Generalized Tangents of Curves and Generalized Vector Fields

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Introduction

The main purpose of this paper is to introduce the notion of generalized tangent of a curve $\gamma$ given by $\varphi : I \rightarrow M$, where $M$ is an $n$-dimensional Paracompact topological manifold with a (fixed) metric $\rho$. Here $\rho$ is assumed to satisfy (*) \[ \rho(x_1, x_2) \leq 1, \] then there exists unique curve $\gamma$ of $M$ which joins $x_1$ and $x_2$ and

\[ \int_I \rho = \rho(x_1, x_2). \]

(For the existence of such metric, see [4]). In the rest, we set

\[ S_a = \{ y | \rho(x, y) = 1 \}. \]

By assumption, $S_a$ is homeomorphic to $S^{n-1}$, the unit $(n-1)$-sphere. Then the generalized tangent of $\gamma$ at $a = \varphi(0)$ is defined to be a positive Radon measure on $S_a$ and we show that for any positive Radon measure $\xi$ on $S_a$, there exists a curve $\gamma$ on $M$ whose generalized tangent at $a$ is $\xi$ (§ 3, theorem 3).

More precisely, to define the generalized tangent of $\gamma$, first we introduce the notion of $\mathcal{F}(S^{n-1})$-smooth function at $a$, where $\mathcal{F}(S^{n-1})$ is a (fixed) function space on $S^{n-1}$ such as $C(S^{n-1})$, $L^p(S^{n-1})$ (the measure on $S^{n-1}$ is the standard volume element, that is given by $\sum_i dx_i$ (cf. [5], [11]) or (if $M$ is smooth or real analytic) $C^0(S^{n-1})$ or $C^\omega(S^{n-1})$, as follows: A function $f$ defined on some neighborhood of $a$ is called to be $\mathcal{F}(S^{n-1})$-smooth if $f$ is written as

\[ f(x) = f(a) + g(\varepsilon, x)\rho(a, x) + o(\rho(a, x)), \quad \rho(a, x) < 1, \]

and $g(y)$ belongs in $\mathcal{F}(S_a)$. Here $\varepsilon, x$ means the point $y$ on $S_a$ such that

\[ x \in r_{a, x}, \]

where $r_{a, x}$ is the curve of $M$ which joins $a$ and $y$ and \[ \int_{r_{a, x}} \rho = 1, \]
and $\mathcal{F}(S_a)$ means the function space on $S_a$ defined similarly as (using the measure induced from $\rho$ (cf.
Then the generalized tangent of \( r \) at \( a \) is defined to be the element \( \xi \) of \( \mathcal{F}^* (S_a) \), the dual space of \( \mathcal{F} (S_a) \) which is determined by
\[
\langle \xi, g \rangle = \lim_{s \to 0} \frac{1}{s} \lim_{h \to 0} \left( \int_0^1 \{ f(\rho(t)) - f(a) \} dt \right),
\]
where \( r \) is given by \( \rho : I \to M \) and \( \rho \) is assumed to satisfy
\[
\begin{align*}
(i) & \quad \rho(0) = a, \quad \rho(t) \neq a, \quad \text{if } t \neq 0, \\
(ii) & \quad \rho(a, \rho(t)) \in (t),
\end{align*}
\]
f is an \( \mathcal{F}(S^{n-1}) \)-smooth function at \( a \) and \( g \in \mathcal{F}(S_a) \) is given by
\[
g(y) = \lim_{t \to 0} \frac{1}{t} (f(r_{a,y,t}) - f(a)),
\]
where \( r_{a,y,t} \) is given by
\[
r_{a,y,t} \in r_{a,y}, \quad \rho(a, r_{a,y,t}) = t.
\]
We denote \( g \) by \( d_s f \) or \( d_s f(a) \) or \( d_s a \).

We note that this definition of the generalized tangent depends on the choice of parameter \( t \) of \( \gamma \)(cf. n°11 of § 3).

If \( \mathcal{F}(S^{n-1}) \) is taken to be \( C(S^{n-1}) \), the Banach space consisted by the continuous functions on \( S^{n-1} \) with the uniform convergence topology, then \( C^*(S^{n-1}) \) is the space of Radon measures on \( S^{n-1} \)(cf. [18]), and we can show that an element of \( C^*(S_a) \) is expressed as a generalized tangent at \( a \) of a curve if and only if it is positive, that is \( \langle \xi, g \rangle \geq 0 \) if \( g(y) \geq 0 \) on \( S_a \). For example, the Dirac measure on \( S_a \) is expressed as the generalized tangent at \( a \) of a curve \( \gamma \) which is smooth at \( a \). Here a curve \( \gamma \) given by \( \rho : I \to M, \quad \rho(0) = a, \) is called smooth at \( a \) if \( \lim_{t \to 0} \frac{1}{t} \rho_{a,\rho(t)} \) exists. The problem to characterize the element of \( C^*(S^{n-1}) \), the space of distributions on \( S^{n-1} \) or \( C^*(S^{n-1}) \), the space of analytic functionals on \( S^{n-1} \), which is expressed as the generalized tangent of some curve, remains open.

We note that although the O.N. -basis of \( L^2(S^{n-1}) \) is given by spherical harmonics (cf.[5], [11]), a smooth function at \( a \) only represents a spherical function of degree 1. Hence, since the usual tangent of a smooth curve is defined only by using smooth functions, the usual tangents of smooth curves corresponds only this part of \( L^2(S^{n-1}) \). But the above result shows, if we use the \( L^2(S^{n-1}) \)-smooth functions, the generalized tangents covers the positive part of \( L^2(S^{n-1}) \).

As in the case of usual tangent vectors (cf.[6], [13]), to set
\[
\hat{X}_s(f) = \lim_{s \to 0} \frac{1}{s} \lim_{h \to 0} \left( \int_0^1 \{ f(\rho(t)) - f(a) \} dt \right),
\]
where \( f \) is \( \mathcal{F}(S^{n-1}) \)-smooth at \( a \), we have.
(i) $\mathcal{X}_\rho(\alpha f_1 + \beta f_2) = \alpha \mathcal{X}_\rho(f_1) + \beta \mathcal{X}_\rho(f_2)$,

(ii) $\mathcal{X}_\rho(f_1 f_2) = f_1(\rho) \mathcal{X}_\rho(f_2) + f_2(\rho) \mathcal{X}_\rho(f_1)$,

and we also have

(iii) $\mathcal{X}_\rho(f) = 0$ if $|f(x) - f(a)| = o(\rho(a, x))$,

(iv) $\mathcal{X}_\rho(f) \geq 0$ if $d_x f \geq 0$.

On the other hand, if the map $\mathcal{X}$ from the space of $\mathcal{F}(S^{n-1})$-smooth functions at $a$ to $\mathbb{R}$, the real number field, satisfies (i), (ii) and (iii), then $\mathcal{X}$ is written as

$f = \langle \xi, d_x f(a) \rangle$,

by some $\xi \in \mathcal{F}(S^n)$. Hence if $\mathcal{X}$ also satisfies (iv), $\mathcal{X}$ is written as $\mathcal{X}_\rho$ by some $\rho : I \to M$. In some part, the globalization of these discussions are possible. To do this, first we construct the associate $\mathcal{F}(S^{n-1})$-bundle of the tangent micro-bundle of $M$, which is denoted by $\mathcal{F}(s(M))$ and its dual bundle, which is denoted by $\mathcal{F}^\ast(s(M))$ (§ 1). (cf. [1], [9], [12]).

Next, we set

$$d_x f(x, y) = \lim_{t \to 0} \frac{1}{t}(f(r_{x,y,t}) - f(x)), \quad y \in S_x.$$

If $d_x f(x)$ is a continuous cross-section of $\mathcal{F}(s(M))$, then we call $f$ is $\mathcal{F}(S^{n-1})$-smooth on $M$ (for $n = 1$, cf. [7], [8], [10]). We can show that the space of $\mathcal{F}(S^{n-1})$-smooth functions on $M$ (denoted by $C_\mathcal{F}(S^{n-1})(M)$) is dense in $C(M)$ or in $L^p_{loc}(M)$ (§ 2, theorem 1). (The measure on $M$ by which $L^p(M)$ or $L^p_{loc}(M)$ is defined, is that of induced from $\rho$ (cf. [3], [4])). Then a linear operator $X$ of $C(M)$ which satisfies the following (i), (ii), (iii) is called an $\mathcal{F}(S^{n-1})$-vector field on $M$.

(i). $X$ is a closed operator from $C_\mathcal{F}(S^{n-1})(M)$ into $C(M)$,

(ii). $(Xf)(a) = 0$, if $|f(x) - f(a)| = o(\rho(a, x))$ at $a$,

(iii). $X(f_1 f_2) = f_1 X(f_2) + f_2 X(f_1)$.

We show that if $X$ is an $\mathcal{F}(S^{n-1})$-vector field on $M$, then $X$ is written as

$Xf(x) = \langle \xi(x), d_x f(x) \rangle$, \quad $x \in M$,

where $\xi$ is a continuous cross-section of $\mathcal{F}^\ast(s(M))$ (§ 2, theorem 2). Therefore, as usual vector field, we may identify $X$ and a continuous cross-section of $\mathcal{F}^\ast(s(M))$. But an $\mathcal{F}(S^{n-1})$-vector field $X$ does not generate a 1-parameter group germ of $M$ in general. For example, the theorem of Hille-Yosida shows that if $M$ is compact and simply connected, the $C(S^{n-1})$-vector field $X$ corresponds to the cross-section $m$ of $C^\ast(s(M))$ given by $m = m(x)$, $m(x)$ is the cannonical measure on
\textit{S}_x \textit{defined from the metric, does not generate any (equi-continuous) 1-parameter semi group of } \textit{C}(\textit{M}) \text{ or } \textit{L}^p(\textit{M}) (\S 2, exemple). \textit{(cf.} [17], [18]). \textit{We note since } m(x) \text{ is positive, there exists a curve } \gamma = \gamma_x \text{ for any } x, \text{ such that } \gamma_x \text{ starts from } x \text{ and whose generalized tangent at } x \text{ is } m(x) \text{ (cf. } \S 3, \text{ exemple 2), if } n = 2, \gamma_x \text{ is given by } r^\theta = 1.

