A Relative Form of Equivariant $K$-Theory

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Introduction. Let $G$ be a compact Lie group, $T$ a maximal torus of $G$, $W(G)$ the Weyl group of $G$ and $X$ a compact $G$-space. Then the following results on the equivariant $K$-theory will be required from [1] and [2].

Theorem (A). (i) We have a ring homomorphism $R(G) \rightarrow R(T)$ (by the restriction map) which is injective. $R(G)$ maps bijectively onto the ring of invariants of $R(T)$ under the action of $W(G)$.

(ii) The sequence

$$0 \rightarrow K_0^G(X) \rightarrow K_0^T(X) \rightarrow 0$$

is split exact.

(iii) is obtained from 4.4 of [1] and (ii) from Proposition (4.9) of [2].

Now the aim of this paper is to prove the following Theorem:

Theorem (B). We have the following split exact sequences:

$$0 \rightarrow K^G_*(X) \rightarrow K^T_*(X) \rightarrow K^*_{G,T}(X) \rightarrow 0$$

$$0 \rightarrow K^G_*(X) \rightarrow K^T_*(X)^W(G) \rightarrow K^*_{G,T}(X)^W(G) \rightarrow 0$$

where $K^*_{G,T}(X)$ is defined in §1 and $K^*_{T}(X)^W(G)$ (respectively, $K^*_{G,T}(X)^W(G)$) is an abelian group of invariants of $K^*_{T}(X)$ (respectively, $K^*_{G,T}(X)$) under the action of $W(G)$.

Note: It has been proved by Mr. Haruo Minami (to appear) that if $G = U(n)$ and $K^*_{T}(X)$ is torsion free, then $K^*_{U(n)}(X)$ and $K^*_{T}(X)^W(U(n))$ are isomorphic. So we predict the following result:

Prediction. If $K^*_{T}(X)$ is torsion free, then $K^*_{T}(X)^W(G)$ and $K^*_{G}(X)$ are isomorphic.

Throughout this paper $G$ will denote a compact Lie group, $H$ a closed subgroup of $G$, $X$ a compact $G$-space and $A$ a closed $G$-invariant subspace of $X$.

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§ 1 Definition of $K_{(G,H)}(X, A)$

1.1 Definition. We define $L_{(G,H)}(X, A)$ to be a category as follows: An object of $L_{(G,H)}(X, A)$ is a pair $(E, F)$ of $G$-vector bundles over $X$, together with an $H$-isomorphism

$$\delta : I \times E|I \times A \cup \{0\} \times X \longrightarrow I \times F|I \times A \cup \{0\} \times X$$

such that $\delta|\{1\} \times A : E|A \longrightarrow F|A$ is a $G$-isomorphism.

The morphism $\varphi : \sigma_0 \longrightarrow \sigma_1$, where $\sigma_i = (E_i, F_i, \delta_i) \ (i = 0, 1)$, is a pair of $G$-homomorphisms $(f, g) : (E_0, F_0) \longrightarrow (E_1, F_1)$ such that the diagram

$$
\begin{array}{ccc}
(I \times E_0)|I \times A \cup \{0\} \times X & \xrightarrow{\delta_0} & (I \times F_0)|I \times A \cup \{0\} \times X \\
\downarrow & & \downarrow \\
(I \times E_1)|I \times A \cup \{0\} \times X & \xrightarrow{\delta_1} & (I \times F_1)|I \times A \cup \{0\} \times X
\end{array}
$$

(1.1.1)

is commutative. From now on, we put $B = I \times A \cup \{0\} \times X$.

An elementary object in $L_{(G,H)}(X, A)$ is an object of the form $(E, E, id)$. If $\sigma_i = (E_i, F_i, \delta_i) \ (i = 0, 1)$ are in $L_{(G,H)}(X, A)$, their sum is defined by

$$\sigma_0 \oplus \sigma_1 = (E_0 \oplus E_1, F_0 \oplus F_1, \delta_0 \oplus \delta_1).$$

(1.1.2)

Two objects $\sigma_0$ and $\sigma_1$ are homotopic in $L_{(G,H)}(X, A)$, in symbols

$$\sigma_0 \sim \sigma_1,$$

if there exists an object $\bar{\sigma} = (\bar{E}, \bar{F}, \bar{\delta})$ of $L_{(G,H)}(X \times I, A \times I)$ such that

$$\bar{\sigma}|\{0\} = \sigma_0 \text{ and } \bar{\sigma}|\{1\} = \sigma_1,$$

i. e. $\bar{E}|X \times \{i\} = E_i$, $\bar{F}|X \times \{i\} = F_i$ and $\bar{\delta}|B \times \{i\} = \delta_i$.

Two objects $\sigma_0$ and $\sigma_1$ are stably homotopic in $L_{(G,H)}(X, A)$, in symbols

$$\sigma_0 \simeq \sigma_1,$$

if there exist elementary objects $\tau_0$ and $\tau_1$ such that

$$\sigma_0 \oplus \tau_0 \sim \sigma_1 \oplus \tau_1.$$

We shall write $[\sigma]$ for the stably homotopic class of $\sigma$. The set of such stably homotopic classes is denoted by $K_{(G,H)}(X, A)$.

If $[\sigma_i] \ (i = 0, 1)$ are in $K_{(G,H)}(X, A)$, their sum is defined by

$$[\sigma_0] + [\sigma_1] = [\sigma_0 \oplus \sigma_1].$$

(1.1.5)
Then $K(G,H)(X, A)$ is a semigroup.

Two objects $\sigma_0$ and $\sigma_1$ are isomorphic in $L(G,H)(X, A)$, in symbols

\[(1.1.6) \quad \sigma_0 \cong \sigma_1,\]

if there exists an isomorphism $\varphi : \sigma_0 \rightarrow \sigma_1$ in $L(G,H)(X, A)$.

