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Note on the Suspension-order of a Some Complex

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1 Introduction

We shall attempt to compute the suspension-order of a complex

$$Y_{k+1} = S^n \cup e^{n+1} \cup e^{n+2(q-1)} \cup e^{n+2(q-1)+1} \cup \cdots\cdots \cup e^{n+2k(q-1)+1}$$

with only homology groups $H^{n+2i(q-1)}(Y_{k+1}) \approx Z_q$ for $i = 0, 1, \cdots\cdots$, k, where p is odd prime.

H. Toda [4] gave the general properties of the suspension-order. We compute $K_O{}^*$ in order to give a lower boundary of the suspension-order of Y_{k+1}. Here, the notation $K_O{}^*$ means the reduced group of original K-theory of vector bundles. But, we shall use only the definitions

$$K_O^{-i}(X) = [S^iX,\ B_O] \text{ and } K_U^{-i}(X) = [S^iX,\ B_U]$$

and Bott's periodicity [1].

2 Suspension-order of complex Y_k

We shall use the following notations. For each topological space, we always associate a base point $*$. Mapping and homotopies considered are base point presevring. The set of the homotopy classes of mappings f: $(X, *) \longrightarrow (Y, *)$ is denoted by $[X,\ Y]$.

Let A be a subspace of a topological space X, then X/A is a space obtained by smashing A to a base point. q: $X \longrightarrow X/A$ indicates the smashing map (projection). A reduced product $A \wedge B$ is $A \times B / A \vee B$, where $A \vee B = A \times * \cup * \times B$. We define n-fold suspension S^nX of X by the formula $S^nX = X \wedge S^n$ $(SX = S^1X)$. We denote by 1_X and ι_X the identity map of space X and its homotopy class and call the order of the homotopy class ι_{SX} the suspension-order of X.

Let A and B be topological spaces, and $f: B \longrightarrow A$ a continuous mapping, then we denote by

$$A \underset{f}{\cup} CB$$

a space obtained from a cone CB by identifying its base B with the mapping f into A.

Lemma 2.1. *Let $Y=Y_k$. Then the class $p\iota_{SY}$ is represented by a mapping $f: SY_k \longrightarrow SY_k$ satisfying the following conditions; $f(SY_k^{n+1}) \subset SY_k^{n+i-1}$ for $i=0$, $1, \cdots\cdots, 2(k-1)(p-1)+1$. A mapping f_j: $S^{n+2j(p-1)+1} \longrightarrow S^{n+2(j-1)(p-1)+2}$ given uniqueiy by the commutativity of the diagram*

$$\begin{array}{ccccc}
SY_k^{n+t} & & \xrightarrow{\ f\ } & & SY_k^{n+t-1} = SY_k^{n+2(j-1)(p-1)+1} \\
\downarrow{\scriptstyle q} & & & & \downarrow{\scriptstyle q'} \\
SY_k^{n+t}/SY_k^{n+t-1} = S^{n+t+1} & & \xrightarrow{\ f_j\ } & & S^{n+2(j-1)(p-1)+2} = SY_k^{n+t-1}/SY_k^{n+t-2}
\end{array}$$

represents a generator of $\pi_{n+2j(p-1)+1}(S^{n+2(j-1)(p-1)+2};\ p)$ *if and only if* $(\Delta\mathfrak{P}^1-\mathfrak{P}^1\Delta)$ $H^{n+2j(p-1)}(Y_k; Z_p)=0$, *where* $t=2j(p-1)$ *and* Δ *is mod. p Bockstein operator.*