\textit{As usual vector field, if } X, Y \text{ are } \mathcal{F}(S^{n-1})\text{-vector fields such that their compositions } XY \text{ and } YX \text{ are both possible, then}

\[ [X, Y] = XY - YX \]

\textit{is also an } \mathcal{F}(S^{n-1})\text{-vector field of } \textit{M}. \text{ But the composition of } \mathcal{F}(S^{n-1})\text{-vector fields may not be possible in general.}

\textit{In } \S 1, \text{ we also construct associate } \mathcal{F}(S^{n-1} \times \cdots \times S^{n-1})\text{-bundle of the tangent microbundle of } \textit{M}. \text{ It is denoted by } \mathcal{F}(s^p(\textit{M})). \text{ We denote by } \mathcal{A}\mathcal{F}(s^p(\textit{M})) \text{ the subbundle of } \mathcal{F}(s^p(\textit{M})) \text{ whose fibre is consisted by those functions } f(y_1, \ldots, y_p), \text{ if } n = 2, \text{ of } \mathcal{F}(S^{n-1} \times \cdots \times S^{n-1}) \text{ such that}

\[ f(y_{a(1)}, \ldots, y_{a(p)}) = \text{sgn}(\sigma)f(y_1, \ldots, y_p), \quad \sigma \in \rho. \]

\textit{The cross-sections of these bundles are considered to be reductions of Alexander-Spanier cochains (cf. [1], [3], [14], [15]).}

\textit{For the cross-sections of } \mathcal{F}(s^p(\textit{M})) \text{ and } \mathcal{A}\mathcal{F}(s^p(\textit{M})), \text{ we define the maps } d_p \text{ and } Ad_p \text{ by}

\[ d_p f(x, y_1, \ldots, y_{p+1}) = \lim_{t \to 0} \frac{1}{t} [f(x, y_1, t, y_2, \ldots, y_{p+1}) - f(x, y_2, \ldots, y_{p+1})], \]

\[ Ad_p f(x, y_1, \ldots, y_{p+1}) = \frac{1}{p+1} \sum_{i=1}^{p+1} (-1)^i \lim_{t \to 0} \frac{1}{t} [f(x, y_1, \ldots, y_i, t, \ldots, y_{p+1}) - f(x, y_1, \ldots, y_{i-1}, y_i, t, \ldots, y_{p+1})]. \]

\textit{We call } f \text{ is } \mathcal{F}(S^{n-1})\text{-smooth if } d_p f \text{ (or } Ad_p f) \text{ defines a continuous cross-section of } \mathcal{F}(s^{p+1}(\textit{M})) \text{ (or } \mathcal{A}\mathcal{F}(s^{p+1}(\textit{M}))). \text{ We note that to define}

\[ \int f(x, y_1, \ldots, y_p) \text{ by}

\[ \int f(x, y_1, \ldots, y_p) = \int f(x, \epsilon x, x_1, \ldots, \epsilon x, x_p) \rho(x, x_1) \cdots \rho(x, x_p), \]

\textit{where } \gamma \text{ is a singular } p\text{-chain of } \textit{M} \text{ and the right hand side is the integration of}
Alexander-Spanier cochain defined in [3], \( \int_r f \) is exists if \( f \) is \( \mathcal{T}(S^{n-1}) \)-smooth and \( \gamma \) is given by \( \varphi : I^p \to M \) where \( \varphi \) satisfies

\[
\rho(\varphi(a_{j+1}), \varphi(a_j)) \leq N|a_{j+1} - a_j|,
\]

\[
a_j = (a_{j_1}, \ldots, a_{j_p}), \quad a_{j+1} = (a_{j_1}, \ldots, a_{j_{i-1}}, a_{j_i+1}, a_{j_{i-1}}, \ldots, a_{j_p}),
\]

for some \( N > 0 \). Since \( Ad_f(Ad_f) = 0 \) if \( Ad_f \) is \( \mathcal{T}(S^{n-1}) \)-smooth, we can obtain the analogy of de Rham's theorem by using the cross-sections of \( A \mathcal{F}(s^p[M]) \) and the Cech cohomology group of \( M \). But the above shows that the analogy of de Rham's theorem is also obtained by using the singular homology group of \( M \) (cf. [15], [16]).

We note that if \( M = \mathbb{R}^1 \), the 1-dimensional euclidean space with the euclidean metric, then

\[
d_f(x) = (D_+ f(x), D_- f(x)),
\]

where \( D_+ \) and \( D_- \) mean the right hand side and the left hand side derivations of \( f \) and the (fibre of \( \mathcal{F}(s(R^1)) \) is \( R \oplus R \). We know that \( f \) is smooth if and only if \( D_+ f = D_- f \) at any point of \( \mathbb{R}^1 \), that is \( d_f \) defines a cross-section of the subbundle of \( \mathcal{F}(s(R^1)) \) whose fibre is the diagonal of \( R \oplus R \).

To generalize this, first we assume the metric \( \rho \) of \( M \) satisfies (*)\. If \( \rho(x_1, x_2) \leq 2 \), then there is unique path \( \gamma \) which joins \( x_1 \) and \( x_2 \) and

\[
\int_{\gamma} \rho = \rho(x_1, x_2).
\]

Under this assumption, for any \( y \in S_x \), there exists unique point \( \hat{y} \) of \( S_x \) such that

\[
\rho(y, \hat{y}) = 2.
\]

We denote the quotient space of \( S_x \) obtained by identifying \( \hat{y} \) and \( y \) by \( P_x \). By definition, \( P_x \) is homeomorphic to \( \mathbb{R}P^{n-1} \), the \((n-1)\)-dimensional real projective space.

For this \( P_x \), if \( f \) is \( \mathcal{T}(S^{n-1}) \)-smooth at \( x \) and

\[
d_f(x, \hat{y}) = d_f(x, y),
\]

for any \( y \in S_x \), then \( d_f \) may be considered to be an element of \( \mathcal{T}(P_x) \). Here \( \mathcal{T}(P_x) \) is defined similarly as \( \mathcal{T}(S_x) \) and it is also considered to be a subspace of \( \mathcal{T}(S_x) \) given by

\[
\mathcal{T}(P_x) = \{ g | g \in \mathcal{T}(S_x), g(\hat{y}) = g(y) \}.
\]

Since \( \mathcal{T}(P_x) \) is isomorphic to \( \mathcal{T}(RP^{n-1}) \), we call \( f \) to be \( \mathcal{T}(RP^{n-1}) \)-smooth in this case. If \( M = \mathbb{R}^n \), the \( n \)-dimensional euclidean space with the euclidean
metric, then $f$ is $M(S^{n-1})$-smooth at $x$ if and only if $f$ is one-sided differentiable at $x$ along any line which ends at $x$ and $f$ is $M(RP^{n-1})$-smooth at $x$ if and only if $f$ is differentiable at $x$ along any line which through $x$.

Since the total spaces of $s(M)$ and $s^p(M)$, the associate $S^{n-1}$ and $S^{n-1} \times \cdots \times S^{n-1}$ bundles of the tangent microbundle of $M$ are given by

\[ s(M) = \{(x, y) | \rho(x, y) = 1, x \in M, (x, y) \in M \times M, \} \]
\[ s^p(M) = \{(x, y_1, \ldots, y_p) | \rho(x, y_i) = 1, i = 1, \ldots, p, x \in M, \}
\]

\[ (x, y_1, \ldots, y_p) \in M \times M \times \cdots \times M, \]

we can construct the associate $RP^{n-1}$-bundle and $RP^{n-1} \times \cdots \times RP^{n-1}$-bundle of $\tau(M)$, the tangent microbundle of $M$, by taking $s(M)/\sim$ and $s^p(M)/\sim$ to be the total spaces. Here the equivalence relations $\sim$ or $\sim_p$ are given by

\[ (x, y) \sim (x', y') \text{ if and only if } x = x' \text{ and } \rho(y, y') = 2, \]
\[ (x, y_1, \ldots, y_p) \sim_p (x', y_1', \ldots, y_p') \text{ if and only if } x = x, \text{ and } \rho(y_i, y_i') = 2, i = 1, \ldots, p, \]

where $\rho$ is assumed to satisfy (*). Then using $s(M)/\sim$ and $s^p(M)/\sim$, we can construct associate $s(RP^{n-1})$-bundle and $s(RP^{n-1} \times \cdots \times RP^{n-1})$-bundle of $\tau(M)$. They are denoted by $s(\tau(M)/\sim)$ and $s^p(\tau(M)/\sim)$. We note that since we have

\[ s(RP^{n-1}) = s(\tau(RP^{n-1})) \oplus s^p(RP^{n-1}), \]
\[ \tilde{s}(RP^{n-1}) = \{g | g(y) = g(y)\}, \]
\[ \tilde{s}(RP^{n-1}) = \{g | g(y) = -g(y)\}, \]

we may consider $\tilde{s}(s(M)/\sim)$ and $\tilde{s}(s^p(M)/\sim)$ are the subbundles of $\tilde{s}(s(M))$ and $\tilde{s}(s^p(M))$ and can be considered to be direct summands of them.

We note that using $\tilde{s}(s(RP^{n-1}))$-smooth functions and the bundles $\tilde{s}(s(M)/\sim)$, $\tilde{s}(s^p(M)/\sim)$ and $\tilde{s}(\tau(M)/\sim)$ (A $\tilde{s}(s^p(M)/\sim)$ is defined similarly as others), we can construct same theories as above.

Similarly, if $M = \mathbb{C}$, the complex number plane with the euclidean metric, $f$ a holomorphic function, then

\[ df(a, y) = \frac{df}{dz}(a). \]

This suggests that if $\text{dim.} M = 2m$, then the condition

\[ (*), \text{ there exists associate } CP^{m-1} \text{-bundle of } \tau(M), \]

may have some meaning for $M$.

The outline of this paper is as follows: In §1, we define the bundles
The generalized tangents of curves and generalized vector fields are stated in § 3.

Added in proof. In $\mathbb{R}^n$ with the euclidean metric, $d, f$ may be considered (one-sided) Gâteaux's differential $Vf$. Here Gâteaux's differential $Vf(x_0, h)$ is defined by

$$Vf(x_0, h) = \lim_{t \to 0} \frac{f(x_0 + th) - f(x_0)}{t}, \quad h \in \mathbb{R}^n,$$

where $f$ is a map from a Banach space $M_1$ to a Banach space $M_2$. For the details and related notions with their applications, see Burysek, S.: On symmetric G-differential and convex functionals in Banach spaces, Publ. Math., (Debrecen), 17, 1970, 145-161.

§ 1. Bundles $\mathcal{F}(s(M))$ and $\mathcal{F}(s'(M))$.

1. We denote by $M$ an $n$-dimensional connected paracompact topological manifold. On $M$, we fix a metric $\rho$ by which the topology of $M$ is given, and assume $\rho$ satisfies the following (i), (ii), (iii) (For the existence of such metric, see [4]).

(i). If $\rho(x_1, x_2) \leq 1$, then there exists unique path $\gamma$ which joins $x_1$ and $x_2$ and

$$\int_\gamma \rho = \rho(x_1, x_2).$$

(ii). $M$ is complete with respect to $\rho$.

(iii). The measure $m(\rho)$ induced from $\rho$ on $M$ is a positive Radon measure and satisfies

$$m(\rho)(E) = 0, \text{ if } E \text{ is measurable and contains some non-empty open set.}$$

For $x \in M$, we set

$$S_x = \{y | y \in M, \rho(x, y) = 1\}.$$  

Since $\dim M = n$, $S_x$ is homeomorphic to $S^{n-1}$, the unit $(n - 1)$-sphere (cf. [4]). We assume that for any $x$, $\rho$ induces a metric $\rho_x$ on $S_x$ which is given by

$$\rho_x(y_1, y_2) = \inf_{\gamma} \rho, \text{ } \gamma \text{ joins } y_1 \text{ and } y_2 \text{ in } S_x \int_\gamma \rho.$$  

The measure on $S_x$ induced from $\rho_x$ (cf. [3]) is denoted by $m = m(x)$. For this $m(x)$, we assume (cf. [4])

(i). A Borel set of $S_x$ is $m(x)$-measurable and if $E$ is $m(x)$ measurable and contains some non-empty open set of $S_x$, then

$$m(x)(E) \neq 0.$$
(ii). $m(x)$ depends continuously on $x$.

Since $S_+$ is compact, $m(x)(S_+)$ is finite. Hence, for the simplicity, we normalize $m(x)$ to satisfy $m(x)(S_+) = 1$.

Note. If $M$ is a smooth manifold, $\rho$ is the geodesic distance defined by a (complete) Riemannian metric on $M$, then $m(x)$ depends differentiably on $x$.

In $M \times M$, we set

$$s(M) = \{(x, y) | x \in M, \rho(x, y) = 1\}.$$

We define $\pi: s(M) \rightarrow M$ by $\pi(x, y) = x$. Then $\{s(M), \pi, M\}$ is the associate unit sphere bundle of the tangent microbundle of $M$ (cf. [1], [9], [12]). We denote the transition function of $s(M)$ by $\{g_{UV}(x)\}$ if we consider the fibre of $s(M)$ at $x$ to be $S_+$. We note that if we consider the fibre of $s(M)$ at $x$ to be the measure space $(S_+, m_\nu)$, then the transition function of $s(M)$ should be replaced by $\{(g_{UV}(x), m_U(x)(g_{UV}(x)^*m_\nu(x))^{-1})\}$, where $m_U(x)$ is given by

$$m(x)(E) = \int_{h_{U, x}(E)} m_U(x) d\Omega.$$

Here $h_{U, x}$ is the local homeomorphism from $\pi^{-1}(U)$ to $U \times S^{n-1}$ and $d\Omega$ is the standard measure on $S^{n-1}$.

