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On Group Rings over Semi-primary Rings II

KAORU MOTOSE

Department of Mathematics, Faculty of Science,
Shinshu University
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In this note, R will represent a ring with 1, and $\bar{R}$ the residue class ring of R modulo its (Jacobson) radical $J(R)$. Further, G will represent a group, H a normal subgroup of G, G^* the residue class group G/H, and RG the group ring of G over R. For an arbitrary ideal I of R and an arbitrary normal subgroup N of G, $RG \to R/I \cdot G/N$ will denote the ring epimorphism given by $\sum a_\sigma \sigma \to \sum (a_\sigma + I)\sigma N$ $(a_\sigma \in R,\ \sigma \in G)$.

In what follows, we shall generalize slightly the previous results obtained in [2]. Our first lemma contains the last assertion of [1, d)].

Lemma 1. *If G^* is locally finite, then $J(RH)G \subseteq J(RG)$ and hence $J(RG) \cap RH = J(RH)$.*

Proof. Evidently, $J(RH)G$ is an ideal of RG. Let $x = \sum_{i=1}^{n} y_i\sigma_i$ $(y_i \in J(RH),\ \sigma_i \in G)$ be an arbitrary element of $J(RH)G$, and $H' = \langle H, \sigma_1, \cdots\cdots, \sigma_n \rangle$. Then $RH' = RH'(1-x) + J(RH)RH'$. Since RH' is a finitely generated RH-module, Nakayama's Lemma implies $RH' = RH'(1-x)$, and hence x is quasi-regular. Thus, $J(RH)G \subseteq J(RG)$. Since RG is a free RH-module, $J(RG) \cap RH = J(RH)$.

If G contains no elements of prime order p, then we call G a p'-group. Concerning a locally finite p'-group, we have the following

Theorem 1. *If G is a locally finite p'-group and R is a semiprimary ring with $p\bar{R} = 0$, then $J(RG) = J(R)G$.*

Proof. Let $x = \sum_{i=1}^{n} a_i\sigma_i$ $(a_i \in R,\ \sigma_i \in G)$ be an arbitrary element of $J(RG)$, and $K = \langle \sigma_1, \cdots\cdots, \sigma_n \rangle$. Since RG is a free RK-module and $(|K|, p) = 1$, x is contained in $RK \cap J(RG) \subseteq J(RK) = J(R)K$ (cf. [2, Th. 1]) and hence $J(RG) = J(R)G$ by Lemma 1.

The next contains [2, Cor. 1].

Theorem 2. *If G is a locally finite p-group and R is a semiprimary ring with $p\bar{R} = 0$, then $J(RG) = Ker(RG \to \bar{R})$.*

Proof. It is clear that $J(RG) \subseteq Ker(RG \to \bar{R})$. Let $x = \sum_{i=1}^{n} a_i\sigma_i$ $(a_i \in R,\ \sigma_i \in G)$

be an arbitrary element of $Ker(RG \to \bar{R})$, and $K = <\sigma_1, \cdots, \sigma_n>$. Then, by [2, Cor. 1], x is contained in $J(RK)$ and x is quasi-regular, which means $Ker(RG \to \bar{R}) \subseteq J(RG)$.

Theorem 3. *Let G be a locally finite group, H a p-group, G^* a p'-group, and R a semi-primary ring with $p\bar{R}=0$. Then $J(RG) = J(RH)G$.*

Proof. Since G^* is a locally finite p'-group, $J(RG^*) = J(R)G^*$ (Th. 1), and hence $(J(RG))(RG \to RG^*) \subseteq J(RG^*) = J(R)G^*$. On the other hand, by Th. 2, $Ker(RH \to R) \subseteq J(RH)$ and $J(RG) \subseteq (Ker(RH \to R))G + J(R)G \subseteq J(RH)G$. Hence, $J(RG) = J(RH)G$ by Lemma 1.

The proof of the following lemma is quite similar to that of [3, Lemma (a)].

Lemma 2. *Let R be primary, and let $G \neq 1$ be periodic. If x is a unit in RG whenever $(x)(RG \to \bar{R})$ is a unit in $\bar{R}$, then G is a p-group and $p\bar{R} = 0$.*

Proof. Let σ be an arbitrary element of G different from 1, and n the order of σ. If $\bar{R}$ is of characteristic 0, then $(\sum_{i=0}^{n-1} \sigma^i)(RG \to \bar{R}) = n$ is a unit in $\bar{R}$. However, we have a contradiction $(\sum_{i=0}^{n-1} \sigma^i)(1-\sigma) = 0$. Hence, $\bar{R}$ must be of prime characteristc p. Now, suppose that $n = p^e \cdot n'$ with $(n', p) = 1$ and $n' > 1$. Then $(\sum_{j=0}^{n'-1} \sigma^{p^e j})(RG \to \bar{R}) = n'$ is a unit in R. While, we have $(\sum_{j=0}^{n'-1} \sigma^{p^e j})(1 - \sigma^{p^e}) = 0$. This contradiction means that G is a p-group.

The assertion of the next Cor. 1 and Th. 4 is a genelization of [4, Th. 1]. Moreover, Th. 4 contains [2, Th. 3], too.

Corollary 1. *If RG is primary then G is a p-group and R is a primary ring with $p\bar{R} = 0$.*

Proof (cf. the proof of [4, Th. 1]). Since $J(RG) = Ker(RG \to \bar{R})$, R is a primary ring and x is a unit in RG whenever $(x)(RG \to \bar{R})$ is a unit in $\bar{R}$. Now, let σ be an arbitrary element of G different from 1. Then, $\sigma - \sigma^2$ is an element of $Ker(RG \to \bar{R}) = J(RG)$. Recalling that RG is a free $R<\sigma>$-module, one will easily see that $1 - \sigma + \sigma^2$ is a unit in $R<\sigma>$, whence it follows at once that σ is of finite order. Hence, G is periodic, and then our assertion is clear by Lemma 2.

Theorem 4. *Let R be a primary ring. If G is locally finite, and $H \neq 1$, then the following conditions are equivalent:*

(1) *H is a p-group and $p\bar{R} = 0$.*

(2) *RH is primary.*

(3) *x is a unit in RG whenever $(x)(RG \to \bar{R}G^*)$ is a unit in $\bar{R}G^*$.*

(4) *x is a unit in RG whenever $(x)(RG \to RG^*)$ is a unit in RG^*.*

Proof. (1)$\Leftrightarrow$(2) is evident by Th. 2 and Cor. 1, and (3)$\Leftrightarrow$(4) is a consequence of $J(R)G^* \subseteq J(RG^*)$ (Lemma 1). Hence, it remains only to prove (2)$\Leftrightarrow$(3).

(2)⇒(3): Since $Ker(RH \to \overline{R}) = J(RH)$, $Ker(RG \to \overline{R}G^*) = (Ker\ (RH \to \overline{R}))G = J(RH)G \subseteq J(RG)$ (Lemma 1). Now, (3) is evident.

(3)⇒(2): Obviously, $Ker(RH \to \overline{R}) \subseteq Ker(RG \to \overline{R}G^*)$ and $Ker(RG \to \overline{R}G^*)$ is quasi-regular ideal. Noting that RG is a free RH-module, we can readily see that $Ker(RH \to \overline{R}) = J(RH)$.

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