Proof. Let u: $S^1 \longrightarrow S^1$ be a mapping of degree p. Then $p\iota_{SY}$ is represented by $1_Y \wedge u$: $SY_k \longrightarrow SY_k$. Let $h=1_Y \wedge u$. Since $h_*(\alpha)=p\alpha=0$ for $\alpha \in \mathrm{H}_*(SY_k)$, we have that h_* : $H_*(SY_k) \longrightarrow H_*(SY_k)$ is trivial for $i>0$. Obviously, $H_{n+t+2}(SY_k, SY_k^{n+t+1}) \approx H_{n+t+2}(SY_k)=0$, and $H_{n+t+1}(SY_k) \approx H_{n+t+1}(SY_k, SY_k^{n+t+1}) \approx Z_p$. Applying lemma 1.4 of [4] for $n+t+1=n+2j(p-1)+1$ $(j=0, 1, \cdots\cdots, k-1)$ in place of n in the lemma 1.4 of [4], we get a homotopy $h_s: SY_k \longrightarrow SY_k$ such that $h=h_0$, $h_s(SY_k^{n+2j(p-1)}) \subset SY_k^{n+2j(p-1)})$, $h_1(SY_k^{n+2j(p-1)}) \subset SY_k^{n+2(j-1)(p-1)+1}$ and h_1 $(SY_k^{n+2j(p-1)+1}) \subset SY_k^{n+2j(p-1)}$. We put $f=h_1$, then the first condition is satisfied.

Consider complexes $K=SY_k \underset{h}{\cup} CSY_k$ and $K_1=SY_k \underset{f}{\cup} CSY_k$. They have the same homotopy type. In fact, we give a homotopy equivalence F: $K \longrightarrow K_1$ as follows. A point of CSY_k is represented by a pair (x, s), $x \in SY_k$, $s \in [0, 1]$. Then F is defined by the fomulas

$$F|SY_k = 1_{SY}$$

and

$$F(x, s)=\begin{cases} (x, 2s-1) & \text{for } 1/2 \leqslant s \leqslant 1, \\ h_{2s}(x) & \text{for } 0 \leqslant s \leqslant 1/2. \end{cases}$$

F is a cellular map and

$$F(SY_k^{n+2j(p-1)+1} \underset{h}{\cup} CSY_k^{n+2j(p-1)+1}) \subset SY_k^{n+2j(p-1)+1} \underset{f}{\cup} CSY_k^{n+2j(p-1)+1}. \tag{1}$$

Let a_i and b_i be chains in the cell complex K represented by the cells Se^{n+i} and CSe^{n+i}, respectively. For suitably chosen orientations of the cells, we have

$$\partial a_{2j(p-1)+1} = pa_{2j(p-1)},$$
$$\partial a_{2j(p-1)} = 0,$$
$$\partial b_{2j(p-1)+1} = pb_{2j(p-1)} - pa_{2j(p-1)},$$
$$\partial b_{2j(p-1)} = pa_{2j(p-1)}.$$

Let a'_i and b'_i be corresponding chains of K_1, then

$$\partial a'_{2j(p-1)+1} = pa'_{2j(p-1)},$$
$$\partial a'_{2j(p-1)} = \partial b'_{2j(p-1)} = 0,$$
$$\partial b'_{2j(p-1)+1} = pb'_{2j(p-1)}.$$

Consider the chain mapping $F_{\#}$: $C(K) \longrightarrow C(K_1)$ induced by F. From the definition of F, $F_{\#}(a_s) = a'_s$ and $F_{\#}(b_s) = b'_s$ mod a'_{s+1}. It follows from the relation (1) that $F_{\#}(b_{2j(p-1)+1}) = b'_{2j(p-1)+1}$. By use of the naturality $\partial_\circ F_{\#} = F_{\#\circ}\partial$, we have that $F_{\#}(b_{2j(p-1)}) = b'_{2j(p-1)} + a'_{2j(p-1)+1}$.

Let α_i, β_i, α'_i, and β'_i be the dual classes mod p of a_i, b_i, a'_i and b'_i, respectively. Then they are independent generators and the induced homomorphism $F^* : H^*(K_1;\ Z_p) \longrightarrow H^*(K;\ Z_p)$ satisfies

$$F^*(\alpha'_{2j(p-1)}) = \alpha_{2j(p-1)},$$
$$F^*(\alpha'_{2j(p-1)+1}) = \alpha_{2j(p-1)+1} + \beta_{2j(p-1)},$$
$$F^*(\beta_i') = \beta_i.$$

It is easy to see that $K = SY_k \underset{h}{\cup} CSY_k = Y_k \wedge P^2$ for $P^2 = S^1 \underset{u}{\cup} CS^1 = * \cup e^1 \cup e^2$. Each cell of Y_k and P^2 represents cohomology class mod p. Then $\Delta e^1 = e^2$, $\Delta e^{n+2j(p-1)} = e^{n+2j(p-1)+1}$.

