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On the Number of Integers Representable as the Sum of Two Squares

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§1. It has been proved by E. Landau [1] [A] that for $B(x)$, the number of integers not exceeding x and representable as the sum of two squares of integers, we have

$$B(x)\sim b_0\frac{x}{\sqrt{\log x}},\qquad b_0=\frac{1}{\sqrt{2}}\prod_r(1-r^{-2})^{-\frac{1}{2}},$$

as $x\to\infty$, where r runs through all the primes $\equiv 3 \pmod 4$.

Recently some authors [2] [3] have investigated certain generalizations of the theorem of Landau, and in the present paper I consider some related problems from a different point of view.

My purpose is to prove the following theorems.

Theorem 1 *If we denote by $C(x)$ the number of integers $\leqq x$ with primitive representations. i. e., which are representable as the sum of two squares of coprime natural numbers, then we have as $x\to\infty$*

$$C(x)\sim\frac{3}{8b_0}\cdot\frac{x}{\sqrt{\log x}},$$

where b_0 is the same constant as Landau's.

Theorem 2 *Let $P(x)$ denote the number of integers $\leqq x$ with precisely one* primitive representation. Then we have*

$$P(x)\sim\frac{3}{4}\cdot\frac{x}{\log x}.$$

Theorem 3 If *$R(x)$ denotes the number of integers $\leqq x$ representable as the sum of two squares of integers in one and only one way,* then we have an asymptotic expansoin*

$$R(x)\sim c_1\frac{x}{\log x}+c_2\frac{x}{\log^2 x}+\cdots\cdots.$$

* Two representations are regarded here as being equal when they differ only in the order of summands.

Theorem 4. *If we denote by $D(x)$ the number of natural numbers $\leqq x$ each of which is not the hypotenuse of any Pythagorean triangle, then*

$$D(x)=\frac{4b_0}{\pi}\cdot\frac{x}{\sqrt{\log x}}+O\left(\frac{x}{(\log x)^{\frac{3}{2}}}\right).$$

§ **2.** Theorem 1 is derived from the following

Lemma 1.* *A natural number n is the sum of two squares of coprime natural numbers if and only if n is divisible neither by 4 nor by a natural number $\equiv 3$ (mod 4).*

We know from Lemma 1 that $C(x)$ is the number of integers $\leqq x$ such that each of them is the product of primes $\equiv 1$ (mod 4) only, or the product of 2 and primes $\equiv 1$ (mod 4).

So, if we put

$$c_n=\begin{cases} 1 & \text{if } n=\text{product of primes}\equiv 1 \text{ (mod 4) only, or 1,} \\ 0 & \text{otherwise,}\end{cases}$$

then we have

$$C(x)=\sum_{n\leqq x} c_n+\sum_{n\leqq\frac{x}{2}} c_n-2.$$

Next we define as usual a Dirichlet series

$$f(s)=\sum_{n=1}^{\infty}\frac{c_n}{n^s},\quad s=\sigma+ti,$$

for $\sigma>1$, which converges absolutely and almost uniformly** in the half-plane. Since c_n is completely multiplicative, we have

$$f(s)=\prod_q(1-q^{-s})^{-1},$$

where q runs through all the primes $\equiv 1$ (mod 4).

But we know that

$$\zeta(s)L(s,\chi_4)=(1-2^{-s})^{-1}\prod_q(1-q^{-s})^{-2}\prod_r(1-r^{-2s})^{-1},$$

where $L(s,\chi_4)$ is the Dirichlet's L-function for the nonprincipal character (mod 4).

Hence we have for $\sigma>1$,

$$f(s)^2=(1-2^{-s})\prod_r(1-r^{-2s})\ \zeta(s)L(s,\ \chi_4).$$

Now it is obvious that we can go farther in quite the same way as that of Landau using the known properties of ζ and L, and we omit the details except

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* See [D] *p.* 362.

** This terminology is that of Saks-Zygmund "*Analytic Functions*".

the final result

$$\sum_{n\leq x} c_n \sim \frac{1}{4b_0}\cdot\frac{x}{\sqrt{\log x}}.$$

This gives at once Theorem 1.

