On the equivariant K-groups
\( K^{\bullet}_{UF(n)}(cs^{n}_{F}, s^{n}_{F}) \) and \( K^{\bullet}_{SU(n)}(cs^{n}_{F}, s^{n}_{F}) \)

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§ 1.

1. Introduction. Let \( F \) denote \( R, C, \) or \( H \). Let \( U_{F}(n) \) denote \( O(n) \) for \( F = R \), \( U(n) \) for \( F = C \), and \( S_{\mu}(n) \) for \( F = H \). Let \( S^{n}_{F} \) denote \( S^{n-1} \) for \( F = R \), \( S^{2n-1} \) for \( F = C \), and \( S^{4n-1} \) for \( F = H \). We can regard \( S^{n}_{F} \) as a \( U_{F}(n) \)-space (respectively \( SU_{F}(n) \)-space) with a natural \( U_{F}(n) \) action (respectively \( SU_{F}(n) \) action) on \( F^{n} \). We can consider \( F^{n} \subset F^{n+1} \) where a vector in \( F^{n+1} \) has zero in its \((n+1)\)-coordinate when viewed in \( F^{n+1} \). Consequently, there exists natural inclusions \( U_{F}(n-1) \subset U_{F}(n) \), \( SU_{F}(n-1) \subset SU_{F}(n) \). Let \( CS^{n}_{F} \) be a cone over \( S^{n}_{F} \). Then \( CS^{n}_{F} \) is a \( U_{F}(n) \)-space (respectively \( SU_{F}(n) \)-space) with group action \( g(y, t) = (gy, t) \).

Now, in this paper we shall prove the following

**Theorem.** For \( F = C, \) or, \( H, \) \( K^{\bullet}_{U(n)}(CS^{n}_{F}, S^{n}_{F}) \) is an \( R(U(n)) \)-module with one generator. \( K^{\bullet}_{U(n)}(CS^{n}_{F}, S^{n-1}_{F}) \) is an \( R(U(n)) \)-module with one generator. \( K^{\bullet}_{SO(2r+1)}(CS^{n}_{F}, S^{2n-1}_{F}) \) is an \( R(SO(2r+1)) \)-module with one generator. \( K^{\bullet}_{SO(2r)}(CS^{2r}_{F}, S^{2r-1}_{F}) \) is an \( R(SO(2r)) \)-module with two generators.

Let \( G \) be a compact Lie group and \( H \) a closed subgroup of \( G \), then from 4.1 and 4.11 of [1], we have the following isomorphism between \( Z_{\ast} \)-graded \( R(G) \)-modules:

\[
K^{\ast}_{(G, H)}(\text{point}) \cong K^{\ast}_{G}(G/H), \ G/H.
\]

From 3.5 Theorem of [1], the sequence

\[
0 \rightarrow K^{\ast}_{(G, H)}(\text{point}) \rightarrow K^{0}_{G}(\text{point}) \rightarrow K^{1}_{H}(\text{point}) \rightarrow K^{1}_{(G, H)}(\text{point}) \rightarrow 0
\]

is exact, and each of the maps is a \( K^{0}_{G}(\text{point}) \)-module map. Therefore, we have the following isomorphisms between \( R(G) \)-modules: (Paticulaly, if \( (G, H) = (U_{F}(n), \ U_{F}(n-1)) \) or \( (SU_{F}(n), \ SU_{F}(n-1)) \), \( G/H = S^{n}_{F} \).
On the equivariant $K$-groups $K^*_U(p)(CS^{*}_{p}, S^*_p)$ and $K^*_{SU(p)}(CS^{*}_{p}, S^*_p)$

\[(1.3)\quad K_G(C(G/H), G/H) \cong \text{Kernel } i^*
\]
\[K_G(C(G/H), G/H) \cong \text{Cokernel } i^*.
\]

§ 2

The following results on the representation theory will be required from [2]. Throughout this paper we use the same notation as in [2].