Assume that $\mathfrak{P}^1 e^{n+s} = x_s e^{n+s+2(p-1)}$ for some integer x_s $(0 \leqq x_s < p)$. Then we have, by Cartan's formula,

$$\mathfrak{P}^1(e^{n+s} \times e^1) = x_s(e^{n+s+2(p-1)} \times e^1)$$

and

$$\mathfrak{P}^1(e^{n+s} \times e^2) = x_s(e^{n+s+2(p-1)} \times e^2).$$

The projection $q: Y_k \times P^2 \longrightarrow Y_k \wedge P^2$ induces isomorphisms of $H^*(Y_k \wedge P^2;\ Z_p)$ into $H^*(Y_k \times P^2;\ Z_p)$ such that

$$q^*(\alpha_s) = e^{n+s} \times e^1 \text{ and } q^*(\beta_s) = e^{n+s} \times e^2.$$

By the naturality of the $\mathfrak{P}^1$ -operation, we have that

$$\mathfrak{P}^1\beta_s = x_s\beta_{s+2(p-1)},$$
$$\mathfrak{P}^1\alpha_s = x_s\alpha_{s+2(p-1)}.$$

By use of the inverse of F^*, we have that

$$\mathfrak{P}^1\beta'_s = x_s\beta'_{s+2(p-1)},$$
$$\mathfrak{P}^1\alpha'_{2i(p-1)} = x_{2i(p-1)}\,\alpha'_{2(i+1)(p-1)},$$
$$\mathfrak{P}^1\alpha'_{2i(p-1)+1} = x_{2i(p-1)+1}\,\alpha'_{2(i+1)(p-1)+1} + (x_{2i(p-1)} + x_{2i(p-1)+1})\beta'_{2(i+1)(p-1)}.$$

The commutativity diagram of the lemma defines a mapping

$$Q: SY_k^{n+2(j-1)(p-1)+1} \underset{f}{\cup} CSY_k^{n+2j(p-1)} \longrightarrow S^{n+2(j-1)(p-1)+2} \underset{f_j}{\cup} CS^{n+2j(p-1)+1},$$

which induces monomorphisms of cohomology groups mod p. $\alpha'_{2(j-1)(p-1)+1}$ and $\beta'_{2j(p-1)}$ are the images of Q^*. It follows that $\mathfrak{P}^1 \neq 0$ in $S^{n+2(j-1)(p-1)+2} \underset{f_j}{\cup} CS^{n+2j(p-1)+1}$ if and only if $x_{2j(p-1)} - x_{2j(p-1)+1} \not\equiv 0$ (mod. p). Its means that $(\Delta\mathfrak{P}^1 - \mathfrak{P}^1\Delta)H^{n+2j(p-1)}(Y_k;\ Z_p) \neq 0$.

Proposition 2. 2. *Let* $n \geqq 2$. *The suspension-order of* $Y_k = S^n \cup e^{n+1} \cup \cdots\cdots \cup e^{n+2(k-1)(p-1)} \cup e^{n+2(k-1)(p-1)+1}$ *is a divisor of* p^k.

Proof. We prove by induction on k. Since we may consider $SY_{k-1} = SY_k^{n+2(k-2)(p-1)+1}$, SY_{k-1} is a subcomplex of SY_k. Then, from Theorem 4. 4 of [4], the suspension-order of $SY_k/SY_{k-1} = S^{n+2(k-1)(p-1)+1} \cup e^{n+2(k-1)(p-1)+2}$ is p. From the assumption of induction and Theorem 1. 2 of [4], we obtain the result.

