JOUR. FAC. SCI., SHINSHU UNIV. Vol. 2, pp. 67–69, Dec. 1969

A Note on Perfect Rings and Semi-perfect Rings

KAORU MOTOSE

Depertment of Mathematics, Faculty of Science,
Shinshu University
(Received Oct. 31, 1969)

In his paper [1], H. Bass gave characterizations of perfect rings and semi-perfect rings. In this paper, we shall give another characterizations of perfect rings and some properties of extension rings over perfect and semi-perfect rings.

Throughout our study, we use the following conventions: Let R be a ring with 1 and J the radical[1)] of R. An R-module means (unital) left R-module. For a set $\Lambda(\neq\phi)$, by $(R)_\Lambda$ and $R^{(\Lambda)}$, we denote the ring of all row finite matrices (x_{ij}) $(i, j\in\Lambda)$ over R and the direct sum of $\sharp\Lambda$[2)]-copies of left R-module R, thus $(R)_\Lambda$ can be regarded as the ring of all linear transformations in $R^{(\Lambda)}$.

1. We shall first prove the following

Theorem 1. *The following conditions are equivalent.*

(1) *J is left T-nilpotent.*

(2) *For every R-module M, JM is small in M.*

(2′) *For every free R-module M, JM is small in M.*

(3) *For every set Λ, the radical of $(R)_\Lambda$ is $(J)_\Lambda$.*

(3′) *For the set Z_+ of natural numbers, the radical of $(R)_{Z+}$ is $(J)_{Z+}$.*

Proof. (1)⇒(2) is due to Bass [1, pp. 473–474.]. Therefore, it suffices that we prove (2′)⇒(3) and (3′)⇒(1).

(2′)⇒(3): Let X be an element of $(J)_\Lambda$. Then $R^{(\Lambda)}=R^{(\Lambda)}(I-X)+J^{(\Lambda)}=R^{(\Lambda)}(I-X)+JR^{(\Lambda)}$.[3)] Hence $R^{(\Lambda)}=R^{(\Lambda)}(I-X)$ and for every $\lambda\in\Lambda$, there exist vectors $(a_{\lambda\nu})$ of $R^{(\Lambda)}$ such that $E_\lambda=(a_{\lambda\nu})(I-X)$, where E_λ represents the vector with element 1 in the λ-position and O's elsewhere. Therefore, $I=Y(I-X)$ where Y is an element of $(R)_\Lambda$ such that the λ-row of Y coincides with the vector $(a_{\lambda\nu})$. Hence X is left quasi-regular. Accordingly, $(J)_\Lambda$ is contained in the radical of $(R)_\Lambda$. By [6, Th. 1] or [8, Th. 1], the assertion is clear.

(3′)⇒(1) (Cf. [6, Th. 5]): Let $\{a_i\}_{i=1,2,\ldots\ldots}$ be any sequence of elements in J.

1) Throughout the present paper, the radical means the Jacobson radical.
2) $\sharp\Lambda$ means the cardinal number of a set Λ.
3) I is the identity of $(R)_\Lambda$.

By the assumption, the matrix $\begin{pmatrix} 1 & -a_1 & & 0 \\ & 1 & -a_2 & \\ & & 1 & \ddots \\ 0 & & & \ddots \end{pmatrix}$ is regular in $(R)_A$.

Since it's inverse element is a row-finite matrix, there exists a natural number s such that $a_1 a_2 \cdots a_s = 0$. Hence, J is left T-nilpotent.

By virture of Th. 1, we have an following characterizations of perfect rings.

Corollary 1. *Let R be a semi-primary ring.*[4] *Then the following conditions are equivalent.*

(1) *R is left perfect.*

(2) *For every R-module M, JM is small in M.*

(3) *For every set A, the radical of $(R)_A$ is $(J)_A$.*

By Th. 1, we give an alternative proof of [7, Th. 1]. In the following Cor. 2, we do not assume that R has the identity.

Corollary 2. *$(R)_A$ has the radical $(J)_A$ if and only if J is left T-nilpotent.*

Proof. Let Z be the ring of integers. Then we can constract a ring $R' = R+Z$ such that $R \cap Z = 0$ and the identity of Z is the identity of R'. If we note that J is the radical of R', [8, Th. 1] implies that the radical of $(R)_A$ is that of $(R')_A$. the rest is clear.