We denote by $\mathcal{S}(S^{n-1})$ a function space over $S^{n-1}$. In the rest, $(S^{n-1})$ means either of $C(S^{n-1})$ or $L^p(S^{n-1})$, $1 \leq p \leq \infty$, regarding them to be Banach spaces. Here $L^p(S^{n-1})$ is defined by $d\Omega$. (If $M$ is smooth, or real analytic, then $C^\infty(S^{n-1})$, or $C^\infty(S^{n-1})$, is also taken as $(S^{n-1})$). Then by identifying $U \times C(S^{n-1}) \equiv (x, f(y))$ and $(x, f(g_{UV}(x)y)) \in V \times C(S^{n-1})$, $x \in U \cap V$, we obtain the associate $C(S^{n-1})$-bundle of $s(M)$. It is denoted by $C(s(M))$. Since $C(s(M))$ is a vector bundle over $M$ with the fibre $C(S^{n-1})$, its dual bundle $C^\ast(s(M))$ is defined. $C^\ast(s(M))$ is a vector bundle over $M$ with the fibre $C^\ast(S^{n-1})$, where $C^\ast(S^{n-1})$ is the space of Radon measures on $S^{n-1}$.

Lemma 1. Regarding $m(x)$ to be a function on $M$, $m(x)$ is a cross-section of $C^\ast(s(M))$.

Corollary. We have

$$m_U(x)(g_{UV}(x)^*m_\nu(x))^{-1} = 1.$$

By this corollary, although $\mathcal{S}(S^{n-1})$ is $L^p(S^{n-1})$, we can construct the associate $\mathcal{S}(S^{n-1})$-bundle of $s(M)$ by identifying $U \times \mathcal{S}(S^{n-1}) \equiv (x, f(y))$ and $(x, g_{UV}(x)^*f(y)) \in V \times \mathcal{S}(S^{n-1})$, $x \in U \cap V$. This bundle is denoted by $\mathcal{S}(s(M))$. The dual bundle of $\mathcal{S}(s(M))$ is denoted by $\mathcal{S}^\ast(s(M))$. By definition, the fibre of $\mathcal{S}^\ast(s(M))$ is $\mathcal{S}^\ast(S^{n-1})$, the dual space of $\mathcal{S}(S^{n-1})$. We denote the fibre of $\mathcal{S}(s(M))$ (and $\mathcal{S}^\ast(s(M))$) at $x$ by $\mathcal{S}(S_x)$ (and $\mathcal{S}^\ast(S_x)$).

Definition. An element of $\mathcal{S}^\ast(S_x)$ is called an $\mathcal{S}(S^{n-1})$-vector at $x$. 
Note. If we regard $S_x$ to be a measure space $(S_x, k(x))$, and define $L^p(S_x)$ by $k(x)$, then to define $K_U(x)$ similarly as $m_U(x)$, we obtain the associate $L^p(S^{n-1})$-bundle of $s(M)$ by identifying $U \times L^p(S^{n-1}) \ni (x, f(y))$ and $(x, [k_U(x)(g_{UV}(x)^*k_U(x))^{-1/\rho} g_{UV}(x)^*f(y)]) \in V \times L^p(S^{n-1}), x \in U \cap V.$

2. In $M \times M \times \cdots \times M$, we set

$$s^p(M) = \{(x, y_1, \ldots, y_p) \mid x \in M, \rho(x, y_i) = 1, i = 1, \ldots, p\}.$$ 

To define $\pi: s^p(M) \to M$ by $\pi(x, y_1, \ldots, y_p) = x$, $\{s^p(M), \pi, M\}$ is associate $S^{n-1} \times \cdots \times S^{n-1}$-bundle over $M$. If the fibre of $s(M)$ at $x$ is considered to be the measure space $(S_x, m(x))$, then we consider the fibre of $s^p(M)$ at $x$ to be the measure space $(S_x \times \cdots \times S_x, m(x) \times \cdots \times m(x))$. The transition functions $\{g_{UV}(x)\} = \{g_{UV}(x)\}$ of $s^p(M)$ is given by $g_{UV}(x)(y_1, \ldots, y_p) = (g_{UV}(x)y_1, \ldots, g_{UV}(x)y_p)$, where $g_{UV}(x)$ in the right hand side is the transition function of $s(M)$.

We denote by $\mathcal{F}(S^{n-1} \times \cdots \times S^{n-1})$ or $\mathcal{F}(S^{n-1})$ the function space over $S^{n-1} \times \cdots \times S^{n-1}$ which is of the same type with $\mathcal{F}(S^{n-1})$. That is $\mathcal{F}(S^{n-1})$ means either of $C(S^{n-1} \times \cdots \times S^{n-1})$ or $L^p(S^{n-1} \times \cdots \times S^{n-1})$ with the measure $m(x) \otimes \cdots \otimes m(x)$ in general and $C^0(S^{n-1} \times \cdots \times S^{n-1})$ or $C^\infty(S^{n-1} \times \cdots \times S^{n-1})$ is also considered if $M$ is smooth or real analytic. By assumption, $\mathcal{F}(S^{n-1})$ is dense in $\mathcal{F}(S^{n-1})$.

As $\mathcal{F}(s(M))$, we construct the associate $\mathcal{F}(s^p(M))$-bundle of $s^p(M)$. It is denoted by $\mathcal{F}(s^p(M))$. The dual bundle of $\mathcal{F}(s^p(M))$ is denoted by $\mathcal{F}^*(s^p(M))$. The fibres of $\mathcal{F}(s^p(M))$ and $\mathcal{F}^*(s^p(M))$ at $x$ are denoted by $\mathcal{F}(S_x \times \cdots \times S_x)$ or $\mathcal{F}^*(S_x)$. $\mathcal{F}(S_x \times \cdots \times S_x)$ is a $\mathcal{F}^*(S_x)$-bundle over $S_x$.

Definition. An element of $\mathcal{F}^*(S_x)$ is called an $\mathcal{F}^*(S^{n-1})$-$p$-vector at $x$.

For any $f \in \mathcal{F}^*(S^{n-1})$ and $\sigma \in \mathcal{F}^p$, we set

$$\sigma(f)(y_1, \ldots, y_p) = \sigma(f)(y_{i(1)}, \ldots, y_{i(p)}), y_i \in S^{n-1}.$$ 

Then, since $\mathcal{F}(S^{n-1}) \otimes \cdots \otimes \mathcal{F}(S^{n-1})$ is dense in $\mathcal{F}^*(S^{n-1})$, $\sigma$ is continuous. Therefore, setting

$$A_{\mathcal{F}^p(S^{n-1})} = \{f \in \mathcal{F}(S^{n-1}), \sigma(f) = \text{sgn}(\sigma)f\},$$

$A_{\mathcal{F}^p(S^{n-1})}$ is a closed subspace of $\mathcal{F}^*(S^{n-1})$. Since $\sigma^*$, the adjoint operator of $\sigma$, is $\sigma^{-1}$, we have

$$(A_{\mathcal{F}^p(S^{n-1}})^* = A_{\mathcal{F}^p(S^{n-1})}.$$ 

As we know
we obtain an \( A_{\mathcal{F}}^{\kappa}(S^{n-1}) \)-bundle over \( M \) to be a subbundle of \( \mathcal{F}(S(M)) \). This bundle is denoted by \( A_{\mathcal{F}}(S(M)) \). Its dual bundle is denoted by \( A_{\mathcal{F}}^{*}(S(M)) \). The fibres of \( A_{\mathcal{F}}(S(M)) \) and \( A_{\mathcal{F}}^{*}(S(M)) \) are denoted by \( A_{\mathcal{F}}(S_x) \) and \( A_{\mathcal{F}}^{*}(S_x) \).

**Note.** Similarly, to set

\[
S_{\mathcal{F}}^{\kappa}(S^{n-1}) = \{ f | f \in \mathcal{F}(S^{n-1}), \sigma(f) = f \},
\]

we can define an \( S_{\mathcal{F}}^{\kappa}(S^{n-1}) \)-bundle \( S_{\mathcal{F}}(S(M)) \) to be a subbundle of \( \mathcal{F}(S(M)) \). Its dual bundle is denoted by \( S_{\mathcal{F}}^{*}(S(M)) \). The fibres of \( S_{\mathcal{F}}(S(M)) \) and \( S_{\mathcal{F}}^{*}(S(M)) \) at \( x \) are denoted by \( S_{\mathcal{F}}(S_x) \) and \( S_{\mathcal{F}}^{*}(S_x) \).

**Definition.** A (continuous) cross-section \( \varphi \) of \( \mathcal{F}(S(M)) \) is called a (continuous) \( \mathcal{F}(S^{n-1}) \)-\( p \)-cochain of \( M \). If \( \varphi \) is a cross-section of \( A_{\mathcal{F}}^{*}(S(M)) \), then \( \varphi \) is called an \( \mathcal{F}(S^{n-1}) \)-\( p \)-form of \( M \).

**Definition.** A (continuous) cross-section of \( \mathcal{F}^{*}(S(M)) \) is called an \( \mathcal{F}(S^{n-1}) \)-\( p \)-vectorfield of \( M \).

In general, we call an element of \( \mathcal{F}(S_x) \otimes \mathcal{F}^{*}(S_x) \) to be an \( \mathcal{F}(S^{n-1}) \)-\( (p, q) \)-tensor at \( x \) and a continuous cross-section of \( \mathcal{F}(S(M)) \otimes \mathcal{F}^{*}(S(M)) \) to be an \( \mathcal{F}(S^{n-1}) \)-\( (p, q) \)-tensorfield of \( M \).

If \( M \) is smooth (or real analytic), then \( \mathcal{F}(S(M)) \) and \( \mathcal{F}^{*}(S(M)) \) allow the structure of smooth (or real analytic) vector bundles. Hence we can define smooth (or real analytic) \( \mathcal{F}(S^{n-1}) \)-\( p \)-cochain, etc.

3. We denote by \( r_{x, y} \) the unique curve which joins \( x \) and \( y \), \( y \in S_x \) and satisfies

\[
\int_{r_{x, y}} \rho = 1.
\]

Then for any \( a, 0 \leq a \leq 1 \), there exists unique point \( z \) in \( r_{x, y} \) such that \( \rho(x, z) = a \). We denote this \( z \) by \( r_{x, y, a} \).

On the other hand, if \( \rho(x, z) < 1 \), then there exists unique point \( y \) of \( S_x \) such that \( z \in r_{x, y} \). Or, in other word, \( x, z \) determines a point \( y \) of \( S_x \). We denote this \( y \) by \( \varepsilon_{x, z} \).

By definition, if \( \rho(x, z) < 1 \), then

\[
r_{x, \varepsilon_{x, z}}(x, z) = z.
\]

For an \( \mathcal{F}(S^{n-1}) \)-\( p \)-cochain \( \varphi = \varphi(x, y_1, \ldots, y_p) \) of \( M \), we set

\[
\tilde{\varphi}(x, x_1, \ldots, x_p) = \varphi(x, \varepsilon_{x, x_1}, \ldots, \varepsilon_{x, x_p}) \rho(x, x_1) \cdots \rho(x, x_p),
\]
Then \( \varphi \) defines an Alexander-Spanier \( p \)-cochain of \( M \). By definition, if \( \varphi \) is an \( \mathcal{F}(S^{n-i}) \)-\( p \)-form, then \( \varphi \) is alternative in \( x_i, \ldots, x_p \).

