On Group Rings over Semi-primary Rings

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In this note, \( R \) will represent a ring with 1, and \( \overline{R} \) the residue class ring of \( R \) modulo its (Jacobson) radical \( J(R) \). Further, \( G \) will represent a group, \( H \) a normal subgroup of \( G \), \( G^* \) the residue class group \( G/H \), and \( RG \) the group ring of \( G \) over \( R \). For an arbitrary ideal \( I \) of \( R \) and an arbitrary normal subgroup \( N \) of \( G \), \( RG \rightarrow R/I \cdot G/N \) will denote the ring epimorphism given by \( \sum a_i \sigma \rightarrow \sum (a_i + I) \cdot \sigma N \) (\( a_i \in R, \sigma \in G \)).

In what follows, we shall generalize slightly the previous results obtained in [2]. Our first lemma contains the last assertion of [1, d)].

Lemma 1. If \( G^* \) is locally finite, then \( J(RH)G \subseteq J(RG) \) and hence \( J(RG) \cap RH = J(RH) \).

Proof. Evidently, \( J(RH)G \) is an ideal of \( RG \). Let \( x = \sum a_i \sigma_i \) (\( a_i \in J(RH), \sigma_i \in G \)) be an arbitrary element of \( J(RH)G \), and \( H' = \langle H, \sigma_1, \ldots, \sigma_n \rangle \). Then \( RH' = RH'(1 - x) + J(RH)RH' \). Since \( RH' \) is a finitely generated \( RH \)-module, Nakayama's Lemma implies \( RH' = RH'(1 - x) \), and hence \( x \) is quasi-regular. Thus, \( J(RH)G \subseteq J(RG) \). Since \( RG \) is a free \( RH \)-module, \( J(RG) \cap RH = J(RH) \).

If \( G \) contains no elements of prime order \( p \), then we call \( G \) a \( p' \)-group. Concerning a locally finite \( p' \)-group, we have the following

Theorem 1. If \( G \) is a locally finite \( p' \)-group and \( R \) is a semi-primary ring with \( pR = 0 \), then \( J(RG) = J(R)G \).

Proof. Let \( x = \sum a_i \sigma_i \) (\( a_i \in R, \sigma_i \in G \)) be an arbitrary element of \( J(RG) \), and \( K = \langle \sigma_1, \ldots, \sigma_n \rangle \). Since \( RG \) is a free \( RK \)-module and \( (|K|, p) = 1 \), \( x \) is contained in \( RK \cap J(RG) \subseteq J(RK) = J(R)K \) (cf. [2, Th. 1]) and hence \( J(RG) = J(R)G \) by Lemma 1.

The next contains [2, Cor. 1].

Theorem 2. If \( G \) is a locally finite \( p \)-group and \( R \) is a semi-primary ring with \( pR = 0 \), then \( J(RG) = \text{Ker}(RG \rightarrow R) \).

Proof. It is clear that \( J(RG) \subseteq \text{Ker}(RG \rightarrow R) \). Let \( x = \sum a_i \sigma_i \) (\( a_i \in R, \sigma_i \in G \))
be an arbitrary element of $\text{Ker}(RG \rightarrow \overline{R})$, and $K = \langle \sigma_1, \ldots, \sigma_n \rangle$. Then, by [2, Cor. 1], $x$ is contained in $J(RK)$ and $x$ is quasi-regular, which means $\text{Ker}(RG \rightarrow \overline{R}) \subseteq J(RG)$.

**Theorem 3.** Let $G$ be a locally finite group, $H$ a $p$-group, $G^*$ a $p'$-group, and $R$ a semi-primary ring with $pR = 0$. Then $J(RG) = J(RH)G$.

**Proof.** Since $G^*$ is a locally finite $p'$-group, $J(RG^*) = J(R)G^*$ (Th. 1), and hence $(J(RG))(RG \rightarrow RG^*) \subseteq J(RG^*) = J(R)G^*$. On the other hand, by Th. 2, $\text{Ker}(RG \rightarrow R) \subseteq J(RH)$ and $J(RG) \subseteq (\text{Ker}(RH \rightarrow R))G + J(R)G \subseteq J(RH)G$. Hence, $J(RG) = J(RHG)$ by Lemma 1.