2.1 Theorem. The ring $R(U(n))$ equals the polynomial ring

\[Z[\lambda_1(n), \ldots, \lambda_n(n), \lambda_{n-1}(n)^{-1}]\]

where as a subring of $R(T(n)) = Z[\alpha_1, \alpha_1^{-1}, \ldots, \alpha_n, \alpha_n^{-1}]$ the relation

\[\lambda_k(n) = \sum_{j(1) < \cdots < j(k)} \alpha_{i(1)} \cdots \alpha_{i(k)}\]

holds. The ring $R(SU(n))$ equals the polynomial ring

\[Z[\lambda_1(n), \ldots, \lambda_{n-1}(n)].\]

2.2 Theorem. The ring $R(Sp(n))$ equals the polynomial ring

\[Z[\lambda_1(n), \ldots, \lambda_n(n)],\]

where, as a subring of $Z[\alpha_1, \alpha_1^{-1}, \ldots, \alpha_n, \alpha_n^{-1}]$ the element $\lambda_k(n)$ is the $k$-th symmetric function in the $2n$ variables $\alpha_1, \alpha_1^{-1}, \ldots, \alpha_n, \alpha_n^{-1}$.

2.3 Theorem. In the case $n = 2r + 1$, $R(Spin(n))$ equals the polynomial ring

\[Z[\lambda_1(p_r), \ldots, \lambda_r(p_r), \lambda_{r-1}(p_r)],\]

and $R(SO(n))$ equals the polynomial ring

\[Z[\lambda_1(p_r), \ldots, \lambda_r(p_r)].\]

The following relation holds :

\[A_n^2 = \lambda(p_r) + \cdots + \lambda_r(p_r) + 1.\]

In the case $n = 2r$, $R(Spin(n))$ equals the polynomial ring

\[Z[\lambda_1(p_r), \ldots, \lambda_r(p_r), \lambda_{r+1}(p_r), \lambda_{r+1}(p_r)],\]

and $R(SO(n))$ equals the polynomial ring

\[Z[\lambda_1(p_r), \ldots, \lambda_{r-1}(p_r), \lambda_r(p_r), \lambda_{r-1}(p_r)].\]
On the equivariant K-groups $K_{U(p,0)}^*(CS_p^n, S_p^n)$ and $K_{SU(p,0)}^*(CS_p^n, S_p^n)$

with one relation

$$(\lambda_{-} + \lambda^{-2} + \lambda^{-4} + \ldots) (\lambda_{-} + \lambda^{-2} + \lambda^{-4} + \ldots)
= (\lambda^{-1} + \lambda^{-3} + \lambda^{-5} + \ldots)^2.$$ 

In $R(\text{Spin}(2r))$ the following relation holds:

$$(A_2, A_{-2}) = 2^{-r}(p_{-2}) + \lambda_{-2}(p_{-2}) + \ldots,$$

$$(A_2, A_{-2}) = 2^{-r}(p_{-2}) + \lambda_{-2}(p_{-2}) + \ldots,$$

From the selection of $\alpha_i$ in [2], throughout this paper we shall write

$R(T(n)) = Z[\alpha_i, \alpha_{i-1}, \ldots, \alpha_n, \alpha_n^{-1}]$ and $R(T(n-1)) = Z[\alpha_i, \alpha_{i-1}, \ldots, \alpha_{n-1}, \alpha_{n-1}^{-1}]$ with the same notations $\alpha_i$.

§ 3

3.1 Lemma. Let $i^*_n : R(U(n)) \to R(U(n-1))$ be a natural homomorphism (by the inclusion $U(n-1) \subset U(n)$), then we have

$$(3.1.1) \quad i^*_n(\lambda_i(n)) = \begin{cases} \lambda_i(n-1) + \lambda_{i-1}(n-1) & \text{for } i \leq n-1 \\ \lambda_{i-1}(n-1) & \text{for } i = n-1. \end{cases}$$

Proof. The diagram

\[
\begin{array}{ccc}
R(U(n)) & \to & R(T(n)) \\
\downarrow i^*_n & & \downarrow i^*_n \\
R(U(n-1)) & \to & R(T(n-1))
\end{array}
\]

is commutative, and we have $i^*_n(\alpha_i) = \alpha_i$ for $i \leq n-1$ and $i^*_n(\alpha_n) = 1.$

since $\lambda_i(n)$ is a $i$-th symmetric function in the $n$ variables $\alpha_1, \ldots, \alpha_n$, we have easily (3.1.1).