**Definition.** If \( \gamma \) is a singular \( p \)-chain of \( M \), then we define the integration
\[
\int_\gamma \varphi \quad \text{of} \quad \varphi, \quad \text{an} \quad \mathcal{F}(S^{n-i}) \)-\( p \)-cochain of \( M \) on \( \gamma \) by
\[
\int_\gamma \varphi = \int_\gamma \varphi.
\]

Here the right hand side means the integral of the Alexander-Spanier cochain \( \varphi \) on \( \gamma \) \([3]\)].

By the definition of the integral (cf. \([3]\)), if \( \varphi \) is a \( C(S^{n-i}) \)-\( p \)-cochain and \( \gamma \) is given by \( f: \{p\} \to M \) where \( f \) satisfies
\[
\rho(f(a_{j+1}), f(a_j)) \leq N|a_{j+1} - a_j|,
\]
\[
a_{j+1} = (a_{j_1}, \ldots, a_{j_i}, a_{j_i+1}, \ldots, a_{j_{i+1}}, \ldots, a_{j_k}), a_j = (a_{j_1}, \ldots, a_{j_k}),
\]
for some \( N < 0 \), then \( \varphi \) is absolutely integrable on \( \gamma \). In fact, since \( S^{n-1} \) and \( \gamma \) both compact, to set
\[
K = \max \left( \max_{x \in \gamma} |\varphi(x, y_1, \ldots, y_p)| \right),
\]
\( K \) is finite, and for any partition \( \Delta \) of \( I \), we have
\[
\left| \sum_\gamma \varphi \right| \leq KN^p \left( \sum_\gamma |a_{j_{i+1}} - a_{j_i}| \cdot |a_{j_{i+1}} - a_{j_i}| \right) \leq KN^p,
\]
\( \Delta \) is given by \( 0 = a_0 < a_1 < \ldots < a_m < 1 \),
which shows the absolute integrability of \( \varphi \) on \( \gamma \).

**Note.** This is also true if \( \varphi \) is an \( M(S^{n-i}) \)-\( p \)-cochain and it seems to be true for \( L(S^{n-i}) \)-\( p \)-cochains if we change the definition of the integral of Alexander-Spanier cochains to the Lebesgue type.

**Definition.** For an \( \mathcal{F}(S^{n-i}) \)-\( p \)-cochain \( \varphi = \varphi(x, y_1, \ldots, y_p) \) of \( M \), we define
\[
d_{\varphi} \varphi(x, y_1, \ldots, y_{p+1})
\]
\[
= \lim_{a \to 0} \frac{1}{a} (\varphi(x, y_1, a, y_2, \ldots, y_{p+1}) - \varphi(x, y_1, y_2, \ldots, y_{p+1})).
\]

By definition, if \( d_\varphi \varphi(x, y_1, \ldots, y_{p+1}) \) exists as an element of \( \mathcal{F}(S_x) \) for any \( x \) and continuous in \( x \), then \( d_\varphi \varphi \) is an \( \mathcal{F}(S^{n-i}) \)-\( (p + 1) \)-cochain of \( M \).

**Definition.** An \( \mathcal{F}(S^{n-i}) \)-\( p \)-cochain \( \varphi \) is called \( \mathcal{F}(S^{n-i}) \)-smooth if \( d_\varphi \varphi \) is a (con-
In general we define \( d^m \varphi \) by

\[
d^m \varphi = d_s(d^{m-1} \varphi),
\]

and call \( \varphi \) to be \( \mathcal{A}(S^n) \)-smooth if \( d^m \varphi \) is a (continuous) \( \mathcal{A}(S^n) \)-(\( p + 1 \))-cochain of \( M \). If \( \varphi \) is \( \mathcal{A}(S^n) \)-smooth for all \( m \), then we call \( \varphi \) to be \( \mathcal{A}(S^n) \)-\( \infty \)-smooth.

**Definition.** For an \( \mathcal{A}(S^n) \)-\( p \)-form \( \varphi(x, y_1, \ldots, y_p) \) of \( M \), we define

\[
Ad_{\varphi}(x, y_1, \ldots, y_{p+1}) = \frac{1}{p+1} \sum_{i=1}^{p+1} (-1)^{i+1} \lim_{a \to 0} \frac{1}{a} \left[ \varphi(x, y_1, a, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{p+1}) - \varphi(x, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{p+1}) \right].
\]

By definition, if \( \varphi \) is \( \mathcal{A}(S^n) \)-smooth, then \( Ad_{\varphi} \) is an \( \mathcal{A}(S^n) \)-(\( p+1 \))-form and if \( \varphi \) is \( \mathcal{A}(S^n) \)-\( 3 \)-smooth, then

\[
Ad_{\varphi}(Ad_{\varphi}) = 0.
\]

By \( (9)' \), denoting \( Cp(M, \mathcal{A}(S^n)) \) the space of \( \mathcal{A}(S^n) \)-\( \infty \)-smooth \( \mathcal{A}(S^n) \)-\( p \)-forms on \( M \), \( \{ \sum p \geq 0 \mathcal{A}(S^n), \mathcal{A}(S^n), Ad_{\varphi} \} \) is a differential complex and we can show the analogy of de Rham’s theorem. Because we know

\[
Ad_{\varphi} = \partial \varphi,
\]

where \( \partial \) is the coboundary homomorphism in the Alexander-Spanier cochain. By \( (9) \), we also have

\[
\int_{S^1} \varphi = \int_{S^1} Ad_{\varphi},
\]

if \( \varphi \) is an \( \mathcal{A}(S^n) \)-\( p \)-form (cf. [3]).

**Note.** By \( (10) \), we have especially

\[
\int \gamma d_{\varphi} = \int \gamma' d_{\varphi}, \text{ if } \gamma \text{ and } \gamma' \text{ start from same point and end at same point}.
\]

Because for a function \( f \), we have

\[
Ad_{\varphi} f = d_{\varphi} f.
\]

Therefore, we may write \( \int_a^x d_{\varphi} f \) if \( \rho(a, x) \) is small and we obtain

\[
(10)' \quad \int_a^x d_{\varphi} f = f(x) - f(a).
\]
2. Generalized vector fields.

Definition. A function \( f \) on some neighborhood of \( x \) is called to be \( \mathcal{F}(S_n) \)-smooth at \( x \) if \( (d_{\rho, x} f)(y) = d_{\rho} f(x, y), \ x \) is fixed, defines a function of \( \mathcal{F}(S_n) \).

By definition, we have

Lemma 2. \( f \) is \( \mathcal{F}(S_n) \)-smooth at \( a \) if and only if \( f \) is written as

\[
(11) \quad f(x) = f(a) + g(x, x) \rho(a, x) - o(\rho(a, x)),
\]

where \( x \) belongs in \( U(a) \), a neighborhood of \( a \) and \( g(y) \) is an element of \( \mathcal{F}(S_n) \).

For example, if \( M = \mathbb{R}^n \), \( n \)-dimensional euclidean space, \( \rho \) is the euclidean metric of \( \mathbb{R}^n \) and \( f \) is smooth at \( a \), then \( f \) is written as

\[
f(x) = f(a) + \left( \sum_i \frac{\partial f(a)}{\partial x_i} (x_i - a_i)/||x - a||||x - a|| + o(||x - a||),
\]

where \( = (x_1, \ldots, x_n), \ a = (a_1, \ldots, a_n) \) and \( ||x|| = \sqrt{\sum_i x_i^2} \). Then since \( g(y) = \sum_i \frac{\partial f(a)}{\partial x_i} y_i \), \( y = (y_1, \ldots, y_n), \ ||y|| = 1 \), belongs for any \( \mathcal{F}(S_n) \), \( f \) is \( \mathcal{F}(S_n) \)-smooth at \( a \) for any \( \mathcal{F}(S_n) \).

Definition. A function \( f \) on some neighborhood of \( x \) is called to be \( \mathcal{F}(S_n) \)-\( m \)-smooth at \( x \) if

\[
(d_{\rho, x}^m f)(y_1, \ldots, y_m) = d_{\rho}^m f(x, y_1, \ldots, y_m), \ x \ is \ fixed,
\]

defines a function of \( \mathcal{F}(S_n) \). If \( f \) is \( \mathcal{F}(S_n) \)-\( m \)-smooth at \( x \) for any \( m \), then we call \( f \) is \( \mathcal{F}(S_n) \)-\( \infty \)-smooth at \( x \).

For example, if \( M = \mathbb{R}^n \), \( \rho \) is the euclidean metric of \( \mathbb{R}^n \) and \( f \) is of class \( C^m \) at \( a \), then \( f \) is \( \mathcal{F}(S_n) \)-\( m \)-smooth at \( a \) for any \( \mathcal{F}(S_n) \). In fact, in this case, we get

\[
(d_{\rho, x}^m f)(y_1, \ldots, y_m) = \frac{1}{m!} \sum_{i,j \leq n} \frac{\partial^m f(a)}{\partial x_i \partial x_j} y_1, i_1, \ldots, y_m, i_m,
\]

\[
y_i = (y_{i_1}, \ldots, y_{i_m}), \ ||y_i|| = 1, \ i = 1, \ldots, m.
\]

We denote by \( \mathcal{F}(M) \) the function space on \( M \) either of \( C(M) \) or \( L^p(M) \), \( 1 \leq p \leq \infty \), if \( M \) is compact and either of \( C(M) \), \( C_0(M) \), the space of bounded continuous functions on \( M \), \( L^p(M) \), \( 1 \leq p \leq \infty \) and \( L^p_{\text{loc}}(M) \), \( 1 \leq p \leq \infty \) if \( M \) is not compact. Here, \( M \) is considered to be a measure space with the measure \( m(\rho) \), the induced measure from the metric.
We assume the manifold structure of $M$ is given by $\{(U, h_U) | h_U : U \to \mathbb{R}^n\}$, then we have

**Lemma 3.** If we have

$$|h_U(a) - h_U(x)| = O(\rho(a, x)),$$

for any $a, x \in M$ and $U \subseteq \{U\}$, where $a$ is regarded to be fixed and $x$ to be a variable, then the space of $\mathcal{F}(S^{n-1})$-smooth functions on $M$ is dense in $\mathcal{F}(M)$.

**Proof.** If $f$ is a smooth function on $\mathbb{R}^n$ with compact carrier, then the function $h_U f$ on $M$ given by

$$h_U f(x) = f(h_U(x)), \quad x \in U,$$

$$h_U f(x) = 0, \quad x \notin U,$$

is an $\mathcal{F}(S^{n-1})$-smooth function on $M$ by (12) and lemma 2. Hence we obtain the lemma since $M$ is paracompact.

**Corollary.** Under the same assumptions about $M$ and $\rho$, for any locally finite open covering $\{V\}$ of $M$, there exists a partition of unity by $\mathcal{F}(S^{n-1})$-smooth functions $\{\varphi(x)\}$ subordinate to $\{V\}$ for any $\mathcal{F}(S^{n-1})$.

**Theorem 1.** A paracompact topological manifold $M$ always has a metric $\rho$ such that the space of $\mathcal{F}(S^{n-1})$-smooth functions by $\rho$ on $M$ is dense in $\mathcal{F}(M)$ if $\mathcal{F}(M)$ is either of $C(M)$, $C_0(M)$ or $L^p_{loc}(M), 1 \leq p \leq \infty$.

**Proof.** We take the metric $\rho$ of $M$ constructed in [4]. Then, since

$$0 < \int_{h_U(a)} ||\xi - \eta|| < \infty, \quad \text{if and only if} \quad 0 < \int \rho < \infty,$$

we set

$$A = \{a \mid a \in M, \ a \ does \ not \ satisfy \ (12)\}.$$

$A$ is a discrete set of $M$. Hence for any $a \in A$, there exists a neighborhood $U(a)$ of $a$ such that $U(a) \cap A = \{a\}$. For this $U(a)$, we set

$$C_0(U(a)) = \{f | f \text{ is continuous on } U(a) \text{ and } f(a) = 0\}.$$

By definition, we have

$$C(U(a)) = \mathbb{R} \oplus C_0(U(a)),$$

where $\mathbb{R}$ is the space of constant functions on $U(a)$.

