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Representations of Loop Spaces and Fibre Bundles

Dedicated to Prof. Keizo Asano for his 60th birth day

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Introduction. It is known that if X is arcwise connected, arcwise locally connected, and semi locally *1*-connected, then the equivalence classes of principal bundles over X with structure group G, a tatally disconnected group, are in *1-1* correspondence with the equivalence classes (under inner automorphisms of G) of homonorphisms of $\pi_1(X)$ into G ([10], § 11).

The purpose of this paper is to extend this result for general bundles. Our result, which is given in § 3 as theorem 1, is stated as follows : *If X is an arcwise connected metric space and satisfies the following condition* (*),

(*) *for any $x \in X$, there exists a neighborhood $U(x)$ of x such that for any $y \in U(x)$, there exists a path $\gamma_{x,y}$ which has finite length and starts from x ends at y and depend continuously on y by the path space topology,*

and G is a topological group such that whose projection onto the space of conjugate classes of G has local sections, then the equivalence classes of principal bundles over X with structure group G are in 1-1 correspondence with the continuous maps from X into the space of equivalence classes (under lnner automorphisms of G) of continuous homomorphisms of $[\Omega_f(X)]$ into G. Here $\Omega_f(X)$ means the space of loops with finite length over X and $[\Omega_f(X)]$ is the group obtained from $\Omega_f(X)$ by the following relations (cf. [3]).

$\alpha \sim \beta$ *if* $\alpha(s) = \beta(h(s))$, $h \in H^+(I)$, *the group of orientation preserving homeomorphisms of* $I = [0, 1]$,

$\alpha_1\beta \cdot \beta^{-1}\alpha_2 \sim \alpha_1\alpha_2$, *where* $\beta^{-1}(s) = \beta(1-s)$.

For arbitrary topological group G, we also obtain the following theorem 1$'$.

Theorem 1$'$. *If X is same as in theorem 1, then the equivalence classes of principal bundles over X with structure group G are in 1-1 correspondence with the set of continuous map χ from X into the space of equivalence classes (under inner automorphisms of G) of continuous homomorphisms of $[\Omega_f(X)]$ into G such that there exists an open covering $\{U\}$ of X (may be depend on χ) and continuous map χ_U from U*

into the space of continuous homomorphisms of $[\Omega_f(X)]$ *into* G *(with compact open topology) for any* $U \in \{U\}$ *such that*

the class of $\chi_U(x)$ *is the value of* $\chi(x)$ *for any* $x \in U$.

This theorem 1′ is also given in § 3.

In § 1, we prove the following lemma 1. **Lemma 1.** *If* ξ *and* η *are principal* G *and* H*-bundles over* X *such that* $\pi^*(\eta)$ *is trivial over* X_ξ, *where* X_ξ *is the tatal space of* ξ *and* π *is the projection from* X_ξ *onto* X. *Then there is a continuous map* χ *from* X *into Hom.* (G, H), *the space of equivalence classes (under inner automorphisms) of continuous homomorphisms from* G *into* H *such that*

(i), *there exists an open covering* $\{U\}$ *of* X *such that for any* $U \cup V$, *there is a continuous map from* $U \cup V$ *into the space of continuous homomorphisms of* G *into* H *(with compact open topology) such that the class of its value at* x *is the value of* χ *for any* $x \in U \cup V$,

(ii), *denoting the transition functions of* ξ *and* η *by* $\{\gamma_{UV}(x)\}$ *and* $\{g_{UV}(x)\}$ *(*$\{U\}$ *also satisfies* (i) *for* χ*), there exists continuous map* $f_U : U \to H$ *for any* $U \in U$ *such that*

$$g_{UV}(x) = f_U(x)\chi_{UV}(x)(\gamma_{UV}(x))f_V(x)^{-1}.$$

In § 2, we prove

(a). $[E_f(X)]$ *is a contractible space.*

(b). $[E_f(X)]$ *is the tatal space of a principal* $[\Omega_f(X)]$*-bundle over* X. Here $[E_f(X)]$ is the quotient space of $E_f(X)$, the space of paths with finite length over X with a basepoint, defined similarly as $[\Omega_f(X)]$.

Combining lemma 1 and the above (a), (b), we obtain theorem 1 and 1′ in § 3.

In § 3, we treat the differentiable case. In fact, the proof of theorem 1 can not applied for differentiable case. Because we do not know whether $\pi^*(\eta)$ is differentiable trivial or not over $[E_f(X)]$ for a differentiable bundle η over X in general. Here π is the projection from $[E_f(X)]$ onto X. But if we use $E_{2,k,0}(X)$ and $\Omega_{2,k,0}(X)$ instead of $E_f(X)$ and $\Omega_f(X)$, we obtain similar result for differentiable bundles. Here $E_{2,k,0}(X)$ and $\Omega_{2,k,0}(X)$ are given by

$$E_{2,k,0}(X) = \{\alpha \mid \alpha : I \to X \textit{ belongs in } k\textit{-th Sobolev space and } \alpha(a) = \alpha(0),$$
$$\alpha(1-a) = \alpha(1) \textit{ if } 0 \leq a \leq \varepsilon \textit{ for some } \varepsilon\},$$
$$\Omega_{2,k,0}(X) = \{\beta \mid \beta \textit{ belongs in } E_{2,k,0}(X) \text{ and } \beta(0) = \beta(1) = *, \textit{ the base point}\},$$

(cf. [3].) In § 3, we also give a differentiable version of lemma 1.

It has been known that hte topological structure of the loop space $\Omega(X)$ over X has deep connection with that of X. But it seems that the results of this paper and [3] sugest the algebraic structure of $\Omega(X)$ also has deep connection with the topological structure of X. In fact, there are some other results to sugest this.

For example, Chen has shown

(i). *If X is a smooth manifold, and $\omega_1, \cdots\cdots, \omega_m$ forms 1 -form basis at any point of X ($\omega_1(x), \cdots\cdots, \omega_m(x)$ need not be linear independent), then setting*

$$\theta(\alpha) = 1 + \sum_{p \geqq 1} \sum_{i_1, \cdots, i_p} a_{i_1 \cdots i_p} X_{i_1} \ldots X_{i_p} \ldots,$$

$$a_{i_1 \cdots i_p} = \int_\alpha \theta_{i_1} \otimes \cdots \otimes \theta_{i_p},$$

where $\{X_1, \cdots, X_m\}$ are non -commutative indeterminants, θ is an into isomorphism from $[\Omega(X)]$ (defined from $\Omega(X)$ similarly as $[\Omega_f(X)]$ into $\mathfrak{T}_{\boldsymbol{R}}[[X_1, \cdots, X_m]]$, the free tensor algebra over $\boldsymbol{R}$ with generators $X_1, \cdots, X_m$ ([4], [5]).

(ii). log $\theta(\alpha)$ *is a Lie element for any α* ([4], [9]) *and denoting the Lie algebras* {log $\theta(\alpha)\}$, $\alpha \in E(X)$ and {leg $\theta(\beta)\}$, $\beta \in \Omega(X)$ *by $\mathfrak{L}[E(X)]$ and $\mathfrak{L}\Omega(X)]$, we have*

$$\mathfrak{L}[\Omega(X)]/\mathfrak{D}^2(\mathfrak{L}[E(X)])]_\cap \mathfrak{L}[\Omega(X)] \cong H^1(X, \boldsymbol{R}),$$

where $\mathfrak{D}^2(\mathfrak{L}[E(X)])$ is the second derived ideal of $\mathfrak{L}[E(X)]$ ([6]).