1.2 Lemma. If $\sigma_0 \cong \sigma_1$ then $\sigma_0 \sim \sigma_1$.

Proof. From (1.1.6), there exists an isomorphism $(f, g) : \sigma_0 \rightarrow \sigma_1$.

We define the $G$-vector bundles $E$, $F$ over $X \times I$ as follows:

\[
E = E_0 \times [0, 1/2] \cup E_1 \times [1/2, 1] \quad \text{and} \quad F = F_0 \times [0, 1/2] \cup F_1 \times [1/2, 1].
\]

Moreover we define an $H$-isomorphism $\tilde{\sigma} : (I \times E)B \times I \rightarrow (I \times F)B \times I$ as follows:

\[
\tilde{\sigma}|B \times [0, 1/2] = \delta_0 \times \text{id}_{[0, 1/2]} \quad \text{and} \quad \tilde{\sigma}|B \times [1/2, 1] = \delta_1 \times \text{id}_{[1/2, 1]}.\]

Then $\tilde{\sigma} = [E, F, \tilde{\sigma}]$ is an object of $L(G,H)(X \times I, A \times I)$ and $\tilde{\sigma}|\{i\} = \sigma_i$ $(i = 0, 1)$. Hence $\sigma_0 \sim \sigma_1$.

1.3 Lemma. If $[E, F, \delta]$, $[F, Q, \gamma]$ are in $K(G,H)(X, A)$, then we have

\[
(1.3.1) \quad [E, F, \delta] + [F, Q, \gamma] = [E, Q, \gamma \delta].
\]

Proof. We define a $G$-isomorphism $\alpha(t) : F \oplus F \rightarrow F \oplus F$ by

\[
\alpha(t) = \begin{pmatrix} \cos \frac{\pi t}{2} & -\sin \frac{\pi t}{2} \\ \sin \frac{\pi t}{2} & \cos \frac{\pi t}{2} \end{pmatrix} \quad \text{for} \quad t \in [0, 1].
\]

Then $(E \oplus F, F \oplus Q, (id \oplus \gamma)\alpha(t)(\delta \oplus \text{id}))$ is an object of $L(G,H)(X, A)$, and we have

\[
(1.3.2) \quad (E \oplus F, F \oplus Q, \delta \oplus \gamma) \sim (E \oplus F, F \oplus Q, (id \oplus \gamma)\alpha(1)(\delta \oplus \text{id})).
\]

Now the diagram

\[
\begin{array}{ccc}
(I \times (E \oplus F))B & \xrightarrow{(id \oplus \gamma)\alpha(1)(\delta \oplus \text{id})} & (I \times (F \oplus Q))B \\
\downarrow \text{id} & & \downarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
(I \times (E \oplus F))B & \xrightarrow{\gamma \delta \oplus \text{id}} & (I \times (Q \oplus F))B
\end{array}
\]

is commutative. Therefore, the result follows from 1.2, (1.3.2) and (1.3.3).

1.4 Lemma. $K(G,H)(X, A)$ is an abelian group.
Proof. From 1.3, if \([E, F, \delta]\) is in \(K_{(G, H)}(X, A)\), we have

\[(1.4.1) \quad [E, F, \delta] + [F, E, \delta^{-1}] = [E, E, id] = 0.\]

Hence \(K_{(G, H)}(X, A)\) is an abelian group.

1.5 Definition. We set

\[
K_0(G, H)(X, A) = K(G, H)(X, A)
\]

\[
K^{-1}(G, H)(X, A) = K(G, H)(X \times I, A \times I \cup X \times S^0)
\]

and inductively

\[
K^{-(n+1)}(G, H)(X, A) = K^{-n}(G, H)(X \times I, A \times I \cup X \times S^0) \quad \text{for } n = 1, 2, 3, 4, \ldots.
\]

We define \(L_0(X, A)\) to be a category as follows: An object of \(L_0(X, A)\) is a pair \((E, F)\) of \(G\)-vector bundles over \(X\), together with a \(G\)-isomorphism over \(A\). The morphism \(\varphi : \sigma_{0} \rightarrow \sigma_{i}\), where \(\sigma_{i} = (E_{i}, F_{i}, \beta_{i})\) \((i = 0, 1)\), is a pair of \(G\)-isomorphisms \((f, g) : (E_{0}, F_{0}) \rightarrow (E_{i}, F_{i})\) such that \((g|A)\beta_{0} = \beta_{i}(f|A)\). Then we can define the equivalence relation \(\sim, \sim_{1}, \sim_{2}\), and the abelian groups \(K^{-n}(X, A)\) in the same way as 1.1 and 1.5.

Note: \(K_{(G, H)}(X, A)\), which is defined in this section, and \(K_{(G, H)}(X, A)\), which is defined in \([3]\), are isomorphic.

§ 2 Properties of the elements of \(K_{(G, H)}(X, A)\) and \(K_{(G, H)}(X, A)\).

2.1 Lemma. An element of \(K_{(G, H)}^{-1}(X, A)\) is represented by an object \((E \times I, E \times I, \beta)\) of \(L_{0}(X \times I, A \times I \cup X \times S^0)\) such that \(\beta|X \times \{i\} = id_E\). (Such an object is called a normalized object.)

Proof. If \([E, F, \delta]\) is in \(K_{(G, H)}^{-1}(X, A)\), there exist the \(H\)-isomorphisms

\[
p : E \rightarrow (E|X \times \{1\}) \times I
\]

\[
q : F \rightarrow (F|X \times \{1\}) \times I
\]

such that \(p|X \times \{1\} = id\) and \(q|X \times \{1\} = id\). We define an \(H\)-isomorphism \(\beta^*\) by the following composition:

\[
E|Y \xrightarrow{\beta} F|Y \xrightarrow{q|Y} (F \times I)|Y \xrightarrow{(g|Y)^{-1}} (E \times I)|Y \rightarrow E|Y
\]

where \(Y = A \times I \cup X \times S^0, F = F|X \times \{1\}, E = E|X \times \{1\}\) and \(g = \beta|X \times \{1\} \times id_I\).