3. Computation of $K_U^*(Y_k)$ and $K_O^*(Y_k)$.

We shall use the following exact sequence in K-theory ($K^n = K_O{}^n$ or $K^n{}_U$);

$$(2)\cdots\cdots \longrightarrow K^n(X/A) \xrightarrow{p^*} K^n(X) \xrightarrow{i^*} K^n(A) \xrightarrow{\delta} K^{n+1}(X/A) \xrightarrow{p^*} \cdots\cdots$$

Note that if $X = A \underset{f}{\cup} CB$, then the diagram

$$\begin{array}{ccc} K^n(A) & \xrightarrow{\quad\delta\quad} & K^{n+1}(X/A) = K^{n+1}(SB) \\ & {\scriptstyle f^*}\searrow \quad \nearrow{\scriptstyle S} & \\ & K^n(B) & \end{array}$$

is commutative, where the suspension homomorphism S is an isomorphism onto.

For a CW-complex Y and a fibering $0 \xrightarrow{p} 0/U\,(=\Omega^2 B_O)$, there is an exact sequence

$$\cdots\cdots \longrightarrow K_U^n(Y) \xrightarrow{i_*} K_O^n(Y) \xrightarrow{p_*} K_O^{n-1}(Y) \xrightarrow{\delta} K_U^{n+1}(Y) \longrightarrow \cdots\cdots \tag{3}$$

The sequence commute with homomorphism induced by a mapping $f: Y' \longrightarrow Y$ and also the homomorphism in (2).

Let B_U^k be a $(k-1)$-connected space such that $\Omega^{k-1}(B_U^k)$ has the same singular homotopy type as B_U.

Lemma 3. 1. (H. Toda [4]) *(i) Let L be a CW-complex such that* $L^{n-1} = *$. *Then we have natural isomorphism* $k_U^{n+i}(L) = [SL,\ B_U^{n+i}\ ]$ *for* $i \geqq 1$.

(ii) We may take B_U^k *as CW-complex such that the* $(k+4)$*-skeleton of* B_U^k *is* S^{k-1} $M_2 = S^{k+1} \cup e^{k+3}$ *if* $k \geqq 2$, *where* M_2 *is the complex projective space.*

The values of $K_U^{n+i}(S^n)$ and $K_O^{n+i}(S^n)$ are as follows

(4)

i	-4	-3	-2	-1	0	1	2	3
$K_U^{n+i}(S^n)$	Z	0	Z	0	Z	0	Z	0
$K_O^{n+i}(S^n)$	Z	0	Z_2	Z_2	Z	0	0	0

Consider $Y_1 = S^n \cup_u e^{n+1}$, where u: $S^n \longrightarrow S^n$ is a mapping of degree p. Then, it follow from the table (4) and exact sequence of (2) that

(5)

i	-4	-3	-2	-1	0	1	2	3
$K_U^{n+i}(Y_1)$	0	Z_p	0	Z_p	0	Z_p	0	Z_p
$K_O^{n+i}(Y_1)$	0	Z_p	0	0	0	Z_p	0	0

Lemma 3. 2. *A generator of* $K_U^{n+1}(S^{2(p-1)}Y_1) \approx [S^{2(p-1)}Y_1,\ B_U^{n+1}] \approx Z_p$ *is represented by a mapping* h: $S^{2(p-1)}Y_1 \longrightarrow B_U^{n+1}$ *satisfying the following condition; Let a mapping* $\bar{h}$: $S^{n+2p-1} \longrightarrow S^{n+2}$ *represent a generator of* $\pi_{n+2p-1}(S^{n+2}: p)$, *then the following diagram*

$$\begin{array}{ccc} S^{2p-1}Y_1 & \xrightarrow{h} & B_U^{n+1} \\ \uparrow i & & \uparrow j \\ S^{n+2p-1} & \xrightarrow{\bar{h}} & S^{n+2} \end{array}$$

is homotopy commutative, where i and j are inclusions $(n \geqq 2p-1)$.

Proof. Let $m = 2p - 1$. From the puppe's exact sequence, we have that the homomorphism $i^* : [S^m Y_1, S^{n+2}] \longrightarrow [S^{n+m}, S^{n+2}]$ induced by inclusion i is an isomorphism of the p-primary components. Since $\pi_{n+2p}(B_U^{n+1}/S^{n+2})$ is an infinite cyclic group, we have that the homomorphism $j_*: [S^m Y_1, S^{n+2}] \longrightarrow [S^m Y_1, B_U^{n+1}]$ is epmorphism of the p-primary components. Therefore we have the required homotopy commutative diagram.