§ **3.** Theorem 2 is an immediate consequence of the following

Lemma 2. * *A natural number n admits precisely one primitive representation if and only if n is one of the numbers*

$$2,\ p^l,\ 2p^l,$$

where p is a prime $\equiv 1 \pmod 4$, *and l a natural number.*

Let $\tilde{P}(x)$ denote the number of natural numbers $\leq x$ of the form p^l where p is any prime $\equiv 1 \pmod 4$ and l any natural number. Then we clearly have

$$\tilde{P}(x) = \sum_{l\leq \log_5 x} \pi(x^{1/l};\ 4,\ 1)$$

$$= \pi(x;\ 4,\ 1) + \sum_{2\leq l\leq \log_5 x} \pi(x^{1/l};\ 4,\ 1)$$

$$= \frac{1}{2}\cdot\frac{x}{\log x} + o\left(\frac{x}{\log x}\right),$$

from the prime number theorem in an arithmetical progression.

Therefore

$$P(x) = 1 + \tilde{P}(x) + \tilde{P}\left(\frac{x}{2}\right) + O(\sqrt{x})$$

$$\sim \frac{3}{4}\cdot\frac{x}{\log x}.$$

§ **4.** In this section we shall prove at a time Theorem 3 and Theorem 4. In fact Theorem 4 is a by-product of Theorem 3. We start from

Lemma 3. *A natural number n admits precisely one representation as the sum of two squares of natural numbers if and only if n is one of the numbers*

$$d^2p,\ 2d^2p,\ 2d^2,\ d^2p^2,$$

where d is any natural number which has no prime divisor $\equiv 1 \pmod 4$ *and p is any prime*$\equiv 1 \pmod 4$.

Proof : Let $n = a^2 + b^2$, $(a,\ b) = d$. Then $\frac{n}{d^2} = a'^2 + b'^2$, $(a',\ b') = 1$. Thus from Lemma 1, we have

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* While the case of odd n is only treated in [D] p. 212, the word 'odd' is dropped there. See also [4].

$$n = d^2 \prod_i p_i, \quad 2d^2 \prod_i p_i, \quad 2d^2,$$

where each p is a prime$\equiv 1 \pmod 4$. On the other hand it is known* that the number $\tau(n)$ of the representations of n as the sum of two squares of integers is equal to $4\prod_i(1+\alpha_i)$ where the α's are the exponents on the primes $\equiv 1 \pmod 4$ in the standard factorization of n. From this fact follows our desired result at once.

Proof of Theorem 3 : Clearly it is sufficient to care for $n=d^2p$ only. Let $\tilde{R}(x)$ denote the number of these n's not exceeding x, and $D(x)$ the number of all such d's $\leqq x$. Then we obtain

$$\tilde{R}(x) = \sum_{\substack{p \leqq x \\ p \equiv 1(4)}} D\left(\sqrt{\frac{x}{p}}\right).$$

If we introduce a Dirichlet series

$$g(s) = \sum_{n=1}^{\infty} \frac{d_n}{n^s},$$

for $\sigma > 1$ with

$$d_n = \begin{cases} 1 \cdots\cdots\cdots\cdots n=d, \\ 0 \cdots\cdots\cdots\cdots \text{otherwise,} \end{cases}$$

then it converges absolutely and almost uniformly in the half plane.

From the definition of d_n we derive

$$g(s) = (1-2^{-s})^{-1} \prod_r (1-r^{-s})^{-1},$$

and hence

$$g(s)^2 = (1-2^{-s})^{-1} \prod_r (1-r^{-2s})^{-1} \zeta(s) L(s, \chi_4)^{-1}.$$

We see from the known properties of ζ and L that $g(s)^2$ has a simple pole at $s=1$, with residue $c_0{}^2 = (\frac{4}{\sqrt{\pi}} b_0)^2$, and is otherwise holomorphic for $\sigma > 1$.

Furthermore there exists a positive constant c such that in the region Γ, i. e.,

$$\sigma \geqq 1 - \frac{c}{\log^7 |t|} > \frac{1}{2} \qquad |t| \geqq 3,$$

$$\sigma \geqq 1 - \frac{c}{\log^7 3} = \sigma_0 > \frac{1}{2} \qquad |t| \leqq 3,$$

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* See e. g. [D] p. 432.

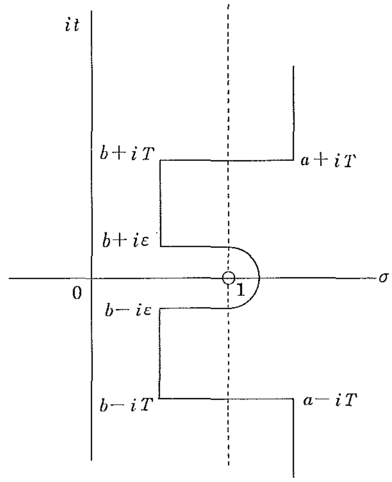
$g(s)^2$ does not vanish and in the region $\widetilde{\Gamma}$ formed from Γ by cutting along the real axis from σ_0 to 1, we have

$$g(s)^2 = O(\log^6|t|), \tag{1}$$

for $|t|\geqq 3$ and s in $\widetilde{\Gamma}$.

