On the Number of Integers Representable as the Sum of Two Squares

TAKESHI KANO

Department of Mathematics, Faculty of Science, Shinshu University
(Received Oct. 31, 1969)

§ 1. It has been proved by E. Landau [1] [A] that for $B(x)$, the number of integers not exceeding $x$ and representable as the sum of two squares of integers, we have

$$B(x) \sim b_0 \frac{x}{\sqrt{\log x}}, \quad b_0 = \frac{1}{\sqrt{2}} \prod \left(1 - r^{-2}\right)^{-\frac{1}{2}},$$

as $x \to \infty$, where $r$ runs through all the primes $\equiv 3 \pmod{4}$.

Recently some authors [2] [3] have investigated certain generalizations of the theorem of Landau, and in the present paper I consider some related problems from a different point of view.

My purpose is to prove the following theorems.

Theorem 1 If we denote by $C(x)$ the number of integers $\leq x$ with primitive representations, i.e., which are representable as the sum of two squares of coprime natural numbers, then we have as $x \to \infty$

$$C(x) \sim \frac{3}{8b_0} \frac{x}{\sqrt{\log x}},$$

where $b_0$ is the same constant as Landau's.

Theorem 2 Let $P(x)$ denote the number of integers $\leq x$ with precisely one* primitive representation. Then we have

$$P(x) \sim \frac{3}{4} \frac{x}{\log x}.$$

Theorem 3 If $R(x)$ denotes the number of integers $\leq x$ representable as the sum of two squares of integers in one and only one way, then we have an asymptotic expansion

$$R(x) \sim c_1 \frac{x}{\log x} + c_2 \frac{x}{\log^2 x} + \cdots.$$

* Two representations are regarded here as being equal when they differ only in the order of summands.
Theorem 4. If we denote by $D(x)$ the number of natural numbers $\leq x$ each of which is not the hypotenuse of any Pythagorean triangle, then

$$D(x) = \frac{4b_9}{\pi} \cdot \frac{x}{\sqrt{\log x}} + O \left( \frac{x}{(\log x)^{\frac{3}{2}}} \right).$$

§ 2. Theorem 1 is derived from the following

Lemma 1.* A natural number $n$ is the sum of two squares of coprime natural numbers if and only if $n$ is divisible neither by 4 nor by a natural number $\equiv 3 \pmod{4}$.

We know from Lemma 1 that $C(x)$ is the number of integers $\leq x$ such that each of them is the product of primes $\equiv 1 \pmod{4}$ only, or the product of 2 and primes $\equiv 1 \pmod{4}$.

So, if we put

$$c_n = \begin{cases} 1 & \text{if } n=\text{product of primes} \equiv 1 \pmod{4} \text{ only, or 1,} \\ 0 & \text{otherwise,} \end{cases}$$

then we have

$$C(x) = \sum_{n \leq x} c_n + \sum_{n \leq \frac{x}{2}} c_n - 2.$$

Next we define as usual a Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}, \quad s = \sigma + ti,$$

for $\sigma > 1$, which converges absolutely and almost uniformly** in the half-plane. Since $c_n$ is completely multiplicative, we have

$$f(s) = \prod_q (1 - q^{-s})^{-1},$$

where $q$ runs through all the primes $\equiv 1 \pmod{4}$.

But we know that

$$\zeta(s)L(s, \chi) = (1 - 2^{-s})^{-1} \prod_q (1 - q^{-s})^{-2} \prod_r (1 - r^{-2s})^{-1},$$

where $L(s, \chi)$ is the Dirichlet's $L$-function for the nonprincipal character $\pmod{4}$.

Hence we have for $\sigma > 1$,

$$f(s)^2 = (1 - 2^{-s}) \prod_r (1 - r^{-2s}) \zeta(s)L(s, \chi).$$

Now it is obvious that we can go farther in quite the same way as that of Landau using the known properties of $\zeta$ and $L$, and we omit the details except

---

* See [D] p. 362.

** This terminology is that of Saka-Zygmund “Analytic Functions”.
the final result

\[ \sum_{n \leq x} c_n \sim \frac{1}{4b_0} \frac{x}{\sqrt{\log x}}. \]

This gives at once Theorem 1.