Now the diagrams

\[
(2.1.1) \quad \begin{array}{ccc}
F|Y & \xrightarrow{(p^{-1}g^{-1})q} & E|Y \\
F|Y & \xrightarrow{id} & F|Y
\end{array}
\]

\[
\begin{array}{ccc}
E|Y & \xrightarrow{\beta^*} & E|Y \\
E|Y & \xrightarrow{(p\beta_{0}p^{-1})Y} & (E \times I)|Y
\end{array}
\]
are commutative. Therefore, from (2.1.1), we have

\[ [E, F, \beta] = [E, F, \beta] + [F, F, \text{id}] \]
\[ = [E, F, \beta] + [F, E, (p^{-1}q)^*Y] \]
\[ = [E, E, \beta^*] = [E \times I, E \times I, (p\beta^*p^{-1})|Y]. \]

Immediately \((E \times I, E \times I, (p\beta^*p^{-1})|Y)\) is a normalized object of \(L_H(X \times I, A \times I \cup X \times S^0)\).

**2.2 Lemma.** An element of \(K^{-1}(G,H)(A)\) is represented by an object \((E \times I, E \times I, \delta)\) of \(L_G(H)(A \times I, A \times S^0)\) such that \(\delta|I \times A \times \{1\} = \text{id}_E\). (Such an object is called a normalized object.)

**Proof.** This follows in the same way as that of 2.1.

**2.3 Lemma.** (i) Let \(E\) be a \(G\)-vector bundle over \(X\), then there exists a complementary \(G\)-vector bundle of \(E\).

(ii) Let \(E\) be an \(H\)-vector bundle over \(X\), then there exist an \(H\)-vector bundle \(E'\) over \(X\) and a \(G\)-vector bundle \(F\) over \(X\) such that \(E \oplus E'\) and \(F\) are \(H\)-isomorphic.

**Proof.** (i) (cf. 2.4 Existence of complementary bundles of \([3]\)).

(ii). From (i), there exist an \(H\)-vector bundle \(E''\) over \(X\) and an \(H\)-module \(N\) such that \(E \oplus E''\) and \(X \times N\) are \(H\)-isomorphic. From Corollary 1.1.4 of \([3]\), there exist an \(H\)-module \(N'\) and a \(G\)-module \(M\) such that \(N \oplus N'\) and \(M\) are \(H\)-isomorphic. Hence the result follows by defining \(E' = E'' \oplus (X \times N')\) and \(F = X \times M\).

**2.4 Lemma.** (i) An element of \(K_H^{-1}(X, A)\) is represented by a normalized object \((E \times I, E \times I, \beta)\) of \(L_H(X \times I, A \times I \cup X \times S^0)\) such that \(E\) has a \(G\)-vector bundle structure over \(X\).

(ii) An element of \(K^{-1}(G,H)(A)\) is represented by a normalized object \((E \times I, E \times I, \delta)\) of \(L_G(H)(A \times I, A \times S^0)\) such that \(E\) is a restriction of a \(G\)-vector bundle over \(X\) to \(A\).

**Proof.** This is clear from 2.3.

**2.5 Lemma.** Let \(M\) be a \(G\)-module and

\[ f : (A \times I \cup X \times \{0\}) \times M \rightarrow (A \times I \cup X \times \{0\}) \times M \]

a \(G\)-isomorphism. Then \(f\) is extendable to a \(G\)-isomorphism \(f^*\) over \(X \times I\).

**Proof.** From Lemma 2.2.1 of \([3]\), there exists a \(G\)-invariant neighbourhood \(U \cup A\), and \(f|A \times I\) is extendable to a \(G\)-isomorphism \(f^*\) over \(U \times \{0\} \cup A \times I\).

Since \(X\) is a compact \(G\)-space and \(G\) is a compact Lie group, so there exists a \(G\)-map \(\phi : X \rightarrow I\) such that \(\phi|U^c = 0\) and \(\phi|A = 1\). Therefore \(f^*\), which is defined by \(f^*(x, t, m) = f(x, t\phi(x), m)\) for \((x, t, m) \in X \times I \times M\), is the required extention.

**2.6 Lemma.** Let \((E, E, \beta)\) be an object of \(L_G(X, A)\). If \([E, E, \beta] = 0\), there exist a \(G\)-vector bundle \(P\) over \(X\) and a \(G\)-isomorphism \(\beta^* : E \oplus P \rightarrow E \oplus P\) such
that

\[ \beta^*|A = \beta \otimes id_P.\]

**Proof.** If \([E, E, \beta] = 0\), from (1.1.4), we have

\[ (E \oplus P, E \oplus P, \beta \otimes id_P) \sim (Q, Q, id) \]

for some \(G\)-vector bundle \(P, Q\) over \(X\). From (1.1.3), there exists an object \(\vec{\alpha} = (E, F, \beta)\) of \(LG(X \times I, A \times I)\) such that

\[ \vec{\alpha}|0 = (E \oplus P, E \oplus P, \beta \otimes id_P) \]

\[ \vec{\alpha}|1 = (Q, Q, id). \]

Now \(E|X \times \{0\} = F|X \times \{0\}\), so there exist the \(G\)-isomorphisms

\[ f : E \longrightarrow (E \oplus P) \times I \]
\[ g : F \longrightarrow (E \oplus P) \times I \]

such that

\[ f|X \times \{0\} = g|X \times \{0\} = id_{E \oplus P}. \]

We define a \(G\)-isomorphism \(\tilde{\beta}^* : (E \oplus P) \times I|A \times I \longrightarrow (E \oplus P) \times I|A \times I\) by the following composition:

\[
\begin{align*}
    (E \oplus P) \times I|A \times I & \xrightarrow{f^{-1}|A \times I} E|A \times I \\
    & \xrightarrow{\beta^*} F|A \times I \\
    & \xrightarrow{g|A \times I} (E \oplus P) \times I|A \times I
\end{align*}
\]

Then we have

\[ (2.6.1) \quad \tilde{\beta}^*|A \times \{0\} = \beta \otimes id_P \]

\[ \tilde{\beta}^*|A \times \{1\} = (gf^{-1}|X \times \{1\})|A \times \{1\}. \]

Now, from 2.3 (i), we can regard \(E \oplus P\) as a trivial \(G\)-vector bundle over \(X\). Therefore the result follows from (2.6.1) and 2.5.

