A Note on Perfect Rings and Semi-perfect Rings

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In his paper [1], H. Bass gave characterizations of perfect rings and semi-perfect rings. In this paper, we shall give another characterizations of perfect rings and some properties of extension rings over perfect and semi-perfect rings.

Throughout our study, we use the following conventions: Let \( R \) be a ring with 1 and \( J \) the radical\(^1\) of \( R \). An \( R \)-module means unital left \( R \)-module. For a set \( A \neq \emptyset \), by \( (R)_A \) and \( R(A) \), we denote the ring of all row finite matrices \( (x_{ij}) \ (i, j \in A) \) over \( R \) and the direct sum of \( \#A \)-copies of left \( R \)-module \( R \), thus \( (R)_A \) can be regarded as the ring of all linear transformations in \( R(A) \).

1. We shall first prove the following

**Theorem 1.** The following conditions are equivalent.

1. \( J \) is left \( T \)-nilpotent.
2. For every \( R \)-module \( M \), \( JM \) is small in \( M \).
2'. For every free \( R \)-module \( M \), \( JM \) is small in \( M \).
3. For every set \( A \), the radical of \( (R)_A \) is \( (J)_A \).
3'. For the set \( Z_+ \) of natural numbers, the radical of \( (R)_{Z_+} \) is \( (J)_{Z_+} \).

**Proof.** (1)\(\Rightarrow\)(2) is due to Bass [1, pp. 473–474]. Therefore, it suffices that we prove (2')\(\Rightarrow\)(3) and (3')\(\Rightarrow\)(1).

(2')\(\Rightarrow\)(3) : Let \( X \) be an element of \( (J)_A \). Then \( R(A)=R(A)(I-X)+J(R(A)(I-X)+JR(A)) \). Hence \( R(A)=R(A)(I-X) \) and for every \( \lambda \in A \), there exist vectors \( (a_{\lambda i}) \) of \( R(A) \) such that \( E_{\lambda}=E_{\lambda}(I-X) \), where \( E_{\lambda} \) represents the vector with element 1 in the \( \lambda \)-position and 0's elsewhere. Therefore, \( I=Y(I-X) \) where \( Y \) is an element of \( (R)_A \) such that the \( \lambda \)-row of \( Y \) coincides with the vector \( (a_{\lambda i}) \). Hence \( X \) is left quasi-regular. Accordingly, \( (J)_A \) is contained in the radical of \( (R)_A \). By [6, Th. 1] or [8, Th. 1], the assertion is clear.

(3')\(\Rightarrow\)(1) (Cf. [6, Th. 5]): Let \( \{a_i\}_{i=1}^{\infty} \) be any sequence of elements in \( J \).

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1) Throughout the present paper, the radical means the Jacobson radical.
2) \( \#A \) means the cardinal number of a set \( A \).
3) \( I \) is the identity of \( (R)_A \).
By the assumption, the matrix \[
\begin{pmatrix}
1 & -a_1 & 0 \\
1 & -a_2 & 0 \\
0 & 1 & 1
\end{pmatrix}
\]
is regular in \((R)_A\).

Since its inverse element is a row-finite matrix, there exists a natural number
\(s\) such that \(a_1, a_2, \ldots, a_s = 0\). Hence, \(J\) is left \(T\)-nilpotent.

By virtue of Th. 1, we have an following characterizations of perfect rings.

**Corollary 1.** Let \(R\) be a semi-primary ring. Then the following conditions are equivalent.

1. \(R\) is left perfect.
2. For every \(R\)-module \(M\), \([M]M\) is small in \(M\).
3. For every set \(A\), the radical of \((R)_A\) is \((J)_A\).

By Th. 1, we give an alternative proof of [7, Th. 1]. In the following Cor. 2, we do not assume that \(R\) has the identity.

**Corollary 2.** \((R)_A\) has the radical \((J)_A\) if and only if \(J\) is left \(T\)-nilpotent.

**Proof.** Let \(Z\) be the ring of integers. Then we can construct a ring \(R' = R + Z\) such that \(R_nZ = 0\) and the identity of \(Z\) is the identity of \(R'\). If we note that \(J\) is the radical of \(R'\), [8, Th. 1] implies that the radical of \((R)_A\) is that of \((R')_A\). The rest is clear.

2. In this section, we shall restrict our attention to the case that \(J\) is left \(T\)-nilpotent.

**Theorem 2.** Let \(S\) be an extension ring of \(R\) with the same identity such that \(JS\) is an ideal of \(S\). Then \(JS\) is contained in the radical of \(S\). In particular, if \(R\) is left perfect and \(S\) is finitely generated as an \(R\)-module, then \(S\) is left perfect.

**Proof.** Let \(x\) be an element of \(JS\). Then \(S = S(1 - x) + JS\). By Th. 1, \(S = S(1 - x)\) and hence \(x\) is left quasi-regular. In the second statement, since \(S/JS\) is left Artinian, there exists a natural number \(k\) such that \(3(S)^k \subseteq JS\).

Let \(M\) be an \(S\)-module and \(N\) a submodule of \(M\) such that \(M = N + 3(S)M\). Then \(M = N + 3(S)M = N + JSN = N + JM\). Since \(R\) is left perfect, \(M = N\). Hence, by Cor. 1, \(S\) is left perfect.

Combining [3, Th. 1.7.4] with Th. 2, we can see the first part of the following.

**Corollary 3.** (1) The radical of the polynomial ring \(R[x]\) is \(J[x]\).

(2) Let \(R\) be a left perfect ring and \(G\) a finite group. Then the group ring \(RG\) is a left perfect ring.

3. Concerning Cor. 3 (2), we establish sufficient conditions for semi-perfectness of a group ring \(RG\). In the first part of the following theorem, we assume that

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4) Cf. [3, pp. 56].
5) Cf. [5, Th. 46.2] and [9, Prop. 3.3(b)].
6) \(3(S)\) means the radical of a ring \(S\).
$R$ is semi-primary, $\bigcap_{n=1}^{\infty} J^n = 0$ and $R^*$ is the completion of $R$ with respect to the metric $d$, where $d(x, y) = \inf_{x, y \in J^n} \|x - y\|$. 

**Theorem 3.** Let $G$ be a finite group. Then we can obtain the following statements.

1. $R^*G$ is semi-perfect.
2. $RG$ is semi-perfect if $G$ is a $p$-group, $R$ is a semi-primary ring and the characteristic of $R/I$ is $p$.

**Proof.**

1. By the same method of [2, Lemma 77.4], idempotents of $R^*G$ modulo $\mathfrak{Z}(R^*G)$ can be lifted. Then since $R$ is semi-primary, $R^*G$ is semi-primary. Hence the assertion is clear.

2. Let $e = \sum_{\alpha G \in G} \alpha G \in \mathfrak{Z}(RG)$ be an idempotent element of $RG$ modulo $\mathfrak{Z}(RG)$. By [4, Cor. 1], $\sum_{\alpha G \in G} \alpha G$ is an idempotent element of $R$ modulo $J$. Hence, there exists an idempotent element $f$ of $R$ such that $f - \sum_{\alpha G \in G} \alpha G \in J$. Then, by [4, Cor. 1], $f - e$ is contained in $\mathfrak{Z}(RG)$. Since $RG$ is semi-primary, the assertion follows.

**References**

1. H. Bass; Finitistic dimension and a homological generalization of semi-primary rings, Trans. Amer. Math. Soc. 95 (1960), 466-488.
3. N. Jacobson; Structure of rings, Providence, 1956.