§ 3. Theorem 2 is an immediate consequence of the following

Lemma 2. * A natural number \( n \) admits precisely one primitive representation if and only if \( n \) is one of the numbers

\[ 2, \ p^4, \ 2p^4, \]

where \( p \) is a prime \( \equiv 1 \pmod{4} \), and \( l \) a natural number.

Let \( \tilde{P}(x) \) denote the number of natural numbers \( \leq x \) of the form \( p^4 \) where \( p \) is any prime \( \equiv 1 \pmod{4} \) and \( l \) any natural number. Then we clearly have

\[ \tilde{P}(x) = \pi \left( x^{1/4}; 4, 1 \right) \]
\[ = \pi \left( x; 4, 1 \right) + \sum_{2 \leq l \equiv 1 \pmod{4}} \pi \left( x^{1/4}; 4, 1 \right) \]
\[ = \frac{1}{2} \cdot \frac{x}{\log x} + o \left( \frac{x}{\log x} \right), \]

from the prime number theorem in an arithmetical progression.

Therefore

\[ P(x) = 1 + \tilde{P}(x) + \tilde{P} \left( \frac{x}{2} \right) + O \left( \sqrt{x} \right) \]
\[ \sim \frac{3}{4} \cdot \frac{x}{\log x}. \]

§ 4. In this section we shall prove at a time Theorem 3 and Theorem 4. In fact Theorem 4 is a by-product of Theorem 3. We start from

Lemma 3. A natural number \( n \) admits precisely one representation as the sum of two squares of natural numbers if and only if \( n \) is one of the numbers

\[ d^2p, \ 2d^2p, \ 2d^2, \ d^2p^2, \]

where \( d \) is any natural number which has no prime divisor \( \equiv 1 \pmod{4} \), and \( p \) is any prime \( \equiv 1 \pmod{4} \).

Proof: Let \( n = a^2 + b^2, \ (a, b) = d \). Then \( \frac{n}{d^2} = a'^2 + b'^2, \ (a', b') = 1 \). Thus

\[ \text{from Lemma 1, we have} \]

\[ * \text{While the case of odd } n \text{ is only treated in } [D] \ p. \ 212, \text{ the word 'odd' is dropped there.} \]

\[ \text{See also } [4]. \]
\[ n = d^2 \prod_{i} p_i, \quad 2d^2 \prod_{i} p_i, \quad 2d^2, \]

where each \( p \) is a prime \( \equiv 1 \pmod{4} \). On the other hand it is known* that the number \( r(n) \) of the representations of \( n \) as the sum of two squares of integers is equal to \( 4 \prod_{i} (1 + \alpha_i) \) where the \( \alpha_i \)'s are the exponents on the primes \( \equiv 1 \pmod{4} \) in the standard factorization of \( n \). From this fact follows our desired result at once.

**Proof of Theorem 3**: Clearly it is sufficient to care for \( n = d^2 p \) only. Let \( \tilde{R}(x) \) denote the number of these \( n's \) not exceeding \( x \), and \( D(x) \) the number of all such \( d's \leq x \). Then we obtain

\[
\tilde{R}(x) = \sum_{\substack{p \leq x \\text{prime} \\equiv 1 \pmod{4} \leq p < x}} D\left( \sqrt{\frac{x}{p}} \right).
\]

If we introduce a Dirichlet series

\[
g(s) = \sum_{n=1}^{\infty} \frac{d_n}{n^s}
\]

for \( \sigma > 1 \) with

\[
d_n = \begin{cases} 
1 & n = d, \\
0 & \text{otherwise},
\end{cases}
\]

then it converges absolutely and almost uniformly in the half plane.

From the definition of \( d_n \) we derive

\[
g(s) = (1 - 2^{-s})^{-1} \prod_{r} (1 - r^{-s})^{-1},
\]

and hence

\[
g(s)^2 = (1 - 2^{-s})^{-2} \prod_{r} (1 - r^{-s})^{-2} \zeta(s) L(s, \chi)^{-1}.
\]

We see from the known properties of \( \zeta \) and \( L \) that \( g(s)^2 \) has a simple pole at \( s = 1 \), with residue \( c_1^2 = \left( \frac{4}{\sqrt{\pi}} b_0 \right)^2 \), and is otherwise holomorphic for \( \sigma > 1 \).