**2.7 Lemma.** Let \((E, F, \beta)\) be an object of \(LG(X, A)\). If \([E, F, \beta] = 0\), there exist a \(G\)-vector bundle \(P\) over \(X\) and a \(G\)-isomorphism \(\beta^* : E \oplus P \longrightarrow F \oplus P\) such that

\[ \beta^*|A = \beta \otimes id_P. \]

**Proof.** If \([E, F, \beta] = 0\), we have

\[ (E \oplus P', F \oplus P', \beta \otimes id_{P'}) \sim (Q', Q', id) \]
for some $G$-vector bundles $P', Q'$ over $X$. So there exist the $G$-isomorphisms $f : E \oplus P' \to Q'$ and $g : F \oplus P' \to Q'$. The diagram

\[
\begin{array}{ccc}
(E \oplus P')|A & \xrightarrow{f|A} & Q'|A \\
\beta \oplus \text{id}_{P'} & \downarrow & \beta' = (g|A)(\beta \oplus \text{id}_{P'})(f|A)^{-1} \\
(F \oplus P')|A & \xrightarrow{g|A} & Q'|A
\end{array}
\]

is commutative.

From 2.6 and (2.7.1), there exists a $G$-vector bundle $P''$ over $X$ such that $\beta' \oplus \text{id}_{P''}$ is extendable to a $G$-isomorphism $\beta^*$ over $X$. Thus $\beta^*$ is the required extension.

2.8 Lemma. Let $(E, F, \delta)$ be an object of $L(G,H)(X, A)$. If $\delta[E, F, \delta] = 0$, there exists an object $(E \oplus P)^* \times I, (E \oplus P)^* \times I, \delta)$ of $L(G,H)(X, A)$ such that

\[\delta | B \times \{0\} = \delta \oplus \text{id}_P\]

is a $G$-isomorphism.

Proof. This can be proved in the same way as in the proof of 2.6.

2.9 Lemma. Let $(E, F, \delta)$ be an object of $L(G,H)(X, A)$. If $\delta[E, F, \delta] = 0$, there exists a $G$-vector bundle $P$ over $X$ and an object $(Q \times I, Q \times I, \delta)$ of $L(G,H)(X \times I, A \times I)$ such that

\[\delta | B \times \{1\} \text{ is a } G\text{-isomorphism.}\]

Proof. This follows from 2.8 in the same way as followed 2.7 from 2.6.

§ 3 Exact sequences.

3.1 Definition. We define the homomorphisms $u, v, i^*, j^*$ as follows:

\[u : K_{(G,H)}(X, A) \to K_d(X, A) \text{ by } u([E, F, \delta]) = [E, F, \delta] \times I \times A,\]

\[v : K_d(X, A) \to K_H(X, A) \text{ induced by an inclusion } H \subset G,\]

\[j^* : K_{(G,H)}(X, A) \to K_{(G,H)}(X) \text{ by } j^*([E, F, \delta]) = [E, F, \delta] \times [0] \times X,\]

\[i^* : K_{(G,H)}(X) \to K_{(G,H)}(A) \text{ by } i^*([E, F, \alpha]) = [E|A, F|A, \alpha|A].\]

Moreover we can define the following homomorphisms:

\[\lambda^*u : K^{-*}_{(G,H)}(X, A) \to K^{-*}_d(X, A),\]

\[\lambda^*v : K^{-*}_d(X, A) \to K^{-*}_H(X, A),\]

\[\lambda^*j^* : K^{-*}_{(G,H)}(X, A) \to K^{-*}_{(G,H)}(X),\]

\[\lambda^*i^* : K^{-*}_{(G,H)}(X) \to K^{-*}_{(G,H)}(A).\]

From 2.4, an element of $K^{-1}_H(X, A)$ is represented by a normalized element.
(E×I, E×I, α) of L_H(X×I, A×I∪X×S^0) such that E has a G-vector bundle structure. So we define a (boundary) homomorphism

\[ \delta : K^{-1}_H(X, A) \longrightarrow K(G,H)(X, A) \]

by

\[ \delta([E×I, E×I, α]) = [E, E, α|I×A∪{0}×X]. \]

From 2.4, an element of K^{-1}(G,H)(A) is represented by a normalized object (E×I, E×I, δ) of L_{G,H} (A×I, A×S^0) such that E is a restriction of a G-vector bundle F over X to A. Now δ is an H-isomorphism over I×A×S^0∪{0}×A×I, so we can regard δ as an H-isomorphism over I×A by an identification I×A×S^0∪0×A×I = I×A. Then we have δ|{0}×A = id and δ|{1}×A is a G-isomorphism. We define an H-isomorphism γ : I×F|B → I×F|B by

\[ γ|I×A = δ \quad \text{and} \quad γ|{0}×X = id. \]

Then we define a (boundary) homomorphism

\[ Λ : K^{-1}(G,H)(A) \longrightarrow K(G,H)(X, A) \]

by

\[ Λ([E×I, E×I, δ]) = [F, F, γ]. \]

Moreover we can define the following boundary homomorphisms:

\[ Λ^α : K^{-α}(X, A) \longrightarrow K^{-α}(G,H)(X, A) \]


### 3.2 Theorem

The sequence

\[ K(G,H)(X, A) → K(H)(X, A) \]

is exact.