It seems that these results may have some relations to algebraic homotopy theory (cf. [7]).

§ 1. Proof of lemma 1.

Let X be a paracompact normal space, then any fibre bundle over X is represented by its transition functions ([8]). Hence we may write a bundle by its transition functions.

Lemma 1. *Let X be a paracompact normal space, $\xi = \{\gamma_{UV}(x)\}$ and $\eta = \{g_{UV}(x)\}$ are principal $\boldsymbol{G}$ and $\boldsymbol{H}$ -bundles over X such that $\pi^*(\eta)$ is trivial over X_ξ, the tatal space of ξ and π is the projection from X_ξ onto X, then there is a continuous map χ from X into Hom. ($\boldsymbol{G}$, $\boldsymbol{H}$), the space of equivalence classes (under inner automorphisms) of the continuous homomorphisms from $\boldsymbol{G}$ into $\boldsymbol{H}$ with the induced toplogy of the compact open topology, and an open covering $\{U\}$ of X such that*

(i), *for any $U \cup V$, U, $V \in \{U\}$, there is a continuous map χ_{UV} from $U \cup V$ into the space of continuous homomorphisms from $\boldsymbol{G}$ nto $\boldsymbol{H}$ (with compact open topology) such that*

$$\text{the class of } \chi_{UV}(x) = \chi(x), \quad x \in U \cup V,$$

(ii), *the transition functions $[\gamma_{UV}(x)]$ of ξ is defined by the open covering $\{U\}$ aod there exists continuous map $f_U : U \to \boldsymbol{G}$ for any $U \in \{U\}$ such that*

$$g_{UV}(x) = f_U(x) \chi_{UV}(x)(\gamma_{UV}(x)) f_V(x)^{-1}, \tag{1}$$

for any $U \cap V$, where $\{g_{UV}(x)\}$ is the transition function of η.

Proof. We denote the elements of X by $x, y, \cdots$, and the elements of G by $\alpha, \beta, \cdots$, then by assumption, we get

$$g_{\pi(U)\pi(V)}(x)=h_U(x, \alpha)h_V(x, \gamma_{\pi(U)\pi(V)}(x)\alpha)^{-1}, \tag{2}$$

where $\{U\}$ is an open covering of X_ξ such that
(a). *ξ is trivial on $\cup_{\pi(U)\cap\pi(V)\neq\phi}\pi(V)$ for any $U\in\{U\}$,*
(b). *each U is written as*

$$U=\pi(U)\times S, \quad S \textit{ is an open set of } \boldsymbol{G}.$$

For the simplicity, we denote $g_{UV}(x)$, etc., instead of $g_{\pi(U)\pi(V)}(x)$, etc., in the rest.

Since $g_{UV}(x)$ does not depend on α, we also have

$$g_{UV}(x)=h_{U'}(x, \beta)h_{V'}(x, \gamma_{UV}(x)\beta)^{-1}, \quad \alpha\neq\beta, \tag{2)'}$$

where U', V' may be different from U, V but must satisfy

$$\pi(U)=\pi(U'), \quad \pi(V)=\pi(V').$$

By (2) and (2)', we get (here β need not be different from α)

$$h_U(x, \alpha)^{-1}h_{U'}(x, \beta)=h_V(x, \gamma_{UV}(x)\alpha)^{-1}h_{V'}(x, \gamma_{UV}(x)\beta).$$

Hence setting

$$\theta(x, \alpha, \beta)\,|\,\hat{\pi}^{-1}(\pi(U))=h_U(x, \alpha)^{-1}h_{U'}(x, \beta),$$

$\theta(x, \alpha, \beta)$ is a continuous map from $X_{\xi,\xi}$ into $\boldsymbol{H}$, where $X_{\xi,\xi}$ is given by

$$X_{\xi,\xi}=\bigcup_U(\pi(U)\times \boldsymbol{G}\times \boldsymbol{G})/\sim,$$
$$\pi(U)\times\boldsymbol{G}\times\boldsymbol{G}\ni x\times\alpha\times\beta\sim x\times\gamma_{UV}(x)\alpha\times\gamma_{UV}(x)\alpha\in\pi(V)\times\boldsymbol{G}\times\boldsymbol{G},$$

and $\hat{\pi}$ is the projection from $X_{\xi,\xi}$ onto X. In the rest, we denote U instead of $\hat{\pi}^{-1}(\pi(U))$.

By (a), ξ has a cross -section $s=s_U(x)$ on $\cup_{\pi(U)\cap\pi(V)\neq\phi}\pi(V)$. Using this s, we set

$$h(x, \alpha)\,|\,\pi^{-1}(\pi(U))=\theta(x, s, \alpha)\,|\,\pi^{-1}(\pi(U)).$$

Although $h(x, \alpha)$ might not be defined on X_ξ but by definition, it is defined on $\pi^{-1}(\cup_{\pi(U)\cap\pi(V)\neq\phi}\pi(V))$. Hence for a fixed x, it is defined for all (x, α), $\alpha\subset\boldsymbol{G}$.

Moreover, since

$$\begin{aligned} h(x, \alpha)^{-1}h(x, \beta) &= \theta(x, s_U)x), \alpha)^{-1}\theta(x, s_U(x), \beta) \\ &= (h_U(x, s_U(x))^{-1}h_{U''}(x, \alpha))^{-1}(h_U(x, s_U(x))^{-1}h_{U'}(x, \beta)) \\ &= h_{U''}(x, \alpha)^{-1}h_U(x, \beta) \\ &= \theta(x, \alpha, \beta), \end{aligned}$$

we get

(3) $$\theta(x, \alpha, \beta) = h(x, \alpha)^{-1}h(x, \beta),$$

on $\hat{\pi}^{-1}(\cap_{\pi(U)\cap\pi(V)\neq\phi}\pi(V))$.

On the other hand, if we get

$$\theta(x, \alpha, \beta) = h'(x, \alpha)^{-1}h'(x, \beta)$$

on some open set W of $\hat{\pi}^{-1}(\cap_{\pi(U)\cap\pi(V)\neq\phi}\pi(V))$, then since

$$h'(x, \alpha)h(x, \beta)^{-1} = h'(x, \alpha)h(x, \beta)^{-1}$$

for arbitrary α, β, we may set

$$h'(x, \alpha) = f(x)h(x, \alpha)$$

where f is a continuous map from $\hat{\pi}(W)$ into $\boldsymbol{H}$. Hence we can set

(4)′ $$h_U(x, \alpha) = f_U(x)h(x, \alpha)$$

Next we set

$$h(x, \alpha\beta) = \chi(x, \alpha)h(x, \beta).$$

Then since

$$\begin{aligned} h(x, \alpha\beta\gamma) &= \chi(x, \alpha\beta)h(x, \gamma) \\ &= \chi(x, \alpha)h(x, \beta\gamma) \\ &= \chi(x, \alpha)\chi(x, \beta)h(x, \gamma), \end{aligned}$$

we have

(4) $$\chi(x, \alpha\beta) = \chi(x, \alpha)\chi(x, \beta).$$

We note that setting $U = \pi(U)\times S$, we can define $\chi_U(x, \alpha)$ by

$$h_U(x, \alpha\beta) = \chi_U(x, \alpha)h_U(x, \beta)$$

if $\alpha\beta$ and β both contained in S. Butsince we get by (4)$'$

$$\chi_U(x, \alpha) = f_U(x)\chi(x, \alpha)f_U(x)^{-1},$$

$\chi_U(x, \alpha)$ is defined for arbitrary $\alpha \in \boldsymbol{G}$.