Furthermore there exists a positive constant \( c \) such that in the region \( \Gamma' \), i. e.,

\[
\sigma \geq 1 - \frac{c}{\log^{2}|t|} > \frac{1}{2}, \quad |t| \geq 3,
\]

\[
\sigma \geq 1 - \frac{c}{\log^{2}|3|} = \alpha_0 > \frac{1}{2}, \quad |t| \leq 3,
\]

* See e. g. [D] p. 432.
$g(s)^2$ does not vanish and in the region $\tilde{\Gamma}'$ formed from $\Gamma'$ by cutting along the real axis from $\sigma_0$ to 1, we have

$$g(s)^2 = O(\log^6 |t|), \quad (1)$$

for $|t| \geq 3$ and $s$ in $\tilde{\Gamma}'$.

On the other hand, according to Perron's formula, we have for non-integral $x$ and $a > 1$,

$$\sum_{n \leq x} d_n = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} g(s) \frac{x^s}{s} \, ds.$$

Let $T > 0$ be fixed and consider the contour described in the figure for

$$a = 1 + \frac{c}{\log^2 T},$$

$$b = 1 - \frac{c}{\log^2 T}.$$

Then by Cauchy's theorem

$$\frac{1}{2\pi i} \int_{a-iT}^{a+iT} g(s) \frac{x^s}{s} \, ds = \frac{1}{2\pi i} \left( \int_{a-iT}^{b+iT} g(s) \frac{x^s}{s} \, ds + \int_{b+iT}^{a+iT} g(s) \frac{x^s}{s} \, ds \right).$$

We can estimate these integrals with the help of (1) in the same way as in a proof* of the prime number theorem and obtain

$$I_1, I_7 = O \left( \frac{x^a}{T(a-1)} + \frac{x \log x}{T} \right),$$

$$I_2, I_8 = O \left( x^{\log^2 T} \right),$$

$$I_3, I_5 = O(x^k \log^4 T).$$

For $I_4$, we have from the property of $g(s)$,

$$I_4 = \int_{cut} g(s) \frac{x^s}{s} \, ds = 2 \int_{b+i}^{a+i} g(s) \frac{x^s}{s} \, ds + \int_{K_1} g(s) \frac{x^s}{s} \, ds,$$

where $K_1$ is a semi-circle of radius $\varepsilon$ drawn around the point 1.

* See e.g. [A] p. 65.
Since $g(s)$, in the neighbourhood of $s=1$, has the expansion
\[ g(s) = \frac{c_0}{\sqrt{s-1}} + a_1(s-1)^{1/2} + a_2(s-1)^{3/2} + \ldots, \]
where $\sqrt{s-1} > 0$ for $s > 1$, we get by putting $s = 1 + \varepsilon e^{i\theta}$,
\[ \int_{K_s} g(s) \frac{x^s}{s} ds = O\left( \frac{1}{\sqrt{\varepsilon}} \cdot \frac{x^{1+\varepsilon}}{1-\varepsilon} \cdot \pi \varepsilon \right) = o(1), \]
as $\varepsilon \to 0$. Consequently, letting $\varepsilon \to 0$, we obtain from the property of $g(s)$,
\[ I_4 = 2\int_{b}^{1} g(s) \frac{x^s}{s} ds, \]
by an extended Cauchy Theorem\textsuperscript{*}.

But on the horizontal line from $b$ to 1 we have
\[ \frac{g(s)}{s} = \frac{c_0 i}{\sqrt{1-s}} \frac{1}{1-(1-s)} + O\left( \sqrt{1-s} \right) = \frac{c_0 i}{\sqrt{1-s}} + O\left( \sqrt{1-s} \right). \]
Accordingly
\[ I_4 = 2c_0 i \int_{b}^{1} \frac{x^t}{\sqrt{1-s}} ds + O \left( \int_{b}^{1} x^{1/2} \sqrt{1-s} ds \right) \]
\[ = 2c_0 i \int_{b}^{1} \frac{x^t}{\sqrt{1-s}} ds \]
\[ = 2c_0 i \left( \frac{1}{2} \int_{(1-b)\log x}^{\infty} e^{-x} v^{1/2} dv \right) + O \left( \frac{x}{\log x} \right) \int_{0}^{\infty} e^{-x} v^{1/2} dv \]
\[ = 2c_0 \sqrt{\pi} i \frac{x}{\log x} + O \left( \frac{x}{\log x} \right) + O \left( \frac{x}{\sqrt{\log x}} \right). \]
Choosing $T = x^{a-b}$, it follows from (2) (3) that
\[ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} g(s) \frac{x^s}{s} ds = c_0 \frac{x}{\sqrt{\log x}} + O \left( \frac{x}{\log x} \right). \]
Therefore we conclude that
\[ D(x) = \frac{4b_0}{\pi} \frac{x}{\sqrt{\log x}} + O \left( \frac{x}{\log x} \right). \]