We take a neighborhood system $\{V_n(a)\}$ of $a$ in $U(a)$ such that

$$V_n(a) \subset V_{n+1}(a), \quad \cap V_n(a) = \{a\}.$$
and denote
\[ C_n(U(a)) = \{ f | f \text{ is continuous on } U(a) \text{ and } f|v_n(a) = 0 \}. \]
Then by lemma 3, \( \mathscr{C}(S^{n-1}) \)-smooth functions are dense in \( C_n(U(a)) \) for any \( n \).
Hence \( \mathscr{C}(S^{n-1}) \)-smooth functions are dense in \( C_n(U(a)) \) because \( \bigcup_n C_n(U(a)) \) is dense in \( C_n(U(a)) \). But, since a constant function is \( \mathscr{C}(S^{n-1}) \)-smooth for any \( \mathscr{C}(S^{n-1}) \), \( \mathscr{C}(S^{n-1}) \)-smooth functions are dense in \( C(U(a)) \) by (13).

For each \( a \in A \), we take a neighborhood \( V(a) \) such that \( V(a) \subset U(a) \) and set
\[
V(A) = \bigcup_{a \in A} V(a), \quad U(A) = \bigcup_{a \in A} U(a).
\]
Then we have
\[
(14) \quad V(A) \subset U(A).
\]
By lemma 3, we know that \( \mathscr{C}(S^{n-1}) \)-smooth functions are dense in \( CM - V(A) \), and by 14, we can set
\[
f = f_1 + f_2, \quad \text{car. } f_1 \subset M - V(A),
\]
\[
f_2 = \sum_{a \in A} f_{a,a}, \quad \text{car. } f_{a,a} \subset U(a),
\]
for any continuous function \( f \) of \( M \). Hence \( \mathscr{C}(S^{n-1}) \)-smooth functions are dense in \( C_\beta(M) \). Since \( C_\beta(M) \) is dense in \( L_p^1(M), 1 \leq p \leq \infty \), we have the theorem.

Note. If the total measure of \( M \) by \( m(\rho) \), the induced measure of \( \rho \), is finite, then \( \mathscr{C}(S^{n-1}) \)-smooth functions are dense in \( L^p(M), 1 \leq p \leq \infty \), although \( M \) is not compact.

5. We denote the space of \( \mathscr{C}(S^{n-1}) \)-smooth functions on \( M \) by \( C_\mathcal{C}(S^{n-1})(M) \).
If \( M \) is not compact, then the subspace of \( C_\mathcal{C}(S^{n-1})(M) \) consisted by bounded \( \mathscr{C}(S^{n-1}) \)-smooth functions on \( M \) is denoted by \( C_\mathcal{C}(S^{n-1}, b(M) \). We assume that \( C_\mathcal{C}(S^{n-1})(M) \) is dense in \( C(M) \).

Lemma 4. \( C_\mathcal{C}(S^{n-1})(M) \) and \( C_\mathcal{C}(S^{n-1}, b(M) \) are both rings with the unit.

Proof. If \( f_1 \) and \( f_2 \) are \( \mathscr{C}(S^{n-1}) \)-smooth at \( a \), then we may set
\[
f_i(x) = f_i(a) + g_i(e_{a,x})\rho(a, x) + o(\rho(a, x)), \quad i = 1, 2, \quad x \in U(a).
\]
Hence we have
\[
f_i(x)f_j(x) = \left\{ f_i(a)f_j(a) + (f_i(a)g_i(e_{a,x}) + f_j(a)g_j(e_{a,x}))\rho(a, x) + o(\rho(a, x)) \right\},
\]
for \( x \in U(a) \). Since \( f_i(a)g_i(e_{a,x}) + f_j(a)g_j(e_{a,x}) \) belongs in \( (S_a) \), \( f_1f_2 \) is \( \mathscr{C}(S^{n-1}) \)-smooth.
at $a$. On the other hand, since we know $d_*1=0$, where $1$ is the constant function with the value $1$, is $\mathcal{F}(S^{n-1})$-smooth for any $\mathcal{F}(S^{n-1})$. Therefore we obtain the lemma.

**Definition.** A closed operator $X$ defined in $C(M)$ with the range in $C(M)$ is called an $\mathcal{F}(S^{n-1})$-vector field of $M$ if it satisfies the following (i), (ii), (iii).

(i). $X$ is defined on $C_{\mathcal{F}(S^{n-1})}(M)$.

(ii). If $|f(a) - f(a)| = o(r(a, x))$ at $a$, then $(Xf)(a)$ is equal to $0$.

(iii). $X(f_1, f_2) = f_1X(f_2) + f_2X(f_1)$.

**Lemma 5.** If $\xi = \xi(x)$ is an $\mathcal{F}(S^{n-1})$-1-vector field of $M$, then to set

$$(Xf)(x) = <\xi(x), d_1f(x)>, \quad x \in M,$$

$X$ is an $\mathcal{F}(S^{n-1})$-vector field of $M$. Here $<\xi, \varphi>, \xi \in \mathcal{F}(S_x), \varphi \in \mathcal{F}(S_x)$, means the value of $\xi$ at $\varphi$.

**Proof.** By the definition of $d_*$, $d_*$ has the following properties.

(i). If $\{f_n\}$ converges to $f$ in $C(M)$ and $\{d_* f_n\}$ converges normally to some $\mathcal{F}(S^{n-1})$-1-cochain $\varphi$, then $f$ is $\mathcal{F}(S^{n-1})$-smooth and $d_* f = \varphi$.

(ii). $(d_* f)(a) = 0$ if $|f(x) - f(a)| = o(r(a, x))$ at $a$.

(iii). If $f_1$ and $f_2$ are both $\mathcal{F}(S^{n-1})$-smooth, then

$$d_* (f_1, f_2) = f_1d_* f_2 + f_2d_* f_1.$$

Hence we have the theorem.

**Note.** A series of $\mathcal{F}(S^{n-1})$-1-cochains $\varphi_m(x, y)$ is called converges normally to $\varphi(x, y)$ if the series of functions on $M$ given by $\{|\varphi_m(x, y) - \varphi(x, y)|\}$ converges uniformly to $0$ on any compact set of $M$. Here $||\varphi(x, y)||_x$ means the norm of $\varphi(x)$, $\varphi(x)(y) = \varphi(x, y)$, in $\mathcal{F}(S_x)$.

By the definition of $\mathcal{F}(S^{n-1})$-vector fields, we have

**Lemma 6.** If $X$ is an $\mathcal{F}(S^{n-1})$-vector field of $M$, then $X$ satisfies the following (14) and (15).

$$Xc = 0, \quad \text{where} \quad c \text{ is a constant function of} \ M.$$

$$<Xf_1, a) = <Xf_2, a), \quad \text{if} \quad |f_1(x) - f_2(x)| = o(r(a, x)).$$

**Theorem 2.** If $X$ is an $\mathcal{F}(S^{n-1})$-vector field of $M$, then there exists an $\mathcal{F}(S^{n-1})$-1-vector field $\xi(x)$ of $M$ such that

$$<Xf(x), d_1f(x)>, \quad x \in M.$$

Such $\xi(x)$ is determined uniquely from $X$ if $A$, the set defined in the proof of theorem 1, is the empty set.
Proof. We use same notations as in the proof of theorem 1 and first assume $x \notin A$. Then the map

$$d_{n,x} : \mathcal{C}^{(S^{n-1})}(M) \rightarrow \mathcal{F}(S^0),$$

given by $(d_{n,x} f)(y) = d_x f(x, y)$, is onto. Then we define

$$(17) \quad \langle \xi(x), g \rangle = (Xf)(x), \quad d_{n,x} f = g, \quad g \in \mathcal{F}(S^{n-1}).$$

By lemma 2 and (15), (17) is well defined and since $d_{n,x}$ is onto, $\xi(x)$ is an element of $\mathcal{F}(S_y)$ by closed graph theorem because $X$ is a closed operator. Since $Xf$ is continuous for any $f \in \mathcal{C}^{(S^{n-1})}(M)$, $\xi(x)$ is continuous in $x$, $x \in M-A$. Moreover, since $M-A$ is dense in $M$ and $Xf$ is continuous on $M$, $\lim_{x_n \to a} \xi(x_n) = \xi(a)$ exists as an element of $d_{n,a}(\mathcal{C}^{(S^{n-1})}(M))^*$ for any $a \in A$. Hence (by the theorem of Hahn-Banach), we may consider $\xi(a)$ to be an element of $\mathcal{F}^*(S_y)$ and $\xi$ is continuous at $a$. Therefore we obtain the theorem.

By lemma 4 and theorem 2, there is a 1 to 1 correspondence between the set of $\mathcal{F}^*(S^n)$-vector fields of $M$ and the set of $\mathcal{F}^*(S^{n-1})$-vector fields of $M$. Hence we identify them.

Note 1. If $X, Y$ are $\mathcal{F}^*(S^{n-1})$-vector fields of $M$ such that their compositions $XY$ and $YX$ are both defined, then $[X, Y] = XY - YX$ also satisfies the conditions (ii), (iii) of $\mathcal{F}^*(S^{n-1})$-vector fields.

Note 2. Let $X$ be a closed operator with the domain $\mathcal{D}(X) \subset C(M)$ and the range is in $C(M)$ such that

(i). $\mathcal{D}(X)$ is a dense subring of $C(M)$ with the unit.
(ii). If $f_1, f_2$ are in $\mathcal{D}(X)$, then $X(f_1 f_2) = f_1 X(f_2) + f_2 X(f_1)$.

Then we call $X$ is a generalized vector field of $M$. If $X$ also satisfies

(iii). $(Xf)(a) = 0$ if $|f(x) - f(a)| = \alpha(p(x, a))$,

for a (fixed) metric $\rho$ of $M$, then we call $X$ is a generalized vector field of $M$ with respect to $\rho$.

Since $X$ is closed, to define the topology of $\mathcal{D}(X)$ by taking

$$U(f, V, W) = \{ g \mid g \in \mathcal{D}(X), \quad g \in V, \quad Xg \in W \},$$

where $V$ and $W$ are the neighborhoods of $f$ and $Xf$ in $C(M)$, as the neighborhood basis of $f \in (X)$, $(X)$ is a complete space and to set

$$\mathcal{Z}_d(X) = \{ f \mid f \in \mathcal{D}(X), \quad f(a) = Xf(a) = 0 \},$$

$\mathcal{Z}_d(X)$ is a closed ideal of $\mathcal{D}(X)$ by this topology. Hence setting

$$\mathcal{F}_d(X) = (X) \cap I_d(M)/\mathcal{Z}_d(X), \quad I_d(M) = \{ f \mid f \in C(M), \quad f(a) = 0 \},$$
we can set

\[ Xf(a) = \langle \xi(a), d_Xf(a) \rangle, \quad \xi(a) \in \mathcal{F}_a(X)^*, \]

where \( d_Xf(a) \) is the class of \( f \cdot f(a) \) in \( \mathcal{F}_a(X) \).

If \( X \) is a generalized vector field of \( M \) with respect to \( \rho \), then we have

\[ \mathcal{F}(X) \ni \{ f \} \in C(X), \quad \| f(x) \| = o(\rho(a, x)). \]

6. For an \( \mathcal{F}(S^{n-1}) \)-vector field \( X \) given by \( Xf = \langle \xi, d_xf \rangle \) and \( t, 0 \leq t \leq 1 \), we set

\[ U_{x,t}(f)(x) = \langle \xi(x), f(r_x,y,t) \rangle. \]

\[
(18) \quad U_{x,t}(f)(x) = \langle \xi(x), f(r_x,y,t) \rangle.
\]

Here \( f(r_x,y,t) \) is regarded to be a function of \( y, y \in S_x \). Since \( f \) is continuous, \( f(r_x,y,t) \) is continuous on \( S_x \). Hence \( U_{x,t}(f) \) is well defined for any \( X \).