We define $\chi_V(x, \alpha)$ similarly as $\chi_U(x, \alpha)$. Then since we get

$$\begin{aligned} h_U(x, \alpha\beta) &= \chi_U(x, \alpha)h_U(x, \beta) \\ = h_V(x, \gamma_{UV}(x)\alpha\beta) &= \chi_V(x, \gamma_{UV}(x))\chi_V(x, \alpha)h_V(x, \beta) \\ &= \chi_V(x, \gamma_{UV}(x))\chi_V(x, \alpha)h_U(x, \gamma_{UV}(x)^{-1}\beta) \\ &= \chi_V(x, \gamma_{UV}(x))\chi_V(x, \alpha)\chi_U(x, \gamma_{UV}(x))^{-1}h_U(x, \beta), \end{aligned}$$

we have

$$\chi_U(x, \alpha) = \chi_V(x, \gamma_{UV}(x))\chi_V(x, \alpha)\chi_U(x, \gamma_{UV}(x))^{-1}.$$

But since by (4)

$$\begin{aligned} &\chi_U(x, \alpha\beta) \\ &= \chi_U(x, \alpha)\chi_U(x, \beta) \\ &= \chi_V(x, \gamma_{UV}(x))\chi_V(x, \alpha)\chi_U(x, \gamma_{UV}(x))^{-1}\chi_V(x, \gamma_{UV}(x))\chi_V(x, \beta)\chi_U(x, \gamma_{UV}(x))^{-1} \\ &= \chi_V(x, \gamma_{UV}(x))\chi_V(x, \alpha\beta)\chi_U(x, \gamma_{UV}(x))^{-1} \\ &= \chi_V(x, \gamma_{UV}(x))\chi_V(x, \alpha)\chi_V(x, \beta)\chi_U(x, \gamma_{UV}(x))^{-1}, \end{aligned}$$

we get

$$\chi_U(x, \gamma_{UV}(x))^{-1}\chi_V(x, \gamma_{UV}(x)) = e,$$

where e is the identity of $\boldsymbol{H}$, for arbitrary V $(U \cap V \neq \phi)$. Hence we can set

$$\chi_V(x, \alpha) = P_{UV}(x)\chi_U(x, \alpha)P_{UV}(x)^{-1}, \tag{5}$$

where $P_{UV}(x)$ is a continuous map from $U \cap V$ into $\boldsymbol{H}$.

Then since

$$\begin{aligned} g_{UV}(x) &= h_U(x, \alpha)h_V(x, \gamma_{UV}(x)\alpha)^{-1} \\ &= f_U(x)h(x, \alpha)(f_V(x)h(x, \gamma_{UV}(x)\alpha))^{-1} \\ &= f_U(x)h(x, \alpha)h(x, \gamma_{UV}(x)\alpha)^{-1}f_V(x)^{-1} \\ &= f_U(x)h(x, \alpha)(\chi(x, \gamma_{UV}(x))h(x, \alpha))^{-1}f_V(x)^{-1} \\ &= f_U(x)\chi(x, \gamma_{UV}(x))^{-1}f_V(x)^{-1}, \end{aligned}$$

we obtain the lemma, because χ is defined on $U \cap V \neq \phi$.

Note. If X_ξ is given by

$$X_\xi = \bigcup_U U\times G/\sim, \quad (x, \alpha)\sim(x, \alpha\gamma_{UV}(x)),$$

then setting

$$g_{UV}(x) = h_U(x, \alpha)^{-1}h_V(x, \alpha\gamma_{UV}(x)),$$
$$h_U(x, \alpha) = h(x, \alpha)f_U(x),$$
$$h(x, \alpha\beta) = h(x, \alpha)\chi(x, \beta),$$

where $h(x, \alpha)$ is defined similarly as above, we obtain

$$(1)' \qquad g_{UV}(x) = f_U(x)^{-1}\chi(x, \gamma_{UV}(x))f_V(x).$$

§ 2. Fibering of path spaces.

We denote by $E(X)$ the path space over X with base point*. The loop space over X is denoted by $\Omega(X)$. As in [3], we define an equivalence relation $\sim$ for the paths α, β, $\cdots$, of $E(X)$ by

$$\alpha(t) \sim \beta(t) \textit{ if and only if } \alpha(t) = \beta(h(t)) \textit{ or}$$
$$\alpha=\alpha_1\alpha_2 \textit{ and } \beta = \alpha_1\alpha_3\alpha_3^{-1}\alpha_2,$$

where h is an orientation preserving homeomorphism of $I = [0, 1]$, $\alpha_1\alpha_2$ means the product of paths and α^{-1} is the inverse path of α.

We denote the class of α by this relation $\sim$ by $[\alpha]$ and the quotient spaces of $E(X)$ and $\Omega(X)$ by this relation are denoted by $[E(X)]$ and $[\Omega(X)]$.

Lemma 2. $[\alpha] \in [E(X)]$ *is uniquely written in the form*

$$(6) \qquad [\alpha] = [\alpha_1]\cdots[\alpha_s], \quad \alpha_1, \cdots, \alpha_s \in \Omega(X) \textit{ if } \alpha \in \Omega(X),$$
$$\alpha_1, \cdots, \alpha_{s-1} \in \Omega(X) \textit{ if } \alpha \notin \Omega(X),$$

where each α_i satisfies

(i) *$\alpha_i^{-1}(*)$ is either I or $\{0\}\cup\{1\}$ if $\alpha_i\in\Omega(X)$ and $\alpha_s^{-1}(*) = \{0\}$ if $\alpha_s \notin \Omega(X)$,*

(ii) *There are no a and ε $(0 < a < 1,\ 0 < \varepsilon < 1)$, such that $\alpha_i\ (a+t) = = a_i(a-t)$ for $t\in [0, \varepsilon]$, unless $a_i = e$, the unit path.*

Proof. Since $\alpha^{-1}(*)$ is compact, we set

$$\alpha^{-1}(*) = I_1\cup\cdots\cup I_s,$$

where each I_k is either a point or a closed interval. We set

$$I_k = [a_{2k-1}], \ a_{2k}, \ a_{2j} < a_{2j+1}, \ a_{2m-1} \leqq a_{2m}, \ a_1 = 0.$$