And in fact this proves Theorem 4 since we have
\textbf{Lemma 4.} \textsuperscript{**} A natural number $n$ is the hypotenuse of a Pythagorean triangle if and only if $n$ has at least one prime divisor $\equiv 1 \pmod{4}$.

Now
\[ \tilde{R}(x) = \sum_{p \leq x} D\left( \sqrt{\frac{x}{p}} \right) + \sum_{p \leq x} \chi_p \frac{x}{p} D\left( \sqrt{\frac{x}{p}} \right) + \sum_{p \leq x} \frac{\chi_p(p)}{D\left( \sqrt{\frac{x}{p}} \right)} - \frac{1}{2} D\left( \sqrt{\frac{x}{2}} \right) \]

Then we shall employ the following

\textsuperscript{*} See e. g. Ahlfors "Complex Analysis".

\textsuperscript{**} See [D] p. 361.
Lemma 5. * If \( B(x) \) is a function of bounded variation in every finite interval for \( x > 0 \), and \( a(n) \) is defined for all natural number \( n \) so that

(i) \( a(n) = O(1) \),
(ii) \( \sum_{n \leq x} a(n) = O(x) \), \( a(x) = O(1), \) \( x \to \infty \)
(iii) \( B(x) = O(x^\alpha), \) \( 0 \leq \alpha < 1, \)

then

\[
\sum_{n \leq x} a(n) B\left( \frac{x}{n} \right) = O(x^{1+\alpha}).
\]

Since

\[
\sum_{p \leq x} \chi(x) = \pi(x) - \pi(x) = O(x e^{-\beta \sqrt{\log x}}),
\]

where \( \pi(x) \) and \( \pi(x) \) denote the number of primes \( \leq x \) of the form \( 4k+1 \) and \( 4k+3 \) respectively, and \( D(\sqrt{x}) = O\left( \sqrt{x} \right) \) by (4),

\[
\sum_{p \leq x} \chi(x) D\left( \frac{x}{p} \right) = O(x e^{-\beta \sqrt{\log x}}),
\]

by taking \( \delta(x) = e^{-\beta \sqrt{\log x}} \), \( B(x) = D(\sqrt{x}) \) and \( \alpha = \frac{1}{2} \) in Lemma 5.

Hence we have

\[
\tilde{R}(x) = \frac{1}{2} \sum_{p \leq x} D\left( \frac{x}{p} \right) + O(x e^{-\frac{\beta^2}{2} \sqrt{\log x}}). \tag{6}
\]

On the other hand the following lemma holds.

Lemma 6** If \( F(u, x) \) is a non-negative function of \( u \) and \( x \) for \( 2 \leq u \leq x \) such that \( F(u, x) / \log u \) is non-increasing when \( u \) increases from 2 to \( x \), then we have as \( x \to \infty \),

\[
\sum_{p \leq x} F(p, x) = \left( 1 + O(e^{-\gamma \sqrt{\log x}}) \right) \int_{\frac{x}{2}}^{x} \frac{F(u, x)}{\log u} du + O(\omega \cdot F(2, x)),
\]

where \( \gamma \) is a certain positive numerical constant and \( \omega = o(x) \) is any function of \( x \) such that \( x \geq \omega + 2 \) for all sufficiently large \( x \) and \( \omega \to \infty \) as \( x \to \infty \).