By definition, \( U_{x,t} \) is defined on \( C(M) \) and a bounded linear operator of \( C(M) \) if \( M \) is compact. We also know that \( \lim_{t \to t_0} U_{x,t_0}(f) \) converges normally to \( U_{x,t_0}(f) \). Therefore, if \( M \) is compact, then \( U_{x,t} \) is strongly continuous in \( t \). Moreover, we know

\[ \lim_{t \to 0} \frac{1}{t}(U_{x,t} - U_{x,0})f = Xf, \quad \text{if } f \in C_0(S^{n-1})(M). \]

We note that

\[ U_{x,0}f(x) = \langle \xi(x), 1 \rangle f(x), \]

where \( 1 \) is the constant function with the value 1 on \( S_x \).

(19) shows that there is a curve in \( L(C(M), C(M)) \), the space of (bounded) linear operators of \( C(M) \) (with the strong topology), such that whose tangent at its starting point is \( X \).

For \( U_{x,t} \), we set

\[ T_{x,t} = \exp_{e} \left( \frac{t}{a}(U_{x,a} - U_{x,0}) \right), \quad t \geq 0. \]

Then \( \{ T_{x,t} \} \) is a 1-parameter semi-group of \( C(M) \) with the generating operator \((1/a)(U_{x,a} - U_{x,0})\). Hence if \( \lim_{a \to 0} T_{x,a} \) exists, then to set its limit by \( T_{x,t} \), \( T_{x,t} \) is a 1-parameter semi-group with the generating operator \( X \). But this limit does not exists in general. In fact, there exists an \( \mathcal{F}(S^{n-1}) \)-vector field which does not generate any 1-parameter semi-group of \( C(M) \) or \( L^p(M) \), \( 1 \leq p \leq \infty \).

**Example.** We assume that \( M \) satisfies

(i). \( H^q(M, \mathbb{R}) \) vanishes.
To define a $C(S^{n-1})$-1-form $\phi(x, y)$ on $M$ by $\phi(x, y) = \lambda$, an (arbitrary) constant, we get

$$d_\phi \varphi = 0.$$ 

Hence by (i), there exists a $C(S^{n-1})$-smooth function $\mu$ on $M$ such that

$$(d, h)(x, y) = \varphi(x, y).$$

Let $X$ be the $C(S^{n-1})$-vector field on $M$ given by

$$Xf(x) = <m(x), d_\phi f(x)>,$$

$m(x)$ is the canonical measure on $S_x$. Then we have for the above $h$,

$$Xh = \lambda, \text{ the constant function with the value } \lambda \text{ on } M.$$ 

For this $h$, we set $k = \exp(h) = \sum_{m(h)/m!}$. Then we get

$$Xk = \lambda k.$$ 

This shows $\lambda$ is a proper value of $X$ in $C(M)$ (or in $L^p(M)$, $1 \leq p \leq \infty$, because $C(M)$ is contained in $L^p(M)$ since $M$ is compact), Since $M$ is compact, $C(M)$ is a Banach space. Then by the theorem of Hille-Yosida ([17], [18]), $X$ can not generate any (equi-continuous) 1-parameter semi-group of $C(M)$ (or $L^p(M)$), because $\lambda$ is arbitrary.

In general, if an $L^2(S^{n-1})$-vector field $X$ is given by

$$Xf = <\xi(x), d_\phi f(x)>,$$

and $M$ is compact, then $X$ does not generate any 1-parameter semi-group of $C(M)$ (or $L^p(M)$, $1 \leq p \leq \infty$). In fact, in this case, we may set

$$L^2(S_x) = (\xi(x))_0 \oplus R\xi(x),$$ 

and denote the projection to $R\xi(x)$ by $P_{\xi(x)}$. Then a cross-section $f$ of the bundle $\bigcup_{x \in X} R\xi(x)$ is considered to be a function $f$ of $M$ by setting

$$f_M(x) = a, \text{ if } f(x) = a - \frac{\xi(x)}{|\xi(x)|}.$$

(We note that this also shows that a function of $M$ always defines a crosssection of $\bigcup_{x \in X} R\xi(x)$). Then by the definition of $X$, we have

$$Xf(x) = |\xi(x)||P_{\xi(x)}d_\phi f|^2(x).$$
We define $P_\varepsilon^d_x f$ by $(P_\varepsilon^d_x f)(x) = P_{\varepsilon(x)}^d_x f$. Then $P_\varepsilon^d_x C L^q(S^{n-1})$ is dense in the space of the cross-sections of $\cup_x \mathbb{R}^2(x)$, for any constant function $\lambda$ and $\varepsilon > 0$, there exists an $L^2(S^{n-1})$-smooth function $f_{\varepsilon, \lambda}$ such that

$$||X_{f_{\varepsilon, \lambda}} - \lambda|| < \varepsilon.$$  

This means $\lambda$ is at least continuous spectre of $X$, because $M$ is compact. Hence by the theorem of Hille-Yosida, we have the assertion.

**Note.** The generating operator of a 1-parameter semi-group $\{T_t\}$ is an $\mathcal{F}(S^{n-1})$-vector field of $M$, if and only if $\{T_t\}$ satisfies

$$T_t(f_1 f_2) - (T_t f_1)(T_t f_2) = 0(t), \text{ if } f_1, f_2 \in C_0(S^{n-1})(M).$$

7. In this note, we give some definitions about $X$, an $\mathcal{F}(S^{n-1})$-vector field on $M$.

**Definition.** $X$ is called to be $0$ at $a$, $a \in M$, if $(Xf)(a) = 0$ for all $\mathcal{F}(S^{n-1})$-smooth functions.

By definition, if $X$ is given by $Xf = \langle \xi(x), d_x f(x) \rangle$, then $X$ is $0$ at $a$ if and only if $\xi(a) = 0$ as an element of $\mathcal{F}^{-1}(S_n)$. As usual, we set

$$\text{car. } X = \{x : X \text{ is not } 0 \text{ at } x \}.$$

**Definition.** For $X$, we set

$$\text{CAR. } X = \bigcup_{x \in M} \text{car. } \xi(x), \text{ if } (Xf)(x) = \langle \xi(x), d_x f(x) \rangle.$$

By definition, CAR. $X$ is a (closed) subset of $s(M)$ and we have

$$\pi(\text{CAR. } X) = \text{car. } X.$$

We note that if $M$ is smooth and $X$ is a usual vector field on $M$ regarded to be a $C(S^{n-1})$-vector field on $M$ and does not vanish at any point of $M$, then CAR. $X$ is a cross-section of $s(M)$ (cf. no9).

**Definition.** $X$ is called to be positive if $X$ is given by $Xf = \langle \xi(x), d_x f(x) \rangle$ and

$$\xi(x) \geq 0 \text{ for any } x \in M.$$

As usual, we call $X \succeq Y$ if $X - Y \succeq 0$. Then since

$$\langle \sup_a \{X_a\} f = \langle \sup_a \{\xi_a(x)\}, d_x f(x) \rangle,$$

if $\{X_a\}$ is upper (or lower) bounded, then $\sup_a \{X_a\}$ (or $\inf_a \{X_a\}$) exists to be an
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\( S^*(S^{n-1}) \)-vector field. Especially, we may define \( X^+ = \max. (X,0) \) and \( X^- = (\neg X)^+ \) for any \( S^*(S^{n-1}) \)-vector field \( X \) and we have

\[
\text{(23)}
X = X^+ - X^-.
\]

We note that if \( Xf = \langle \xi(x), d_\rho f(x) \rangle \), then

\[
\langle X^+f(x) \rangle = \langle (\xi(x))^+, d_\rho f(x) \rangle, \quad \langle X^-f(x) \rangle = \langle (\xi(x))^-, d_\rho f(x) \rangle,
\]

where \((\xi(x))^+ = \max. (\xi(x), 0)\) and \((\xi(x))^- = (\xi(x))^+\).

**Note.** Since the space of \( S^*(S^{n-1}) \)-vector field of \( M \) is a vector space, these shows that this space has the structure of (complete) vector lattice. Hence to fix an \( S^*(S^{n-1}) \)-vector field \( Y \), \( \langle Yf(x) \rangle = \langle \gamma(x), d_\rho f(x) \rangle \), the Radon-Nykodim partition of any \( S^*(S^{n-1}) \)-vector field \( X \), \( Xf = \langle \xi(x), d_\rho f \rangle \) with respect to \( Y \) is possible. It corresponds to the Radon-Nykodim partition of \( \xi(x) \) with respect to \( \gamma(x) \).

**Definition.** If \( S^*(S^{n-1}) \)-vector fields \( X_1 \) and \( X_2 \) are given by \( \langle X_if(x) \rangle = \langle \xi_i(x), d_\rho f \rangle \), \( i = 1, 2 \), and \( Y = [X_1, X_2] \) is defined to be an \( S^*(S^{n-1}) \)-vector field of \( M \), then we denote

\[
\text{(24)}
\gamma(x) = [\xi(x), \xi_2(x)].
\]

Here \( Y \) is given by \( \langle Yf(x) \rangle = \langle \gamma(x), d_\rho f(x) \rangle \).

We note that if \( x \) is fixed in (24), then (24) defines the bracket product for some elements of \( S^*(S^{n-1}) \). Or, in other word, \( S^*(S^{n-1}) \) contains (as a dense subset), a Lie pseudoalgebra.

**§ 3. Generalized tangent of a curve.**

8. We denote the set of germs of \( S^*(S^{n-1}) \)-smooth functions of \( M \) at \( a \), \( a \in M \), by \( C_{S^*(S^{n-1})}^{s, a}(M) \).

**Lemma 7.** If \( S^*(S^{n-1}) \)-smooth functions \( f_1 \) and \( f_2 \) defines same germ in \( C_{S^*(S^{n-1})}^{s, a}(M) \) and \( |f_1(x) - f_1(a)| = o(\rho(x, a)) \), then \( |f_2(x) - f_2(a)| \) is also \( o(\rho(x, a)) \).

By this lemma, we can say \( |f(x) - f(a)| \) is \( o(\rho(x, a)) \) although \( f \) is regarded to be an element of \( C_{S^*(S^{n-1})}^{s, a}(M) \).

**Definition.** A linear map \( \xi \) from \( C_{S^*(S^{n-1})}^{s, a}(M) \) to \( R \) is called an \( S^*(S^{n-1}) \)-vector of \( M \) at \( a \) if it satisfies the following (i), (ii), (iii).

(i. \( \xi(f_1 f_2) = f_1(a) \xi(f_2) + f_2(a) \xi(f_1) \)).

(ii. \( \xi(f) = 0, \) if \( |f(x) - f(a)| = o(\rho(x, a)) \)).

(iii. \( \xi(f) = (Xf)(a), \) where \( X \) is an \( S^*(S^{n-1}) \)-vector field of \( U(a), \) a neighborhood of \( a \).

By (iii) and theorem 2, we have
**Theorem 2'.** For any $\mathcal{T}(S^{n-1})$-vector $\xi$ of $M$ at $a$, there exists an element $\xi$ of $\mathcal{T}^*(S_a)$ such that

$$\hat{\xi}(f) = \langle \xi, d_{n,a}f \rangle,$$

and such $\xi$ is determined uniquely by $\hat{\xi}$. Conversely, if $\xi \in \mathcal{T}^*(S_a)$, then $\langle \xi, d_{n,a}f \rangle$ is an $\mathcal{T}(S^{n-1})$-vector of $M$ at $a$.

Let $\gamma$ be a curve of $M$ given by $\varphi : I \to M$ such that

\begin{align*}
(25) & \quad \varphi(0) = a, \quad \varphi(t) \neq a \text{ if } t > 0, \\
(25)' & \quad \rho(a, \varphi(t)) = 0(t).
\end{align*}

Then we set

\begin{align*}
(26) & \quad \hat{\xi}_\varphi(f) = \lim_{s \to 0} \frac{1}{s} \left[ \lim_{h \to 0} \frac{1}{h} \int_{h}^{s} f(\varphi(t)) - f(\varphi(0)) \, dt \right],
\end{align*}

where $f$ is an $\mathcal{T}(S^{n-1})$-smooth function at $a$.