If $a_2\neq a_1$, then we set $\alpha_1 = e$, the unit path. If $a_1 = a_2$, then we set

$$\alpha_1'(t) = \alpha(a_3 t).$$

If α_1' satisfies (ii), then we take α_1' as α_1. If α_1' does not satisfy (ii), then we set

$$I = J_1 \cup J_2,\ \ J_1 \cup J_2 = \phi,\ \ J_2 = \bigcup_i [a_i - \varepsilon_i,\ \ a_i + \varepsilon_i],$$

where $a_i + \varepsilon_i < v_{i+1} - \varepsilon_{i+1}$ and $\alpha_1'(a_i - t) = \alpha_1'(a_i + t)$, $t \in [0,\ \varepsilon_i]$. Then we set the lower bound and upper bound of $\{a_i\}$ by a and a and set

$$\begin{aligned}
&\alpha_1(t) = \alpha_1'(t),\ \ 0 \leqq t \leqq a,\ \ if\ a \notin \{a_i\},\\
&\alpha_1(t) = \alpha_1'(((a_1 - \varepsilon_1)/a_1)t),\ \ 0 \leqq t \leqq a_1,\ \ if a = a_1,\\
&\alpha_1(t) = \alpha_1'(a_{i-1} + \varepsilon_{i-1} + ((a_i - a_{i-1} + \varepsilon_{i-1} - \varepsilon_i)/(a_i - a_{i-1}))(t - a_{i-1})),\\
&\qquad a_{i-1} \leqq t \leqq a_i,\\
&\alpha_1(t) = \alpha_1'(t),\ \ a \leqq t \leqq 1,\ \ if\ a \notin \{a_i\},\\
&\alpha_1(t) = \alpha_1'(a_m + \varepsilon_m + ((1 - a_m - \varepsilon_m)/(1 - a_m))(t - a_m)),\\
&\qquad a_m \leqq t \leqq 1,\ \ if\ a = a_m.
\end{aligned}$$

Repeating this method, we obtain the decomposition (6). The uniqueness of the decomposition (6) follows from the definitions of α_i $(1 \leqq i \leqq s)$.

If X is a metric space, then denoting the distance of $x,\ y \in X$ by *dis.* $(x,\ y)$, we can define the length of a path by

$$\lim_{|a_i - a_{i-1}| \to 0} \sum_{i=1}^{m} dis.\ (\alpha(a_i),\ \alpha(a_{i-1})),$$
$$0 = a_0 < a_1 < \cdots < a_{m-1} < a_m = 1,$$

if the limit exists. Then if α_1 and α_2 are the paths of X with finite length, the product $\alpha_1\ \alpha_2$ also has finite length. Hence denoting the subspaces of $E(X)$ and $\Omega(X)$ consisted by the paths of finite length by $E_f(X)$ and $\Omega_f(X)$ and their quotient spaces defined similarly as $[E(X)]$ and $[\Omega(X)]$ are by $[E_f(X)]$ and $[\Omega_f(X),]$ $[\Omega_f(X)]$ is a group and operates on $[E_f(X)]$ (cf. [4]).

Lemma 3. *$[E_f(X)]$ is a contractible space.*

Proof. By lemma 2, if $[\alpha] \in [E_f(X)]$, then we can set

$$[\alpha] = [\alpha_1] \cdots [\alpha_s],$$

and by the proof of lemma 2, each α_i has finite length. Hence we can define the canonical parameter of α_i by its arclength, that is

$$|\alpha_{i,t}| = t|\alpha_i|,$$

where $|\alpha|$ means the length of α and $\alpha_{i,t}$ is given by $\alpha_{i,t}(u) = \alpha_i(tu)$, $0 \leqq t \leqq 1$.

We denote this path by $\hat{\alpha}_i$. Then we define the canonical representation $\hat{\alpha}$ of α by

$$\hat{\alpha}(u) = \hat{\alpha}_i(su - i + 1), \quad \frac{i-1}{s} \leqq u \leqq \frac{i}{s}.$$

Then for α, the path $\hat{\alpha}_t$ given by $\hat{\alpha}_t u) = \hat{\alpha}(tu)$ is uniquely determined for $0 \leqq t \leqq 1$. Hence the map

$$[E_f(X)] \times \boldsymbol{I} \ni ([\alpha], \ t) \to [\hat{\alpha}_t] \in [E_f(X)]$$

gives the contraction of $[E_f(X)]$.

Lemma 4. *If X is an arcwise connected metric space and satisfies (*), for each $x \in X$, there exists a neighborhood $U(x)$ of x such that for each $y \in U(x)$, there is a canonicalpath $\gamma_{x,y}$, $\gamma_{x,y}(0) = x$, $\gamma_{x,x}(1) = y$, and has finite length and as the map from $U(x)$ into the path space, $\gamma_{x,y}$ is continuous in y, then $[E_f(X)]$ is the tatal space of a principal bundle over X with structure group $[\Omega_f(X)]$.*

Proof. For each $x \in X$, we fix a path γ_x with finite length such that

$$\gamma_x(0) = *, \quad \gamma_x(1) = x.$$

Such path exists by the arcwise connectedness of X and (*).

We denote the projection from $[E_f(X)]$ onto X by π. Then to define the continuous mappings $\gamma_x^* : [\Omega_f(X)] \to \pi^{-1}(x)$ and $\gamma_{x*} : \pi^{-1}(x) \to [\Omega_f(X)]$ by

$$(7)' \qquad \gamma_x^*([\alpha]) = [\alpha\gamma_x], \quad [\alpha] \in [\Omega_f(X)],$$

$$(7)'' \qquad \gamma_x^*([\alpha]) = [\gamma_x^{-1}\alpha,] \quad [\alpha] \in \pi^{-1}(x),$$

we get $\gamma_x^*(\gamma_{x*}([\gamma_x\alpha\gamma_x^{-1}])) = [\alpha]$ and $\gamma_{x*}(\gamma_x^*([\alpha])) = [\alpha]$. Hence $\pi^{-1}(x)$ is homeomorphic to $[\Omega_f(X)]$ for each x.

If $U(x)$ is a neighborhood of x which satisfies the assumption (*), then to define $\gamma_{Ux}^* : \pi^{-1}(U(x)) \to U(x) \times [\Omega_f(X)]$ by

$$(7) \qquad \gamma_{Ux}^*([\alpha]) = (\pi(\alpha), \ \gamma_{x*}(\alpha\gamma_{x,\pi\alpha}^{-1})),$$

γ_{Ux}^* is a homeomorphism from $\pi^{-1}(U(x))$ onto $U(x) \times [\Omega_f(X])$.

Since $\{U(x), \ x \in X\}$ forms an open covering of X, we take a subcovering $\{U\}$ of $\{U(x), \ x \in X\}$ and denote $\gamma_{U,y}$, γ_U, $\bar{\gamma}_U^*$, γ_{U*} and γ_U^* instead of $\gamma_{x,y}$, γ_x, γ_{Ux}^*, γ_{x*} and γ_x^* if $U = U(x)$. Then on $U \cap V$, the homeomorphism $\bar{\gamma}_V^* \bar{\gamma}_U^{*-1} : (U \cap V) \times [\Omega_f(X)] \to (U \cap V)[\times \Omega_f(X)]$ is given by

$$\bar{\gamma}_V^* \bar{\gamma}_U^{*-1}(\pi([\alpha]), \ \gamma_{U*}([\alpha\gamma_{U,\pi(\alpha)}^{-1}]))$$

$$= (\pi([\alpha]),\ \gamma_{V*}([\alpha\gamma_{V,\pi(\alpha)}{}^{-1}])).$$

Hence we get

$$\bar{\gamma}_V{}^*\bar{\gamma}_U{}^{*-1}(y,\ \alpha) = (y,\ \alpha\gamma_U\gamma_{U,y}\gamma_{V,y}{}^{-1}\gamma_V{}^{-1}). \tag{8}$$

Then since $\gamma_U\gamma_{U,y}\gamma_{V,y}{}^{-1}\gamma_V{}^{-1}$ is a loop, γ_U and γ_V are fixed on $U\cap V$ and $\gamma_{U,y}$ and $\gamma_{U,y}$ both depends continuously on y, we have the lemma.