On the other hand, according to Perron's formula, we have for non-integral x and $a>1$,

$$\sum_{n\leq x} d_n = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{a-iT}^{a+iT} g(s)\frac{x^s}{s}ds.$$



Let $T>0$ be fixed and consider the contour described in the figure for

$$a = 1 + \frac{c}{\log^7 T},$$

$$b = 1 - \frac{c}{\log^7 T}.$$

Then by Cauchy's theorem

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} g(s)\frac{x^s}{s}ds$$

$$=\frac{1}{2\pi i}\left(\int_{a-i\infty}^{a-iT} + \int_{a-iT}^{b-iT} + \int_{b-iT}^{b-i\varepsilon} + \int_{cut} + \int_{b+i\varepsilon}^{b+iT} + \int_{b+iT}^{a+iT} + \int_{a+iT}^{a+i\infty}\right)$$

$$\overset{def}{=}\frac{1}{2\pi i}(\mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4 + \mathrm{I}_5 + \mathrm{I}_6 + \mathrm{I}_7).$$

We can estimate these integrals with the help of (1) in the same way as in a proof* of the prime number theorem and obtain

$$\begin{aligned} &\mathrm{I}_1,\ \mathrm{I}_7 = O\left(\frac{x^a}{T(a-1)} + \frac{x\log x}{T}\right), \\ &\mathrm{I}_2,\ \mathrm{I}_6 = O\left(x^a\frac{\log^3 T}{T}\right), \\ &\mathrm{I}_3,\ \mathrm{I}_5 = O(x^b\log^4 T). \end{aligned} \tag{2}$$

For I_4, we have from the property of $g(s)$,

$$\mathrm{I}_4 = \int_{cut} g(s)\frac{x^s}{s}ds = 2\int_{b-i\varepsilon}^{1-i\varepsilon} g(s)\frac{x^s}{s}ds + \int_{K_\varepsilon} g(s)\frac{x^s}{s}ds,$$

where K_ε is a semi-circle of radius ε drawn around the point 1.

* See e. g. [A] *p.* 65.

Since $g(s)$, in the neighbourhood of $s=1$, has the expansion

$$g(s)=\frac{c_0}{\sqrt{s-1}}+\alpha_1(s-1)^{\frac{1}{2}}+\alpha_2(s-1)^{\frac{3}{2}}+\cdots\cdots,$$

where $\sqrt{s-1}>0$ for $s>1$, we get by putting $s=1+\varepsilon e^{i\theta}$,

$$\int_{K_\varepsilon} g(s)\frac{x^s}{s}ds=O\Big(\frac{1}{\sqrt{\varepsilon}}\cdot\frac{x^{1+\varepsilon}}{1-\varepsilon}\cdot\pi\varepsilon\Big)=o(1),$$

as $\varepsilon\to 0$. Consequently, letting $\varepsilon\to 0$, we obtain from the property of $g(s)$,

$$\mathrm{I}_4=2\int_b^1 g(s)\frac{x^s}{s}ds,$$

by an extended Cauchy Theorem*.

But on the horizontal line from b to 1 we have

$$\frac{g(s)}{s}=\frac{c_0 i}{\sqrt{1-s}\ \ (1-(1-s))}+O(\sqrt{1-s})=\frac{c_0 i}{\sqrt{1-s}}+O(\sqrt{1-s}).$$

Accordingly

$$\begin{aligned}\mathrm{I}_4&=2c_0 i\int_b^1\frac{x^s}{\sqrt{1-s}}ds+O\left(\int_b^1 x^s\sqrt{1-s}\,ds\right)\\&=2c_0 i\left(\Gamma(\frac{1}{2})-\int_{(1-b)\log x}^{\infty} e^{-v}v^{\frac{1}{2}}dv\right)+O\Big(\frac{x}{(\log x)^{\frac{3}{2}}}\int_0^{\infty}e^{-v}v^{\frac{1}{2}}dv\Big)\\&=2c_0\sqrt{\pi}\,i\frac{x}{\sqrt{\log x}}+O\left(\frac{x}{(\log x)^{\frac{3}{2}}}\right)+O(\Big(\frac{x^b}{\sqrt{\log x}}\Big).\end{aligned}\tag{3}$$