3.2 Lemma. In $R(U(n))$, we set

$\bar{\lambda}_i(n) = \sum_{i=0}^l(-1)^i \lambda_{i-i}(n),$

then $R(U(n))$ equals the polynomial ring

$Z[\bar{\lambda}_i(n), \ldots, \bar{\lambda}_n(n), \bar{\lambda}_n(n) + \bar{\lambda}_{n-1}(n)^{-1}].$

Proof. Since $\bar{\lambda}_i(n), \ldots, \bar{\lambda}_n(n)$ are algebraically independent, and $\bar{\lambda}_n(n) + \bar{\lambda}_{n-1}(n) = \bar{\lambda}_n$, it is trivial.

3.3 Theorem. We have the following isomorphism between $Z_\infty$-graded $R(U(n))$-
modules and \( R(SU(n))\)-modules:

\[
K_{U(0)}(C^{2n-1}, S^{2n-1}) \cong \{\lambda(n)\}
\]
\[
K_{SU(0)}(C^{2n-1}, S^{2n-1}) \cong \{\lambda_{n-1}(n) - 1\}
\]

where \( \{\lambda(n)\} \) is an \( R(U(n))\)-module generated by \( \lambda(n) \) and \( \{\lambda_{n-1}(n) - 1\} \) is an \( R(SU(n))\)-module generated by \( \lambda_{n-1}(n) - 1 \).

**Proof.** From the definitions of \( \lambda(n) \), we have

\[
i^*_n(\lambda(n)) = \begin{cases} 
\lambda_{n-1}(n-1) & \text{(for } l \leq n - 1 \text{)} \\
0 & \text{(for } l = n \text{).}
\end{cases}
\]

Therefore, from the last statement in § 1, it is sufficient to determine Kernel \( i^*_n \).

From (3.3.1), Kernel \( i^*_n \) is an ideal generated by \( \lambda(n) \) in \( R(U(n)) \), so kernel \( i^*_n \) is an \( R(U(n))\)-module generated by \( \lambda(n) \). For \( R(SU(n)) \), observe that the ring \( R(ST(n)) \) is an ideal generated by \( (\lambda - 1) \). Therefore Kernel \( i^*_n \) (\( i^*_n : R(SU(n)) \to R(SU(n-1)) \)) is an \( R(SU(n))\)-module generated by \( \lambda(n) \).

**Remark.** For \( A^e, A^o \) we take the representations of \( U(n) \) on the even and odd parts of the exterior algebra \( A^e(C^n) \), and we identify these two parts by exterior multiplication with the \( n \)-th basic vector \( e_n \) of \( C^n \). Since \( U(n-1) \) keeps \( e_n \) fixed, this identification is compatible with the action of \( U(n-1) \). So we obtain an element \( [\gamma] \) of \( K(U(0), U(n-1)) \) (point) such that \( j^*(\gamma) = \lambda(n) \). Therefore \( K(U(0), U(n-1)) \) (point) equals the \( R(U(n))\)-module generated by \( \lambda(n) \).

### § 4

4.1 Let \( \lambda(n) \) be a \( l \)-th symmetric function in the \( 2n \) variables \( a_1, a_1^{-1}, \ldots, a_n, a_n^{-1} \), and \( \lambda'(n) \) be a \( l \)-th symmetric function in the \( (2n + 1) \) variables \( a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}, 1 \) and \( \lambda'^*(n) \) be a \( l \)-th symmetric function in the \( 2n \) variables \( a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}, 1, 1 \). Then

\[
H(x - \alpha_i)(x - \alpha^{-1}_i) = \sum_{i=0}^{N} (-1)^i \lambda_i(n)x^{2n-1} + \sum_{i=1}^{n-1} (-1)^i \lambda_i(n)x^i
\]

implies

\[
\lambda_i(n) = \lambda_i(n-1) + \lambda_i(n-1) + \lambda_{i-1}(n-1) = 2\lambda_{i-1}(n-1)
\]

\[
\lambda'(n) = \lambda(n) + \lambda(n) - 1
\]

\[
i^*(\lambda(n)) = \lambda(n) + 2\lambda'(n-1) + \lambda'(n-1) = 2\lambda'(n-1) + \lambda'(n-1)
\]
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where $\lambda(n) = 1$ and $\lambda_{-1}(n) = 0$.