Note. If there is a neighborhood $U(\Delta(X))$ of $\Delta(X)$, the diagonal of $X\times X$, in $X\times X$ such that for any $(x,\ y)\in U(\Delta(X))$, there corresponds a unique path $t_{x,y}$ with finite length which starts from x ends at y and depends continuously on $(x,\ y)$, then setting

$$s_U(x,\ y) = [\gamma_U\gamma_{U,x}t_{x,y}\gamma_{U,y}{}^{-1}\gamma_U{}^{-1}],$$

$\{s_U(x,\ y)\}$ is a topological connection of the bundle $([E_f(X)],\ [\Omega_f(X)],\ \pi,\ X)$ (cf. [1]). On the other hand, if the bundle $([E_f(X)],\ [\Omega_f(X)],\ \pi,\ X)$ has a topological connection $\{s_U(x,\ y)\}$, then denoting the canonical representation of $s_U\{(x,\ y)\in\{s_U(x,\ y)\}$ given by lemma 3 also by $s_U(x,\ y)$ for each U, the path $t_{x,y}$ given by

$$t_{x,y} = \gamma_{U,x}{}^{-1}\gamma_U{}^{-1}s_U(x,\ y)\gamma_U\gamma_{U,y},$$

does not depend on U because we have

$$\begin{aligned}
&\gamma_{V,x}{}^{-1}\gamma_V{}^{-1}s_V(x,\ y)\gamma_V\gamma_{V,y}\\
&= \gamma_{V,x}{}^{-1}\gamma_V{}^{-1}(\gamma_{V,x}\gamma_V\gamma_{U,x}{}^{-1}\gamma_U{}^{-1}s_U(x,\ y)\gamma_U\gamma_{U,y}\gamma_{V,y}{}^{-1}\gamma_V{}^{-1})\gamma_V\gamma_{V,y}\\
&= \gamma_{U,x}{}^{-1}\gamma_U{}^{-1}s_U(x,\ y)\gamma_U\gamma_{U,y}
\end{aligned}$$

if each $\gamma_U,\ \gamma_{U,x},\ \cdots,$ is given by ts canonical form. Since $t_{x,y}$ is continuous in $(x,\ y)$ and starts from x ends at y, the existence of above $t_{x,y}$ (and $U(\Delta(X))$), is the necessary and sufficient condition for the existence of the topological connection of the bundle $([E_f(X)],\ [\Omega_f(X)],\ \pi,\ X)$.

§ 3 Proof of theorem 1.

Theorem 1. *If X satisfies the assumptions of lemma 4, then there is a 1 to 1 correspondence between the set of the equivalence classes of G -bundles over X and the set of continuous maps from X into Hom. $([\Omega_f(X)],\ G)$, the space of equivalence classes (under inner automorphisms) of continuous homomorphisms from $[\Omega_f(X)]$ into G (the toplogies of $[\Omega_f(X)]$ and Hom. $[(\Omega_f(X)],\ G)$ are both induced toplogies from the compact open toplogies of $\Omega_f(X)$ and the space of continuous homomorphisms from $[\Omega_f(X)]$ into G), if the projection from G onto the space of conjugate classes* *of G has local sections.*

Proof. If ξ is a G -bundle over X, then $\pi^*(\xi)$ is trivial by lemma 3. Hence $\pi^*(\xi)$ is obtained by some continuous map from X into the space of continuous homomorphisms from $[\Omega_f(X)]$into G with the equivalence relation under the inner automorphisms of G by lemma 1 and lemma 4.

On the other hand, if χ is a continuous map from X into *Hom.* $([\Omega_f(X)], G)$, then by assumption, there exists an open covering $\{U_i, \ i \in I\}$ of X such that for any $\cup_{U_i \cap U_j \neq} U_j$, there exists a continuous map χ_i from $\cap_{U_i \cap U_j \neq ,} U_j$ into the space of continuous homomorphisms from $[\Omega_f(X)]$ into G (with compact open toplogy), such that whose class at x, $x \in \cap_{U_i \cap U_j \neq .} U_j$ is equal to the value of χ at x.

We denote $\chi_{i_1 i_2}'$ either of i_1 or i_2 for any (i_1, i_2). Then by assumption, we get

$$\chi_{i_2 i_3}' = P_{i_2 i_3}^{i_1 i_2} \chi_{i_1 i_2}' (P_{i_2 i_3}^{i_1 i_2})^{-1},$$

if $U_{i_1} \cap U_{i_2} \cap U_{i_3} \neq \phi$. Here P_{jk}^{ij} is a continuous map from $U_i \cup U_j \cup U_k$ into G. Then since we may consider $\{P_{jk}^{ij}\}$ only on $\cup_{U_j \cup U_k \neq ,} U_k$, we can take P_{jk}^{ij} to satisfy

$$P_{i_2 i_3}^{i_1 i_2} P_{i_3 i_1}^{i_2 i_3} P_{i_1 i_2}^{i_3 i_1} = e, \text{ the identity of } G,$$

on $U_{i_1} \cap U_{i_2} \cap U_{i_3}$. Hence denoting $\mathscr{B}$ the sheaf with base space I (with discreet topology), stalk at i is the space of continuous maps from U_i into G, $\overline{P}_{jk}^{ij}$ is a 1 -cocycle of $\mathscr{B}$ with covering system $\{(i, j)\}$, where $\overline{P}_{jk}^{ij}$ is given by

$$\overline{P}_{jk}^{ij}(j) = \overline{P}_{jk}^{ij} | U_j.$$

Then since I is discreet, we get

$$\overline{P}_{i_2 i_3}^{i_1 i_2}(i_2) = \overline{f}_{i_2 i_3}(i_2)(\overline{f}_{i_1 i_2}(i_2))^{-1}. \tag{9}$$

Then we set

$$f_{ij}(x) = (f_{ij}(i))(x), \ \ x \in U_i,$$

and define

$$\chi_{ij} = f_{ij}^{-1} \chi_{ij}' f_{ij},$$

on $U_i \cap U_j$.

On the other hand, since we may assume the bundle $([E_f(X)], [\Omega_f(X)], \pi, X)$ is trivial on each U_i, $i \in I$, we denote its transition functions with covering system U_i, $i \in I$ by $\{\gamma_{ij}(x)\}$. Then setting

$$g_{ij}(x) = \chi_{ij}(x)(\gamma_{ij}(x)),$$

$\{g_{ij}(x)\}$ defines a G -bundle by (9).