**Proof.** It is clear that Image \( u \subset \) Kernel \( v \), so it is sufficient to prove that Image \( u \supset \) Kernel \( v \). Let [E, F, β] be an element of K_0(X, A) such that \( v([E, F, β]) = 0 \). From 2.7, there exist an H-vector bundle P over X and an H-isomorphism \( β^* : E ⊕ P → F ⊕ P \) such that \( β^*|A = β ⊕ id_P \). We define an H-isomorphism \( δ : (I×(E⊕P))|B → (I×(F⊕P))|B \) by \( δ|{0}×X = β^* \) and \( δ|I×A = id_I×(β ⊕ id_P) \). Now, from 2.3 (ii), we can regard P as a G-vector bundle over X. Therefore we have

\[ u([E⊕P, F⊕P, δ]) = [E⊕P, F⊕P, δ|{1}×A] = [E⊕P, F⊕P, β⊕id_P] = [E, F, β]. \]
3.3 Theorem. The sequence
\[ K^{-1}_H(X, A) \xrightarrow{\partial} K_{(G,H)}(X, A) \xrightarrow{u} K_G(X, A) \]
is exact.

Proof. It is clear that Image \( \partial \subset \text{Kernel } u \), so it is sufficient to prove that Image \( \partial \subset \text{Kernel } u \). Let \([E, F, \delta]\) be an element of \( K_{(G,H)}(X, A) \) such that \( u([E, F, \delta]) = 0 \). From 2.7, there exist a \( G \)-vector bundle \( P \) over \( X \) and a \( G \)-isomorphism \( \beta : E \oplus P \rightarrow F \oplus P \) such that \( \beta|A = (\delta|\{1\} \times A) \oplus id_P \). We define an \( H \)-isomorphism \( \alpha : (I \times (E \oplus P))|Y \rightarrow (I \times (E \oplus P))|Y \)
where \( Y = I \times A \cup S^0 \times X \), by the following composition:
\[
(I \times (E \oplus P))|Y \xrightarrow{\beta^*} (I \times (F \oplus P))|Y \xrightarrow{(id_Y \times \beta^{-1})|Y} (I \times (E \oplus P))|Y
\]
where \( \beta^* \) is defined by \( \beta^*|\{1\} \times X = \beta \) and \( \beta^*|I \times A \cup \{0\} \times X = \delta \oplus id_P \).
Then \((I \times (E \oplus P), I \times (E \oplus P), \alpha)\) is a normalized object of \( L_H(I \times X, I \times A \cup S^0 \times X) \).
Now the diagram
\[
\begin{array}{ccc}
(I \times (E \oplus P))|B & \xrightarrow{\alpha|B} & (I \times (E \oplus P))|B \\
\downarrow{id} & & \downarrow{(id_Y \times \beta)\mid B} \\
(I \times (E \oplus P))|B & \xrightarrow{\delta \oplus i_P} & (I \times (F \oplus P))|B
\end{array}
\]
is commutative. So, from (3.3.2), we have
\[
\partial([I \times (E \oplus P), I \times (E \oplus P), \alpha]) = [E \oplus P, E \oplus P, \alpha|B] \\
= [E \oplus P, F \oplus P, \delta \oplus id_P] = [E, F, \delta].
\]

3.4 Theorem. The sequence
\[ K^{-1}_G(X, A) \xrightarrow{A^\psi} K^{-1}_H(X, A) \xrightarrow{\partial} K_{(G,H)}(X, A) \]
is exact.

Proof. It is clear that Image \( A^\psi \subset \text{Kernel } \partial \), so it is sufficient to prove that Image \( A^\psi \subset \text{Kernel } \partial \). Let \((I \times E, I \times E, \alpha)\) be a normalized object of \( L_H(I \times X, I \times A \cup S^0 \times X) \) such that \( \partial([I \times E, I \times E, \alpha]) = [E, E, \alpha|\{0\} \times X \cup I \times A] = 0 \). From 2.8, there exist a \( G \)-vector bundle \( P \) over \( X \) and an object \((E \oplus P) \times I, (E \oplus P) \times I, \beta)\) of \( L_{(G,H)}(X \times I, A \times I) \) such that \( \beta|B \times \{1\} \) is a \( G \)-isomorphism and \( \beta|B \times \{0\} = (\alpha|\{0\} \times X \cup I \times A) \oplus id_P \). Now, from 2.3, we can regard \( E \oplus P \) as a trivial \( G \)-vector bundle over \( X \), since \( \beta|\{1\} \times A \times I \) is a \( G \)-isomorphism and \( \beta|\{1\} \times A \times \{0\} = id \).
so from 2.5, there exists a $G$-isomorphism

$$\delta^* : (I \times (E \oplus P) \times I) \to (I \times (E \oplus P) \times I)$$

such that $\delta^* \cdot [1] \times X \times I = \delta \cdot [1] \times X \times I$ and $\delta^* \cdot [1] \times X \times \{0\} = \text{id}$. We define an $H$-isomorphism

$$\alpha : (I \times (E \oplus P) \times I) \to (I \times (E \oplus P) \times I)$$

by

$$\alpha \cdot [1] \times X \times I = \delta^* \cdot [1] \times X \times I$$

Then $(I \times (E \oplus P) \times I, I \times (E \oplus P) \times I, \alpha)$ is an object of $L_H(I \times X \times I, (I \times A \cup S^0 \times X) \times I)$ and $\alpha \cdot [I \times A \cup S^0 \times X] \cdot \{0\} = \alpha \cdot \text{id}_P$. So we have

$$\nu^*([I \times (E \oplus P), I \times (E \oplus P), \alpha \cdot [I \times A \cup S^0 \times X] \cdot \{1\}])$$

$$= [I \times (E \oplus P), I \times (E \oplus P), \alpha \cdot \text{id}_P]$$

$$= [I \times E, I \times E, \alpha]$$

3.5 Theorem. The sequence

$$\ldots \to K^-(G, H)(X, A) \to K^-(G, H)(X, A) \to \ldots$$

$$\partial \quad \text{u} \quad \text{v}$$

is exact.