2. In this section, we shall restrict our attention to the case that J is left T-nilpotent.

Theorem 2.[5] *Let S be an extension ring of R with the same identity such that JS is an ideal of S. Then JS is contained in the radical of S. In paticular, if R is left perfect and S is finitely generated as an R-module, then S is left perfect.*

Proof. Let x be an element of JS. Then $S = S(1-x) + JS$. By Th. 1, $S = S(1-x)$ and hence x is left quasi-regular. In the second statement, since S/JS is left Artinian, there exists a natural number k such that $\mathfrak{J}(S)^k \subseteq JS$.[6] Let M be an S-module and N a submodule of M such that $M = N + \mathfrak{J}(S)M$. Then $M = N + \mathfrak{J}(S)^k M = N + JSM = N + JM$. Since R is left perfect, $M = N$. Hence, by Cor. 1, S is left perfect.

Combining [3, Th. 1.7.4] with Th. 2, we can see the first part of the following

Corollary 3. (1) *The radical of the polynomial ring $R[x]$ is $J[x]$.*

(2) *Let R be a left perfect ring and G a finite group. Then the group ring RG is a left perfect ring.*

3. Concerning Cor. 3 (2), we establish sufficient conditions for semi-perfectness of a group ring RG. In the first part of the follwing theorem, we assume that

4) Cf. [3, pp. 56].
5) Cf. [5, Th. 46.2] and [9, Prop. 3.3 (b)].
6) $\mathfrak{J}(S)$ means the radical of a ring S.

R is semi-primary, $\bigcap_{n=1}^{\infty} J^n = 0$ and R^* is the completion of R with respect to the metric d, where $d(x, y) = \inf_{x-y \in J^n} 2^{-n}$ $(x, y \in R)$.

Theorem 3. *Let G be a finite group. Then we can obtain the following statements.*

(1) *R^*G is semi-perfect.*

(2) *RG is semi-perfect if G is a p-group, R is a semi-perfect ring and the characteristic of R/J is p.*

Proof. (1) By the same method of [2, Lemma 77.4], idempotents of R^*G modulo $\mathfrak{J}(R^*G)$ can be lifted. Then since R is semi-primary, R^*G is semi-primary. Hence the assertion is clear.

(2) Let $e = \sum_{g \in G} \alpha_g g (\alpha_g \in R)$ be an idempotent element of RG modulo $\mathfrak{J}(RG)$. By [4, Cor. 1], $\sum_{g \in G} \alpha_g$ is an idempotent element of R modulo J. Hence, there exists an idempotent element f of R such that $f - \sum_{g \in G} \alpha_g \in J$. Then, by [4, Cor. 1], $f - e$ is contained in $\mathfrak{J}(RG)$. Since RG is semi-primary, the assertion follows.

References

(1) H. BASS; Finitistic dimension and a homological generalization of semi-primary rings, Trans. Amer. Math. Soc. 95 (1960), 466–488.

(2) C. W. CURTIS and I. REINER; *Representation theory of finite groups and associative algebras*, Interscience, 1962.

(3) N. JACOBSON; *Structure of rings*, Providence, 1956.

(4) K. MOTOSE; On group rings over semi-primary rings, Math. J. Okayama Univ. 14 (1969), 23–26.

(5) T. NAKAYAMA and G. AZUMAYA; *Algebra II (Theory of rings)*, Tokyo, 1954 (in Japanese).

(6) E. M. PATTERSON; On the radicals of certain rings of infinite matrices, Proc. Royal Soc. Edinburgh 65 (1961), 263–271.

(7) ————; On the radicals of rings of row-finite matrices, Proc. Royal Soc. Edinburgh 66 (1962), 42–46.

(8) A. D. SANDS; Primitive rings of infinite matrices, Proc. Edinburgh Math. Soc. 14 (1964–65), 47–53.

(9) H. TOMINAGA; Some results on normal bases, Math. J. Okayama Univ. 13 (1968), 111–118.