Moreover, if $\chi(\xi)$ is the map from X into *Hom.* $([\Omega_f(X)], G)$ obtained from ξ, a G -bundle over X, then we get

$$\{(\chi)_{UV}(x)(\gamma_{UV}(x))\} \sim \xi, \tag{10}$$

$$\chi(\chi_{UV}(x)(\gamma_{UV}(x))) = \chi(x), \tag{10$'$}$$

we have the theorem.

Corollary ([10]). *If X satisfies the assumptions of lemma 4, G is a tatally disconnected group, then there is a 1-1 correspondence between the set of equivalence calsses of G -bundles over X and Hom. $(\pi_1(X), G)$, the set of equivalence classes (under inner automorphisms of G) of homomorphisms from $\pi_1(X)$ into G.*

Proof. If G is tatally disconnected, then we have

$$Hom.([\Omega_f(X)], G) = Hom.(\pi_1(X), G) \tag{11}$$

and since $Hom.(\pi_1(X), G)$ is also tatally disconnected, any continuous map from X into $Hom.(\pi_1(X), G)$ is always a constant map. Hence we have the corollary.

Note. In theorem 1, we assume that G satisfies

(*), *the projection from G onto the space of conjugate classes of G has local sections.*

But (*) is used only to show

($\substack{*\\*}$), *if χ is a continuous map from X into Hom. $([\Omega_f(X)], G)$, then there exists an open covering $\{U\}$ of X and a set of continuous maps (from U into the space of continuous homomorphisms from $[\Omega_f(X)]$ into G (with compact open toplogy)) $\{\chi_U\}$ such that*

the class of $\chi_U(x)$ is the value of χ at x, $x \in U$,

for any U, $U \in \{U\}$.

Hence for arbitrary topological group G, we obtain

Theorem 1$'$. *If X satisfies the assumptions of lemma 4, then there is a 1 to 1 correspondence between the set of equivalence classes of G -bundles over X and the set of continuous maps from X into Hom. $([\Omega(X)], G)$, such that there exists an open covering $\{U\}$ of X and a set of continuous maps (from U into the space of continuous homomorphisms from $[\Omega_f(X)]$ into G (with compact open topology)) $\{\chi_U\}$ such that*

the class of $\chi_U(x)$ is the value of χ at x, $x \in U$, *for any* U, $U\in U$.

§ 4. The differentiable case.

If X is a smooth manifold, ξ and η are smooth G and H -bundles over X, the lemma 1 is also true replacing continuous map χ by smooth map χ if $\pi^*(\eta)$ is differentiably trivial. But since $[E_f(X)]$ is not C^∞-smooth although it is a smooth Banach manifold, we can not know whether $\pi^*(\xi)$ is differentiably trivial or not on $[E_f(X)]$. Therefore theorem 1 is not extended for smooth bundles.

On the other hand, if X_ξ is the tatal space of a differentiable G -bundle over X, then the cotangent bundle $T^*(X_\xi)$ of X_ξ is written as

$$(12)' \qquad T^*(X_\xi)=\pi^*(T^*(X))+T^*_F,$$

where π is the projection from X_ξ onto X, $T^*(X)$ is the cotangent bundle of X. We denote by p_1 and p_2 the projections from $T^*(X_\xi)$ onto the first and the second components of the right hand side of (12)'. Then for a smooth function f on X_ξ, we can write

$$df=p_1(df)+p_2(df),$$

where d is the exterior differential of X_ξ. We set

$$\pi^*(d)f=p_1(df), \quad d_F f=p_2(df).$$

Then $\pi^*(d)$ and d_F are defined for arbitrary form φ on X_ξ, because φ is written as $\sum_I f_I dx_I$, $I=(i_1, \cdots, i_n)$, locally, and we may set

$$\pi^*(d)(\sum_I f_I dx_I)=\sum_I \pi^*(d)f_I\wedge dx_I,$$

$$d_F(\sum_I f_I dx_I)=\sum_I d_F f_I\wedge dx_I.$$

Hence the exterior differential d of X_ξ is written as

$$(12) \qquad d=\pi^*(d)+d_F.$$

Although this decomposition is derived from (12)' and in (12)', T_F^* is not determined uniquely, d_F is determined uniquely because we have

$$d_F=d-\pi^*(d),$$

and d and $\pi^*(d)$ are both determined uniquely.

If $\eta=\{g_{UV}(x)\}$ is a smooth $\boldsymbol{H}$ -bundle over X such that $\pi^{*}(\eta)$ is (differentiably) trivial, then we can write on X

$$g_{\pi(U)\pi(V)}(x)=h_{U}(x,\ \alpha)h_{V}(x,\ \gamma_{\pi(U)\pi(V)}(x)\alpha)^{-1},$$

where α is an element of $\boldsymbol{G}$, $\{\gamma_{\pi(U)\pi(V)}(x)\}$ is the transition functions of ξ and $h_{U}(x,\ \alpha)$ is a smooth map. As in the proof of lemma 1, we denote $g_{UV}(x)$, $\cdots$, instead of $g_{\pi(U)\pi(V)}(x)$, $\cdots$,. Then since $d_{F}g_{UV}(x)$ is equal to 0, we have

$$\begin{aligned}&d_{F}(h_{U}(x,\ \alpha))h_{V}(x,\ \gamma_{UV}(x)\alpha)^{-1}\\&=h_{U}(x,\ \alpha)h_{V}(x,\ \gamma_{UV}(x)\alpha)^{-1}d_{F}(h_{V}(x,\ \gamma_{UV}(x)v))h_{V}(x,\ \gamma_{UV}(x)\alpha)^{-1}.\end{aligned}$$

Hence we can define a smooth $\mathfrak{L}(\boldsymbol{H})$ -valued functon $\theta(x,\ \alpha)$ on X_{ξ} ($\mathfrak{L}(\boldsymbol{H})$ is the Lie algebra of $\boldsymbol{H}$) by

$$\theta(x,\ \alpha)\,|\,U=h_{U}(x,\ \alpha)^{-1}d_{F}(h_{U}(x,\ \alpha)).$$

Then if X_{ξ} is simply connected and $T_{F}(X_{\xi})$, the complement bundle of $\pi^{*}(T(X))$ in $T(X_{\xi})$ ($T(X)$ and $T(X_{\xi})$ are the tangent bundles of X and X_{ξ}), has a non-trivial cross -section, there exists an $\boldsymbol{H}$ -valued function $h(x,\ \alpha)$ on X_{ξ} such that

$$\theta(x,\ \alpha)=h(x,\ \alpha)^{-1}d_{F}(h(x,\ \alpha)), \tag{13}$$

because we get

$$d_{F}\theta=-\theta\wedge\theta,$$

on X_{ξ} (cf. [2]).

Moreover, if $h'(x,\ \alpha)$ also satisfies $\theta(x,\ \alpha)=h'(x,\ \alpha)^{-1}d_{F}(h'(x,\ \alpha))$, then since

$$d_{F}(h^{-1}h')=-h^{-1}d_{F}(h)h^{-1}h'+h^{-1}d_{F}(h')=0,$$

we have

$$h'(x,\ \alpha)=f(x)h(x,\ \alpha),\ f(x) \textit{ is a smooth map from } X \textit{ into } \boldsymbol{H}. \tag{14}$$

Because a smooth function on X_{ξ} depends only on x, $x\in X$, if and only if $d_{F}f$ is equal to 0.