By (25) and (25)', we have

\begin{align*}
(26)' & \quad \hat{\xi}_\varphi(f) = \lim_{s \to 0} \frac{1}{s} \left[ \lim_{h \to 0} \frac{1}{h} \int_{h}^{s} \rho(a, \varphi(t))(d_{n,a}f(\varphi(t))) \, dt \right].
\end{align*}

**Lemma 8.** If $\hat{\xi}_\varphi(f)$ exists for all $\mathcal{T}(S^{n-1})$-smooth functions at $a$, then $\hat{\xi}_\varphi$ is an $\mathcal{T}(S^{n-1})$-vector of $M$ at $a$.

**Proof.** By (26)', we only need to show (i). But, since we know

\begin{align*}
(d_{n,a}(f_1 f_2))(\varphi_1, \varphi_2) &= f_1(\varphi_1)(d_{n,a}f_2)(\varphi_1, \varphi_2) + f_2(\varphi_2)(d_{n,a}f_1)(\varphi_1, \varphi_2),
\end{align*}

we have (i) by (26)'.

**Definition.** If $\hat{\xi}_\varphi$ is defined on $C_{\mathcal{T}^*(S^{n-1})}, \gamma, a(M)$, then $\gamma$ is called $\mathcal{T}(S^{n-1})$-smooth at $a$.

By theorem 2' and lemma 8, if $\hat{\xi}_\varphi$ is defined on the space of $\mathcal{T}(S^{n-1})$-smooth functions at $a$, then there exists an element $\xi = \hat{\xi}(\varphi)$ of $\mathcal{T}^*(S_a)$ such that

$$\hat{\xi}_\varphi(f) = \langle \hat{\xi}(\varphi), d_{n,a}f \rangle.$$
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which is dense in $C(S^{n-1})$ or in $L^p(S^{n-1})$ (cf. [5], [11]).

9. In this no, we give some examples of the generalized tangent.

**Example 1.** We assume $\gamma$ is smooth at $a$, that is

$$\lim_{t \to 0} \varepsilon_{a, \gamma(t)} = y, \ y \in S_n,$$

$$\lim_{t \to 0} \frac{\rho(a, \varphi(t))}{t} = c, \ c \ is \ a \ (positive) \ real \ number,$$

both exists and $f$ is $C(S^{n-1})$-smooth at $a$, then we have by the mean value theorem

$$\int_{h}^{s} \frac{(\rho(a, \varphi(t))}{t} (d_{a, \varphi(t)} f)(x_{a, \varphi(t)}) \, dt$$

$$= \frac{\rho(a, \varphi(s))}{s} (d_{a, \varphi(s)} f)(x_{a, \varphi(s)}) (s - h), \ h < s_0 < s.$$ 

Hence we have

$$\varepsilon_{\gamma}(f) = c(d_{a, f}(y)).$$ 

Therefore, denoting the Dirac measure of $S_n$ concentrated at $y$ by $\delta_y$, we get

$$\varepsilon_{\gamma}(f) = \langle c \delta_y, d_{a, f} \rangle.$$ 

We note that if $f$ is smooth at $a$, then $\varepsilon_{\gamma}(f)$ coincide to the usual definition of the (one-sided) derivation of $f$ along $\gamma$.

**Note.** If $M$ is smooth and $X$ is a usual vector field of $M$ which does not vanish at any point of $M$, then at any point $a$ of $M$, $X$ has a smooth integral curve $\gamma_a$ given by $\varphi_a: I \to M, \ \varphi_a(0) = a$, and

$$(Xf)(a) = \varepsilon_{\gamma_a}(f).$$ 

Hence we have by (27)

$$(Xf)(a) = \langle c(a) \delta_{\gamma(a)}, d_{a, f} \rangle.$$ 

Hence we have

$$\text{CAR.} X = \bigcup_{a \in \mathcal{M}} \gamma(a).$$ 

Since $\gamma(a)$ depends continuously on $a$, \text{CAR.} X is a (continuous) cross-section of $s(M)$.

In the following two examples, we need the following

$$C\alpha^{\varepsilon}(S^{n-1}) = R[x_1, \ldots, x_n]/(x_1^2 + \ldots + x_n^2 - 1),$$
Lemma 9. If \( g(t) \) is a continuous periodic function on \( \mathbb{R} \) with the period \( T \), then

\[
\lim_{s \to \infty} \int_{s}^{\infty} \frac{g(t)}{t^2} \, dt = \frac{1}{T} \int_{0}^{T} g(t) \, dt.
\]

Proof. We define a periodic function \( e_{[a, b]}(t) \), \( 0 \leq a < b \leq T \), with the period \( T \) by

\[
e_{[a, b]}(t) = 1, \quad t \in [a + nT, b + nT], \text{ for some integer } n, \\
= 0, \quad \text{otherwise}.
\]

Then for \( 0 \leq a' \leq a < b \leq b' \leq T \), to set

\[
v_{m, a', b'}^{a, b} = \frac{b' - a'}{b - a} (t - \{mT + a\}) + mT + a', \quad mT \leq v_{m, a', b'}^{a, b} \leq (m + 1)T,
\]

we have

\[
e_{[a, b]}(v_{m, a', b'}^{a, b}) = e_{[a', b']}^{a', b'}(t), \quad mT \leq v_{m, a', b'}^{a, b} \leq (m + 1)T.
\]

Hence we get

\[
\int_{mT}^{\infty} \frac{e_{[a, b]}(t)}{t^2} \, dt = \frac{b - a}{b' - a'} \int_{mT}^{\infty} \frac{e_{[a', b']}^{a', b'}(t)}{t^2} \, dt.
\]

Then, since we know

\[
\lim_{a' \to a, b' \to b} \int_{mT}^{\infty} \frac{e_{[a', b']}^{a', b'}(t)}{t^2} \, dt = s \int_{s}^{\infty} \frac{dt}{t^2},
\]

we obtain

\[
\lim_{s \to \infty} s \int_{s}^{\infty} \frac{e_{[a, b]}(t)}{t^2} \, dt = \frac{|b - a|}{T}.
\]

Then, since \( g(t) \) is bounded and (uniformly) continuous, we have

\[
\lim_{s \to \infty} s \int_{s}^{\infty} \frac{g(t)}{t^2} \, dt
\]

\[
= \lim_{s \to \infty} \left[ \lim_{s \to \infty} \sum_{|a_i+1 - a_i| \to 0} g(a_i) \left( \int_{s}^{\infty} \frac{e_{[a_i, a_i+1]}(t)}{t^2} \, dt \right) \right]
\]

\[
= \lim_{|a_i+1 - a_i| \to 0} \sum_{i} g(a_i) \left[ \lim_{s \to \infty} \int_{s}^{\infty} \frac{e_{[a_i, a_i+1]}(t)}{t^2} \, dt \right].
\]
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\[ \lim_{|a_{i+1} - a_i| \to 0} \sum_{i} g((a_i) \frac{|a_{i+1} - a_i|}{T}) \]

Here, \( 0 = a_0 < a_1 < \ldots < a_m < a_{m+1} = T \) is a partition of \([0, T]\).

**Example 2.** Let \( M \) be \( \mathbb{R}^2 \) with the euclidean metric, \( a \) the origin \( O=(0, 0) \) of \( \mathbb{R}^2 \) and \( r \) is given by \( \varphi : I \to \mathbb{R}^2 \), where \( \varphi \) is given by

\[ \varphi(t) = (t \cos \left( \frac{1}{t} \right), t \sin \left( \frac{1}{t} \right)), \quad t > 0, \]

\[ \varphi(0) = 0. \]

Hence, if we use the polar coordinate \((r, \theta)\) of \( \mathbb{R}^2 \), \( r = \sqrt{x^2 + y^2} \) and \( \theta = \tan^{-1}(y/x) \), then \( r \) is given by

\[ r(0) = 1, \quad r > 0. \]

Then, if \( S^1 = \{(x, y)|x^2 + y^2 = 1\} \) is parametrized by \( \theta \) and \( g \) is continuous on \( S^1 \), we get

\[ \lim_{s \to 0} \frac{1}{s} \left[ \lim_{h \to 0} \int_{h}^{s} g\left( \frac{1}{t} \right)dt \right] \]

\[ = \lim_{s \to 0} \frac{1}{s} \left[ \lim_{h \to 0} \int_{h}^{s} g\left( \frac{1}{t} \right)dt \right] = \lim_{h \to 0} \int_{\pi/\cos h}^{\pi} \frac{g(v)}{v^2} dv. \]

Hence by lemma 9, we have

\[ \chi_\rho(f) = \frac{1}{2\pi} \int_{0}^{2\pi} (d_{r, \rho}f)(\theta)d\theta. \]

Or, in other word, the generalized tangent of the curve \( r\theta = 1 \) at \( 0 \) is the standard measure of \( S^1 \).

**Example 3.** We take \( M \) and \( \rho \) same as above and take \( \varphi \) to be

\[ \varphi(t) = (t, t \sin(\frac{1}{t})), \quad t > 0, \quad \varphi(0) = 0, \quad \text{the origin of} \ \mathbb{R}^2. \]

By definition, we have

\[ \rho(0, \varphi(t)) = \sqrt{1 + \sin^2(\frac{1}{t})}, \quad \varepsilon_{0, \varphi(t)} = \tan^{-1}(\sin(\frac{1}{t})). \]

Hence we have by lemma 9,
\[
\lim_{s \to 0} \frac{1}{s} \left[ \lim_{h \to 0} \int_{h}^{s} \frac{\rho(0, y(t))}{t} g(x, y(t)) \, dt \right] = \frac{1}{2\pi} \int_{0}^{2\pi} \sqrt{1 + \sin^{2} \phi \left( \tan^{-1}(\sin \phi) \right)} \, d\phi
\]

Therefore, the generalized tangent of the curve \(x \sin(1/x)\) at the origin is the measure on \(S^{1}\) concentrated on \(-\pi/4 \leq \theta \leq \pi/4\) with the weight \((1/\pi)(1/\cos^{2} \theta)\). 

Note. If \(r\) is given by \((-t, t \sin(1/t)), t > 0\), then the generalized tangent of \(r\) at the origin is similar as above but has carrier on \(3\pi/4 \leq \theta \leq 5\pi/4\).

10. **Lemma 10.** The generalized tangent of a curve at \(a\) is a positive measure on \(S_{n}\).

**Proof.** If \(\xi\) is the generalized tangent of \(\varphi: I \to M\), then we have

\[
\int_{S_{a}} g(y) d\xi = \lim_{s \to 0} \frac{1}{s} \left[ \lim_{h \to 0} \int_{h}^{s} \frac{\rho(a, y(t))}{t} g(x, y(t)) \, dt \right].
\]

Hence if \(g \geq 0\) on \(S_{a}\), then \(\int_{S_{a}} g(y) d\xi \geq 0\). Therefore \(\xi\) is a positive measure.

**Lemma 11.** If the parameter of \(r\) is changed to \(ct\) instead of \(t\), \(c\) is a constant, then the generalized tangent \(\xi\) of \(r\) at \(a\) is changed to \(c\xi\). In general, if the parameter of \(r\) is changed to \(a(t)\) and

\[
\lim_{t \to 0} \frac{a(t)}{t} = c,
\]

then the generalized tangent \(\xi\) of \(r\) at \(a\) changes to \(c\xi\).

By this lemma, we may assume the generalized tangent \(\xi\) of \(r\) at \(a\) satisfies

\[
(31) \quad \xi(S_{a}) = 1.
\]

**Theorem 3.** If \(\xi\) is a positive measure on \(S_{n}\), then there exists a curve of \(M\) starts from \(a\) such that whose generalized tangent at \(a\) is \(\xi\).