Proof. It follows from 3.2, 3.3 and 3.4.

3.6 Theorem. The sequence

$$\ldots \to K^-(G, H)(X) \to K^-(G, H)(A) \to K^-(G, H)(X, A) \to \ldots$$

$$\partial \quad \text{u} \quad \text{v}$$

is exact.

Proof. This can be proved by the same methods as in the proof of 3.5.

3.7 Definition. Let $H'$ be a closed subgroup of $H$, then we define the natural homomorphisms

$$A^\mu \ : \ K^-{n'\times H}(X, A) \to K^-{n'\times H}(X, A)$$

induced by $(H, H') \subseteq (G, H)$,

$$A^\nu \ : \ K^-{n'\times H}(X, A) \to K^-{n'\times H}(X, A)$$

induced by $(G, H') \subseteq (G, H)$.

Let $\Lambda^d$ be a boundary homomorphism which defined by the following composition:
\[ A^d : K^{-((n+1)H)}(X, A) \xrightarrow{\varDelta^{(n+1)H}} K^{-((n+1)H)}(X, A) \xrightarrow{A^d} K^{-n(G, H)}(X, A) \]

**3.8 Lemma.** The following diagram is commutative.

Proof. From the definitions of homomorphisms, this is clear.

**3.9 Theorem.** The sequence

\[ \cdots \rightarrow K^{-((n+1)H)}(X, A) \xrightarrow{A^d} K^{-n(G, H)}(X, A) \xrightarrow{\lambda^p} K^{-n(G, H)}(X, A) \rightarrow \cdots \]

is exact.

Proof. From 3.5 and 3.8, this is clear.

**3.10 Theorem.** The following diagrams are commutative, and each row and
each column are exact.

Proof. From the above arguments, this is clear.

§ 4 \((C(G/H), G/H)\) coefficient \(K\)-theory.

**4.1 Theorem.** We obtain the following isomorphism:

\[ K_{(G, H)}(X, A) \cong K_0(C(G/H), G/H) \times (X, A), \]

where \(C(G/H)\) is a cone over \(G/H\).

The proof of the Theorem will be broken down into a series of Lemmas.

**4.2 Let \(Y\) be an \(H\)-space. Let \(G \times Y\) denote the identification space obtained from \(G \times Y\) by the equivalence relation:

\[ (g_1, y_1) \sim (g_2, y_2) \text{ if and only if } g_2 = g_1 h^{-1} \text{ and } y_2 = h y_1 \text{ for some } h \in H. \]
Then $G \times Y$ admits a $G$-space structure: we define

$$g(g_i, y) = (gg_i, y)$$

and note that

$$g(g_i h^{-1}, hy) = (gg_i h^{-1}, hy) = (g h^{-1}, y) = g(g_i, y).$$

Let $E$ be an $H$-vector bundle over $Y$, then $G \times E$ admits a $G$-vector bundle structure over $G \times Y$.

If $Y$ is a $G$-space, $f: (G/H) \times Y \to G \times Y$, which defined by

(4.2.1) $$f(gH, y) = [g, g^{-1}y],$$

is a $G$-homeomorphism.

Let $E$, $F$ be $H$-vector bundle over $Y$ and $\alpha: E \to F$ an $H$-isomorphism. Then

(4.2.2) $$\tilde{\alpha}(gH, e) = [g, \alpha(e)],$$

is a $G$-isomorphism.

If $Y$ is a $G$-space and $E$, $F$ are $G$-vector bundles over $Y$, then $\tilde{\alpha}: (G/H) \times E \to (G/H) \times F$, which defined by

(4.2.3) $$\tilde{\alpha}(gH, e) = [gH, g\alpha(g^{-1}e)],$$

is a $G$-isomorphism. Moreover we have

(4.2.4) $$f^*(G \times E) = (G/H) \times E \text{ and } f^*(\tilde{\alpha}) = \tilde{\alpha}.$$

**4.3 Lemma.** Let $l_1: K_H(X, A) \to K_G(G/H) \times X, (G/H) \times A)$ be a following composition:

$$K_H(X, A) \xrightarrow{l'_1} K_G(G \times X, G \times A) \xrightarrow{f^*} K_G(G/H) \times X, (G/H) \times A),$$

where $l'_1$ is defined by $l'_1([E, F, \alpha]) = [G \times E, G \times F, \alpha]$. Then $l_1$ is an isomorphism.

**Proof.** This follows directly from Proposition 1.1.3 of [3].

**4.4 Lemma.** We define $l_2: K_G(C(G/H) \times X, C(G/H) \times A)$ by

$$l_2([E, F, \alpha]) = [C(G/H) \times E, C(G/H) \times F, \text{id} \times \alpha].$$

Then $l_2$ is an isomorphism.