On the other hand, setting

$$h(x,\ \alpha\beta)=\chi(x,\ \alpha)h(x,\ \beta),$$

we have

$$\chi(x, \alpha\beta) = \chi(x, \alpha)\chi(x, \beta),$$

and

$$g_{UV}(x) = \chi(x, \gamma_{UV}(x))^{-1}.$$

Moreover, we know that

(i). *χ is a smooth map from X into the space of smooth homomorphisms from G into H (with C^1 -topology).*

(ii). *If the coordinates of η is changed, then χ is changed as*

$$\chi'(x, \alpha) = P(x)\chi(x, \alpha)P(x)^{-1}, \quad \alpha \in G,$$

where $P(x)$ is a smooth map from X into H.

Hence we have the following version of lemma 1 for the smooth bundles.

Lemma 1'. *Let $\xi = \{\gamma_{UV}(x)\}$ and $\eta = \{g_{UV}(x)\}$ are smooth principal G and H -bundles over X and the tatal space X_ξ of ξ satisfies*

(i). *X_ξ is a simply connected space.*

(ii). *$T_F(X_\xi)$, the complement bundle of $\pi^*(T(X))$ in $T(X_\xi)$ (π is the projection from X_ξ onto X and $T(X)$ and $T(X_\xi)$ are the tangent bundles of X and X_ξ) has a non -trivial cross -section.*

Then if $\pi^*(\eta)$ is trivial over X_ξ, there exists a smooth map χ from X into the space of smooth homomorphisms from G into H (with C^1 -topology) such that

$$g_{UV}(x) = \chi(x)(\gamma_{UV}(x)).$$

Moreover, if χ and χ' are the maps from X into the space of smooth homomorphisms from G into H obtained from η, then there exists a smooth map P from X into H such that

$$\chi'(x, \alpha) = P(x)\chi(x, \alpha)P(x)^{-1}, \quad \alpha \in G.$$

To use lmma 1' for smooth bundles over a smooth manifold X, we use

$$\begin{aligned} E_{2,k,0}(X) = \{\alpha \mid \alpha : I \to X \ &\text{belongs in } k\text{-th Sobolev space and there} \\ &\text{exists } \varepsilon > 0 \text{ such that } \alpha(a) = *, \ 0 \leqq a \leqq \varepsilon, \\ &\alpha(1-b) = \alpha(1), \ 0 \leqq b \leqq \varepsilon\}, \\ \Omega_{2,k,0}(X) = \{\alpha \mid \alpha \in E_{2,k,0}(X), \ &\alpha(0) = \alpha(1) = *\}, \end{aligned}$$

where $k > n/2$, $n = dim.\ X$ (cf. 3), instead of $E_f(X)$ and $\Omega_f(X)$. We note that by Sobolev's lemma, $E_{2,k,0}(X) \subset E_f(X)$ and $\Omega_{2,k,0}(X) \subset \Omega_f(X)$ if the metric of X is a Riemannian metric of X.

In $E_{2,k,0}(X)$, we define an equivalence relation $\sim$ for the paths $\alpha, \cdots$, by

$$\alpha(t) \sim \beta(t), \textit{ if and only if } \alpha(t) = \beta(h(t)) \textit{ or}$$
$$\alpha = \alpha_1\alpha_2 \textit{ and } \beta = \alpha_1\alpha_3\alpha_3^{-1}\alpha_2,$$

where h is an orientation preserving C^k-diffeomorphism of $I = [0, 1]$, α_1, α_2 and α_3 are the elements of $E_{2,k,0}(X)$.

The class of α by this relation is denoted by $\{\alpha\}$ and the quotient spaces of $E_{2,k,0}(X)$ and $\Omega_{2,k,0}(X)$ by this relation are denoted by $\{E_{2,k,0}(X)\}$ and $\Omega_{2,k,0}(X)$.

For $\alpha \in E_{2,k,0}(X)$, we set

$$\overline{int.\ \alpha^{-1}(*)} = I_0 \cup \cdots \cup I_s,$$
$$I_k = [a_{2k-1}, a_{2k}],\ a_{2p} < a_{2p+1},\ a_{2q-1} < a_{2q},\ a_{-1} = 0,$$

Then we define $\alpha_1' \in \Omega_{2,k,0}(X)$(or $\in E_{2,k,0}(X)$ if $s = 1$ and $\alpha \notin \Omega_{2,k,0}(X)$) by

$$\alpha_1'(t) = \alpha(((a_1 + a_2)/2)t).$$

If α_1' satisfies

(ii)$'$. *There are no* a *and* ε, ε' $(0 < a < 1,\ 0 < \varepsilon < 1,\ 0 < \varepsilon' < 1)$ *such that*
$\alpha_1'(a + t) = \alpha_1'(a - t)'$ *for* $t \in [0, \varepsilon + \varepsilon']$,
for some $t \in [0, \varepsilon]$, $\alpha_1'(a + t) \neq \alpha_1(a)$, *and*
$\alpha_1'(a + t) = \alpha_1'(a + \varepsilon + \varepsilon')$ *for* $t \in [\varepsilon, \varepsilon + \varepsilon']$,

then we set $\alpha_1' = \alpha_1$. If α_1' does not satisfy (ii)$'$, then setting

$$I = J_1 \cup J_2,\ J_1 \cap J_2 = \phi,$$
$$J_2 = \bigcup_i [a_i - \varepsilon_i - \varepsilon_i',\ a_i + \varepsilon_i + \varepsilon_i'],$$
$$a_i + \varepsilon_i + \varepsilon_i' < a_{i+1} - \varepsilon_{i+1} - \varepsilon_{i+1}',$$
$$\alpha_1'(a_i - t) = \alpha_1'(a_i + t),\ \textit{if } t \in [0,\ \varepsilon_i + \varepsilon_i'],$$
$$\alpha_1'(a_i - t) \neq \alpha_1'(a_i),\ \textit{for some } t \in [0,\ \varepsilon_i],$$
$$\alpha_1'(a_i - t) = \alpha_1'(a_i - \varepsilon_i - \varepsilon_i'),\ \textit{if } t \in [\varepsilon_i,\ \varepsilon_i + \varepsilon_i'],$$

we denote by $\underline{a}$ the lower bound of $\{a_i\}$ and by $\bar{a}$ the upper bound of $\{a_i\}$. Then we define $\alpha_1 \in \Omega_{2,k,0}(X)$ ($\in E_{2,k,0}(X)$ if $\alpha_1' \notin \Omega_{2,k,0}(X)$ by

$$\alpha_1(t) = \alpha_1'(t),\ 0 \leqq t \leqq a,\ \textit{if } a \notin \{a_i\},$$
$$\alpha_1(t) = \alpha_1'(((a_1 - \varepsilon_1 - \varepsilon_1'/2)/a_1)t),\ 0 \leqq t \leqq a_1,\ \textit{if } a_1 = a,$$
$$\alpha_1(t) = \alpha_1'((a_{i-1} + \varepsilon_{i-1} + \varepsilon_{i-1}'/2) +$$
$$+ ((a_i - a_{i-1} + \varepsilon_{i-1}' - \varepsilon_i + (\varepsilon_{i-1}' - \varepsilon_i')/2))/(a_i - a_{i-1}))(t - a_{i-1})),$$
$$a_{i-1} \leqq t \leqq a_i,$$

$$\alpha_1(t) = \alpha_1'(t), \quad \bar{a} \leqq t \leqq 1, \quad \textit{if } \bar{a} \notin \{a_i\},$$
$$\alpha_1(t) = a_1'((a_m + \varepsilon_m + \varepsilon_m'/2) + ((1 - a_m - \varepsilon_m'/2)/(1 - a_m))(s - a_m)),$$
$$a_m \leqq t \leqq 1, \quad \textit{if } a_m = \bar{a}.$$