**Proof.** Since the proof for \(n = 1\) is similar, we assume \(n \geq 2\).

First we note that the problem is local, we may assume \(M = R^{n}\) with the euclidean metric and \(a\) is the origin \(0 = (0, \ldots, 0)\) of \(R^{n}\). Hence \(S_{n}\) is the unit \((n-1)\)-sphere \(S^{n-1}\).

We take a positive measure \(\xi\) of \(S^{n-1}\) such that \(\xi(S^{n-1}) = 1\). By lemma 11, this is not restrictive.

We choose a countable dense subset \(\{y_{j}\}\) of \(S^{n-1}\) such that
(32) \( y_p \neq \pm y_q \), if \( p \neq q \).

For this \( \{y_p\} \), we divide \( S^{n-1} \) by Borel sets \( \{E_p^q\} \) as follows:

(33) \( S^{n-1} = \bigcup_{p \leq q} E_p^q, E_p^q \cap E_{p'}^q = \emptyset, \) if \( p' \neq p'' \), \( y_p \in E_p^q \).

(33)\(^{\prime}\) \( \lim_{q \to \infty} \text{dia.} \ (E_p^q) = 0 \).

Here \( \text{dia.} \ (E_p^q) \) means the diameter of \( E_p^q \). Hence, if \( g(y) \) is a continuous function of \( S^{n-1} \), then

(34) \( \int_{S^{n-1}} g(y) d\zeta = \lim_{q \to \infty} \sum_{p \leq q} g(y_p) E_p^q \).

On the other hand, for the above \( \{E_p^q\} \) and \( \zeta \), we take a series of (positive) real numbers \( \{t_{q,p}\}, \ p \leq q \), as follows:

(35) \( t_{q,p} > t_{q,p+1}, \) if \( p + 1 \leq q, \ t_{q,q} > t_{q+1,1} \),

(35)\(^{\prime}\) \( \lim_{q \to \infty} t_{q,p} = 0 \),

(35)\(^{\prime\prime}\) \( \sum_{q \geq t_{q,p}, p \leq q} \frac{1}{s} |(t_{q,p} - t_{q,p+1}) - \xi(E_p^q)| \leq \frac{s}{2\rho}, \ s > 0 \).

This is possible because \( \xi(S^{n-1}) = 1 \) and \( \sum_{p} \sum_{q \geq t_{q,p}, p \leq q} (1/s) |t_{q,p} - t_{q,p+1}| = 1 - (s - t_{q,p})/s \) is sufficiently near to 1. Here, \( t_{q,p} \) is the largest \( t_{q,p} \) which is smaller than \( s \).

Using this \( \{t_{q,p}\} \), we set

\[
\begin{align*}
\Psi(t_{q,p}) &= t_{q,p} y_p, \\
\Psi(t) &= \frac{t_{q,p} - t}{t_{q,p} - t_{q,p+1}} \Psi(t_{q,p+1}) + \frac{t - t_{q,p+1}}{t_{q,p} - t_{q,p+1}} \Psi(t_{q,p}), \\
&\text{if } t_{q,p} > t > t_{q,p+1}, \\
\Psi(t) &= \frac{t_{q,q} - t}{t_{q,q} - t_{q+1,1}} \Psi(t_{q+1,1}) + \frac{t - t_{q+1,1}}{t_{q,q} - t_{q+1,1}} \Psi(t_{q,q}), \\
&\text{if } t_{q,q} > t > t_{q+1,1}, \\
\Psi(0) &= 0.
\end{align*}
\]

Then since \( ||y_p|| = 1 \), we have the definition of \( \Psi(t) \) and (32),

(36)\(^{\prime\prime}\) \( ||\Psi(t)|| \leq ||t|| \),

(36)\(^{\prime}\) \( \Psi(t) \neq 0, \) if \( t \neq 0 \).

We also note that by the definition of \( \Psi(t) \), \( \Psi(t) \) is continuous for all \( t, \ 0 \leq t \leq 1 \).
By (36)', to define \( \varphi(t) \) by

\[
\varphi(t) = \frac{\psi(t)}{||\psi(t)||}, \quad t > 0, \quad \psi(0) = 0,
\]

\( \varphi(t) \) is also continuous in \( t \) and satisfies similar conditions as (36)' and.

\[
||\varphi(t)|| = t.
\]

By (36) and the mean value theorem, if \( \{y_p\} \) satisfies

\[
\lim_{p \to \infty} \frac{y_{p+1} - y_p}{t} = 0,
\]

then we have for this \( \varphi(t) \),

\[
\int_{t_{q,p}}^{t_{q, p+1}} \frac{||\varphi(t)||}{t} g(y_p(\varphi(t)))dt = g(y_q(t_{q,p} - t_{q, p+1})) + o(|t_{q,p} - t_{q, p+1}|).
\]

Hence we have

\[
\lim_{s \to 0} \frac{1}{s} \left[ \lim_{h \to 0} \int_{h}^{s} \frac{||\varphi(t)||}{t} g(y_p(\varphi(t)))dt \right]
\]

\[
= \lim_{s \to 0} \frac{1}{s} \sum_p g(y_p) \sum_{q, t_{q,p} \leq s} (t_{q,p} - t_{q, p+1})
\]

On the other hand, by (35)''', we obtain

\[
\sum_{p \leq s} \sum_{t_{q,p} \leq s} g(y_p) \xi(E_{p}) - \frac{1}{s} \sum_p g(y_p) \sum_{q, t_{q,p} \leq s} (t_{q,p} - t_{q, p+1})
\]

\[
\geq \sum_p \frac{s}{2^p} = s.
\]

Then, by (34) and (38), we get

\[
\int_{S^{n-1}} g(\gamma) d\tilde{\xi} = \lim_{s \to 0} \frac{1}{s} \left[ \lim_{h \to 0} \int_{h}^{s} \frac{||\varphi(t)||}{t} g(y_p(\varphi(t)))dt \right],
\]

for this \( \varphi(t) \). Therefore the curve \( \gamma \) given by \( \varphi: I \to M \), has the generalized tangent at the origin and it is equal to \( \tilde{\xi} \). Hence we have the theorem.

Note. Since \( C^*(S^{n-1}) \) contains \( L^s(S^{n-1}) \), a positive linear functional of \( L^s(S^{n-1}) \) always expressed as the generalized tangent of some curve.
Example 1. If $\xi$ is the Dirac measure of $S^{n-1}$ concentrated at $y_i$, $y_i \in S^{n-1}$, then \{t_{q, p}\} is given by

$$t_{q, 1} = \frac{1}{2^q}, \quad t_{q, p} = \frac{1}{2^q} - \left(1 - \frac{1}{2^{p-1}}\right) \frac{1}{2^q}, \quad 2 \leq p \leq q.$$ 

Example 2. If $\xi$ is the standard measure of $S^{n-1}$, then we take $E_{p, q}$ to satisfy $\xi(E_{p, q}) = 1/q$. Then we can take \{t_{q, p}\} to be

$$t_{q, p} = \frac{1}{q + 1} + \frac{q + 1 - p}{p + 1} \left(-\frac{1}{q(q + 1)}\right).$$

We note that although the curve $\varphi(t) = y_it$ has the generalized tangent $\partial y_i$, it is not given by the above method.

11. We denote by $H^*(I)$ the group of orientation preserving homeomorphisms of $I = [0, 1]$. The subgroup of $H^*(I)$ consisted by those homeomorphisms that are the identity map on $[0, \varepsilon]$ for some $\varepsilon > 0$, is denoted by $H_\varepsilon(I)$. Then we set

$$H^*_{\varepsilon}(I) = H^*(I)/H_\varepsilon(I),$$

$H^*_{\varepsilon}(I)$ is the group of germs of the (orientation preserving) homeomorphisms of $I$ (cf. [2]).

If $\alpha \in H^*(I)$, then by the theorem of Radon-Nykodim, there exists a (positive) measurable function $m_\alpha$ on $I$ which does not vanish almost everywhere on $I$, such that

$$\int_a^b \rho(\alpha(t))dt = \int_{\alpha(a)}^{\alpha(b)} \rho(u)m_\alpha(u)du,$$

where $\rho(t)$ is an (arbitrary) measurable function on $I$. We note that this $m_\alpha(t)$ also satisfies

$$\int_0^1 m_\alpha(t)dt = 1.$$

Conversely, if $m(t)$ is a positive measurable function on $I$ such that to satisfy (40) and does not vanish almost everywhere on $I$, then $\int_0^1 m(u)du$ is an element of $H^*(I)$. Moreover, we know that

(i). If $\alpha_1$, $\alpha_2 \in H^*(I)$ and $\alpha_2\alpha_1$ is the composition of $\alpha_1$ and $\alpha_2$ in $H^*(I)$, then

$$m_{\alpha_1\alpha_2} = \alpha_2^*(m_{\alpha_1})m_{\alpha_2}, \quad \alpha^*(m(t)) \text{ means } m(\alpha(t)).$$

(ii). $\alpha$ belongs in $H_\varepsilon(I)$ if and only if $m_\alpha(t) = 1$, $0 \leq t < \varepsilon$, for some $\varepsilon > 0$.

Hence to denote the set of all positive measurable functions on $I$ which do...
not vanish almost everywhere on \( I \) and satisfy (40) by \( \mathcal{M}^*(I) \) and to define a multiplication \( m_1 \cdot m_2 \) for \( m_1, m_2 \in \mathcal{M}^*(I) \) by

\[
m_1 \cdot m_2 = \alpha_2^*(m_1)m_2, \quad \alpha_2(t) = \int_0^t m_2(u)du,
\]

\( \mathcal{M}^*(I) \) is isomorphic to \( H^*(I) \) and to set

\[
\mathcal{M}_a(I) = \{ m \mid m \in \mathcal{M}^*(I), m(t) = 1, \ 0 \leq t < \varepsilon, \text{ for some } \varepsilon > 0 \},
\]

we have

\[
\mathcal{M}_a(I) \cong H^*_a(I), \quad \mathcal{M}_a(I) = \mathcal{M}(I)/\mathcal{M}_a(I).
\]

For \( \varphi: I \to M, \) and \( \alpha \in H^*(I) \), we set

\[
\alpha^*(\varphi)(t) = \varphi(\alpha(t)).
\]

Then the image of \( \varphi \) and \( \alpha^*(\varphi) \) is same. Moreover, we know if \( \alpha \in H_a(I) \), then \( \varphi \) has the generalized tangent at its starting point if and only if \( \alpha^*(\varphi) \) has the generalized tangent at its starting point and we have by lemma 10,

\[
\mathcal{E}_\varphi(f) = \mathcal{E}_{\alpha^*(\varphi)}(f).
\]

By (44), we have

\[
\mathcal{E}_{\alpha^*(\varphi)} = \mathcal{E}_{\beta^*(\varphi)}, \text{ if } \alpha \equiv \beta \mod H_a(I).
\]

By (43), (44)' and theorem 3, we can define an operation of the element \( m \) of \( \mathcal{M}_a(I) \) to \( \mathcal{D}^*_a(S^{n-1}) \), the set of positive linear functionals of \( \mathcal{D}(S^{n-1}) \) by

\[
\langle m(\xi), g \rangle = \mathcal{E}_{\alpha^*(\varphi)}(f),
\]

where, assuming the starting point of \( \varphi \) is \( a \), \( d_{\kappa, a}f = g \), \( \mathcal{E}_\kappa(f) = \langle \xi, g \rangle \) and the class of \( m \) in \( \mathcal{M}_a(I) \) is \( m \). Then, since the change of parameter of \( \gamma \) corresponds to the operation of \( \mathcal{M}_a(I) \), we may consider the generalized tangent of \( \gamma \) to be an element of \( \mathcal{D}^*_a(S^{n-1})/\mathcal{M}_a(I) \).
References.