Consider $K_U{}^*$ of a complex Y_2.

Proposition 3.3. *Let n be sufficientlr large* ($n > 2p - 2$).

(i) $K_U^n(Y_2) = 0$ *and* $K_U^{n+1}(Y_2)$ *has* p^2*-elements,*

(ii) *If* $\Delta\mathfrak{P}^1 - \mathfrak{P}^1\Delta \neq 0$ *in* Y_2, *then* $K_U^{n+1}(Y_2) \approx Z_{p^2}$,

Proof. We apply the exact sequence (2) for projection $q: Y_2 \longrightarrow Y_2/Y_2^{n+1}$ where we may consider that $Y_2/Y_2^{n+1} = S^{2(p-1)} Y_1$ and $Y_2^{n+1} = Y_1$. Then the following sequence is exact

$$0 \longrightarrow K_U^{n+1}(S^{2p-2}Y_1) \xrightarrow{q^*} K_U^{n+1}(Y_2) \xrightarrow{i^*} K_U^{n+1}(Y_1) \longrightarrow 0.$$

An element $\alpha \in K_U^{n+1}(Y_2) \approx [SY_2, B_U^{n+1}]$ of $i^*(\alpha) \neq 0$ is represented by a composition $g \circ p_0 : SY_2 \longrightarrow SY_2/S^{n+1} \longrightarrow B_U^{n+1}$ where $g|SY_2/S^{n+1} = g|S^{n+2} : S^{n+2} \longrightarrow B_U^{n+1}$ is the injection. Now $p\alpha = \alpha \circ p\iota_{SY}$ for $Y = Y_2$. By Lemma 2.1, $p\iota_{SY}$ is represented by a mapping f: $SY_2 \longrightarrow SY_2$ such that $f(SY_2^{n+2p-2}) \subset SY_2^{n+1}$ and $f(SY_2^{n+1}) \subset SY_2^n = S^{n+1}$. Then there is a mapping f': $SY_2/SY_1 \longrightarrow SY_2/S^{n+1}$ such that the following diagram is commutative

$$\begin{array}{ccccc} SY_2 & \xrightarrow{f} & SY_2 & & \\ \downarrow Sq & & \downarrow q_0 & & \\ SY_2/SY_1 & \xrightarrow{f'} & SY_2/S^{n+1} & \xrightarrow{g} & B_U^{n+1} \end{array}$$

where q_0: $SY_2 \longrightarrow SY_2/S^{n+1}$ is projection.

Thus $p\alpha$ is represented by $g \circ q_0 \circ f = g \circ f' \circ Sq$. Thus $p\alpha = q^*(\gamma)$ for an element γ of $K_U^{n+1}(Y_2/Y_1) \approx [SY_2/SY_1, B_U^{n+1}]$ represented by $g \circ f'$. We may assume that g is cellular. Then $(g \circ f')(SY_2^{n+2p-2}/SY_1) \subset g(SY_2^{n+1}/S^{n+1}) = g(S^{n+2}) \subset S^{n+2} \subset B_U^{n+1}$.

Consider the restriction $g \circ f' | S^{n+2p-1}$. Then $g \circ f' | S^{n+2p-1} = f' | S^{n+2p-1} : S^{n+2p-1} \longrightarrow S^{n+2}$. By Lemma 2.1. $f' | S^{n+2p-1}$ is essential if and only if $\Delta\mathfrak{P}^1 - \mathfrak{P}^1\Delta \neq 0$ in Y_2. It follows from Lemma 3.2 that γ is a generator of $K_U^{n+1}(Y/Y_2)$, the order of α is

p^2, and $K_U^{n+1}(Y_2) \approx Z_{p^2}$ if $\Delta\mathfrak{P}^1 - \mathfrak{P}^1\Delta \neq 0$ in Y_2.

Theorem 3.4. (i) $K_U^n(Y_k) = 0$,

(ii) *The group* $K_U^{n+1}(Y_k)$ *has* p^k*-elements,*

(iii) *If* $(\Delta\mathfrak{P}^1 - \mathfrak{P}^1\Delta)H^{n+2i(p-1)}(Y_k; Z_p) = 0$, *for integer* $i = 0, 1, \cdots\cdots, k\text{-}1$, *then* $K_U^{n+1}(Y_k) \simeq Z_{pk}$.