4.2 Lemma. Let $i_n^*: R(S_p(n)) \to R(S_p(n-1))$ be a natural homomorphism (by the inclusion $S_p(n-1) \subset S_p(n)$), then we have

$$i_n^* (\lambda_i(n)) = \begin{cases} 2(\lambda_{n-1}(n-1) + \lambda_{n-2}(n-1)) & \text{(for } l = n) \\ \lambda_i(n-1) + 2\lambda_{n-1}(n-1) + \lambda_{n-2}(n-1) & \text{(for } l < n) \end{cases}.$$

Proof. From (4.1.4), it is trivial.

4.3 Lemma. We set $a_{-1} = 0$, $a_0 = 1$, $a_1 = -2$, $a_2 = 3$, and inductively $a_l = -2a_{l-1} - a_{l-2}$ for $l = 3, 4, \ldots, n - 1$. (In fact, $a_l = (-1)^l(l + 1)$). We set

$$\bar{\lambda}_i(n) = \sum_{i=0}^{l} (-1)^{i+1} \lambda_i(n) \quad \text{(for } l < n)$$

$$\bar{\lambda}_i(n) = \lambda_i - 2\sum_{i=0}^{n} (-1)^{i+1} \lambda_{n-1}(n)),$$

then we have

$$i_n^* (\lambda_i(n)) = \begin{cases} \lambda_i(n-1) & \text{(for } l < n) \\ 0 & \text{(for } l = n) \end{cases}.$$

Proof. From (4.1.4), it is trivial.

4.4 Theorem. We have the following isomorphism between $\mathbb{Z}_2$-graded $R(S_p(n))$-modules :

$$K^*_S\text{U}(\text{CS}_p^n, S_p^n) \cong \{ \lambda_i(n) \},$$

where $\lambda_i(n)$ is as in 4.3, and $\{ \}$ is the same notation as in 3.3.

Proof. Since $\bar{\lambda}_i(n), \cdots, \bar{\lambda}_i(n)$ are algebraically independent and $i_n^*: R(S_p(n)) \to R(S_p(n-1))$ is an onto homomorphism, the result follows at once.

§ 5

5.1 Lemma. Let $i_r^*: R(SO(2r)) \to R(SO(2r-1))$ be a natural homomorphism (by the inclusion $SO(n-1) \subset SO(n)$), then we have

$$i_r^* (\lambda^r(p_{2r})) = \begin{cases} 2\lambda^r(p_{2r-1}) + \lambda^{r-1}(p_{2r-1}) & \text{(for } l < r) \\ 2\lambda^{r-1}(p_{2r-1}) & \text{(for } l = r) \end{cases}.$$

Proof. For (5.1.1), from (4.1.4), it is trivial. For (5.1.2), from 2.3 we have
\[ \chi^\lambda_{\pm}(p_{2r}) = (d_{2r}^{\pm})^2 - (\chi^{-\lambda}(p_{2r}) + \chi^{-\lambda}(p_{2r}) + \cdots ...).
\]

\( i^r ((d_{2r}^{\pm})^2) = (d_{2r-1}^{\pm})^2 \quad \text{(c.f, Proposition 9.4 of [2])} \)

\( (d_{2r-1}^{\pm})^2 = \chi^{-\lambda}(p_{2r-1}) + \cdots + \lambda(\rho_{2r}) + 1. \)

Therefore (5.1.1) implies (5.1.2).