**Proof.** Since $C(G/H)$ is $G$-contractible, so the result follows at once.
4.5 Definition. We define \( l_3 : K_{(G,H)}(X, A) \longrightarrow K_{d}(C(G/H), (G/H) \times (X, A)) \) as follows: Let \([E, F, \delta]\) be an element of \( K_{(G,H)}(X, A)\). From 1.1, we can construct a \( G \)-isomorphism

\[
\tilde{\delta} : (G/H) \times ((I \times E) \cup B) \longrightarrow (G/H) \times ((I \times F) \cup B)
\]

as (4.2.3). Now \( \tilde{\delta}(gH, (1, e)) = (gH, \delta(1, g^{-1}e)) \) and \( \tilde{\delta}|1 \times A \) is a \( G \)-isomorphism, so we have

\[
(4.5.1) \quad \tilde{\delta}(gH, (1, e)) = (gH, \delta(1, e)).
\]

From (4.5.1), we can regard \( \tilde{\delta} \) as a \( G \)-isomorphism

\[
\tilde{\delta} : C(G/H) \times E \cup C(G/H) \times A \cup (G/H) \times X \longrightarrow C(G/H) \times F \cup C(G/H) \times A \cup (G/H) \times X.
\]

So we define \( l_3 \) by

\[
l_3([E, F, \delta]) = (C(G/H) \times E, C(G/H) \times F, \tilde{\delta}).
\]

4.6 Lemma. We obtain the following exact sequence:

\[
\cdots \longrightarrow K_{d}(C(G/H), (G/H) \times (X, A)) \xrightarrow{\partial} K_{d}(C(G/H) \times X, C(G/H) \times A) \xrightarrow{j_1^*} K_{d}(C(G/H), (G/H) \times (X, A)) \xrightarrow{i_1^*} K_{d}(C(G/H) \times X, (G/H) \times A) \longrightarrow \cdots
\]

Proof. For a triple \((C(G/H) \times X, C(G/H) \times A \cup (G/H) \times X, C(G/H) \times A)\), we have the exact sequence:

\[
\cdots \longrightarrow K_{d}(C(G/H), (G/H) \times (X, A)) \xrightarrow{\partial} K_{d}(C(G/H) \times X, C(G/H) \times A) \xrightarrow{j^*} K_{d}(C(G/H) \times X, (G/H) \times A) \xrightarrow{i^*} K_{d}(C(G/H) \times A \cup (G/H) \times X, C(G/H) \times A) \longrightarrow \cdots
\]

Now, from the Excision Theorem, we have the following isomorphism \( \tilde{i}^* \):

\[
K_{d}(C(G/H) \times A \cup (G/H) \times X, C(G/H) \times A) \xrightarrow{\tilde{i}^*} K_{d}(C(G/H) \times X, (G/H) \times A).
\]

So the result follows by defining \( \tilde{\partial}_1 = \partial(\tilde{i}^*)^{-1} \), \( i_1^* = i^* \tilde{i}^* \) and \( j_1^* = j^* \).

4.7 Lemma. The diagram

\[
\begin{array}{ccc}
K_{d}(X, A) \xrightarrow{v} & & K_{d}(X, A) \\
\downarrow i_2 & & \downarrow i_1 \\
K_{d}(C(G/H) \times X, C(G/H) \times A) \xrightarrow{i_1^*} & & K_{d}(C(G/H) \times X, (G/H) \times A)
\end{array}
\]

is commutative.
Proof. Let \([E, F, \alpha]\) be an element of \(K(X, A)\). Then we have
\[
liv([E, F, \alpha]) = l([E, F, \alpha]) = \frac{f^*(G \times E), f^*(G \times F), f^*(\alpha)}{G/H} \times E, (G/H) \times F, id \times \alpha}.
\]
and
\[
iil([E, F, \alpha]) = i_\#([C(G/H) \times E, C(G/H) \times F, id \times \alpha] = [(G/H) \times E, (G/H) \times F, id \times \alpha].
\]
Therefore \(l_i = i_\#l_s\).

4.8 Lemma. The diagram

\[
K_{(G,H)}(X, A) \xrightarrow{u} K_{(G,H)}(X, A) \\
\downarrow{I_1} \quad \downarrow{I_1}
\]

\[
K_{(G,H)}((C(G/H), G/H) \times (X, A)) \xrightarrow{J_1} K_{(G,H)}((C(G/H), \times X, C(G/H) \times A)
\]
is commutative.

Proof. Let \([E, F, \delta]\) be an element of \(K_{(G,H)}(X, A)\). Then we have
\[(4.8.1) j^*l_s([E, F, \delta]) = l([E, F, \delta]) = \left[\frac{C(G/H) \times E, C(G/H) \times F, \delta C(G/H) \times A}{C(G/H) \times E, C(G/H) \times F, \delta C(G/H) \times A}\right].
\]
and
\[(4.8.2) j^*l_s([E, F, \delta]) = i_\#([C(G/H) \times E, C(G/H) \times F, \delta C(G/H) \times A] = [C(G/H) \times E, C(G/H) \times F, \delta C(G/H) \times A].
\]
Now we define a \(G\)-isomorphism \(\gamma : C(G/H) \times (E \times A) \times I \rightarrow C(G/H) \times (F \times A) \times I\) by
\[
\gamma([t, gH], e, s) = ([t, gH], g\delta((1-t)s+t, g^{-1}e), s),
\]
where \(\delta'\) is defined by \(\delta'(t, e) = (t, \delta'(t, e))\). Then we have
\[
\gamma(s = 0) = \delta C(G/H) \times A \quad \text{and} \quad \gamma(s = 1) = id \times (\delta \{1\} \times A).
\]
Therefore, from (4.8.1) and (4.8.2), we have \(l_i = j^*l_s\).

4.9 Lemma. The diagram

\[
K_{H^{-1}}(X, A) \xrightarrow{u} K_{H^{-1}}(X, A) \\
\downarrow{I_1} \quad \downarrow{I_1}
\]