Proof. It is obvious for $k=1$, by table (5). Then (i) and (ii) are proved by induction on k, using the exact sequence (2) for projection $q: Y_k \longrightarrow Y_k/Y_k^{n+2(k-2)(p-1)+1}$ where we may consider that $Y_k/Y_k^{n+2k(k-2)(p-)+1} = S^{2(k-2)(p-1)+1}Y_1$ and $Y_k^{n+2(k-2)(p-1)+1} = Y_{k-1}$. In fact, the following sequence is exact;

$$0 \longrightarrow K_U^{n+1}(Y_k/Y_{k-1}) \xrightarrow{p^*} K_U^{n+1}(Y_k) \longrightarrow K_U^{n+1}(Y_{k-1}) \longrightarrow 0.$$

(iii) Assume that $K_U^{n+1}(Y_k)$ is not cyclic. Let i be the maximal intger such that $K_U^{n+1}(Y_k/Y_i)$ is not cyclic. Then it follows easily that $K_U^{n+1}(Y_{i+2}/Y_i)$ is not cyclic and $(\Delta\mathfrak{P}^1 - \mathfrak{P}^1\Delta)H^{n+2(i-1)(p-1)}(Y_k; Z_p) \neq 0$.

Theorem 3.5. (i) *The group* $K_O^{n+i}(Y_k) = 0$ *for* $i \not\equiv -3$ *and* 1 (*mod* 8), (ii) *The group* $K_0^{n+i}(Y_k)$ has p^k*–elements if* $i \equiv -3$ *or* 1 (*mod* 8), (iii) *If* $(\Delta\mathfrak{P}^1 - \mathfrak{P}^1\Delta)H^{n+2j(p-1)}(Y_k; Z_p) = 0$ *for integers* $j = 0, 1, \cdots\cdots, k-1$, *then the groups* $K_O^{n+1}(Y_k)$ *and* $K_O^{n-3}(Y_k)$ *are cyclic.*

Proof. (i), (ii). We prove by induction on k. It is obvious for $k = 1$, by table (5). Suppose that (i) and (ii) are true for $k-1$. Using the exact sequence (2), we have the following sequence

$$\cdots\cdots \longrightarrow K_O^{n+i-1}(Y_{k-1}) \longrightarrow K_O^{n+i}(Y_k/Y_{k-1}) \longrightarrow K_O^{n+i}(Y_k) \longrightarrow K_O^{n+i}(Y_{k-1}) \longrightarrow \cdots\cdots$$

is exact. We may consider that $Y_k/Y_{k-1} = S^{2(k-1)(p-1)}Y_1$. From the table (5) and Bott's periodicity, $K_O^{n+i}(Y_k/Y_{k-1}) \approx Z_p$ for $i \equiv -3$ or 1 (mod 8) and $K_O^{n+i}(Y_k/Y_{k-1}) = 0$ for $i \not\equiv -3$ and 1 (mod 8).

By the inductive assumption and the above exact sequence, (i) and (ii) are true for k.

(iii). Consider the exact sequence (3)

$$\cdots\cdots \longrightarrow K_O^{n+i-2}(Y_k) \longrightarrow K_U^{n+i}(Y_k) \xrightarrow{i_*} K_O^{n+i}(Y_k) \longrightarrow K_O^{n+i-1}(Y_k) \longrightarrow \cdots\cdots.$$

From (i) and (ii), $i_*: K_U^{n+i}(Y_k) \longrightarrow K_O^{n+i}(Y_k)$ is an isomorphism for $i \equiv -3$ or 1 (mod 8).

Corollary. *Let* $n \geqq 2p-1$. *If* $(\Delta\mathfrak{P}^1 - \mathfrak{P}^1\Delta)\, H^{n+2j(p-1)}(Y_k;\ Z_p) \neq 0$ *for integers* $j = 0, 1, \cdots\cdots, k-1$, *then the suspension-order of* Y_k *is* p^k.

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