Choosing $T=x^{a-b}$, it follows from (2) (3) that

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}g(s)\frac{x^s}{s}ds=\frac{c_0}{\sqrt{\pi}}\cdot\frac{x}{\sqrt{\log x}}+O\left(\frac{x}{(\log x)^{\frac{3}{2}}}\right).$$

Therefore we conclude that

$$D(x)=\frac{4b_0}{\pi}\cdot\frac{x}{\sqrt{\log x}}+O\left(\frac{x}{(\log x)^{\frac{3}{2}}}\right).\tag{4}$$

And in fact this proves Theorem 4 since we have

Lemma 4.** *A natural number n is the hypotenuse of a Pythagorean triangle if and only if n has at least one prime divisor* $\equiv 1 \pmod 4$.

Now

$$\widetilde{R}(x)=\sum_{\substack{p\le x\\ p\equiv 1(4)}}D\Big(\sqrt{\frac{x}{p}}\Big)=\frac{1}{2}\sum_{p\le x}D\Big(\sqrt{\frac{x}{p}}\Big)+\frac{1}{2}\sum_{p\le x}\chi_4(p)D\Big(\sqrt{\frac{x}{p}}\Big)-\frac{1}{2}D\Big(\sqrt{\frac{x}{2}}\Big)\tag{5}$$

Then we shall employ the following

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* See e. g. Ahlfors "*Complex Anlyasis*".

** See [D] *p.* 361.

Lemma 5.* *If $B(x)$ is a function of bounded variation in every finite interval for $x>0$, and $a(n)$ is defined for all natural number n so that*

(i) $a(n)=O(1)$, (ii) $\sum_{n\leqq x} a(n) = O(x\delta(x))$, $\delta(x)=o(1)$, $x\delta(x)\to\infty$

(iii) $B(x)=O(x^{\alpha})$, $0\leqq\alpha<1$,

then

$$\sum_{n\leqq x} a(n)\,B\left(\frac{x}{n}\right)=O(x\delta(x)^{1-\alpha}).$$

Since

$$\sum_{p\leqq x}\chi_4(p)=\pi^{(1)}(x)-\pi^{(3)}(x)=O(xe^{-\beta\sqrt{\log x}}),$$

where $\pi^{(1)}(x)$ and $\pi^{(3)}(x)$ denote the number of primes $\leqq x$ of the form $4k+1$ and $4k+3$ respectively, and $D(\sqrt{x})=O\left(\sqrt{\frac{x}{\log x}}\right)$ by (4),

$$\sum_{p\leqq x}\chi_4(p)\,D\left(\sqrt{\frac{x}{p}}\right)=O(xe^{-\frac{\beta}{2}\sqrt{\log x}}),$$

by taking $\delta(x)=e^{-\beta\sqrt{\log x}}$, $B(x)=D(\sqrt{x})$ and $\alpha=\frac{1}{2}$ in Lemma 5.

Hence we have

$$\widetilde{R}(x)=\frac{1}{2}\sum_{p\leqq x} D\left(\sqrt{\frac{x}{p}}\right)+O(xe^{-\frac{\beta}{2}\sqrt{\log x}}). \tag{6}$$

On the other hand the following lemma holds.

Lemma 6** *If $F(u,x)$ is a non-negative function of u and x for $2\leqq u\leqq x$ such that $F(u,x)/\log u$ is non-increasing when u increases from 2 to x, then we have as $x\to\infty$,*

$$\sum_{p\leqq x} F(p,\ x)=\{1+O(e^{-\gamma\sqrt{\log\omega}})\}\int_2^x\frac{F(u,x)}{\log u}du+O(\omega\cdot F(2,\ x)),$$

where γ is a certain positive numerical constant and $\omega=\omega(x)$ is any function of x such that $x\geqq\omega+2$ for all sufficiently large x and $\omega\to\infty$ as $x\to\infty$.