Since \( F(u, x) = D\left( \sqrt{\frac{x}{u}} \right) \) satisfies the conditions of Lemma 6,

\[
\sum_{p \leq x} D\left( \frac{x}{p} \right) = \left( 1 + O(e^{-\gamma \sqrt{\log x}}) \right) \int_{\frac{x}{2}}^{x} \frac{D\left( \sqrt{\frac{x}{u}} \right)}{\log u} du + O\left( \frac{x^5}{\sqrt{\log x}} \right)
\]

\[
= \left( x + O(x e^{-\gamma \sqrt{\log x}}) \right) \int_{\frac{x}{2}}^{x} \frac{D\left( \sqrt{\frac{x}{u}} \right)}{\sqrt{u^2}} du + \int_{\frac{x}{2}}^{x} \frac{du}{\log u} + O\left( \frac{x^5}{\sqrt{\log x}} \right). \tag{7}
\]

* This is a modification of Auer's theorem (cf. [A], p. 113).
** This is merely a variant of the lemma in [B], S. 203.
by taking \( \omega = x^{\frac{1}{2}} \).

We shall now show that
\[
\int_{\frac{1}{2}}^{x} D\left(\sqrt{v}\right) \frac{dv}{v^2 \log \frac{x}{v}} \sim \beta_1 \frac{1}{\log x},
\]
\[\tag{8}\]
as \( x \to \infty \), i.e.,
\[
\lim_{x \to \infty} (\log x) \int_{\frac{1}{2}}^{x} D\left(\sqrt{v}\right) \frac{dv}{v^2 \log \frac{x}{v}} = \beta_1 < \infty,
\]
exists.

Writing \( f(x, v) = \frac{D(\sqrt{v}) \log x}{v^2 \log \frac{x}{v}} \), \( f(\infty, v) = \frac{D(\sqrt{v})}{v^2} \), we have
\[
\int_{\frac{1}{2}}^{x} f(x, v) dv = \left(\int_{\frac{1}{2}}^{x} + \int_{\frac{1}{2}}^{x} \right) f(x, v) dv = \int_{\frac{1}{2}}^{x} f(x, v) dv + o\left(\frac{1}{\sqrt{x}}\right),
\]
and
\[
\left| \int_{\frac{1}{2}}^{x} \left\{ f(x, v) - f(\infty, v) \right\} dv \right| = \left| \int_{\frac{1}{2}}^{x} D\left(\sqrt{v}\right) \frac{dv}{v^2} \right| \frac{\log v}{\log x - \log v} dv
\]
\[= O\left(\frac{1}{\log x} \int_{\frac{1}{2}}^{x} \frac{D(\sqrt{v}) dv}{v^2} \right) = O\left(\frac{1}{\log x}\right).
\]
Consequently
\[
\lim_{x \to \infty} \int_{\frac{1}{2}}^{x} \left\{ f(x, v) - f(\infty, v) \right\} dv = 0.
\]

But \( \int_{\frac{1}{2}}^{x} \frac{D(\sqrt{v}) dv}{v^2} \) clearly exists and is finite. Thus we conclude that
\[
\lim_{x \to \infty} \int_{\frac{1}{2}}^{x} f(x, v) dv = \int_{\frac{1}{2}}^{x} D\left(\sqrt{v}\right) dv = \beta_1 < \infty,
\]
which in view of (9) proves (8).

And we can prove that
\[
\lim (\log x) \left\{ \int_{\frac{1}{2}}^{x} D\left(\sqrt{v}\right) \frac{dv}{v^2 \log \frac{x}{v}} - \beta_1 \frac{1}{\log x} \right\} = \beta_2 < \infty,
\]
exists, i.e.,
\[
\int_{\frac{1}{2}}^{x} D\left(\sqrt{v}\right) \frac{dv}{v^2 \log \frac{x}{v}} = \beta_1 \frac{1}{\log x} + \beta_2 \frac{1}{(\log x)^2} + o\left(\frac{1}{(\log x)^2}\right),
\]
as \( x \to \infty \), and furthermore for any \( m \geq 1 \),
by repeating the similar process and argument as above.

Hence from (6) (7) (10) we have an asymptotic expansion

$$
\bar{R}(x) \sim \sum_{n=1}^{\infty} \beta_n \frac{x}{(\log x)^n},
$$

in Poincaré's sense.

Thus

$$
R(x) = \bar{R}(x) + \bar{R}\left(\frac{x}{2}\right) + O(\sqrt{x})
$$

$$
\sim \sum_{n=1}^{\infty} c_n \frac{x}{(\log x)^n}.
$$

This proves Theorem 3.

REFERENCES