The proof of the following lemma is quite similar to that of [3, Lemma (a)].

**Lemma 2.** Let $R$ be Primary, and let $G \neq 1$ be periodic. If $x$ is a unit in $RG$ whenever $(x)(RG \rightarrow \overline{R})$ is a unit in $\overline{R}$, then $G$ is a $p$-group and $pR = 0$.

**Proof.** Let $a$ be an arbitrary element of $G$ different from 1, and $n$ the order of $a$. If $R$ is of characteristic 0, then $\sum_{i=0}^{n-1} a^i (RG \rightarrow \overline{R}) = n$ is a unit in $\overline{R}$. However, we have a contradiction $(\sum_{i=0}^{n-1} a^i)(1-a) = 0$. Hence, $\overline{R}$ must be of prime characteristic $p$. Now, suppose that $n = p^j n'$ with $(n', p) = 1$ and $n' > 1$. Then $(\sum_{j=0}^{n'-1} a^{p^j})(RG \rightarrow \overline{R}) = n'$ is a unit in $R$. While, we have $(\sum_{j=0}^{n'-1} a^{p^j})(1-a^{p^j}) = 0$. This contradiction means that $G$ is a $p$-group.

The assertion of the next Cor. 1 and Th. 4 is a generalization of [4, Th. 1]. Moreover, Th. 4 contains [2, Th. 3], too.

**Corollary 1.** If $RG$ is Primary then $G$ is a $p$-group and $R$ is a Primary ring with $pR = 0$.

**Proof** (cf. the proof of [4, Th. 1]). Since $J(RG) = \text{Ker}(RG \rightarrow \overline{R})$, $R$ is a primary ring and $x$ is a unit in $RG$ whenever $(x)(RG \rightarrow \overline{R})$ is a unit in $\overline{R}$. Now, let $\sigma$ be an arbitrary element of $G$ different from 1. Then, $\sigma - \sigma^2$ is an element of $\text{Ker}(RG \rightarrow \overline{R}) = J(RG)$. Recalling that $RG$ is a free $R<\sigma>$-module, one will easily see that $1 - \sigma + \sigma^2$ is a unit in $R<\sigma>$, whence it follows at once that $\sigma$ is of finite order. Hence, $G$ is periodic, and then our assertion is clear by Lemma 2.

**Theorem 4.** Let $R$ be a primary ring. If $G$ is locally finite, and $H \neq 1$, then the following conditions are equivalent:

1. $H$ is a $p$-group and $pR = 0$.
2. $RG$ is primary.
3. $x$ is a unit in $RG$ whenever $(x)(RG \rightarrow \overline{RG^*})$ is a unit in $\overline{RG^*}$.
4. $x$ is a unit in $RG$ whenever $(x)(RG \rightarrow RG)$ is a unit in $RG^*$.

**Proof.** (1) $\Rightarrow$ (2) is evident by Th. 2 and Cor. 1, and (3) $\Leftrightarrow$ (4) is a consequence of $J(R)G^* \subseteq J(RG^*)$ (Lemma 1). Hence, it remains only to prove (2) $\Rightarrow$ (3).
(2)→(3): Since \( \text{Ker}(RH \to \overline{R}) = J(RH) \), \( \text{Ker}(RG \to \overline{RG}^*) = (\text{Ker} \ (RH \to \overline{R}))G = J(RH)G \subseteq J(RG) \) (Lemma 1). Now, (3) is evident.

(3)→(2): Obviously, \( \text{Ker}(RH \to \overline{R}) \subseteq \text{Ker}(RG \to \overline{RG}^*) \) and \( \text{Ker}(RG \to \overline{RG}^*) \) is quasi-regular ideal. Noting that \( RG \) is a free \( RH \)-module, we can readily see that \( \text{Ker} \ (RH \to \overline{R}) = J(RH) \).

References