Since $F(u,\ x)=D\left(\sqrt{\frac{x}{u}}\right)$ satisfies the conditions of Lemma 6,

$$\begin{aligned}\sum_{p\leqq x} D\left(\sqrt{\frac{x}{p}}\right)&=\{1+O(e^{-\gamma'\sqrt{\log x}})\}\int_2^x\frac{D\left(\sqrt{\frac{x}{u}}\right)}{\log u}du+O\left(\frac{x^{\frac{5}{6}}}{\sqrt{\log x}}\right)\\&=\{x+O(xe^{-\gamma'\sqrt{\log x}})\}\int_2^{\frac{x}{2}}\frac{D(\sqrt{v})}{v^2}dv+\int_{\frac{x}{2}}^x\frac{du}{\log u}+O\left(\frac{x^{\frac{5}{6}}}{\sqrt{\log x}}\right),\end{aligned} \tag{7}$$

* This is a modification of Axer's theorem (*cf.* [A] *p.* 113).

** This is merely a variant of the lemma in [B] S. 203.

by taking $\omega = x^{\frac{1}{3}}$.

We shall now show that

$$\int_2^{\frac{x}{2}} \frac{D(\sqrt{v})}{\log \frac{x}{v}} \cdot \frac{dv}{v^2} \sim \beta_1 \frac{1}{\log x}, \tag{8}$$

as $x \to \infty$, i. e.,

$$\lim_{x\to\infty} (\log x) \int_2^{\frac{x}{2}} \frac{D(\sqrt{v})}{\log \frac{x}{v}} \cdot \frac{dv}{v^2} = \beta_1 < \infty,$$

exists.

Writing $f(x, v) = \dfrac{D(\sqrt{v}) \log x}{v^2 \log \frac{x}{v}}$, $f(\infty, v) = \dfrac{D(\sqrt{v})}{v^2}$, we have

$$\int_2^{\frac{x}{2}} f(x, v) dv = \left(\int_2^{\sqrt{x}} + \int_{\sqrt{x}}^{\frac{x}{2}} \right) f(x, v) dv = \int_2^{\sqrt{x}} f(x, v) dv + o\left(\frac{1}{\sqrt{x}}\right), \tag{9}$$

and

$$\left| \int_2^{\sqrt{x}} \{f(x, v) - f(\infty, v)\} dv \right| = \left| \int_2^{\sqrt{x}} \frac{D(\sqrt{v})}{v^2} \cdot \frac{\log v}{\log x - \log v} dv \right|$$

$$= O\left(\frac{1}{\log x} \int_2^{\sqrt{x}} \frac{\sqrt{\log v}}{v^{\frac{3}{2}}} dv \right) = O\left(\frac{1}{\log x}\right).$$

Consequently

$$\lim_{x\to\infty} \int_2^{\sqrt{x}} \{ f(x, v) - f(\infty, v) \} dv = 0.$$

But $\int_2^{\infty} \frac{D(\sqrt{v})}{v^2} dv$ clearly exists and is finite. Thus we conclude that

$$\lim_{x\to\infty} \int_2^{\sqrt{x}} f(x, v) dv = \int_2^{\infty} \frac{D(\sqrt{v})}{v^2} dv = \beta_1 < \infty,$$

which in view of (9) proves (8).

And we can prove that

$$\lim_{x\to\infty} (\log x)^2 \left\{ \int_2^{\frac{x}{2}} \frac{D(\sqrt{v})}{\log \frac{x}{v}} \cdot \frac{dv}{v^2} - \beta_1 \frac{1}{\log x} \right\} = \beta_2 < \infty,$$

exists, i. e.,

$$\int_2^{\frac{x}{2}} \frac{D(\sqrt{v})}{\log \frac{x}{v}} \cdot \frac{dv}{v^2} = \beta_1 \frac{1}{\log x} + \beta_2 \frac{1}{(\log x)^2} + o\left(\frac{1}{(\log x)^2} \right),$$

as $x \to \infty$, and furthermore fore any $m \geqq 1$,

$$\int_2^{\frac{x}{2}} \frac{D(\sqrt{v})}{\log \frac{x}{v}} \cdot \frac{dv}{v^2} = \beta_1 \frac{1}{\log x} + \beta_2 \frac{1}{(\log x)^2} + \cdots + o\left(\frac{1}{(\log x)^m}\right), \tag{10}$$

by repeating the similar process and argument as above.

Hence from (6) (7) (10) we have an asymptotic expansion

$$\widetilde{R}(x) \sim \sum_{n=1}^{\infty} \beta'_n \frac{x}{(\log x)^n},$$

in Poincaré's sense.

Thus

$$R(x) = \widetilde{R}(x) + \widetilde{R}\left(\frac{x}{2}\right) + O(\sqrt{x})$$

$$\sim \sum_{n=1}^{\infty} c_n \frac{x}{(\log x)^n}.$$

This proves Theorem 3.

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