\[
K_{H^{-1}}((G/H) \times X, (G/H) \times X) \xrightarrow{\delta_1} K_{H^{-1}}((G/H), G/H) \times (X, A)
\]
is commutative.
Proof. Let \( x = [I \times E, I \times E, \alpha] \) be an element of \( KH^{-i}(X, A) \). Then we have
\[
(i^* - l_i)(x) = (i^* - l_i)([G/H \times I \times E, (G/H) \times I \times E, \tilde{\alpha}])
\]
\[
= [C(G/H) \times I \times (E | A) \cup (G/H) \times I \times E, C(G/H) \times I \times (E | A) \cup (G/H) \times I \times E, \tilde{\alpha}'],
\]
where \( \tilde{\alpha}' \) is defined by
\[
\tilde{\alpha}'([t, gH], s, e) = ([t, gH], s, g\alpha'((1-t)s + t, g^{-1}e)),
\]
where \( \alpha' \) is defined by \( \alpha(s, e) = (t, \alpha'(s, e)) \). Since \( (i^* - l_i)(x) \) is also a normalized object, we have
\[
\partial l_i(x) = [C(G/H) \times E, C(G/H) \times E, \beta],
\]
where \( \beta = \tilde{\alpha}'|C(G/H) \times \{0\} \times A \cup (G/H) \times \{0\} \times X \). On the other hand we have
\[
l_i\tilde{\beta}(x) = [C(G/H) \times E, C(G/H) \times E, \tilde{\beta}],
\]
where \( \tilde{\beta} \) is defined by \( \tilde{\beta}([t, gH], e) = ([t, gH], s, g\tilde{\beta}(t, g^{-1}e)) \). Then, from the definitions of \( \beta \) and \( \tilde{\beta} \), \( \beta = \tilde{\beta} \). So we have \( l_i\beta = \partial l_i \).

4.10 Proof of 4.1 Theorem.
From the above Lemmas, the following diagram is commutative and each row is exact.
\[
\begin{array}{ccccccccc}
\vdots & K_{H^{-i}}(X, A) & \partial & K_{G, H}(X, A) & \rightarrow & K_{H^{-i}}((G/H) \times X, (G/H) \times A) & \rightarrow \\
& l_1 & & l_3 & & & \\
& \downarrow & & \downarrow & & & \\
\vdots & K_{G^{-i}}((G/H) \times X, (G/H) \times A) & \rightarrow & K_{G}(C(G/H), (G/H) \times (X, A)) & \rightarrow \\
& & & & & & & \\
\rightarrow & K_{G}(X, A) & \rightarrow & K_{H}(X, A) & \rightarrow \\
& l_2 & & l_1 & & & \\
& \downarrow & & \downarrow & & & \\
K_{G}(C(G/H) \times X, C(G/H) \times A) & \rightarrow & K_{H}((G/H) \times X, (G/H) \times A).
\end{array}
\]

Therefore the result follows from Five Lemma.

4.11 Corollary. We obtain the following isomorphism:
\[
K^{-n}_{G, H}(X, A) \cong K^{-((n+2) \times G, H}(X, A) \quad (Complex \ case)
\]
Proof. From 4.1 Theorem, this is clear.

§ 5 Wyel group operations.

5.1 Let \( G \) be a compact connected Lie group, \( T \) a maximal torus of \( G \) and \( W(G) = N(T)/T \) the Wyel group. Let \( E \) be a \( T \)-vector bundle over \( X \). For each \( n \in N(T), n^*E \) admits a \( T \)-vector bundle structure (we regard \( n \) as a continuous map \( n : X \rightarrow X \) by its action on \( X \)): we define \( h : (n^*E)_x \rightarrow (n^*E)_{hx} \) by \( nhn^{-1} : \)
For all \( h \in T \). If \( n \) is in \( T \), \( n^{*}E \) and \( E \) are isomorphic by \( T \)-isomorphism \( n^{-1} \). If \( E \) is a \( G \)-vector bundle, \( n^{*}E \) admits a \( G \)-vector bundle structure, and \( n^{*}E \) and \( E \) are isomorphic by \( G \)-isomorphism \( n^{-1} \). So the following operation is well defined:

\[
(5.1.1) \quad K_{T}(X) \times W(G) \longrightarrow K_{T}(X) \\
([E, F], [n]) \mapsto [n^{*}E, n^{*}E].
\]

Let \( E, F \) be \( G \)-vector bundle over \( X \) and \( \alpha : E \longrightarrow F \) a \( T \)-isomorphism. In general the diagram

\[
\begin{array}{ccc}
n^{*}E & \xrightarrow{n^{*}\alpha} & n^{*}F \\
\downarrow n^{-1} & & \downarrow n^{-1} \\
E & \xrightarrow{\alpha} & F
\end{array}
\]

is not commutative, but if \( n \) is in \( T \), the diagram is commutative. So the following operation is well defined:

\[
(5.1.2) \quad K_{(G, T)}(X) \times W(G) \longrightarrow K_{(G, T)}(X) \\
([E, F, \alpha], [n]) \mapsto [n^{*}E, n^{*}F, n^{*}\alpha].
\]

Similarly, we can define the following operations:

\[
(5.1.3) \quad K_{T}^{*}(X) \times W(G) \longrightarrow K_{T}^{*}(X) \\
K^{*}(G, T)(X) \times W(G) \longrightarrow K^{*}(G, T)(X).
\]

Let \( K_{T}^{*}(X)^{W(G)} \) (respectively \( K^{*}(G, T)(X)^{W(G)} \)) be an abelian group of invariants of \( K_{T}^{*}(X) \) (respectively \( K^{*}(G, T)(X) \)) under the action of \( W(G) \). Then we have

\[
(5.1.4) \quad v(K_{G}^{*}(X)) \subseteq K_{T}^{*}(X)^{W(G)} \\
\partial(K_{T}^{*}(X)^{W(G)}) \subseteq K^{*}(G, T)(X)^{W(G)},
\]

and the commutative diagram

\[
\begin{array}{ccc}
K_{G}^{*}(X) & \xrightarrow{v} & K_{T}^{*}(X) \\
\downarrow w & & \downarrow w \\
K_{T}^{*}(X) & \xrightarrow{\partial} & K^{*}(G, T)(X)
\end{array}
\]

for all \( w \in W(G) \).

5.2 Proof of Maine Theorem (B).

By 3.5 Theorem, Theorem (A), 4.11 Corollary and (5.1.5), the proof will be carried out directly. Note: From 4.11 Corollary, the exact sequences of 3.5, 3.6
and 3. 9 are extendable to the right side.

References