Repeating this, we obtain

Lemma 2′. *$\{\alpha\} \in \{E_{2,k,0}(X)\}$ is uniquely written as*

(6)′
$$\{\alpha\} = \{\alpha_1\}\cdots\{\alpha_s\}, \ \alpha_1, \ \cdots, \ \alpha_s \in \Omega_{2,k,0}(X) \ \textit{if} \ \alpha \in \Omega_{2,k,0}(X),$$
$$\alpha_1, \ \cdots, \ \alpha_{s-1} \in \Omega_{2,k,0}(X) \ \textit{if} \ \alpha \notin \Omega_{2,k,0}(X),$$

where each α_i satisfies (ii)′) and

(i)′
$$\alpha_i^{-1}(*) \textit{ is either } \boldsymbol{I} = [0, \ 1], \ [0, \ a] \textit{ or } [0, \ a] \cup [b, \ 1],$$
$$0 < a < b < 1.$$

Since X is smooth X allows a Riemannian metric, and by this metric, any path in $E_{2,k,0}(X)$ has finite length, we obtain by lemma 2′

Lemma 3′. *$\{E_{2,k,0}(X)\}$ is a contractible space.*

We fix a Riemannian metric on X, then for any $x \in X$, there exists a neighborhood $U(x)$ of x such that

(*). *For vny $y \in U(x)$, there exists a unique geodesic $\beta_{x,y}$ (by the given Riemannian metric) which starts from x and ends at y.*

Then since $\beta_{x,y}$ depends differentiably on y (as the map from $U(x)$ into $E_{2,k}(X)$ regarded x to be the basepoint, where $E_{2,k}(X)$ is the space of paths which belongs in k-th Sobolev space). Although $\beta_{x,y}$ does not belong in $E_{2,k,0}(X)$, if we fix a C^∞-class function $f: \boldsymbol{I} \to \boldsymbol{I}$ such that

$$f(t) = 0, \quad \textit{if } 0 \leqq t \leqq a,$$
$$f(t_1) < f(t_2), \quad \textit{if } a < t_1 < t_2 < b,$$
$$f(t) = 1, \quad \textit{if } b \leqq t \leqq 1.$$

Then $\gamma_{x,y}$ given by $\gamma_{x,y}(t) = \beta_{x,y}(f(t))$ belongs in $E_{2,k,0}(X)$, and also depends differentiably on y. Similarly, we can take the path γ_x starts from $*$ ends at x to be an element of $E_{2,k,0}(X)$. Hence we can take the loop $\gamma_U \gamma_{U,y} \gamma_{V,y}^{-1} \gamma_V^{-1}$ defined similarly as in § 2, to be an element of $\Omega_{2,k,0}(X)$ for any U, V and y, and to depend differentiably on y. Therefore we obtain

Lemma 4′. *$\{E_{2,k,0}(X)\}$ is a smooth principal bundle over X with structure group $\Omega_{2,k,0}(X)$.*

Note. We set $U(\Delta(X))$ the neighborhood of $\Delta(X)$ in $X \times X$ such that if (x, y) is in $U(\Delta(X))$, then there exists a unique geodesic $t'_{x,y}$ which joins x and y (with respect to the given Riemannian metric). Then setting

$$s_U(x, y) = \{\gamma_U \gamma_{U,x} t_{x,y} \gamma_{U,y}^{-1} \gamma_U^{-1}\},$$
$$t_{x,y}(t) = t'_{x,y}(f(t)),$$

$\{s_U(x, y)\}$ is a topological connection of the bundle $(\{E_{2,k,0}(X)\}, \{\Omega_{2,k,0}(X)\}, \pi X)$.

For the bundle $(\{E_{2,k,0}(X)\}, \{\Omega_{2,k,0}(X)\}, \pi, X)$, we know

(a). $\{E_{2,k,0}(X)\}$ *is a* C^∞*-smooth manifold* (cf. [3], [5], [9]).

(b). $\{E_{2,k,0}(X)\}$ *is a simply connected space.*

(c). *codim.* $\pi^*(T(X))$ *in* $T(\{E_{2,k,0}(X)\})$ *is* ∞.

Since $T(\{E_{2,k,0}(X)\})$ is a trivial bundle, there exists a non-trivial vector field of $\{E_{2,k,0}(X)\}$ which is not in $\pi^*(T(X))$. Hence by (c), we obtain (c)′. *The complement bundle of* $\pi^*(T(X))$ *in* $T(\{E_{2,k,0}(X)\})$ *has a non-trivial cross-section.*

By lemma 1′, lemma 3′, lemma 4′ and the above (a), (b), (c)′, we obtain

Theorem 2. *If X is a smooth manifold, then there is a 1 to 1 correspondence between the set of equivalence classes of smooth G-bundles over X and the set of equivalence classes of smooth maps from X into the space of smooth homomorphisms from* $\{\Omega_{2,k,0}(X)\}$ *into G (with* C^1*-topology) with the equivalence relation*

$$\chi(x, \alpha) \sim P(x)\chi(x, \alpha)P(x)^{-1}, \quad \alpha \in G \tag{15}$$

where $P(x)$ *is a smooth map from X into G.*

Note. By (15), as the map from X into the space of equivalence classes (under inner automorphisms of G) of smooth homomorphisms from $\{\Omega_{2,k,0}(X)\}$ into G, χ is uniquely determined.

References

[1] ASADA, A. : Connection of topological fibre bundles, Proc. Japan Acad., 42 (1966), 13–18.

[2] ASADA, A. : Connection of flat vector bundles, J. Fac. Sci. Shinshu Univ., 2 (1967), 109–116.

[3] ASADA, A. : Algebraic cohomology of loop spaces, J. Fac. Sci. Shinshu Univ., 4 (1969), 1–23.

[4] CHEN, K. T. : Integration of paths, geomteric invariants and a generalized Baker Hausdorff formula, Ann. of Math., 65 (1957), 163–178.

[5] CHEN, K. T. : Integration of paths, A faithful representation of paths by noncommutative formal power series, Trans. Amer. Math. Soc., 89 (1958), 395–407.

[6] CHEN, K. T. : Formal differential equations, Ann. of Math., 73 (1961), 110–133.

[7] CURTIS, E. B. : Some relations between homotopy and homology, Ann. of Math., 82 (1965), 386–413.

[8] HIRZEBRUCH, F. : *Topological methods in algebraic geometry*, Berlin, 1966.

[9] REE, R. : Lie elements and an algebra associated with shuffles, Ann. of Math., 68 (1958), 210–220.

[10] STEENROD, N. E. : *The topology of fibre bundles*, Princeton, 1951.