5.2 Lemma. We set:

\[ \tilde{\chi}^l(p_{2r}) = \sum_{l=0}^{l=1} (-1)^l \chi^{-l}(p_{2r}) \quad \text{(for $l=1, 2, 3, \ldots, r-1$),} \]

then we have:

\[ \tilde{\chi}^l(p_{2r}) = \tilde{\chi}^l(p_{2r}) + \chi^{-l}(p_{2r}) \]

(5.2.1) \( i^r (\tilde{\chi}^l(p_{2r})) = \chi^l(p_{2r-1}) \)

\( i^r (\tilde{\chi}^l(p_{2r})) = \chi^{-l}(p_{2r-1}). \)

Proof. From 5.1, it is trivial.

5.3 Theorem. We have the following isomorphism between \( \mathbb{Z}_2 \)-graded \( R(\text{SO}(2r)) \)-modules:

\[ \mathcal{K}_{\text{SO}(2r)}(CS^{2r-1}, S^{2r-1}) \cong \{ (\tilde{\chi}^{-l}(p_{2r}) - \chi^l(p_{2r})), \ (\tilde{\chi}^{-l}(p_{2r}) - \chi^l(p_{2r})) \} \]

, where \{ \} is the same notation as in 3.3.

Proof. From 5.2, \( i^r : R(\text{SO}(2r)) \rightarrow R(\text{SO}(2r-1)) \) is an onto homomorphism, so it is sufficient to determine Kernel \( i^r \). Now, from the definition of \( \tilde{\chi}^l(p_{2r}), \)

\( \tilde{\chi}^l(p_{2r}), \ldots, \tilde{\chi}^{-l}(p_{2r}) \) are algebraically independent. Therefore (5.2.2) and (5.2.3) implies the result.

§ 6

6.1 Lemma. Let \( i^r : R(\text{SO}(2r+1)) \rightarrow R(\text{SO}(2r)) \) be a natural homomorphism (by the inclusion \( \text{SO}(2r) \subset \text{SO}(2r + 1) \)) then we have:

\[ i^r (\tilde{\chi}^l(p_{2r+1})) = \tilde{\chi}^l(p_{2r}) + \chi^{-l}(p_{2r}). \]

Proof. From (4.1.4), it is trivial.

6.2 Theorem. We have the following isomorphism between \( \mathbb{Z}_2 \)-graded \( R(\text{SO}(2r + 1)) \)-modules:

\[ \mathcal{K}_{\text{SO}(2r+1)}(CS^{2r}, S^{2r}) \cong \{ \chi^l(p_{2r}) \} \]

Proof. From 6.1, \( i^r : R(\text{SO}(2r + 1)) \rightarrow R(\text{SO}(2r)) \) is a monomorphism, and
On the equivariant K-groups $K^*_{U^p(B)}(CS^*, S^*)$ and $K^*_{SU(B)}(CS^*, S^*)$

Image $i^*_r$ equals the subring

$$Z[\lambda^1(p_2), \ldots, \lambda^{r-1}(p_2), (\lambda^r_+ (p_2) + \lambda^r_-(p_2))]$$

in $R(SO(2r))$. So it is sufficient to determine Cokernel $i^*_r$. From the relation in $R(SO(2r))$, $(\lambda^r_+(p_2)) (\lambda^r_-(p_2))$ is an element of Image $i^*_r$. We set $x = \lambda^r_+(p_2)$, $y = \lambda^r_-(p_2)$ and $B = \text{Image } i^*_r$. Since

$$x^2 = (x + y)x - yx$$
$$x^k = (x+y)x^{k-1} - (yx)x^{k-2}, \quad (\text{for } k \geq 3),$$

we can write

(6.1.1) \hspace{1cm} x^k = \sum a_i x \quad (\text{mod } B)

where $a_i$ is an element of $B$. Since

$$y^k = (x + y - x)^k = \sum_{i=0}^{k} \binom{k}{i} (x+y)^i (-1)^{k-i} x^{k-i},$$

from (6.1.1), we can write

(6.1.2) \hspace{1cm} y^k = \sum b_i x \quad (\text{mod } B),

where $b_i$ is an element of $B$. Therefore Cokernel $i^*_r$ is an $R(SO(2r + 1))$-module generated by $\lambda^r_+(p_2)$.

References
