(n)$ denotes, as usual in number theory, the number of primes not exceeding n.

Since $\pi(n)$ is monotonely increasing with n and we know the so-called Chebyshov estimate

$$A\frac{n}{\log n}\leqq\pi(n)\leqq B\frac{n}{\log n}, \tag{2}$$

we see that

$$\sum_{n=2}^{\infty}\frac{1}{n\,\pi(n)}<\infty,$$

and hence it follows that (1) is a Fourier series of $s(x)$ which is integrable [1] [2] [3] [7].

If we are going to apply directly the known theorem of Salem ([1] vol. II, [5]) in purpose to estimate $s(x)$ near $x=0$, we encounter the defect that it demands the convexity of $\frac{1}{\pi(n)}$ and the monotonity of $\frac{n}{\pi(n)}$ which both do not hold. But if we assume the Prime Number Theorem with remainder term, i. e.

$$\pi(n)=\frac{n}{\log n}+O\left(\frac{n}{(\log n)^{d}}\right),\ (\forall d>1) \tag{3}$$

we can then obtain as $x \to 0^+$,

$$C \log \frac{1}{x} \leqq s(x) \leqq D \log \frac{1}{x}, \tag{4}$$

by Salem's theorem.

§ 2 Now let us investigate more generally the following sine series

$$\sum_{n=1}^{\infty} b_n \sin nx, \quad (0 < |x| \leqq \pi) \tag{5}$$

with $b_n \downarrow 0$ as $n \to \infty$.

For the series of this type we can prove the more general theorems below than that of Salem.

Theorem 1 *There exists a positive absolute constant c_1 independent of both n and x such that for the partial sum $S_n(x)$ of* (5),

$$|S_n(x)| \leqq \frac{c_1}{|x|} b\left(\frac{\pi}{|x|}\right),$$

provided that

$$\sum_{k=1}^{n} k\, b_k = O(n^2 b_n),$$

or $n\, b_n = O(1)$.

Theorem 2 *If b_n is convex, then we have for the sum $f(x)$ of* (5),

$$f(x) \geqq \frac{c_2}{x} b\left(\frac{\pi}{x}\right), \quad (0 < x \leqq \pi)$$

where c_2 is a positive constant.

The proof of Theorem 1 is easily obtained if we take care for the process of [1] vol. I p. 91, but on the contrary that of Theorem 2 seems to be not so easy and in fact it is a consequence of the following more general theorem [4] of which the proof depends on the well-known van der Corput's lemma [5] [7].

Theorem 3 *We have for* $0 < x \leqq \pi$,

$$f(x) \leqq \int_0^{\frac{\pi}{x}} b(u) \sin(xu)\, du + O(1).$$

If moreover b_n is convex, then we have for $0 < x \leqq \pi$,

$$f(x) \geqq \frac{1}{2} \int_0^{\frac{\pi}{x}} b(u) \sin(xu)\, du + O(1).$$

Thus we obtain (4) from Theorem 1 and Theorem 2 assuming only (2) instead of (3). Another direct consequence of them is the following

Theorem 4 *If b_n is convex the necessary and sufficient condition that $f(x)$ should be bounded from above is*

$$b_n = O\left(\frac{1}{n}\right),$$

and hence we know this is equivalent to the uniform boundedness of $S_n(x)$.

Finally we remark that on assuming (3) and the classical result of Kummer ([6] p. 250) which is connected with Gamma Function we may reach the best estimate for (1), but the detail will be left to the reader.

References

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